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On the order of prime powers dividing $\binom{2n}{n}$

by

JÜRGEN W. SANDER(1)

Universität Hannover
Hannover, Germany

Contents

1. Introduction

A number of conjectures originating from the 1970's and before is related to the prime decomposition of middle binomial coefficients, i.e. binomial coefficients of the form

$$\binom{2n}{n}.$$

Most of these problems were raised by Paul Erdős and some of his co-authors (see [3], [4], [5], and [6, Problems B31, B33]). Apart from being interesting in itself, the prime factorization of middle binomial coefficients has an important application in elementary number theory, namely the distribution of primes.

Chebyshev was the first mathematician who (around 1850) could prove that the prime counting function $\pi(x)$ satisfies

$$\pi(x) \asymp \frac{x}{\log x},$$

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which means that $\pi(x)$ is bounded from above and below by the function on the right. This can be obtained by studying middle binomial coefficients which—for two reasons—are an appropriate tool: They have nice multiplicative properties, most of all many distinct prime factors, and a simple additive property, that is, their recursion formula.

With this in mind, Erdős and others investigated more closely the prime decomposition of middle binomial coefficients. Despite the fact that these are "almost" squarefree, i.e. they have only very few prime factors of order at least 2, it was conjectured that for any integer a and sufficiently large $n>n_0(a)$, there is always a prime p such that

$$p^a \,\Big|\, \binom{2n}{n}. \tag{1}$$

This conjecture was settled for the case $a=2$ by Sárközy [13] in 1985. Recently, the general conjecture was confirmed by the author [11]. In fact, much more was shown:

— One can choose $p>p_0$ for any p_0 in (1), if $n>n_0(a,p_0)$ [11].

— These results extend to binomial coefficients of the form

$$\binom{2n\pm d}{n}, \tag{2}$$

if d is "not too large" compared with n [11].

— Let $s_a(n,d)$ denote the largest ath power dividing the binomial coefficient in (2). Then for small d, we have asymptotically

$$\log s_a(n,d) \sim C(a) n^{1/a}$$

with an explicitly given constant $C(a)$ [12].

An open problem, which has not been dealt with so far, pertains to the function

$$E(n) := \max\left\{ J : p^J \,\Big|\, \binom{2n}{n} \text{ for some prime } p \right\} = \max\left\{ e\left(\binom{2n}{n}; p\right) : p \in \mathbf{P} \right\},$$

where

$$e(n;p) := \max\{e : p^e | n\}, \tag{5}$$

i.e. $E(n)$ is the largest exponent in the prime factorization of $\binom{2n}{n}$. Clearly, the conjecture mentioned above is equivalent to

$$E(n) \to \infty, \quad n \to \infty,$$

which follows from the result in [11]. In this paper, we shall present a lower bound for the function $E(n)$, which gives the first answer to a question of Erdős (see [6, Problem B31]). It shows that $E(n)$ is at least of order $(\log n)^\delta$ for some positive constant δ. More precisely, we obtain the following

THEOREM. *For sufficiently large n, we have*

$$E(n) \gg \left(\frac{\log n}{(\log\log n)^3}\right)^{1/10}.$$

We like to mention that one can easily show

$$E(n) \ll \log n, \tag{4}$$

and

$$E(n) \gg \log n \quad \text{for almost all } n. \tag{5}$$

Concerning the true size of $E(n)$, we make the

CONJECTURE. *For $n\to\infty$, we have*

$$E(n) \asymp \log n.$$

In §2, we shall prove the upper bound (4) of the conjecture, as well as (5). Moreover, some heuristic argument for the lower bound in our conjecture will be given. §§3–7 will be devoted to the proof of the theorem. A major effort is made to keep all the results along the way explicit with respect to certain parameters. We did, however, disregard any constants that are not important for our final result.

The following notation will be used throughout the paper. As widely accepted, $\mathbf{N}$, $\mathbf{Z}$, and $\mathbf{C}$ designate the sets of natural numbers $1, 2, 3, \ldots$, integers, and complex numbers, respectively. By $\mathbf{P}$ we denote the set of primes $2, 3, 5, 7, 11, \ldots$, while the letter p with or without subscript will always be restricted to be an element of $\mathbf{P}$. For real x, we define $e(x) = \exp(2\pi i x)$. All the explicit and implicit constants (as in $O(\,\cdot\,)$ or, equivalently, $\ll$) are absolute and positive unless otherwise indicated. We adopt the convention that the constants c and C, which always are assumed to be absolute and positive, may change their values within inequalities. This enables us to write

$$x^{1-c} \log x \ll x^{1-c},$$

for instance. While c is supposed to denote small constants, C will be used for large ones.

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2. A heuristic consideration

The upper bound (4) of our conjecture is simply proved by applying the familiar decomposition of factorials, namely

$$e\left(\binom{2n}{n};p\right)=\sum_{a>0}\left(\left[\frac{2n}{p^a}\right]-2\left[\frac{n}{p^a}\right]\right)=\sum_{1\leqslant a\leqslant \log 2n/\log p}\left(\left[\frac{2n}{p^a}\right]-2\left[\frac{n}{p^a}\right]\right)$$
$$\leqslant \sum_{1\leqslant a\leqslant \log 2n/\log p} 1 \leqslant \frac{\log 2n}{\log p}\leqslant\frac{\log 2n}{\log 2}.$$

No improvement over (4) is known (see [6, Problem B31]). In fact, it is quite easy to show that we have $\log n$ also as a lower bound, at least for almost all n, i.e. (5) holds.

In order to see this, we first recollect that for almost all $n<N$ the sum $S_2(n)$ of the digits of n in binary expansion satisfies $S_2(n)\approx\frac{1}{2}\log_2 N$; this is obtainable by a straightforward counting argument. More precisely, for any $\varepsilon>0$,

$$\operatorname{card}\{n<N:(\tfrac{1}{2}-\varepsilon)\log_2 N<S_2(n)<(\tfrac{1}{2}+\varepsilon)\log_2 N\}=(1+o(1))N. \tag{6}$$

Using this and Lemma 12, we obtain

$$\operatorname{card}\left\{n<N:e\left(\binom{2n}{n};2\right)\leqslant\tfrac{1}{3}\log N\right\}=\operatorname{card}\{n<N:S_2(n)\leqslant\tfrac{1}{3}\log N\}=o(N).$$

This means that for almost all $n<N$,

$$E(n)\geqslant e\left(\binom{2n}{n};2\right)>\tfrac{1}{3}\log N\geqslant\tfrac{1}{3}\log n,$$

which proves (5).

In order to tackle our conjecture, one is tempted to make use of the bounds given in [12]. For this reason, let $s_J(n)$ be the largest Jth power dividing $\binom{2n}{n}$. The main result in [12] is that for any $\varepsilon>0$ and sufficiently large $n\geqslant n_0(J,\varepsilon)$,

$$\exp((C(J)-\varepsilon)n^{1/J})\leqslant s_J(n)\leqslant\exp((C(J)+\varepsilon)n^{1/J}), \tag{7}$$

where

$$C(J)=2^{1/J}\left(\tfrac{1}{2}\right)^{J-1}\sum_{k=1}^{\infty}\left(\left(\frac{1}{2k-1}\right)^{1/J}-\left(\frac{1}{2k}\right)^{1/J}\right).$$

Now

$$C(J)\leqslant\left(\tfrac{1}{2}\right)^{J-2}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)=\left(\tfrac{1}{2}\right)^{J-2}\log 2\leqslant\left(\tfrac{1}{2}\right)^{J-2}.$$

We set

$$\varepsilon := \left(\tfrac{1}{2}\right)^{J-2}.$$

By (7), it follows for sufficiently large n that

$$s_J(n) \leqslant \exp\left(\left(\tfrac{1}{2}\right)^{J-1} n^{1/J}\right). \tag{8}$$

For

$$\log n < \tfrac{1}{2}(\log 2)J^2,$$

we have

$$\log n < J(J-1)\log 2 + J \log\log 2,$$

which is equivalent to

$$\exp\left(\left(\tfrac{1}{2}\right)^{J-1} n^{1/J}\right) < 2.$$

By (8), we get $s_J(n)<2$, i.e. $s_J(n)=1$. Therefore, we have shown that for sufficiently large n

$$s_J(n) > 1 \;\Rightarrow\; \log n \geqslant \tfrac{1}{2}(\log 2)J^2. \tag{9}$$

On the other hand,

$$C(J) > 2^{1/J}\left(\tfrac{1}{2}\right)^{J-1}\left(1-\left(\tfrac{1}{2}\right)^{1/J}\right) = \left(\tfrac{1}{2}\right)^{J-1}(2^{1/J}-1). \tag{10}$$

We have for all $x \geqslant 0$

$$f(x) := 2^x - x\log 2 - 1 \geqslant 0,$$

since $f(0)=0$ and $f'(x) \geqslant 0$ for $x \geqslant 0$. Hence (10) implies

$$C(J) > \left(\tfrac{1}{2}\right)^{J-1}\frac{1}{J}\log 2 > \left(\tfrac{1}{2}\right)^{2J-1}.$$

With

$$\varepsilon := \left(\tfrac{1}{2}\right)^{2J},$$

we obtain by (7) for sufficiently large n

$$s_J(n) \geqslant \exp\left(\left(\tfrac{1}{2}\right)^{2J} n^{1/J}\right). \tag{11}$$

The inequality

$$\log n \geqslant 2(\log 2)^2 J^2$$

is equivalent to

$$\exp\left(\left(\tfrac{1}{2}\right)^{2J} n^{1/J}\right) \geqslant 2.$$

With (11) we conclude that for sufficiently large n

$$\log n \geqslant 2(\log 2)^2 J^2 \;\Rightarrow\; s_J(n) > 1. \tag{12}$$

By definition,

$$E(n) = \max\left\{ J : p^J \,\middle|\, \binom{2n}{n} \text{ for some prime } p \right\} = \max\{J : s_J(n) > 1\}.$$

If (9) and (12) held for sufficiently small n, they would imply

$$E(n) \asymp \sqrt{\log n},$$

which, however, contradicts (5).

The reason why this argument apparently fails is that the exponential sums which are used in [12] can be bounded non-trivially only if the summation variable ranges over large intervals of primes. On the other hand, the above proof for the lower bound in (5) suggests that small primes are the ones to look at. In fact, (5) was shown by taking just the prime $p=2$ into account. Formulae similar to (6) do hold for any prime p. Assuming that the p-ary expansions of an integer with respect to different primes p are independent of each other, the conjecture seems to be reasonable.

3. An exponential sum estimate of Karacuba

We will make use of the following result due to Vinogradov.

