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The Wiener test and potential estimates for quasilinear elliptic equations

by

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1. Introduction

Let Ω be an open set in $\mathbf{R}^n$ and let $1<p\leqslant n$ be a fixed number. Consider the quasilinear partial differential operator

$$Tu=-\operatorname{div}\mathcal{A}(x,\nabla u),$$

where $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $\mathcal{A}(x,\xi)\cdot\xi\approx|\xi|^p$; the precise assumptions on $\mathcal{A}$ are listed in Section 2. The principal model operator is the p-Laplacian

$$Tu=-\Delta_p u=-\operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

and so the ordinary Laplacian $\Delta=\Delta_2$ is included in our study.

A boundary point x_0 of bounded Ω is *regular* if the solution u to the Dirichlet problem

$$\begin{cases} Tu=0 & \text{in } \Omega \\ u-f\in W^{1,p}_0(\Omega) \end{cases}$$

has the limit value $f(x_0)$ at x_0 whenever $f\in W^{1,p}(\Omega)$ is continuous in the closure of Ω. In [23] Wiener proved that in the case of the Laplacian the regularity of a boundary point $x_0\in\partial\Omega$ can be characterized by a so called Wiener test, where one measures the thickness of the complement of Ω near x_0 in terms of capacity densities; we soon come to the precise formulation of this test. In the fundamental work [17] Littman, Stampacchia, and Weinberger showed that the same Wiener test identifies the regular boundary points whenever T is a uniformly elliptic linear operator with bounded measurable coefficients; then the regularity of a boundary point is independent of the particular operator.

For general nonlinear operators the classical Wiener test has to be modified so that the type p of the operator T is involved. Maz'ya [18] established in 1970 that the boundary point x_0 is regular if

$$W_p(\mathbf{R}^n\setminus\Omega, x_0)=+\infty,$$

where $W_p(\mathbf{R}^n\setminus\Omega, x_0)$ is a Wiener type integral defined for an arbitrary set E by

$$W_p(E,x_0)=\int_0^1\left(\frac{\operatorname{cap}_p(B(x_0,t)\cap E, B(x_0,2t))}{\operatorname{cap}_p(B(x_0,t), B(x_0,2t))}\right)^{1/(p-1)}\frac{dt}{t},$$

and $\operatorname{cap}_p(E,G)$ is the p-capacity of a set E in G (see Section 3 for the definition). Later Gariepy and Ziemer [5] extended this result to a very general class of equations.

The question whether regular boundary points of Ω can be characterized by using the Wiener test has been a well known open problem in nonlinear potential theory; see e.g. [1]. The problem was partly solved in the affirmative when Lindqvist and Martio [16] proved that if p equals n, the dimension of the underlying space, the divergence of the integral $W_n(\mathbf{R}^n\setminus\Omega, x_0)$ is not only sufficient but also necessary for the regularity of x_0. Unfortunately, their method cannot be extended to cover all values $1<p\leqslant n$; it worked only for $p>n-1$.

In this paper we establish the necessity part of the Wiener test for all $p\in(1,n]$ and prove:

THEOREM 1.1. *A finite boundary point $x_0\in\partial\Omega$ is regular if and only if*

$$W_p(\mathbf{R}^n\setminus\Omega, x_0)=\infty.$$

An immediate corollary is:

COROLLARY 1.2. *The regularity depends only on n and p, not on the operator T itself.*

Note that no boundedness assumption on Ω was made in the theorems above, for we extend the definition of regularity for boundary points of unbounded sets in Section 5.3 below. Also observe that the similar question could be asked also for $p>n$. However, then all points are regular and the corresponding Wiener integral always diverges because singletons are of positive p-capacity; see [10, Chapter 6 or 9].

The uniformly elliptic linear equations are included in our presentation; hence we extend the result in [17]; no Green's function is involved in our proof. Let us also point out that our methods can be applied to the equations with weights so that the results of this paper are easily generalized to cover the equations considered in [10].

There is another variant of the Wiener criterion problem, known among specialists in nonlinear potential theory. A set $E\subset\mathbf{R}^n$ is said to be *p-thin* at a point $x_0\in\mathbf{R}^n$ if

$W_p(E,x_0)<+\infty$. This concept of thinness was first considered in nonlinear potential theory by Adams and Meyers [3]. See also [2], [6], [9], and the references therein. Note that because each singleton is of p-capacity zero it does not have any effect on the p-thinness of E whether or not the point x_0 is in E. Also it is trivial that E is p-thin at each point in the complement of $\overline{E}$. How is p-thinness related to $\mathcal{A}$-superharmonic functions (defined in Section 2)? An interesting answer to this question was given in [9], where the sets that are p-thin at x_0 were characterized as those sets whose complements are $\mathcal{A}$-fine neighborhoods of x_0; here $\mathcal{A}$-fine refers to the fine topology of $\mathcal{A}$-superharmonic functions. However it remained unsolved if the p-thinness is equivalent to the so called Cartan property: "there is an $\mathcal{A}$-superharmonic function u in a neighborhood of x_0 such that

$$\liminf_{\substack{x\to x_0\\ x\in E\setminus\{x_0\}}} u(x)>u(x_0)."$$

(The sufficiency part was established in [8].) We answer affirmatively to this in the following result.

THEOREM 1.3. *Let $E\subset\mathbf{R}^n$ and $x_0\in\overline{E}\setminus E$. Then E is p-thin at x_0 if and only if there is an $\mathcal{A}$-superharmonic function u in a neighborhood of x_0 such that*

$$\liminf_{\substack{x\to x_0\\ x\in E}} u(x)>u(x_0). \tag{1.4}$$

The proofs of Theorems 1.1 and 1.3 are based on pointwise estimates of solutions to

$$Tu=\mu \tag{1.5}$$

with a Radon measure μ on the right side. In [14] we established estimates for $\mathcal{A}$-superharmonic solutions of (1.5) in terms of the Wolff potential

$$\mathbf{W}^{\mu}_{1,p}(x_0,r)=\int_0^r\left(\frac{\mu(B(x_0,t))}{t^{n-p}}\right)^{1/(p-1)}\frac{dt}{t}.$$

One easily infers that $\mathbf{W}^{\mu}_{1,2}(x_0,\infty)$ is the Newtonian potential of μ. This estimation gives a solid link between the two nonlinear potential theories; cf. [2], [6], and [10].

In [14] we were able to control the solution from above only when $p>n-1$. In our second main theorem we dispense with this restriction and derive an estimate which improves that in [14] even for $p>n-1$.

THEOREM 1.6. *Suppose that u is a nonnegative $\mathcal{A}$-superharmonic function in $B(x_0,3r)$. If $\mu=Tu$, then*

$$c_1\mathbf{W}^{\mu}_{1,p}(x_0;r)\leqslant u(x_0)\leqslant c_2\inf_{B(x_0,r)}u+c_3\mathbf{W}^{\mu}_{1,p}(x_0;2r),$$

where c_1, c_2, *and* c_3 *are positive constants, depending only on* n, p, *and the structural constants* α *and* β.

In particular, $u(x_0)<\infty$ *if and only if* $\mathbf{W}^{\mu}_{1,p}(x_0;r)<\infty$.

In [12] it was indicated that the necessity of the Wiener test follows from an estimate like that in Theorem 1.6. In the present paper we choose another route, more natural and direct.

Moreover, we deduce from Theorem 1.6 a Harnack inequality for positive solutions to (1.5), where the measure μ satisfies for some positive constants c and ε

$$\mu(B(x,r))\leqslant cr^{n-p+\varepsilon} \tag{1.7}$$

whenever $B(x,r)$ is a ball. Iterating the Harnack inequality in a standard way one sees that the solutions are Hölder continuous; moreover, we show that if the solution of $Tu=\mu$ is Hölder continuous, then μ satisfies a restriction like (1.7). That (1.7) is almost equivalent to Hölder continuity was first observed by Rakotoson and Ziemer [20]. Our result extends theirs, for they imposed an additional strong monotonicity assumption on the operator T. While writing up the manuscript we learned that Gary Lieberman independently has arrived at a Harnack inequality for solutions to (1.5), (1.7).(1)

As a further consequence of Theorem 1.6 we characterize continuous $\mathcal{A}$-superharmonic functions in terms of the corresponding Wolff potentials.

Our method is applicable to other problems as well. To illustrate this we apply our results and verify that the regular points for the obstacle problem coincide with the Wiener points of the obstacle (Theorem 5.7); this result was partially proved in [19] and [8]. The similar problem for double obstacle problems (cf. [15]) can also be treated so that the main result of [4] is extended to nonlinear operators.

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Notation. Our notation is standard. Throughout the paper we let Ω be an open set in $\mathbf{R}^n$ and $1<p\leqslant n$ a fixed number. The letter c stands for various constants. For an open (closed) ball $B=B(x_0,r)$ ($\overline{B}=\overline{B}(x_0,r)$) with radius r and and center x_0 and $\sigma>0$, we write σB ($\sigma\overline{B}$) for the open (closed) ball with radius σr and center x_0. The barred integral sign $⨍_E f\,dx$ stands for the integral average $|E|^{-1}\int_E f\,dx$, where $|E|$ is Lebesgue measure of E.