LEMMA 1 ([16] or [17]). *For $n \geqslant 12$ and a positive integer l, let*

$$k_l = nl + \left[\tfrac{1}{4} n(n+1) + 1\right]$$

and

$$D_l = (20n)^{\frac{1}{2} n(n+1) l}.$$

Then for positive integers $k \geqslant k_l$ and P, we have

$$\int_0^1 \cdots \int_0^1 \left| \sum_{x=1}^{P} e(\alpha_n x^n + \ldots + \alpha_1 x) \right|^{2k} d\alpha_1 \ldots d\alpha_n < D_l P^{2k - \frac{1}{2} n(n+1) + \frac{1}{2} n(n+1)(1-1/n)^l}.$$

The next lemma which is crucial for our principal result is mainly due to Karacuba [8], but sharpens and simplifies it in a way which suits our purpose; it was also obtained in a slightly weaker form by G. J. Rieger. The proof follows Karacuba's ideas.

LEMMA 2. *Let N, $0\leqslant P'<P$, n, $t>0$ and $2\leqslant s_1<\ldots<s_t\leqslant n$ be integers, let $c_0<1$ and $c_3<c_2<c_1<1$ be positive real numbers satisfying*

$$P\geqslant 8^{1/c_3}, \tag{13}$$

$$n\geqslant \max\left(12, \frac{1}{\sqrt{1-c_1}}, \frac{1}{\sqrt[3]{c_1-c_2}}\right) \tag{14}$$

and

$$c_0 n\leqslant t<n. \tag{15}$$

Furthermore, let $f(x)$ be a real function having a continuous $(n+1)$-st derivative in $N\leqslant x\leqslant N+2P$ such that for $N\leqslant x\leqslant N+2P$

$$\left|\frac{1}{(n+1)!}f^{(n+1)}(x)\right|\leqslant P^{-c_1(n+1)}, \tag{16}$$

and for $j=1,\ldots,t$ and $N\leqslant x\leqslant N+P$

$$P^{-c_2 s_j}\leqslant\left|\frac{1}{s_j!}f^{(s_j)}(x)\right|\leqslant P^{-c_3 s_j}. \tag{17}$$

Then, for

$$S=\sum_{x=N}^{N+P'} e(f(x)),$$

we have

$$S\ll P^{1-\gamma/n^2}(\log P)^{4/n^3},$$

where the constant implied by $\ll$ is absolute (in particular does not depend on the c_i), and

$$\gamma=\varrho c_0^2\left(1+\log\frac{3c_1}{\varrho c_0}\right)^{-2}$$

with

$$\varrho=\min\left(c_3, c_1-c_2-\frac{1}{n^2(n+1)}\right).$$

Proof. Let

$$P_2:=[P^{\frac{1}{2}(c_1-1/n^2(n+1))}].$$

By (14), we get immediately

$$1\leqslant P_2\leqslant\sqrt{P}. \tag{18}$$

By definition of S, we have for non-negative integers y and z

$$S=\sum_{a=N-yz}^{N-yz+P'} e(f(a+yz)).$$

Thus,

$$\begin{aligned}(P_2+1)^2S &= \sum_{y=0}^{P_2}\sum_{z=0}^{P_2}\sum_{a=N-yz}^{N-yz+P'} e(f(a+yz))\\ &= \sum_{a=N-P_2^2}^{N+P'} \sum_{0\leqslant y\leqslant P_2} \sum_{\substack{0\leqslant z\leqslant P_2\\ N-a\leqslant yz\leqslant N+P'-a}} e(f(a+yz))\\ &= \sum_{a=N}^{N+P'} \sum_{0\leqslant y\leqslant P_2} \sum_{\substack{0\leqslant z\leqslant P_2\\ yz\leqslant N+P'-a}} e(f(a+yz))+\theta_1 P_2^2(P_2+1)^2\\ &= \sum_{a=N}^{N+P'} W(a)+\theta_2 P_2^2(P_2+1)^2,\end{aligned} \tag{19}$$

where θ_i are absolute constants with $|\theta_i|\leqslant i$, $i=1,2$, and

$$W(a)=\sum_{y=0}^{P_2}\sum_{z=0}^{P_2} e(f(a+yz)).$$

By Taylor's formula,

$$f(a+yz)=\sum_{s=0}^{n} b_s(yz)^s+R,$$

where for $s=0,1,\dots,n$

$$b_s=\frac{1}{s!}f^{(s)}(a),$$

and for some ξ with $N\leqslant\xi\leqslant N+P'+P_2^2<N+2P$,

$$|R|\leqslant\frac{1}{(n+1)!}|f^{(n+1)}(\xi)|P_2^{2n+2}.$$

By (16) and the definition of P_2, we get

$$|R|\leqslant P^{-c_1(n+1)}P_2^{2n+2}\leqslant P^{-1/n^2}.$$

For real ϕ, $|e(\phi)-1|\leqslant 2\pi\phi$ holds, hence

$$W(a)=W_1(a)+2\pi\theta_3(P_2+1)^2P^{-1/n^2}, \tag{20}$$

where $|\theta_3|\leqslant 1$, and

$$W_1(a)=\sum_{y=0}^{P_2}\sum_{z=0}^{P_2} e(b_0+b_1yz+\dots+b_n(yz)^n).$$

For any non-negative numbers u_ν, v_ν, and a positive integer k, we have by Hölder's inequality

$$\left(\sum_{\nu=0}^{P} u_\nu v_\nu\right)^k = \left(\sum_{\nu=0}^{P} u_\nu^{(k-1)/k}(u_\nu^{1/k}v_\nu)\right)^k \leqslant \left(\sum_{\nu=0}^{P} u_\nu\right)^{k-1} \sum_{\nu=0}^{P} u_\nu v_\nu^k. \tag{21}$$

Setting all $u_\nu = 1$, we get

$$\begin{aligned} |W_1(a)|^{2k} &\leqslant (P_2+1)^{2k-1} \sum_{y=0}^{P_2} \left|\sum_{z=0}^{P_2} e(b_1 yz + \ldots + b_n (yz)^n)\right|^{2k} \\ &= (P_2+1)^{2k-1} \sum_{y=0}^{P_2} \sum_{\lambda_1,\ldots,\lambda_n} J_k(\lambda_1,\ldots,\lambda_n) e(b_1\lambda_1 y + \ldots + b_n \lambda_n y^n) \\ &\leqslant (P_2+1)^{2k-1} \sum_{\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)} J_k(\boldsymbol{\lambda}) \left|\sum_{y=0}^{P_2} e(b_1\lambda_1 y + \ldots + b_n\lambda_n y^n)\right|, \end{aligned} \tag{22}$$

where

$$\begin{aligned} J_k(\boldsymbol{\lambda}) &= \operatorname{card}\{(z_1,\ldots,z_{2k}) : z_1^j + \ldots + z_k^j - z_{k+1}^j - \ldots - z_{2k}^j = \lambda_j,\ 1 \leqslant j \leqslant n\} \\ &= \int_0^1 \ldots \int_0^1 \left|\sum_{z=0}^{P_2} e(\alpha_1 z + \ldots + \alpha_n z^n)\right|^{2k} e(-(\alpha_1\lambda_1 + \ldots + \alpha_n\lambda_n))\, d\alpha_1 \ldots d\alpha_n. \end{aligned}$$

Obviously, in (22) we have for $\boldsymbol{\lambda}$ in $\sum_{\boldsymbol{\lambda}}$

$$|\lambda_j| \leqslant k P_2^j, \quad 1 \leqslant j \leqslant n. \tag{23}$$

Clearly,

$$\sum_{\boldsymbol{\lambda}} J_k(\boldsymbol{\lambda}) = (P_2+1)^{2k} \tag{24}$$

and

$$|J_k(\boldsymbol{\lambda})| \leqslant \int_0^1 \ldots \int_0^1 \left|\sum_{z=0}^{P_2} e(\alpha_1 z + \ldots + \alpha_n z^n)\right|^{2k} d\alpha_1 \ldots d\alpha_n = J_k(\mathbf{0}). \tag{25}$$

For any positive integers l, $n \geqslant 12$ and $k = nl + \left[\frac{1}{4}n(n+1)+1\right]$, Lemma 1 implies

$$J_k(\mathbf{0}) < D_l P_2^{2k - \frac{1}{2}n(n+1) + \frac{1}{2}n(n+1)(1-1/n)^l}. \tag{26}$$

Raising (22) to the $(2k)$th power, applying (21), and then (24) and (25), we obtain

$$\begin{aligned} |W_1(a)|^{4k^2} &\leqslant (P_2+1)^{4k^2-2k} \left(\sum_{\boldsymbol{\lambda}} J_k(\boldsymbol{\lambda})\right)^{2k-1} \sum_{\boldsymbol{\lambda}} J_k(\boldsymbol{\lambda}) \left|\sum_{y=0}^{P_2} e(b_1\lambda_1 y + \ldots + b_n\lambda_n y^n)\right|^{2k} \\ &\leqslant (P_2+1)^{8k^2-4k} J_k(\mathbf{0}) \sum_{\boldsymbol{\lambda}} \left|\sum_{y=0}^{P_2} e(b_1\lambda_1 y + \ldots + b_n\lambda_n y^n)\right|^{2k} \\ &= (P_2+1)^{8k^2-4k} J_k(\mathbf{0}) \sum_{\boldsymbol{\lambda}} \sum_{\boldsymbol{\mu}} J_k(\boldsymbol{\mu}) e(b_1\lambda_1\mu_1 + \ldots + b_n\lambda_n\mu_n). \end{aligned}$$

For any positive integer U, we have (see for instance [10, p. 189])

$$\sum_{m=1}^{U} e(\alpha m) \leqslant \min\left(U, \frac{1}{2\|\alpha\|}\right),$$

where $\|\alpha\| = \min_{a\in\mathbf{Z}} |\alpha - a|$. This implies

$$\begin{aligned} |W_1(a)|^{4k^2} &\leqslant (P_2+1)^{8k^2-4k} J_k(\mathbf{0}) \sum_{\boldsymbol{\mu}} J_k(\boldsymbol{\mu}) \prod_{s=1}^{n} \min\left(2kP_2^s+1, \frac{1}{2\|b_s\mu_s\|}\right) \\ &\leqslant (P_2+1)^{8k^2-4k} J_k(\mathbf{0})^2 T \end{aligned} \tag{27}$$

with

$$T := \prod_{s=1}^{n} T_s$$

and

$$T_s = \sum_{|\mu_s| \leqslant kP_2^s} \min\left(2kP_2^s+1, \frac{1}{2\|b_s\mu_s\|}\right).$$

Since $P_2 \geqslant 1$ by (18), trivially for $1 \leqslant s \leqslant n$,

$$T_s \leqslant (2kP_2^s+1)^2 \leqslant 5k^2P_2^{2s}. \tag{28}$$

For $s = s_j$, we have by definition of b_s

$$b_s = \frac{1}{q_s} + \frac{\theta_s}{q_s^2}$$

for some $|\theta_s| \leqslant 1$, where the integers $q_s = [b_s^{-1}]$ satisfy by (17) and (13)

$$1 < \tfrac{1}{2}P^{c_3 s} \leqslant P^{c_3 s} - 1 < q_s \leqslant P^{c_2 s} \leqslant P^n. \tag{29}$$