(1) See *Comm. Partial Differential Equations*, 18 (1993), 1191–1212

2. Preliminaries

We assume throughout this paper that $\mathcal{A}:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ is a mapping which satisfies the following assumptions for some constants $0<\alpha\leqslant\beta<\infty$:

$$\begin{aligned}&\text{the function } x\mapsto\mathcal{A}(x,\xi) \text{ is measurable for all } \xi\in\mathbf{R}^n, \text{ and}\\ &\text{the function } \xi\mapsto\mathcal{A}(x,\xi) \text{ is continuous for a.e. } x\in\mathbf{R}^n;\end{aligned}\tag{2.1}$$

for all $\xi\in\mathbf{R}^n$ and a.e. $x\in\mathbf{R}^n$:

$$\mathcal{A}(x,\xi)\cdot\xi\geqslant\alpha|\xi|^p,\tag{2.2}$$

$$|\mathcal{A}(x,\xi)|\leqslant\beta|\xi|^{p-1},\tag{2.3}$$

$$(\mathcal{A}(x,\xi)-\mathcal{A}(x,\zeta))\cdot(\xi-\zeta)>0\tag{2.4}$$

whenever $\xi\neq\zeta$, and

$$\mathcal{A}(x,\lambda\xi)=\lambda|\lambda|^{p-2}\mathcal{A}(x,\xi)\tag{2.5}$$

for all $\lambda\in\mathbf{R}$, $\lambda\neq0$.

The operator T is defined such that for each $\varphi\in C_0^\infty(\Omega)$

$$Tu(\varphi)=\int_\Omega\mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,dx,$$

where $u\in W^{1,p}_{\text{loc}}(\Omega)$. In other words

$$Tu=-\operatorname{div}\mathcal{A}(x,\nabla u)$$

in the sense of distributions.

A solution $u\in W^{1,p}_{\text{loc}}(\Omega)$ to the equation

$$Tu=0\tag{2.6}$$

always has a continuous representative; we call continuous solutions $u\in W^{1,p}_{\text{loc}}(\Omega)\cap C(\Omega)$ of (2.6) *$\mathcal{A}$-harmonic* in Ω.

A lower semicontinuous function $u:\Omega\to(-\infty,\infty]$ is called *$\mathcal{A}$-superharmonic* if u is not identically infinite in each component of Ω, and if for all open $D\subset\subset\Omega$ and all $h\in C(\overline{D})$, $\mathcal{A}$-harmonic in D, $h\leqslant u$ on ∂D implies $h\leqslant u$ in D. A function v is *$\mathcal{A}$-subharmonic* if $-v$ is $\mathcal{A}$-superharmonic.

Clearly, $\min(u,v)$ and $\lambda u+\sigma$ are $\mathcal{A}$-superharmonic if u and v are, and $\sigma,\lambda\in\mathbf{R}$, $\lambda\geqslant0$. The following proposition connects $\mathcal{A}$-superharmonic functions with supersolutions of (2.6).

PROPOSITION 2.7 [7]. (i) *If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is such that $Tu \geqslant 0$, then there is an $\mathcal{A}$-superharmonic function v such that $u=v$ a.e. Moreover,*

$$v(x) = \operatorname{ess} \liminf_{y \to x} v(y) \quad \textit{for all } x \in \Omega. \tag{2.8}$$

(ii) *If v is $\mathcal{A}$-superharmonic, then* (2.8) *holds. Moreover, $Tv \geqslant 0$ if $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$.*

(iii) *If v is $\mathcal{A}$-superharmonic and locally bounded, then $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $Tv \geqslant 0$.*

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be an $\mathcal{A}$-superharmonic function in Ω. Then it follows from Proposition 2.7 that $\mu = Tu$ is a nonnegative Radon measure on Ω. If Ω' is an open subset of Ω with $u \in W^{1,p}(\Omega')$, the restriction ν of μ to Ω' belongs to the dual space $(W^{1,p}_0(\Omega'))'$ of $W^{1,p}_0(\Omega')$. By a standard approximation we see that

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega'} \varphi \, d\mu$$

for any test function $\varphi \in W^{1,p}_0(\Omega')$, where the last integral is the duality pairing between $\varphi \in W^{1,p}_0(\Omega')$ and $\nu \in (W^{1,p}_0(\Omega'))'$.

For the reader's convenience we record here an appropriate form of Trudinger's weak Harnack inequality (see [14, 3.2], [10, 3.59] or [22], and Proposition 2.7 above).

LEMMA 2.9. *Let $B = B(x_0, r)$ and let u be a nonnegative $\mathcal{A}$-superharmonic function in $3B$. If $q > 0$ is such that $q(n-p) < n(p-1)$, then*

$$\left(\fint_{2B} u^q \, dx \right)^{1/q} \leqslant c \inf_B u,$$

where $c = c(n, p, q, \alpha, \beta) > 0$.

3. $\mathcal{A}$-potentials and capacity estimates

In this section we recall the definition of p-capacity and $\mathcal{A}$-potentials, and discuss their relations.

3.1. *p-capacity.* First we define the p-capacity and record some facts that can be found e.g. from [10, Chapters 2 and 4].

For a compact subset K of Ω we let

$${}_*\mathrm{cap}_p(K, \Omega) = \inf \int_\Omega |\nabla u|^p \, dx,$$

where u runs through all $u\in C_0^\infty(\Omega)$ with $u\geqslant 1$ on K. The *p-capacity* of an arbitrary set $E\subset\Omega$ in Ω is

$$\mathrm{cap}_p(E,\Omega)=\inf_{\substack{G\subset\Omega \text{ open}\\ E\subset G}}\ \sup_{\substack{K\subset G\\ K \text{ compact}}}\ {}_*\mathrm{cap}_p(K,\Omega).$$

Then $\mathrm{cap}_p(\cdot,\Omega)$ is a Choquet capacity and

$${}_*\mathrm{cap}_p(K,\Omega)=\mathrm{cap}_p(K,\Omega)$$

if K is compact.