It is well-known that for $U > 0$, $q \geqslant 1$ and $\alpha = a/q + \theta/q^2$ with $(a, q) = 1$ and $|\theta| \leqslant 1$

$$\sum_{x=1}^{P} \min\left(U, \frac{1}{2\|\alpha x + \beta\|}\right) \ll (U + q\log q)\left(\frac{P}{q} + 1\right)$$

holds (see [7, p. 23] or [10, p. 189]). Applying this for $s = s_j$, we get with (29)

$$\begin{aligned} T_s &\ll (kP_2^s + q_s \log q_s)\left(\frac{kP_2^s}{q_s} + 1\right) \ll k^2(P_2^s + q_s)^2 \frac{\log q_s}{q_s} \\ &\ll k^2(P_2^{2s} + q_s^2)\frac{\log q_s}{q_s} \ll k^2 n (\log P) P_2^{2s}\left(\frac{1}{q_s} + \frac{q_s}{P_2^{2s}}\right). \end{aligned}$$

By (18) and (28), we conclude that for some absolute constant C_1

$$T \leqslant C_1^n k^{2n} n^n (\log P)^n P_2^{n(n+1)} \prod_{j=1}^{t} \left(\frac{1}{q_{s_j}} + \frac{q_{s_j}}{P_2^{2s_j}} \right).$$

By (29),

$$\frac{1}{q_{s_j}} < 2P^{-c_3 s_j}$$

and

$$\frac{q_{s_j}}{P_2^{2s_j}} \leqslant \frac{P^{c_2 s_j}}{P_2^{2s_j}} = \left(\frac{P^{c_2}}{P_2^2} \right)^{s_j} \leqslant \left(\frac{P^{c_2}}{\left(\frac{1}{2}(P_2+1)\right)^2} \right)^{s_j} \leqslant 4^{s_j} P^{(c_2 - c_1 + 1/n^2(n+1)) s_j}.$$

Therefore,

$$T \leqslant C_1^n k^{2n} n^n (\log P)^n P_2^{n(n+1)} 4^{n^2} P^{-\varrho \sum s_j}.$$

Together with (27) and (26), this implies

$$\begin{aligned} |W_1(a)|^{4k^2} &\leqslant (P_2+1)^{8k^2-4k} D_l^2 P_2^{4k-n(n+1)+n(n+1)(1-1/n)^l} \\ &\quad \times C_1^n k^{2n} n^n 4^{n^2} (\log P)^n P_2^{n(n+1)} P^{-\varrho \sum s_j} \\ &\leqslant 28k^2 D_l^2 C_1^n k^{2n} n^n 4^{n^2} (\log P)^n P_2^{8k^2+n(n+1)(1-1/n)^l} P^{-\varrho \sum s_j} \\ &= B_1(P) P_2^{8k^2+n(n+1)(1-1/n)^l} P^{-\varrho \sum s_j}, \end{aligned}$$

say.

Clearly, since $k \geqslant \frac{1}{4} n(n+1)$,

$$B_2(P) := B_1(P)^{1/4k^2} \leqslant 2^2 (20n)^{1/n^2} C_1^{1/n} k^{1/k} n^{1/n} (\log P)^{4/n^3} 4^{4/n^2} \leqslant C_2 (\log P)^{4/n^3} \quad (30)$$

with a constant C_2 not depending on any of the parameters. Hence

$$|W_1(a)| \leqslant B_2(P) P_2^{2+n(n+1)(1-1/n)^l/4k^2} P^{-\varrho (\sum s_j)/4k^2},$$

thus by (20),

$$|W(a)| \leqslant B_2(P) P_2^{2+n(n+1)(1-1/n)^l/4k^2} P^{-\varrho(\sum s_j)/4k^2} + 2\pi\theta_1 (P_2+1)^2 P_2^{2n+2} P^{-c_1(n+1)}.$$

By (19), we get

$$\begin{aligned} |S| &\leqslant B_2(P) P_2^{n(n+1)(1-1/n)^l/4k^2} P^{-\varrho(\sum s_j)/4k^2} (P'+1) \\ &\quad + 2\pi\theta_1 P_2^{2n+2} P^{-c_1(n+1)} (P'+1) + \theta_2 P_2^2 \\ &\leqslant B_3(P) \left(P^{1-(1/4k^2)(\varrho(\sum s_j) - \frac{1}{2} c_1 n(n+1)(1-1/n)^l)} + P^{1-1/n^2} + P^{c_1 - 1/n^2(n+1)} \right), \end{aligned} \quad (31)$$

where $B_3(P)=\max(B_2(P), 2\pi\theta_1, \theta_2)$.

Now choose l in such a way that

$$c_1 n(n+1)\left(1-\frac{1}{n}\right)^l < \varrho \sum_{j=1}^{t} s_j \leqslant c_1 n(n+1)\left(1-\frac{1}{n}\right)^{l-1}. \tag{32}$$

This is possible since by assumption on the s_j and (15), we have

$$\sum_{j=1}^{t} s_j \geqslant 2+3+\ldots+(t+1) = \tfrac{1}{2}t(t+3) \geqslant \tfrac{1}{2}c_0^2 n(n+1). \tag{33}$$

Together with (14) which ensures that $1-1/n^2 \geqslant c_1$, (31) and (32) give

$$|S| \leqslant B_3(P)(P^{1-\varrho(\sum s_j)/8k^2} + 2P^{1-1/n^2}). \tag{34}$$

Inequalities (32) and (33) imply

$$\frac{\varrho c_0^2}{2} \leqslant \frac{\varrho c_0^2}{2c_1} \leqslant \frac{\varrho \sum s_j}{c_1 n(n+1)} \leqslant \left(1-\frac{1}{n}\right)^{l-1},$$

thus

$$\log\frac{2c_1}{\varrho c_0^2} \geqslant (l-1)\log\left(1-\frac{1}{n}\right)^{-1} \geqslant (l-1)\frac{1}{n}.$$

Since $n \geqslant 12$ by (14), this in turn yields

$$l \leqslant 1+n\log\frac{2c_1}{\varrho c_0^2} \leqslant n\left(\log e^{1/12}+\log\frac{2c_1}{\varrho c_0^2}\right) \leqslant n\log\frac{3c_1}{\varrho c_0^2} = c_4 n,$$

say. Notice that $c_4 \geqslant 1$. Thus by definition of k,

$$k \leqslant nl+\tfrac{1}{4}n(n+1)+1 \leqslant c_4 n^2+\tfrac{1}{4}(n^2+n+4) \leqslant (c_4+1)n^2.$$

Therefore, (33) gives

$$\frac{\varrho \sum s_j}{8k^2} \geqslant \frac{\varrho c_0^2 n(n+1)}{16(c_4+1)^2 n^4} \geqslant \frac{\varrho c_0^2}{(c_4+1)^2}\cdot\frac{1}{n^2}.$$

Now (34) finally implies

$$|S| \leqslant 3B_3(P)P^{1-\gamma/n^2},$$

where

$$\gamma = \frac{\varrho c_0^2}{(c_4+1)^2}.$$

With regard to (30), this proves Lemma 2.

In the sequel, let for $r>0$ real numbers h_i $(1\leqslant i\leqslant r)$ and positive integers j_i $(1\leqslant i\leqslant r)$ be given such that

$$h=h_1\geqslant 1, \tag{35}$$

$$H=\max\{|h_i|:1\leqslant i\leqslant r\}$$

and

$$1\leqslant j=j_1<j_2<\ldots<j_r\leqslant J, \tag{36}$$

where

$$J>8 \tag{37}$$

is a real number. We define

$$\Lambda(X,Y):=\left(\frac{\log X}{\log Y}\right)^2$$

and

$$v_i(j):=j^{-i}(\log 2j)^{-2}.$$

LEMMA 3. *Let $T'>2$, $T=T'h$ such that*

$$0\leqslant P'<P<T^{1/(j+1)+1/100j^3}, \tag{38}$$

$$P(\log P)^J>JH(3\log T)^J \tag{39}$$

and

$$P>\exp(CJ(\log 2J)^3), \tag{40}$$

where C is a sufficiently large absolute constant. Then

$$\sum_{x=P}^{P+P'} e\left(T'\left(\frac{h_1}{x^{j_1}}+\ldots+\frac{h_r}{x^{j_r}}\right)\right)\ll P^{1-cv_5(j)\Lambda(P,T)}.$$

Proof. We intend to apply Lemma 2 and put $N:=P$,

$$f(x):=T'\left(\frac{h_1}{x^{j_1}}+\ldots+\frac{h_r}{x^{j_r}}\right),$$

$$n:=\left[12j\frac{\log T}{\log P}\right],$$

$$c_0:=\frac{1}{12j},\quad c_1:=1-\frac{1}{6j},\quad c_2:=1-\frac{1}{5j},\quad c_3:=\frac{1}{3},$$

and

$$\{s_i:1\leqslant i\leqslant t\}:=\left\{s\in\mathbf{N}:2\frac{\log T}{\log P}\leqslant s\leqslant 4\frac{\log T}{\log P}\right\}. \tag{41}$$

By (38), we have

$$\frac{\log T}{\log P} > \left(\frac{1}{j+1}+\frac{1}{100j^3}\right)^{-1} > j+1-\frac{1}{2j}. \tag{42}$$

By (41), this yields

$$\begin{aligned} t &\geqslant 2\frac{\log T}{\log P}-1 = \left(2-\frac{\log P}{\log T}\right)\frac{\log T}{\log P} \\ &\geqslant \left(2-\frac{1}{j+1}-\frac{1}{100j^3}\right)\frac{\log T}{\log P} \geqslant \frac{\log T}{\log P} \geqslant c_0 n. \end{aligned} \tag{43}$$

By definition of n and (42), we have

$$n \geqslant \left[12j\left(j+1-\frac{1}{2j}\right)\right] \geqslant 12j^2, \tag{44}$$

hence

$$n \geqslant 12, \tag{45}$$

$$n \geqslant \sqrt{6j} = \frac{1}{\sqrt{1-c_1}} \tag{46}$$

and

$$n \geqslant \sqrt[3]{30j} = \frac{1}{\sqrt[3]{c_1-c_2}}. \tag{47}$$

Summing up so far, we have chosen n, t, $c_0<1$ and $c_3<c_2<c_1<0$ satisfying conditions (13), (14) and (15) of Lemma 2 by virtue of (40), (45), (46), (47) and (43). It remains to check conditions (16) and (17).