If $r>0$ and $2r\leqslant R\leqslant 100r$, then there is a positive constant c, depending only on n and p such that for all $x\in\mathbf{R}^n$

$$c^{-1}r^{n-p}\leqslant \mathrm{cap}_p(B(x,r),B(x,R))\leqslant cr^{n-p}.$$

We say that a set E is of *p-capacity zero* if

$$\mathrm{cap}_p(E\cap B,2B)=0$$

whenever B is an open ball in $\mathbf{R}^n$. Equivalently, E is of p-capacity zero if and only if

$$\mathrm{cap}_p(E\cap\Omega,\Omega)=0$$

for all open sets Ω. Moreover, for $p<n$ this is further equivalent to

$$\mathrm{cap}_p(E,\mathbf{R}^n)=0.$$

We say that a property holds *p-quasieverywhere* in Ω if it holds in Ω except on a set of p-capacity zero.

It is well known that each function $u\in W^{1,p}(\Omega)$ has a representative for which the limit

$$\lim_{r\to 0} \fint_{B(x,r)} u\,dy \tag{3.2}$$

exists and equals $u(x)$ p-quasieverywhere in Ω [24]. These representatives are called p-refined. In what follows we usually consider only the p-refined representatives of functions in $W^{1,p}(\Omega)$; note that for a locally bounded $\mathcal{A}$-superharmonic function u, the limit in (3.2) exists and is equal to $u(x)$ for every x [10, 3.65]. Moreover, we use the fact that for $E\subset\Omega$

$$\mathrm{cap}_p(E,\Omega)=\inf\int_\Omega |\nabla u|^p\,dx,$$

where the infimum is taken either over all $u\in W_0^{1,p}(\Omega)$ such that $u=1$ in an open neighborhood of E, or over all p-refined $u\in W_0^{1,p}(\Omega)$ such that $u\geqslant 1$ p-quasieverywhere on E.

3.3. *$\mathcal{A}$-potentials.* Suppose that E be a subset of Ω. For $x \in \Omega$ let

$$R_E^1(\Omega;\mathcal{A})(x) = \inf u(x),$$

where the infimum is taken over all nonnegative $\mathcal{A}$-superharmonic functions u in Ω such that $u \geqslant 1$ on E. The lower semicontinuous regularization

$$\widehat{R}_E^1(\Omega;\mathcal{A})(x) = \lim_{r\to 0} \inf_{B(x,r)} R_E^1(\Omega;\mathcal{A})$$

of $R_E^1(\Omega;\mathcal{A})$ is called the *$\mathcal{A}$-potential* of E in Ω. The $\mathcal{A}$-potential $\widehat{R}_E^1(\Omega;\mathcal{A})$ is $\mathcal{A}$-superharmonic in Ω and $\mathcal{A}$-harmonic in $\Omega\setminus\overline{E}$.

If Ω is bounded and $E \subset\subset \Omega$, then the $\mathcal{A}$-potential u of E belongs to $W_0^{1,p}(\Omega)$ and

$$\operatorname{cap}_p(E,\Omega) \leqslant \int_\Omega |\nabla u|^p\,dx \leqslant \left(\frac{\beta}{\alpha}\right)^p \operatorname{cap}_p(E,\Omega);$$

(see the proof of [8, 2.2], [10, 9.35, 9.38]).

3.4. *A dual approach to capacity.* Let Ω be bounded. If μ is a Radon measure in the dual $(W_0^{1,p}(\Omega))'$ of $W_0^{1,p}(\Omega)$, we write u_μ for the $\mathcal{A}$-superharmonic function in Ω such that $u_\mu \in W_0^{1,p}(\Omega)$ and $Tu_\mu = \mu$. The existence and uniqueness of u_μ are well known; cf. [18, Proposition 1] and Proposition 2.7. The function u_μ can be regarded as the *$\mathcal{A}$-potential of μ.*

For $E \subset \Omega$ we define

$$\mathcal{C}_{\mathcal{A}}(E,\Omega) = \sup\{\mu(\Omega) : \mu \in (W_0^{1,p}(\Omega))',\ \operatorname{supp}\mu \subset E \text{ and } u_\mu \leqslant 1\}.$$

THEOREM 3.5. *Suppose that Ω is bounded and $E \subset \Omega$ is a Borel (or capacitable) set. Then*

$$\operatorname{cap}_p(E,\Omega) \leqslant \frac{1}{\alpha}\mathcal{C}_{\mathcal{A}}(E,\Omega) \leqslant \left(\frac{\beta}{\alpha}\right)^p \operatorname{cap}_p(E,\Omega).$$