For $x>0$ and $m\in\mathbf{N}$, we have

$$\frac{1}{m!}f^{(m)}(x) = (-1)^m\frac{T'}{m!}\sum_{i=1}^{r}\frac{j_i(j_i+1)\ldots(j_i+m-1)h_i}{x^{j_i+m}}. \tag{48}$$

Now (39) and (38) give $P>H$. By (35), we thus get

$$\frac{h}{P^j} \geqslant \frac{|h_i|}{P^{j_i}}, \quad 1\leqslant i\leqslant r, \tag{49}$$

which implies for $P=N\leqslant x\leqslant N+2P=3P$ by (48)

$$\begin{aligned} \left|\frac{1}{(n+1)!}f^{(n+1)}(x)\right| &\leqslant J^{n+1}T'\left(\frac{|h_1|}{P^{j_1+n+1}}+\ldots+\frac{|h_r|}{P^{j_r+n+1}}\right) \leqslant \frac{J^{n+1}T'}{P^{n+1}}r\frac{h}{P^j} \\ &\leqslant \frac{J^{n+2}T}{P^{j+n+1}} = P^{(n+2)\log J/\log P+\log T/\log P-(j+n+1)} = P^{-c_1'(n+1)}, \end{aligned}$$

where

$$c_1' = 1 + \frac{j}{n+1} - \frac{(n+2)\log J}{(n+1)\log P} - \frac{\log T}{(n+1)\log P}.$$

By (40), we may assume that $P > J^{24j}$. This implies by the definition of n

$$c_1' > 1 - \frac{(n+2)\log J}{(n+1)\log P} - \frac{\log T}{(n+1)\log P} > 1 - 2\frac{1}{12j} = c_1,$$

which proves (16).

In the same way, we get for $P \leqslant x < 2P$ and $s \in \{s_i\}$

$$\left|\frac{1}{s!} f^{(s)}(x)\right| \leqslant \frac{J^{s+1}T}{P^{j+s}} = P^{-c_3' s},$$

where

$$c_3' = 1 + \frac{j}{s} - \frac{(s+1)\log J}{s\log P} - \frac{\log T}{s\log P}.$$

Using again $P > J^{24j}$, the definition of the s_i gives

$$c_3' > 1 - 2\frac{\log J}{\log P} - \frac{\log T}{s\log P} > 1 - \frac{1}{12j} - \frac{1}{2} > \frac{1}{3} = c_3.$$

This proves the upper bound in (17).

By (48), we have for $P \leqslant x \leqslant 2P$ and $s \in \{s_i\}$ using (35)

$$\left|\frac{1}{s!} f^{(s)}(x)\right| \geqslant T'\left(\frac{h}{x^{j+s}} - (r-1)\binom{J-1+s}{s}\frac{H}{x^{j_2+s}}\right). \tag{50}$$

We apply a weak form of Stirling's formula, namely

$$\left(n+\tfrac{1}{2}\right)\log n - n + \tfrac{3}{2}\left(1 - \log\tfrac{3}{2}\right) \leqslant \log n! \leqslant \left(n+\tfrac{1}{2}\right)\log n - n + 1.$$

For $J-1 < s$, this implies together with (37)

$$\binom{J-1+s}{s} < s^{J-1}. \tag{51}$$

For $J-1 \geqslant s$, we have trivially

$$\binom{J-1+s}{s} < 4^{J-1}. \tag{52}$$

By (50), (51), (52), (35) and (36), we get for $s_0 = \max(s, 4)$

$$\begin{aligned}\left|\frac{1}{s!} f^{(s)}(x)\right| &\geqslant \frac{T'}{x^{j+s}}\left(h - \frac{J s_0^{J-1} H}{x^{j_2-j}}\right)\\ &\geqslant \frac{T'}{(2P)^{j+s}}\left(h - \frac{J s_0^{J-1} H}{P^{j_2-j}}\right)\\ &\geqslant \frac{T'}{(2P)^{j+s}}\left(h - \frac{J s_0^{J-1} H}{P}\right).\end{aligned} \tag{53}$$

By (39), (42) and (41),

$$P > 2Js_0^J H.$$

Therefore, (53) yields

$$\left|\frac{1}{s!}f^{(s)}(x)\right| \geqslant \frac{T'h}{2(2P)^{j+s}} = P^{-c_2' s}, \tag{54}$$

where

$$c_2' = \left(1+\frac{j}{s}\right)\left(1+\frac{\log 2}{\log P}\right)+\frac{\log 2}{s\log P}-\frac{\log T}{s\log P}.$$

We have

$$c_2' = 1+\frac{\log 2}{\log P}-\frac{1}{s}G,$$

with

$$G = \frac{\log T}{\log P}-j-\frac{(j+1)\log 2}{\log P}.$$

By (40), $P > 2^{2(j+1)}$. With (42), we get

$$G > j+1-\frac{1}{2j}-j-\frac{1}{2} \geqslant 0.$$

Hence, by (41) and (42), we get for $P > 2^{200j}$, which follows from (40),

$$\begin{aligned} c_2' &\leqslant 1+\frac{\log 2}{\log P}-\frac{\log P}{4\log T}G \\ &= 1-\frac{1}{4}\left(1-j\frac{\log P}{\log T}\right)+\frac{\log 2}{\log P}+\frac{(j+1)\log 2}{4\log T} \\ &\leqslant 1-\frac{1}{4}\left(1-j\left(\frac{1}{j+1}+\frac{1}{100j^3}\right)\right)+\frac{2\log 2}{\log P} \\ &\leqslant 1-\frac{1}{4}\left(\frac{1}{j+1}-\frac{1}{100j^2}\right)+\frac{1}{100j} \leqslant 1-\frac{1}{4j}+\frac{1}{50j} \leqslant c_2. \end{aligned}$$

Hence, by (54) the lower bound in (17) also holds.

Thus we have checked all the conditions of Lemma 2. Its application provides the estimate

$$\sum_{x=P}^{P+P'} e\left(T'\left(\frac{h_1}{x^{j_1}}+\ldots+\frac{h_r}{x^{j_r}}\right)\right) \ll P^{1-\gamma/n^2}(\log P)^{4/n^3}, \tag{55}$$

where by definition of γ and ϱ in Lemma 2

$$\frac{\gamma}{n^2} = \frac{\varrho c_0^2}{(1+\log(3c_1/\varrho c_0))^2}\left[12j\frac{\log T}{\log P}\right]^{-2} \geqslant \frac{\varrho c_0^2}{144j^2(1+\log(3c_1/\varrho c_0))^2}\Lambda(P,T) \tag{56}$$

and

$$\varrho = \min\left(c_3, c_1-c_2-\frac{1}{n^2(n+1)}\right) > \min\left(\frac{1}{3}, \frac{1}{30j}-\frac{1}{1000j^6}\right) > \frac{1}{31j}.$$

Therefore

$$1+\log\frac{3}{\varrho c_0}=\log\frac{3e}{\varrho c_0}<\log(3e\cdot 31j\cdot 12j)<\log(56j)^2\leqslant 12\log 2j.$$

Then (56) implies

$$\frac{\gamma}{n^2}\geqslant\frac{c}{j^5(\log 2j)^2}\Lambda(P,T)=c\cdot v_5(j)\,\Lambda(P,T). \tag{57}$$

For a given positive constant $\tilde{c}$ and $P=\exp(CJ(\log 2J)^3)$, where $C=C(\tilde{c})$ is sufficiently large, we have

$$\log P=CJ(\log 2J)^3\leqslant(2J)^{C\tilde{c}}=(\exp(CJ(\log 2J)^3))^{\tilde{c}v_1(j)}=P^{\tilde{c}v_1(j)}.$$

Hence, by (40) we get for any absolute constant $\tilde{c}$

$$\log P\leqslant P^{\tilde{c}v_1(j)}. \tag{58}$$

By (42), the definition of n, and (57),

$$v_1(j)\leqslant v_2(j)\frac{\log T}{\log P}\leqslant v_5(j)\Lambda(P,T)\left(j\frac{\log T}{\log P}\right)^3\leqslant\frac{1}{c}\cdot\frac{\gamma}{n^2}n^3=\frac{1}{c}\gamma n.$$

With $\tilde{c}:=\frac{1}{8}c$ in (58), we thus obtain

$$\log P\leqslant P^{\gamma n/8}$$

or

$$(\log P)^{4/n^3}\leqslant P^{\gamma/2n^2}.$$

By (55) and (57), we therefore have

$$\sum_{x=P}^{P+P'}e\left(T'\left(\frac{h_1}{x^{j_1}}+\ldots+\frac{h_r}{x^{j_r}}\right)\right)\ll P^{1-\gamma/2n^2}\leqslant P^{1-cv_5(j)\Lambda(P,T)}.$$

4. Application of van der Corput's method

The next two lemmas may be found in [14]. Lemma 5 is obtained by following the proof there and keeping track of the constants.

LEMMA 4 ([14, Lemma 4.2]). *Let $f(x)$ be a real differentiable function with monotonic $f'(x)$ and $f'(x)\geqslant m>0$ or $f'(x)\leqslant -m<0$ on the interval $[a,b]$. Then*

$$\left|\int_a^b e(f(x))\,dx\right|\leqslant\frac{2}{\pi m}.$$

LEMMA 5 ([14, Lemma 4.8]). *Let $f(x)$ be a real differentiable function with monotonic $f'(x)$ and $|f'(x)|\leqslant\theta<1$ on the interval $[a,b]$. Then*

$$\left|\sum_{a<n\leqslant b} e(f(n))-\int_a^b e(f(x))\,dx\right|\leqslant\frac{8}{\pi}\cdot\frac{1}{1-\theta}+\frac{4}{3}\pi+1.$$

LEMMA 6. *Let $T'>2$, $T=T'h$ and $0\leqslant P'<P$ such that*

$$P\geqslant T^{1/(j+1)+1/100j^3}, \tag{59}$$

$$P>2J^3H \tag{60}$$

and

$$P>(2J)^{200J}. \tag{61}$$

Then

$$\left|\sum_{x=P}^{P+P'} e\left(T'\left(\frac{h_1}{x^{j_1}}+\ldots+\frac{h_r}{x^{j_r}}\right)\right)\right|\leqslant 2^{5j}\frac{P^{j+1}}{T}.$$

Proof. We use van der Corput's well-known method. Again let

$$f(x)=T'\left(\frac{h_1}{x^{j_1}}+\ldots+\frac{h_r}{x^{j_r}}\right).$$

Since by (35), (36) and (60) for $P\leqslant x\leqslant 2P$,

$$\begin{aligned} f''(x)&=-T'\left(\frac{j_1(j_1+1)h_1}{x^{j_1+2}}+\ldots+\frac{j_r(j_r+1)h_r}{x^{j_r+2}}\right)\\ &\geqslant\frac{T'}{x^{j_2+2}}(j(j+1)hx^{j_2-j}-(r-1)J(J+1)H)\\ &\geqslant\frac{T'}{(2P)^{j_2+2}}(2hP-J^3H)>0, \end{aligned}$$