Proof. We may clearly assume that E is compact. To prove the first inequality of the assertion, let $u = \widehat{R}_E^1(\Omega;\mathcal{A})$ be the $\mathcal{A}$-potential of E in Ω and $\mu = Tu$. Then we have that $u \in W_0^{1,p}(\Omega)$, $0 \leqslant u \leqslant 1$, and hence

$$\begin{aligned}\mathcal{C}_{\mathcal{A}}(E,\Omega) \geqslant \mu(\Omega) \geqslant \int_\Omega u\,d\mu &= \int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla u\,dx \\ &\geqslant \alpha \int_\Omega |\nabla u|^p\,dx \geqslant \alpha \operatorname{cap}_p(E,\Omega).\end{aligned}$$

For the second inequality, suppose that μ is a measure in $(W_0^{1,p}(\Omega))'$ such that $u_\mu \leqslant 1$ and $\operatorname{supp}\mu \subset E$. Let $v \in C_0^\infty(\Omega)$ be such that $v=1$ on E. Then

$$\begin{aligned}\mu(\Omega) &= \int_\Omega v\,d\mu = \int_\Omega \mathcal{A}(x,\nabla u_\mu)\cdot\nabla v\,dx \\ &\leqslant \beta\left(\int_\Omega |\nabla u_\mu|^p\,dx\right)^{(p-1)/p}\left(\int_\Omega |\nabla v|^p\,dx\right)^{1/p}.\end{aligned} \tag{3.6}$$

Let $v_1=\max(v,u_\mu)$. Because u_μ is $\mathcal{A}$-superharmonic in Ω and $\mathcal{A}$-harmonic in $\Omega\setminus E$, it follows that

$$\begin{aligned}\int_\Omega \mathcal{A}(x,\nabla u_\mu)\cdot\nabla(v-u_\mu)\,dx &= \int_\Omega \mathcal{A}(x,\nabla u_\mu)\cdot\nabla(v_1-u_\mu)\,dx + \int_{\Omega\setminus E}\mathcal{A}(x,\nabla u_\mu)\cdot\nabla(v-v_1)\,dx \\ &= \int_\Omega \mathcal{A}(x,\nabla u_\mu)\cdot\nabla(v_1-u_\mu)\,dx \geqslant 0,\end{aligned}$$

for $v-v_1\in W_0^{1,p}(\Omega\setminus E)$ and $v_1-u_\mu\in W_0^{1,p}(\Omega)$ is nonnegative. Hence

$$\begin{aligned}\alpha\int_\Omega |\nabla u_\mu|^p\,dx &\leqslant \int_\Omega \mathcal{A}(x,\nabla u_\mu)\cdot\nabla u_\mu\,dx \leqslant \int_\Omega \mathcal{A}(x,\nabla u_\mu)\cdot\nabla v\,dx \\ &\leqslant \beta\left(\int_\Omega |\nabla u_\mu|^p\,dx\right)^{(p-1)/p}\left(\int_\Omega |\nabla v|^p\,dx\right)^{1/p},\end{aligned}$$

so that

$$\int_\Omega |\nabla u_\mu|^p\,dx \leqslant \left(\frac{\beta}{\alpha}\right)^p \int_\Omega |\nabla v|^p\,dx.$$

Taking the infimum over all v's we infer from (3.6) that

$$\frac{1}{\alpha}\mu(E) \leqslant \left(\frac{\beta}{\alpha}\right)^p \operatorname{cap}_p(E,\Omega),$$

and the theorem follows.

The measure μ in the following lemma can be regarded as the $\mathcal{A}$-distribution of the set E. See [2] for an analogous result for another type of capacitary distributions.

LEMMA 3.7. *Suppose that Ω is bounded and $E\subset\subset\Omega$. Let $u=\widehat{R}_E^1(\Omega;\mathcal{A})$ be the $\mathcal{A}$-potential of E in Ω and $\mu=Tu$. Then*

$$\mu(U) \leqslant \frac{\beta^p}{\alpha^{p-1}}\operatorname{cap}_p(E\cap U,\Omega)$$

whenever $U\subset\Omega$ is open.

Proof. Let $G\subset\Omega$ be an open set containing $E\cap U$ and choose an increasing sequence of compact sets K_j such that $G=\bigcup_j K_j$. Let u_j be the $\mathcal{A}$-potential of $(E\cap K_j)\cup(E\setminus U)$

in Ω and $\mu_j=Tu_j$. If ν_j is the restriction to U of μ_j, then the support of ν_j is contained in G and $u_{\nu_j}\leqslant 1$ in Ω. Hence we have by Theorem 3.5 that

$$\mu_j(U)=\nu_j(U)\leqslant \frac{\beta^p}{\alpha^{p-1}}\operatorname{cap}_p(G,\Omega).$$

Since u_j increase to u (see [8, 2.2]), the measure μ is the weak limit of μ_j and therefore

$$\mu(U)\leqslant \frac{\beta^p}{\alpha^{p-1}}\operatorname{cap}_p(G,\Omega).$$

Taking the infimum over all open sets $G\supset E\cap U$ we obtain

$$\mu(U)\leqslant \frac{\beta^p}{\alpha^{p-1}}\operatorname{cap}_p(E\cap U,\Omega).$$

COROLLARY 3.8. *Suppose that Ω is bounded and $E\subset\subset\Omega$. Let $u=\widehat{R}_E^1(\Omega;\mathcal{A})$ be the $\mathcal{A}$-potential of E in Ω and $\mu=Tu$. Then*

$$\alpha\operatorname{cap}_p(E,\Omega)\leqslant\mu(\Omega)\leqslant \frac{\beta^p}{\alpha^{p-1}}\operatorname{cap}_p(E,\Omega).$$

Proof. The second inequality of the assertion follows from Lemma 3.7. For the first inequality the reader is asked to mimic the proof of the first part of Theorem 3.5.

We conclude this section with a simple lemma that is needed later.

LEMMA 3.9. *Suppose that $u\in W_0^{1,p}(\Omega)$ is $\mathcal{A}$-superharmonic with $Tu=\mu$. Then for $\lambda>0$ it holds that*

$$\lambda^{p-1}\operatorname{cap}_p(\{x\in\Omega: u(x)>\lambda\},\Omega)\leqslant\frac{\mu(\Omega)}{\alpha}.$$

Proof. Since

$$\begin{aligned}\alpha\int_\Omega|\nabla\min(u,\lambda)|^p\,dx &\leqslant\int_\Omega\mathcal{A}(x,\nabla u)\cdot\nabla\min(u,\lambda)\,dx\\ &=\int_\Omega\min(u,\lambda)\,d\mu\leqslant\lambda\mu(\Omega),\end{aligned}$$

the lemma follows, for $\min(u,\lambda)/\lambda$ is admissible to test the capacity.

4. Potential estimates

In this section we derive estimates for $\mathcal{A}$-superharmonic functions in terms of their Wolff potentials. In particular we prove Theorem 1.6. As examples of its consequences we establish a Harnack inequality for a class of positive $\mathcal{A}$-superharmonic functions, and we give necessary and sufficient conditions for $\mathcal{A}$-superharmonic solutions of $Tu=\mu$ to be Hölder continuous or continuous.

Because an $\mathcal{A}$-superharmonic function does not necessarily belong to $W^{1,p}_{\mathrm{loc}}(\Omega)$, we extend the definition for the operator T: If u is an $\mathcal{A}$-superharmonic function in Ω, then we define

$$Tu(\varphi)=\int_\Omega \lim_{k\to\infty}\mathcal{A}(x,\nabla\min(u,k))\cdot\nabla\varphi\,dx,$$

$\varphi\in C_0^\infty(\Omega)$. By [14, 1.15]

$$\lim_{k\to\infty}\mathcal{A}(x,\nabla\min(u,k))$$

is locally integrable and hence $-Tu$ is its divergence. (Since $\min(u,k)\in W^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\nabla\min(u,k)=\nabla\min(u,j)$$

a.e. in $\{u<\min(k,j)\}$, the limit exists. It is equal to $\mathcal{A}(x,\nabla u)$ if $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$, which is always the case if $p>2-1/n$.) Our definition of Tu overrides the difficulty that arises from the fact that for $p\leqslant 2-1/n$ the distributional gradient ∇u need not be a function. Indeed, the above definition of Tu is merely a technical tool to treat all p's simultaneously. We refer to [14] or [10, Chapter 7] for details.

In [14] we showed that if u is $\mathcal{A}$-superharmonic in Ω, there is a nonnegative Radon measure μ such that

$$Tu=\mu$$

in Ω, and conversely, given a finite measure μ in bounded Ω, there is an $\mathcal{A}$-superharmonic function u such that $Tu=\mu$ in Ω and $\min(u,k)\in W^{1,p}_0(\Omega)$ for all integers k.

We start with an auxiliary estimate.

LEMMA 4.1. *Suppose that u is $\mathcal{A}$-superharmonic in a ball $2B=B(x_0,2r)$ and $\mu=Tu$. If a is a real constant, $d>0$ and $p-1<\gamma<n(p-1)/(n-p+1)$, then there are constants $q=q(p,\gamma)>p$ and $c=c(n,p,\alpha,\beta,\gamma)>0$ such that*

$$\left(d^{-\gamma}r^{-n}\int_{B\cap\{u>a\}}(u-a)^\gamma\,dx\right)^{p/q}\leqslant cd^{-\gamma}r^{-n}\int_{2B\cap\{u>a\}}(u-a)^\gamma\,dx+cd^{1-p}r^{p-n}\mu(2B),$$

provided that

$$|2B\cap\{u>a\}|<\frac{1}{2}d^{-\gamma}\int_{B\cap\{u>a\}}(u-a)^\gamma\,dx. \tag{4.2}$$