$f'(x)$ obviously is an increasing function. Because of (49), we get for $P\leqslant x\leqslant P+P'$ by (59) and (61)

$$\begin{aligned} |f'(x)|&=T'\left|\frac{j_1h_1}{x^{j_1+1}}+\ldots+\frac{j_rh_r}{x^{j_r+1}}\right|\\ &\leqslant\frac{T'J}{P}\left(\frac{h_1}{P^{j_1}}+\ldots+\frac{h_r}{P^{j_r}}\right)\leqslant\frac{T'J}{P}r\frac{h}{P^j}\leqslant\frac{J^2T}{P^{j+1}}\\ &<J^2P^{-1/100j}<J^2(2J)^{-2J/j}\leqslant\tfrac{1}{4}. \end{aligned}$$

Thus Lemma 5 implies

$$\left|\sum_{x=P}^{P+P'} e\left(T'\left(\frac{h_1}{x^{j_1}}+\ldots+\frac{h_r}{x^{j_r}}\right)\right)-\int_P^{P+P'} e\left(T'\left(\frac{h_1}{x^{j_1}}+\ldots+\frac{h_r}{x^{j_r}}\right)\right) dx\right| \tag{62}$$
$$\leqslant \frac{32}{3\pi}+\frac{4\pi}{3}+1 \leqslant 9.$$

The function $f'(x)$ is increasing in $[P, P+P']$ and, by (36) and (60), we have in this interval

$$-f'(x) \geqslant T'\left(\frac{jh}{x^{j+1}}-(r-1)\frac{JH}{x^{j_2+1}}\right) \geqslant \frac{T'}{(2P)^{j+1}}\left(jh-\frac{J^2H}{P^{j_2-j}}\right)$$
$$\geqslant \frac{1}{2}\cdot\frac{T'h}{(2P)^{j+1}} \geqslant \frac{1}{2}\cdot\frac{T}{(2P)^{j+1}}.$$

Hence Lemma 4 gives

$$\left|\int_P^{P+P'} e\left(T'\left(\frac{h_1}{x^{j_1}}+\ldots+\frac{h_r}{x^{j_r}}\right)\right) dx\right| \leqslant \frac{2^{j+3}}{\pi}\cdot\frac{P^{j+1}}{T} \leqslant 2^{j+2}\frac{P^{j+1}}{T}.$$

By (62) and (59), the desired result follows.

LEMMA 7. *Let* $T'>2$, $0 \leqslant P' < P$,

$$P > JH(3\log T'H)^J \tag{63}$$

and

$$P > \exp(CJ(\log 2J)^3). \tag{64}$$

Then

$$\left|\sum_{x=P}^{P+P'} e\left(T'\left(\frac{h_1}{x^{j_1}}+\ldots+\frac{h_r}{x^{j_r}}\right)\right)\right| \leqslant C^j\left(P^{1-cv_5(j)\Lambda(P,T'H)}+\frac{P^{j+1}}{T'}\right).$$

Proof. For $P \geqslant T'$, the lemma obviously holds. In case $P<T'$, (63) implies (39) and (60). By (64), the conditions (40) and (61) are also satisfied. Hence the proof is completed by Lemma 3 and Lemma 6.

LEMMA 8. *Let* $x>2$, $0 \leqslant M' \leqslant M$ *and*

$$M > \exp(CJ(\log 2J)^3). \tag{65}$$

Then

$$\left|\sum_{M'<m\leqslant M} e\left(x\left(\frac{h_1}{m^{j_1}}+\ldots+\frac{h_r}{m^{j_r}}\right)\right)\right|$$
$$\leqslant C^j(M^{1-cv_5(J)\Lambda(M,xH)}+M^{j+1}x^{-1}+H^2(\log xH)^{2J}).$$

Proof. For $M \leqslant J^2H^2(3\log xH)^{2J}$, the lemma is obvious. Thus we assume

$$M > J^2H^2(3\log xH)^{2J}. \tag{66}$$

Let $\frac{1}{2} \leqslant \varkappa < 1$. Then

$$\left|\sum_{M'<m\leqslant M} e\left(x\left(\frac{h_1}{m^{j_1}}+\ldots+\frac{h_r}{m^{j_r}}\right)\right)\right| \leqslant \left|\sum_{M'<m\leqslant M^\varkappa}\right| + \sum_{\substack{\nu\geqslant 0\\ M^\varkappa 2^\nu\leqslant M}} \left|\sum_{\substack{M^\varkappa 2^\nu<m\leqslant M^\varkappa 2^{\nu+1}\\ M'<m\leqslant M}}\right|$$
$$\leqslant M^\varkappa + \sum_\nu R_\nu,$$

where

$$R_\nu = \left|\sum_{\substack{M^\varkappa 2^\nu<m\leqslant M^\varkappa 2^{\nu+1}\\ M'<m\leqslant M}} e\left(x\left(\frac{h_1}{m^{j_1}}+\ldots+\frac{h_r}{m^{j_r}}\right)\right)\right|.$$

Since $\varkappa \geqslant \frac{1}{2}$, (66) implies (63). Moreover, (65) yields (64). Therefore, we get from Lemma 7

$$R_\nu \leqslant C^J((M^\varkappa 2^\nu)^{1-cv_5(j)\Lambda(M^\varkappa 2^\nu, xH)} + (M^\varkappa 2^\nu)^{j+1}x^{-1})$$
$$\leqslant C^J(2^\nu M^{\varkappa - cv_5(j)\varkappa^3\Lambda(M,xH)} + 2^{(j+1)\nu}M^{(j+1)\varkappa}x^{-1}).$$

In $\sum R_\nu$, the variable ν runs through the interval $0 \leqslant \nu \leqslant (1-\varkappa)\log M/\log 2$, hence

$$\sum_\nu 2^\nu \leqslant 2M^{1-\varkappa}$$

and

$$\sum_\nu 2^{(j+1)\nu} \leqslant 2^{j+1}M^{(j+1)(1-\varkappa)}.$$

For sufficiently large $\varkappa < 1$, the lemma follows.

LEMMA 9. *Let* $2 \leqslant M \leqslant M' \leqslant \min(2M, N) \leqslant N \leqslant x$, $B \geqslant 2$ *and*

$$M > \exp(CJ(\log 2J)^3). \tag{67}$$

For

$$T := \sum_{M<m\leqslant M'} \left|\sum_{B<n\leqslant N/m} \Lambda(n) e\left(x\left(\frac{h_1}{(mn)^{j_1}}+\ldots+\frac{h_r}{(mn)^{j_r}}\right)\right)\right|^2,$$

we then have

$$T \leqslant C^J(N^2M^{-1-cv_5(J)\Lambda(M,xH)} + N^{j+2}(Mx)^{-1} + N + N^2M^{-2}H)(\log xH)^{J+2},$$

where $\Lambda(n)$ denotes von Mangoldt's function.

Proof. For $M \leqslant JH(3\log xH)^J$, we obviously have

$$\begin{aligned} T &\leqslant JH(3\log xH)^J \left(\frac{N}{M}\log N\right)^2 \\ &\leqslant C^J H(\log xH)^J \left(\frac{N}{M}\right)^2 (\log x)^2 \\ &\leqslant C^J N^2 M^{-2} H(\log xH)^{J+2}, \end{aligned}$$

which proves the lemma in this case. Hence, let

$$M > JH(3\log xH)^J. \tag{68}$$

Clearly

$$\begin{aligned} T &= \sum_{M<m\leqslant M'} \sum_{B<n_1\leqslant N/m} \sum_{B<n_2\leqslant N/m} \Lambda(n_1)\Lambda(n_2) e\left(x\sum_{i=1}^{r}\left(\frac{h_i}{(mn_1)^{j_i}} - \frac{h_i}{(mn_2)^{j_i}}\right)\right) \\ &= \sum_{B<n_1\leqslant N/M} \sum_{B<n_2\leqslant N/M} \Lambda(n_1)\Lambda(n_2) \sum_{\substack{M<m\leqslant M' \\ m\leqslant N/n_1, m\leqslant N/n_2}} e\left(x\left(\frac{h_1\Delta_1}{m^{j_1}} + \ldots + \frac{h_r\Delta_r}{m^{j_r}}\right)\right) \end{aligned}$$

with

$$\Delta_i = \left(\frac{1}{n_1^{j_i}} - \frac{1}{n_2^{j_i}}\right), \quad 1 \leqslant i \leqslant r.$$

Thus

$$\begin{aligned} T &\leqslant (\log N)^2 \sum_{n_1\leqslant N/M} \sum_{n_2\leqslant N/M} \left| \sum_{\substack{M<m\leqslant M' \\ m\leqslant N/n_1, m\leqslant N/n_2}} e\left(x\left(\frac{h_1\Delta_1}{m^{j_1}} + \ldots + \frac{h_r\Delta_r}{m^{j_r}}\right)\right)\right| \\ &\leqslant (\log N)^2 \left(N + 2 \sum\sum_{0<n_1<n_2\leqslant N/M} \left| \sum_{M<m\leqslant M'} e\left(x\left(\frac{h_1\Delta_1}{m^{j_1}} + \ldots + \frac{h_r\Delta_r}{m^{j_r}}\right)\right)\right|\right) \\ &= (\log N)^2 (N + 2T_1), \end{aligned} \tag{69}$$

say.

By the mean value theorem, we have for fixed $1 < n_1 < n_2 \leqslant N/M$

$$0 < \Delta_r < \ldots < \Delta_1 \leqslant 1.$$

By this and (68), Lemma 7 implies

$$\begin{aligned} T_1 &= \sum\sum_{0<n_1<n_2\leqslant N/M} \left| \sum_{M<m\leqslant M'} e\left(x\Delta_1\left(\frac{h_1}{m^{j_1}} + \frac{h_2\Delta_2/\Delta_1}{m^{j_2}} + \ldots + \frac{h_r\Delta_r/\Delta_1}{m^{j_r}}\right)\right)\right| \\ &\leqslant C^J \sum\sum_{0<n_1<n_2\leqslant N/M} (M^{1-cv_5(J)\Lambda(M,xH)} + M^{j+1}(x\Delta_1)^{-1}) \\ &\leqslant C^J \left(N^2 M^{-1-cv_5(J)\Lambda(M,xH)} + M^{j+1}x^{-1} \sum\sum_{0<n_1<n_2\leqslant N/M} \frac{1}{\Delta_1}\right). \end{aligned} \tag{70}$$

We set $\Delta_0 = n_2 - n_1$. For $1 < n_1 < n_2$ and $1 \leqslant i \leqslant r$, we get

$$\Delta_i = \frac{n_2^{j_i} - n_1^{j_i}}{(n_1 n_2)^{j_i}} = \Delta_0 \frac{n_2^{j_i-1} + n_2^{j_i-2} n_1 + \ldots + n_2 n_1^{j_i-2} + n_1^{j_i-1}}{(n_1 n_2)^{j_i}} \geqslant \frac{\Delta_0 j_i}{n_2^{j_i+1}}.$$

Therefore,

$$\sum_{0<n_1<n_2\leqslant N/M}\sum \frac{1}{\Delta_1} = \sum_{0<\Delta_0<N/M} \sum_{\substack{0<n_1<n_2\leqslant N/M \\ n_2-n_1=\Delta_0}}\sum \frac{1}{\Delta_1}$$
$$\leqslant \frac{1}{j_1} \sum_{0<\Delta_0<N/M} \frac{1}{\Delta_0} \sum_{0<n_2\leqslant N/M} n_2^{j+1} \leqslant 2\left(\frac{N}{M}\right)^{j+2} \log N.$$

By (70),

$$T_1 \leqslant C^J (N^2 M^{-1-cv_5(J)\Lambda(M,xH)} + N^{j+2}(Mx)^{-1} \log N).$$

This and (69) yield the desired result.

5. Application of Vaughan's identity

As a corollary to Vaughan's identity (see for instance [15] or [2, p. 138–140]), we have

LEMMA 10. *Let $U \geqslant 2$, $V \geqslant 2$, $UV \leqslant N$, and let $f(x)$ be a complex-valued function satisfying $|f(x)| = 1$ for real x. Then*

$$\sum_{n\leqslant N} \Lambda(n) f(n) \ll V + (\log N) S_1 + S_2,$$

where $\Lambda(n)$ denotes von Mangoldt's function, and

$$S_1 = \sum_{t\leqslant UV} \max_{w>0} \left| \sum_{w\leqslant s\leqslant N/t} f(st) \right|,$$
$$S_2 = \sum_{U<m<N/V} \sum_{V<n\leqslant N/m} \sum_{\substack{d\leqslant U \\ d|m}} \mu(d)\Lambda(n) f(mn).$$

The constant implied by $\ll$ is absolute.

LEMMA 11. *Let $x > 0$ and*

$$\exp(CJ(\log 2J)^3) \leqslant N \leqslant x^{1/j}. \tag{71}$$

Then

$$\Big|\sum_{n\leqslant N}\Lambda(n)e\Big(x\Big(\frac{h_1}{n^{j_1}}+...+\frac{h_r}{n^{j_r}}\Big)\Big)\Big|$$
$$\leqslant C^J(N^{1-cv_5(J)\Lambda(N,xH)}+N^{(j+2)/2}x^{-1/2}+N^{5/6}H^2)(\log xH)^{4J}.$$

Proof. We apply Lemma 10 with $U=V=N^{1/3}$ and

$$f(n)=e\Big(x\Big(\frac{h_1}{n^{j_1}}+...+\frac{h_r}{n^{j_r}}\Big)\Big).$$

First consider S_2. By splitting up $\sum_m$ into intervals $M\leqslant m<2M$, we get

$$|S_2|\leqslant 3(\log N)\max_{U<M<M'\leqslant\min(2M,N/V)}\Big|\sum_{M\leqslant m<M'}\Big(\sum_{V<n\leqslant N/m}\Lambda(n)f(mn)\Big)\Big(\sum_{\substack{d\leqslant U\\ d|m}}\mu(d)\Big)\Big|.$$

Cauchy's inequality implies

$$\Big|\sum_{M\leqslant m<M'}\Big|\leqslant\Big(\sum_m\Big|\sum_{V<n\leqslant N/m}\Lambda(n)f(mn)\Big|^2\Big)^{1/2}\Big(\sum_m\Big(\sum_{\substack{d\leqslant U\\ d|m}}\mu(d)\Big)^2\Big)^{1/2}=T_1^{1/2}T_2^{1/2},$$

say. The definition of U and (71) guarantee that

$$U\geqslant(2J)^CJ(\log 2J)^2.$$

Thus we may use Lemma 9 and obtain

$$|T_1|\leqslant C^J(N^2M^{-1-cv_5(J)\Lambda(M,xH)}+N^{j+2}(Mx)^{-1}+N+N^2M^{-2}H)(\log xH)^{J+2}.$$

Moreover,

$$\begin{aligned}|T_2|&\leqslant\sum_{M\leqslant m<M'}\Big(\sum_{\substack{d\leqslant U\\ d|m}}1\Big)^2=\sum_{d_1\leqslant U}\sum_{d_2\leqslant U}\sum_{\substack{M\leqslant m<M'\\ m\equiv 0\ (\mathrm{mod}\ d_1),\, m\equiv 0\ (\mathrm{mod}\ d_2)}}1\\ &\leqslant 2M\sum_{d_1\leqslant U}\sum_{d_2\leqslant U}\frac{(d_1,d_2)}{d_1d_2}\leqslant 2M\sum_{b\leqslant U}b\sum_{\substack{d_1\leqslant U\\ d_1\equiv 0\ (\mathrm{mod}\ b)}}\sum_{\substack{d_2\leqslant U\\ d_2\equiv 0\ (\mathrm{mod}\ b)}}\frac{1}{d_1d_2}\\ &\leqslant 2M\Big(\sum_{u\leqslant U}\frac{1}{u}\Big)^3\leqslant M(\log N)^3.\end{aligned}$$

Together, we get

$$
\begin{aligned}
|S_2| &\leqslant 3(\log N) \max_{U<M\leqslant N/V} (T_1T_2)^{1/2} \\
&\leqslant C^J(\log xH)^{(J+7)/2} \max_{U<M\leqslant N/V} (NM^{-cv_5(J)\Lambda(M,xH)} + N^{(j+2)/2}x^{-1/2} \\
&\quad + (NM)^{1/2} + NM^{-1/2}H^{1/2}) \\
&\leqslant C^J(NU^{-cv_5(J)\Lambda(U,xH)} + N^{(j+2)/2}x^{-1/2} + NV^{-1/2} \\
&\quad + NU^{-1/2}H^{1/2})(\log xH)^{4J} \\
&\leqslant C^J(N^{-cv_5(J)\Lambda(N,xH)} + N^{(j+2)/2}x^{-1/2} + N^{5/6}H^{1/2})(\log xH)^{4J}.
\end{aligned} \tag{72}
$$

It remains to bound S_1. For t in S_1, we have by the choice of U and V and (71) that

$$\frac{N}{t} \geqslant N^{1/3} \geqslant \exp(CJ(\log 2J)^3).$$

For $1 \leqslant w \leqslant N/t$, we have by Lemma 8

$$
\begin{aligned}
&\left| \sum_{w\leqslant s\leqslant N/t} e\left(x\left(\frac{h_1}{(st)^{j_1}} + \ldots + \frac{h_r}{(st)^{j_r}}\right)\right)\right| \\
&\quad = \left| \sum_{w\leqslant s\leqslant N/t} e\left(\frac{x}{t^{j_1}}\left(\frac{h_1}{s^{j_1}} + \frac{h_2t^{j_1-j_2}}{s^{j_2}} + \ldots + \frac{h_rt^{j_1-j_r}}{s^{j_r}}\right)\right)\right| \\
&\quad \leqslant C^J\left(\left(\frac{N}{t}\right)^{1-cv_5(J)\Lambda(N/t,xH)} + \left(\frac{N}{t}\right)^{j+1}\left(\frac{x}{t^j}\right)^{-1} + H^2(\log xH)^{2J}\right) \\
&\quad = C^J\left(\left(\frac{N}{t}\right)^{1-cv_5(J)\Lambda(N/t,xH)} + N^{j+1}(xt)^{-1} + H^2(\log xH)^{2J}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\max_{w>0}\left| \sum_{w\leqslant s\leqslant N/t} e\left(x\left(\frac{h_1}{(st)^{j_1}} + \ldots + \frac{h_r}{(st)^{j_r}}\right)\right)\right| \\
&\qquad\qquad \leqslant C^J\left(\left(\frac{N}{t}\right)^{1-cv_5(J)\Lambda(N/t,xH)} + N^{j+1}(xt)^{-1} + H^2(\log xH)^{2J}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|S_1| &\leqslant C^J\left(\sum_{t\leqslant UV}\left(\left(\frac{N}{t}\right)^{1-cv_5(J)\Lambda(N/t,xH)} + N^{j+1}(xt)^{-1} + H^2(\log xH)^{2J}\right)\right) \\
&\leqslant C^J\Bigg(N^{1-cv_5(J)\Lambda(N/UV,xH)} \sum_{t\leqslant UV} t^{-1+cv_5(J)\Lambda(N/UV,xH)}
\end{aligned}
$$

$$+N^{j+1}x^{-1}\sum_{t\leqslant UV}\frac{1}{t}+UVH^2(\log xH)^{2J}\Big)$$

$$\leqslant C^J\left(N^{1-cv_5(J)\Lambda(N/UV,xH)}cv_5(J)\Lambda\left(\frac{N}{UV},xH\right)^{-1}(UV)^{c\Lambda(N/UV,xH)}\right.$$

$$\left.+N^{j+1}x^{-1}\log UV+UVH^2(\log xH)^{2J}\right)$$

$$\leqslant C^J(N^{1-cv_5(J)\Lambda(N,xH)}(\log xH)^2+N^{j+1}x^{-1}\log N+N^{2/3}H^2(\log xH)^{2J}).$$

For $N\leqslant x^{1/j}$, this together with (72) implies according to Lemma 10

$$\left|\sum_{n\leqslant N}\Lambda(n)e\left(x\left(\frac{h_1}{n^{j_1}}+\ldots+\frac{h_r}{n^{j_r}}\right)\right)\right|$$
$$\leqslant C^J(N^{1-cv_5(J)\Lambda(N,xH)}+N^{(j+2)/2}x^{-1/2}+N^{5/6}H^2)(\log xH)^{4J}.$$

This is the desired bound.