Proof. Without loss of generality we may assume that $a=0$. We first assume that u is locally bounded and hence $u\in W^{1,p}_{\mathrm{loc}}(2B)$. We shall estimate the left hand side in several steps. Set

$$q=\frac{p\gamma}{p-\gamma/(p-1)}.$$

Notice that $p<q<p^*$, where p^* denotes the critical Sobolev embedding exponent. Using (4.2) we obtain

$$d^{-\gamma}\int_{B\cap\{0<u<d\}} u^\gamma\,dx \leqslant |B\cap\{u>0\}| \leqslant |2B\cap\{u>0\}| \leqslant \frac{1}{2}d^{-\gamma}\int_{B\cap\{u>0\}} u^\gamma\,dx;$$

therefore

$$d^{-\gamma}\int_{B\cap\{u>0\}} u^\gamma\,dx \leqslant 2d^{-\gamma}\int_{B\cap\{u\geqslant d\}} u^\gamma\,dx \leqslant c\int_B w^q\,dx, \tag{4.3}$$

where

$$w=\left(1+\frac{u^+}{d}\right)^{\gamma/q}-1.$$

Note that

$$\nabla w=\frac{\gamma}{qd}\left(1+\frac{u^+}{d}\right)^{\gamma/q-1}\nabla u^+.$$

Pick a cut-off function $\eta\in C_0^\infty(2B)$ such that $0\leqslant\eta\leqslant 1$, $\eta=1$ on B and $|\nabla\eta|\leqslant 2/r$. The Sobolev inequality yields

$$\begin{aligned}\left(r^{-n}\int_B w^q\,dx\right)^{p/q} &\leqslant \left(r^{-n}\int_{2B}(w\eta)^q\,dx\right)^{p/q}\\ &\leqslant cr^{p-n}\int_{2B}|\nabla w|^p\eta^p\,dx+cr^{p-n}\int_{2B}w^p|\nabla\eta|^p\,dx.\end{aligned} \tag{4.4}$$

By substituting the test function

$$v=\left(1-\left(1+\frac{u^+}{d}\right)^{1-\tau}\right)\eta^p,$$

where

$$\tau=\frac{\gamma}{p-1},$$

we obtain

$$\begin{aligned}&\int_{2B\cap\{u>0\}}\frac{|\nabla u|^p}{(1+u/d)^\tau}\eta^p\,dx\\
&\quad\leqslant \alpha^{-1}\int_{2B\cap\{u>0\}}\mathcal{A}(x,\nabla u)\cdot\frac{\nabla u}{(1+u/d)^\tau}\eta^p\,dx\\
&\quad= -\frac{pd}{\alpha(\tau-1)}\int_{2B\cap\{u>0\}}\mathcal{A}(x,\nabla u)\cdot\left(1-\left(1+\frac{u}{d}\right)^{1-\tau}\right)\eta^{p-1}\nabla\eta\,dx\\
&\qquad+\frac{d}{\alpha(\tau-1)}\int_{2B}v\,d\mu\\
&\quad\leqslant cd\int_{2B\cap\{u>0\}}|\nabla u|^{p-1}\eta^{p-1}|\nabla\eta|\,dx+cd\int_{2B}\eta^p\,d\mu\\
&\quad\leqslant \frac{1}{2}\int_{2B\cap\{u>0\}}\frac{|\nabla u|^p}{(1+u/d)^\tau}\eta^p\,dx+c\left(\frac{d}{r}\right)^p\int_{2B\cap\{u>0\}}\left(1+\frac{u}{d}\right)^\gamma dx\\
&\qquad+cd\int_{2B}\eta^p\,d\mu,\end{aligned}$$

where in the last inequality we employed Young's inequality. Hence

$$\begin{aligned}\int_{2B}|\nabla w|^p\eta^p\,dx &\leqslant cd^{-p}\int_{2B\cap\{u>0\}}\frac{|\nabla u|^p}{(1+u/d)^\tau}\eta^p\,dx\\ &\leqslant cr^{-p}\int_{2B\cap\{u>0\}}\Big(1+\frac{u}{d}\Big)^\gamma dx+cd^{1-p}\mu(\operatorname{supp}\eta).\end{aligned}\tag{4.5}$$

Keeping (4.2) in mind we obtain

$$\int_{2B\cap\{u>0\}}\Big(1+\frac{u}{d}\Big)^\gamma dx\leqslant cd^{-\gamma}\int_{2B\cap\{u>0\}}u^\gamma\,dx\tag{4.6}$$

and, consequently, because $w^q\leqslant(1+u/d)^\gamma$,

$$\begin{aligned}r^p\int_{2B}w^p|\nabla\eta|^p\,dx &\leqslant c\int_{2B}w^p\,dx\\ &\leqslant c\Big(\int_{2B}w^q\,dx\Big)^{p/q}|2B\cap\{u>0\}|^{1-p/q}\\ &\leqslant cd^{-\gamma}\int_{2B\cap\{u>0\}}u^\gamma\,dx.\end{aligned}\tag{4.7}$$