PROPOSITION 1. *Let*

$$\exp(CJ(\log 2J)^3)\leqslant N\leqslant x^{1/j}. \tag{73}$$

Then

$$\left|\sum_{p\leqslant N}e\left(x\left(\frac{h_1}{p^{j_1}}+\ldots+\frac{h_r}{p^{j_r}}\right)\right)\right|$$
$$\leqslant C^J(N^{1-cv_5(J)\Lambda(N,xH)}+N^{(j+2)/2}x^{-1/2}+N^{5/6}H^2)(\log xH)^{4J}.$$

Proof. By Chebyshev's theorem ([2, p. 55]),

$$\left|\sum_{n\leqslant N}\Lambda(n)e\left(x\left(\frac{h_1}{n^{j_1}}+\ldots+\frac{h_r}{n^{j_r}}\right)\right)-\sum_{p\leqslant N}\log p\, e\left(x\left(\frac{h_1}{p^{j_1}}+\ldots+\frac{h_r}{p^{j_r}}\right)\right)\right|$$
$$=\left|\sum_{\substack{p\\ p^a\leqslant N}}\sum_{a\geqslant 2}\log p\, e\left(x\left(\frac{h_1}{p^{aj_1}}+\ldots+\frac{h_r}{p^{aj_r}}\right)\right)\right| \tag{74}$$
$$\leqslant \pi(\sqrt{N})\log N\leqslant 4\sqrt{N}.$$

Put

$$g(N)=(N^{1-cv_5(J)\Lambda(N,xH)}+N^{(j+2)/2}x^{-1/2}+N^{5/6}H^2)(\log xH)^{4J}.$$

By partial summation, Lemma 11 with (73) and (74) gives, using $\sum_{n\leqslant N}\Lambda(n)\leqslant 2N$ on the way,

$$
\begin{aligned}
&\left|\sum_{p\leqslant N} e\left(x\left(\frac{h_1}{p^{j_1}}+\ldots+\frac{h_r}{p^{j_r}}\right)\right)\right| \\
&\quad\leqslant \left|\sum_{p\leqslant N} \log p\, e\left(x\left(\frac{h_1}{p^{j_1}}+\ldots+\frac{h_r}{p^{j_r}}\right)\right)\right|\frac{1}{\log N} \\
&\qquad+\left|\int_2^N \sum_{p\leqslant t} \log p\, e\left(x\left(\frac{h_1}{p^{j_1}}+\ldots+\frac{h_r}{p^{j_r}}\right)\right)\frac{dt}{t(\log t)^2}\right| \\
&\quad\leqslant \frac{1}{\log N}\left|\sum_{n\leqslant N} \Lambda(n) e\left(x\left(\frac{h_1}{n^{j_1}}+\ldots+\frac{h_r}{n^{j_r}}\right)\right)\right|+\frac{4\sqrt{N}}{\log N} \\
&\qquad+\int_2^N\left|\sum_{n\leqslant t} \Lambda(n) e\left(x\left(\frac{h_1}{n^{j_1}}+\ldots+\frac{h_r}{n^{j_r}}\right)\right)\frac{1}{t(\log t)^2}\right| dt+\int_2^N \frac{4\,dt}{\sqrt{t}\,(\log t)^2} \\
&\quad\leqslant C^J\frac{g(N)}{\log N}+\frac{20\sqrt{N}}{\log N}+2\int_2^{\exp(CJ(\log 2J)^3)} \frac{dt}{(\log t)^2} \\
&\qquad+C^J\int_{\exp(CJ(\log 2J)^3)}^N \frac{g(t)}{t(\log t)^2}\,dt \\
&\quad\leqslant C^J g(N)+C^J(\log xH)^{4J}\left(\int_2^N t^{-cv_5(J)\Lambda(t,xH)}\,dt\right. \\
&\qquad\left.+x^{-1/2}\int_2^N t^{j/2}\,dt+H^2\int_2^N t^{-1/6}\,dt\right) \\
&\quad\leqslant C^J g(N)+C^J(\log xH)^{4J}\left(\sqrt{N}+\int_{\sqrt{N}}^N t^{-cv_5(J)\Lambda(t,xH)}\,dt\right. \\
&\qquad\left.+N^{(j+2)/2}x^{-1/2}+N^{5/6}H^2\right) \\
&\quad\leqslant C^J g(N)+C^J(\log xH)^{4J}(\sqrt{N}+N^{-cv_5(J)\Lambda(\sqrt{N},xH)}N) \\
&\quad\leqslant C^J g(N).
\end{aligned}
$$

This completes the proof of Proposition 1.

6. Vinogradov's Fourier series method

The following method may be found in [16, p. 32] or [1, Lemma 2.1].

Let $0<\Delta<\frac{1}{4}$. For $J\in\mathbf{N}$ and real numbers A_j and B_j $(1\leqslant j\leqslant J)$ with $0\leqslant B_j-A_j\leqslant 1-2\Delta$, there are 1-periodic functions $\psi_j(z)$, satisfying

$$
\psi_j(z)=\begin{cases} 1 & \text{for } A_j\leqslant z\leqslant B_j, \\ 0 & \text{for } B_j+\Delta\leqslant z\leqslant 1+A_j-\Delta, \end{cases}
$$

and $0\leqslant\psi_j(z)\leqslant 1$ for all z such that

$$\psi_j(z)=B_j-A_j+\Delta+\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} a_{m,j}e(mz), \tag{75}$$

where $a_{m,j}\in\mathbf{C}$ and, for $|m|>0$ and $1\leqslant j\leqslant J$,

$$|a_{m,j}|\leqslant\frac{1}{m^2\Delta}. \tag{76}$$

PROPOSITION 2. *Let* $\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_J)$ *with* $0<\sigma_j\leqslant 1$ *for* $1\leqslant j\leqslant J$,

$$\exp(CJ(\log 2J)^3)\leqslant P\leqslant x^{1/J} \tag{77}$$

and

$$D(\boldsymbol{\sigma}):=D(\boldsymbol{\sigma};P,x):=\operatorname{card}\left\{p\leqslant P:\left\{\frac{x}{p^j}\right\}<\sigma_j,\ 1\leqslant j\leqslant J\right\}.$$

Then we have for arbitrary ε, $0<\varepsilon\leqslant\frac{1}{12}$,

$$|D(\boldsymbol{\sigma})-\sigma_1\cdot\ldots\cdot\sigma_J\pi(P)|\leqslant C^J(P^{1-c\varepsilon v_6(J)\Lambda(P,x)}+P^{(J+2)/2+\varepsilon}x^{-1/2})(\log x)^{4J}.$$

Proof. For $\mathbf{A}=\{A_1,\ldots,A_J\}$, $\mathbf{B}=\{B_1,\ldots,B_J\}$, let

$$T(\mathbf{A},\mathbf{B})=\operatorname{card}\left\{p\leqslant P:A_j\leqslant\left\{\frac{x}{p^j}\right\}\leqslant B_j,\ 1\leqslant j\leqslant J\right\}.$$

Then

$$T(\mathbf{A},\mathbf{B})\leqslant\sum_{p\leqslant P}\left(\prod_{j=1}^{J}\psi_j\left(\frac{x}{p^j}\right)\right)\leqslant T(\mathbf{A}-\boldsymbol{\Delta},\mathbf{B}+\boldsymbol{\Delta}), \tag{78}$$

where $\boldsymbol{\Delta}=(\Delta,\ldots,\Delta)$.

By (75),

$$\begin{aligned}\prod_{j=1}^{J}\psi_j\left(\frac{x}{p^j}\right)&-\prod_{j=1}^{J}(B_j-A_j+\Delta)\\ &=\sum_{\varnothing\neq\Gamma\subseteq\{1,\ldots,J\}}\prod_{j\notin\Gamma}(B_j-A_j+\Delta)\prod_{j\in\Gamma}\left(\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}a_{m,j}e\left(\frac{mx}{p^j}\right)\right).\end{aligned} \tag{79}$$

By (76),

$$\begin{aligned}\left|\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}a_{m,j}e\left(\frac{mx}{p^j}\right)-\sum_{0<|m|<\Delta^{-2}}a_{m,j}e\left(\frac{mx}{p^j}\right)\right|&\leqslant\left|\sum_{|m|\geqslant\Delta^{-2}}a_{m,j}e\left(\frac{mx}{p^j}\right)\right|\\ &\leqslant\left|\sum_{|m|\geqslant\Delta^{-2}}\frac{1}{m^2\Delta}\right|\leqslant 2\Delta.\end{aligned} \tag{80}$$

Define L to be the right hand side of (79). The preceding inequality yields for some $|\tilde{c}|\leqslant 2$

$$L=\sum_{r=1}^{J}\sum_{1\leqslant j_1<\ldots<j_r\leqslant J}\prod_{i=1}^{r}\left(\tilde{c}\Delta+\sum_{0<|m_i|<\Delta^{-2}}a_{m_i,j_i}e\left(\frac{m_i x}{p^{j_i}}\right)\right),$$

hence

$$\left|L-\sum_{r=1}^{J}\sum_{1\leqslant j_1<\ldots<j_r\leqslant J}\prod_{i=1}^{r}\left(\sum_{0<|m_i|<\Delta^{-2}}a_{m_i,j_i}e\left(\frac{m_i x}{p^{j_i}}\right)\right)\right|\leqslant C^J\Delta.$$

We get by (76), (79) and Proposition 1

$$\begin{aligned}&\left|\sum_{p\leqslant P}\left(\prod_{j=1}^{J}\psi_j\left(\frac{x}{p^j}\right)\right)-\sum_{p\leqslant P}\left(\prod_{j=1}^{J}(B_j-A_j+\Delta)\right)\right|\\&\quad\leqslant C^J\max_{1\leqslant j_1<\ldots<j_r\leqslant J}\left|\sum_{0<|m_1|<\Delta^{-2}}\cdots\sum_{0<|m_r|<\Delta^{-2}}a_{m_1,j_1}\cdots a_{m_r,j_r}\right.\\&\qquad\left.\times\sum_{p\leqslant P}e\left(x\left(\frac{m_1}{p^{j_1}}+\ldots+\frac{m_r}{p^{j_r}}\right)\right)\right|+C^JP\Delta\\&\quad\leqslant C^J((P^{1-cv_5(J)\Lambda(P,x\Delta^{-2})}+P^{(J+2)/2}x^{-1/2}+P^{5/6}\Delta^{-4})\\&\qquad\times(\log x\Delta^{-2})^{4J}\Delta^{-J}+P\Delta).\end{aligned}\tag{81}$$

Choose

$$\Delta=P^{-\gamma\Lambda(P,x)},$$

where

$$\gamma:=\gamma(J):=cv_6(J)\varepsilon,$$

and here c is the constant occurring in (81). Then by (77),

$$x\Delta^{-2}=xP^{2\gamma\Lambda(P,x)}\leqslant x^{1+(2\gamma/J)\Lambda(P,x)}\leqslant x^2,$$

thus

$$(\log x\Delta^{-2})^{4J}\leqslant C^J(\log x)^{4J}$$

and

$$P^{1-cv_5(J)\Lambda(P,x\Delta^{-2})}\leqslant P^{1-\frac{1}{4}cv_5(J)\Lambda(P,x)}.$$

Since $\varepsilon\leqslant\frac{1}{12}$, we get

$$\begin{aligned}P^{1-cv_5(J)\Lambda(P,x\Delta^{-2})}\Delta^{-J}&\leqslant P^{1-\frac{1}{4}cv_5(J)\Lambda(P,x)+J\gamma\Lambda(P,x)}\\&\leqslant P^{1-v_5(J)\Lambda(P,x)(c/4-c/12)}\leqslant P^{1-\frac{1}{6}cv_5(J)\Lambda(P,x)}.\end{aligned}$$

Clearly,

$$P^{(J+2)/2}\Delta^{-J} \leqslant P^{(J+2)/2+\varepsilon}$$

and

$$P^{5/6}\Delta^{-(J+4)} \leqslant P^{5/6+\varepsilon} \leqslant P^{11/12} \leqslant P^{1-c_5(J)\Lambda(P,x)}.$$

Finally,

$$P\Delta \leqslant P^{1-c\varepsilon v_6(J)\Lambda(P,x)}.$$

Applying all these estimates in (81), we obtain

$$\left|\sum_{p\leqslant P}\left(\prod_{j=1}^{J}\psi_j\left(\frac{x}{p^j}\right)\right)-\left(\prod_{j=1}^{J}(B_j-A_j+\Delta)\right)\pi(P)\right| \leqslant C^J\left(P^{1-c\varepsilon v_6(J)\Lambda(P,x)}+P^{(J+2)/2+\varepsilon}x^{-1/2}\right)(\log x)^{4J}.$$

Let R denote the right hand side of the last inequality. Then, by (78),

$$T(\mathbf{A},\mathbf{B}) \leqslant \pi(P)\prod_{j=1}^{J}(B_j-A_j+\Delta)+R \tag{82}$$

and

$$T(\mathbf{A}-\boldsymbol{\Delta},\mathbf{B}+\boldsymbol{\Delta}) \geqslant \pi(P)\prod_{j=1}^{J}(B_j-A_j+\Delta)-R. \tag{83}$$

Replacing $\mathbf{A},\mathbf{B}$ by $\mathbf{A}-\boldsymbol{\Delta},\mathbf{A}$ and $\mathbf{B},\mathbf{B}+\boldsymbol{\Delta}$, respectively, (82) implies

$$T(\mathbf{A}-\boldsymbol{\Delta},\mathbf{A}) \leqslant (2\Delta)^J\pi(P)+R$$

and

$$T(\mathbf{B},\mathbf{B}+\boldsymbol{\Delta}) \leqslant (2\Delta)^J\pi(P)+R,$$

respectively. Thus, by (83),

$$\begin{aligned} T(\mathbf{A},\mathbf{B}) &= T(\mathbf{A}-\boldsymbol{\Delta},\mathbf{B}+\boldsymbol{\Delta})-T(\mathbf{A}-\boldsymbol{\Delta},\mathbf{A})-T(\mathbf{B},\mathbf{B}+\boldsymbol{\Delta}) \\ &\geqslant \pi(P)\prod_{j=1}^{J}(B_j-A_j+\Delta)-3R-2(2\Delta)^J\pi(P) \\ &\geqslant \pi(P)\prod_{j=1}^{J}(B_j-A_j)-6R. \end{aligned}$$

Similarly, we get by (82)

$$T(\mathbf{A},\mathbf{B}) \leqslant \pi(P)\prod_{j=1}^{J}(B_j-A_j)+6R.$$

Together, we have

$$\left| T(\mathbf{A},\mathbf{B}) - \pi(P) \prod_{j=1}^{J} (B_j - A_j) \right| \leqslant 6R.$$

Setting $A_j = 0$, $B_j = \sigma_j$ $(1 \leqslant j \leqslant J)$, the desired result follows by observing that

$$D(\boldsymbol{\sigma}; P, x) = T(\mathbf{A},\mathbf{B}).$$

7. Proof of the theorem

Let m and n be positive integers, and p a prime. We define $U_p(m,n)$ to be the number of "carries" which occur when adding m and n in p-ary notation. Let $e(n;p)$ be defined as in (3). An old result of Kummer is the following

LEMMA 12 ([9, p. 116]).

$$e\left(\binom{m+n}{m}; p\right) = U_p(m,n).$$

PROPOSITION 3. *Let $J > C_0$ and*

$$N_0 = C^{J^{10}(\log J)^3}, \tag{84}$$

where C_0 and C are some absolute constants. For all $n \geqslant N_0$, there is a prime p such that

$$p^J \left| \binom{2n}{n} \right..$$

Proof. We apply Proposition 2 with $x := n \geqslant N_0$, $\varepsilon := \frac{1}{12}$, $P := n^{1/(J+1)}$, and obtain for

$$n \geqslant \exp(CJ^2(\log 2J)^3),$$

which is guaranteed by (84), that

$$K_J(n) := \operatorname{card}\left\{ p < n^{1/(J+1)} : \frac{2}{3} < \left\{ \frac{n}{p^j} \right\},\ 1 \leqslant j \leqslant J \right\}$$

satisfies

$$\begin{aligned} &\left| K_J(n) - \left(\tfrac{1}{3}\right)^J \pi\left(n^{1/(J+1)}\right) \right| \\ &\quad \leqslant C^J \left(n^{1/(J+1) - c\varepsilon v_7(J) \Lambda(n^{1/(J+1)}, n)} + n^{(J+2+2\varepsilon)/2(J+1)} n^{-1/2} \right) (\log n)^{4J} \\ &\quad \leqslant C^J \left(n^{1/(J+1) - c v_9(J)} + n^{7/12(J+1)} \right) (\log n)^{4J} \\ &\quad \leqslant C^J n^{1/(J+1) - c v_9(J)} (\log n)^{4J}. \end{aligned} \tag{85}$$

By Chebyshev's theorem (see [2, p. 54]),

$$\pi(n^{1/(J+1)}) \geqslant \tfrac{1}{2}(J+1)\frac{n^{1/(J+1)}}{\log n}.$$

Together with (85), we have

$$K_J(n) \geqslant c^J \frac{n^{1/(J+1)}}{\log n} - C^J n^{1/(J+1)-cv_9(J)}(\log n)^{4J}. \tag{86}$$

We wish to show that $K_J(n)$ is positive for sufficiently large n. Obviously, it suffices to prove that

$$C\log n < n^{\tilde{c}v_{10}(J)} = n^{2\mu}, \tag{87}$$

where

$$\mu := \tfrac{1}{2}\tilde{c}v_{10}(J) < \frac{1}{e}, \tag{88}$$

without loss of generality. For $n > C^{1/\mu}$, which is guaranteed by (84), we clearly have

$$C < n^{\mu}. \tag{89}$$

Assume that for some $y > e$, there is a γ such that

$$\frac{\log\log y}{\log y - 1} \leqslant \gamma < 1. \tag{90}$$

Then

$$\log(1+\gamma) + \log\log y \leqslant \gamma + \log\log y < \gamma\log y,$$

hence

$$y^{1+\gamma} \leqslant \exp(y^{\gamma}).$$

By (88), $y := 1/\mu$ and

$$\gamma := \gamma(\mu) := \frac{\log\log 1/\mu}{\log 1/\mu - 1}$$

satisfy (90). Thus

$$\left(\frac{1}{\mu}\right)^{1+\gamma(\mu)} \leqslant \exp\left(\left(\frac{1}{\mu}\right)^{\gamma(\mu)}\right),$$

in other words, for

$$N_1 := \exp\left(\frac{1}{\mu^{1+\gamma(\mu)}}\right),$$

we have

$$\log N_1 = \left(\frac{1}{\mu}\right)^{1+\gamma(\mu)} \leqslant \exp\left(\frac{1}{\mu^{\gamma(\mu)}}\right) = N_1^{\mu}.$$

Hence the function

$$f(x)=\frac{x^{\mu}}{\log x}$$

satisfies $f(N_1)\geqslant 1$. Moreover, $f'(x)\geqslant 0$ for $x\geqslant\exp(1/\mu)$, thus $f(x)$ is increasing in this range. Since $N_1\geqslant\exp(1/\mu)$, we conclude that for $n\geqslant N_1$

$$\log n\leqslant n^{\mu}.$$

Then (89) and (88) imply that (87) holds for

$$n>C^{1/\mu^{1+\gamma(\mu)}}. \tag{91}$$

If (91) holds, we then have by (86) that $K_J(n)>0$, which means that there is a prime p satisfying

$$p^{J+1}<n$$

and

$$\left\{\frac{n}{p^j}\right\}>\frac{2}{3}, \quad 1\leqslant j\leqslant J. \tag{92}$$

In order to make sure that the last conclusion holds without additional assumptions, it suffices to show that (91) is satisfied. For this reason let $\tilde{c}$ and $\widetilde{C}$ be arbitrary positive constants. For a sufficiently large constant C only depending on $\tilde{c}$ and $\widetilde{C}$, we have

$$\begin{aligned}\log(J^{10}(\log J)^3\log C)&\geqslant\left(1+\frac{\log\log J^{10}}{\log J^{10}}\right)\log\left(\frac{2}{\tilde{c}}J^{10}(\log 2J)^2\log\widetilde{C}\right)\\&\geqslant(1+\gamma(\mu))\log\left(\frac{2}{\tilde{c}}J^{10}(\log 2J)^2\log\widetilde{C}\right).\end{aligned}$$

This means

$$C^{J^{10}(\log J)^3}\geqslant\widetilde{C}^{((2/\tilde{c})J^{10}(\log 2J)^2)^{1+\gamma(\mu)}}=\widetilde{C}^{(1/\mu)^{1+\gamma(\mu)}}.$$

Hence, (84) yields (91).

Now write n in p-ary notation, namely

$$n=n_Jp^J+n_{J-1}p^{J-1}+\ldots+n_1p+n_0, \quad 0\leqslant n_j<p.$$

For $1\leqslant j\leqslant J$, we have by (92)

$$\frac{2}{3}<\left\{\frac{n}{p^j}\right\}=\frac{n_{j-1}p^{j-1}+\ldots+n_0}{p^j},$$

thus

$$\frac{n_{j-1}}{p}>\frac{2}{3}-(p-1)\left(\frac{1}{p^2}+\ldots+\frac{1}{p^j}\right)>\frac{2}{3}-\frac{1}{p}.$$

This implies for $p \geqslant 7$

$$n_{j-1} > \tfrac{1}{2}p,$$

i.e.

$$n_j > \tfrac{1}{2}p, \quad 0 \leqslant j \leqslant J-1. \tag{93}$$

It follows from this that we get at least J carries when adding $n+n$ in p-ary notation. Hence, by Lemma 12,

$$e\left(\binom{2n}{n}; p\right) \geqslant J,$$

which means that there is a p satisfying $p^J \bigm| \binom{2n}{n}$. This completes the proof of Proposition 3.

Proof of the theorem. For C and C_0 being the constants of Proposition 3, let n be large enough such that for some $a > C_0$

$$C^{a^{10}(\log a)^3} \leqslant n < C^{(a+1)^{10}(\log(a+1))^3}. \tag{94}$$

Then we obtain by Proposition 3

$$E(n) = \max\left\{J : p^J \,\middle|\, \binom{2n}{n} \text{ for some prime } p\right\} \geqslant a. \tag{95}$$

By (94),

$$a \gg \left(\frac{\log n}{(\log a)^3}\right)^{1/10}$$

and

$$\log a \ll \log\log n.$$

Thus (95) yields

$$E(n) \gg \left(\frac{\log n}{(\log\log n)^3}\right)^{1/10},$$

which proves the theorem.

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JÜRGEN W. SANDER
Institut für Mathematik
Universität Hannover
Welfengarten 1
D-30167 Hannover
Germany
sander@math.uni-hannover.de

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