Now we remove the assumption that u is locally bounded. For $k>d$ we write

$$u_k=\min(u,k)$$

and

$$\mu_k=Tu_k.$$

Then (4.2) holds for u_k if k is large enough. Hence by collecting the estimates (4.3)–(4.7) we arrive at the estimate

$$\Big(d^{-\gamma}r^{-n}\int_{B\cap\{u>0\}}u_k^\gamma\,dx\Big)^{p/q}\leqslant cd^{-\gamma}r^{-n}\int_{2B\cap\{u>0\}}u_k^\gamma\,dx+cd^{1-p}r^{p-n}\mu_k(\operatorname{supp}\eta),$$

where $c=c(n,p,\alpha,\beta,\gamma)>0$. Now letting $k\to\infty$ and using the weak convergence of μ_k's to μ [14, 2.2] we conclude the proof.

THEOREM 4.8. *Suppose that u is a nonnegative $\mathcal{A}$-superharmonic function in $B(x_0,2r)$. If $\mu=Tu$, then for all $\gamma>p-1$ we have that*

$$u(x_0)\leqslant c\Big(\fint_{B(x_0,r)}u^\gamma\,dx\Big)^{1/\gamma}+c\mathbf{W}^\mu_{1,p}(x_0;2r),$$

where $c=c(n,p,\alpha,\beta,\gamma)>0$.

Proof. By Hölder's inequality we may assume that

$$\gamma<\frac{n(p-1)}{n-p+1}.$$

We fix a constant $\delta\in(0,1)$ to be specified later. Let $B_j=B(x_0,r_j)$, where $r_j=2^{1-j}r$. We define a sequence a_j recursively. Let $a_0=0$ and for $j\geqslant 0$ let

$$a_{j+1}=a_j+\delta^{-1}\left(r_j^{-n}\int_{B_{j+1}\cap\{u>a_j\}}(u-a_j)^\gamma\,dx\right)^{1/\gamma}.$$

Note that $a_j<\infty$ for all j (see Lemma 2.9 or [10, 7.46]). We first derive the estimate

$$\delta^{p\gamma/q}\leqslant c\delta^\gamma\left(\frac{a_j-a_{j-1}}{a_{j+1}-a_j}\right)^\gamma+c(a_{j+1}-a_j)^{1-p}\,\frac{\mu(B_j)}{r_j^{n-p}},\tag{4.9}$$

if $j\geqslant 1$ is such that $a_{j+1}>a_j$ and $q=(p(p-1)\gamma)/(p(p-1)-\gamma)$ is as in the proof of Lemma 4.1. From now on we assume that $\delta>0$ is so small that

$$\delta^\gamma\leqslant 2^{-n-1}r_j^{-n}|B_j|.$$

Since

$$\begin{aligned}|B_j\cap\{u>a_j\}|&\leqslant(a_j-a_{j-1})^{-\gamma}\int_{B_j\cap\{u>a_j\}}(u-a_{j-1})^\gamma\,dx\\&\leqslant(a_j-a_{j-1})^{-\gamma}\int_{B_j\cap\{u>a_{j-1}\}}(u-a_{j-1})^\gamma\,dx=\delta^\gamma r_{j-1}^n\\&=2^n r_j^n\delta^\gamma=2^n(a_{j+1}-a_j)^{-\gamma}\int_{B_{j+1}\cap\{u>a_j\}}(u-a_j)^\gamma\,dx,\end{aligned}\tag{4.10}$$

we have that

$$|B_j\cap\{u>a_j\}|\leqslant\tfrac12|B_j|\tag{4.11}$$

and the hypothesis (4.2) holds with

$$d_j=2^{-(n+2)/\gamma}(a_{j+1}-a_j).$$

Hence Lemma 4.1 yields

$$\left(d_j^{-\gamma}r_j^{-n}\int_{B_{j+1}\cap\{u>a_j\}}(u-a_j)^\gamma\,dx\right)^{p/q}\leqslant cd_j^{-\gamma}r_j^{-n}\int_{B_j\cap\{u>a_j\}}(u-a_j)^\gamma\,dx+cd_j^{1-p}r_j^{p-n}\mu(B_j).$$

Finally, because

$$d_j^{-\gamma} r_j^{-n} \int_{B_j \cap \{u>a_j\}} (u-a_j)^\gamma \, dx \leqslant d_j^{-\gamma} r_j^{-n} \int_{B_j \cap \{u>a_{j-1}\}} (u-a_{j-1})^\gamma \, dx = c(d_{j-1}/d_j)^\gamma \delta^\gamma,$$

we arrive at

$$\begin{aligned} \delta^{p\gamma/q} &\leqslant c\left(d_j^{-\gamma} r_j^{-n} \int_{B_{j+1} \cap \{u>a_j\}} (u-a_j)^\gamma \, dx \right)^{p/q} \\ &\leqslant c d_j^{-\gamma} r_j^{-n} \int_{B_j \cap \{u>a_j\}} (u-a_j)^\gamma \, dx + c d_j^{1-p} r_j^{p-n} \mu(B_j) \\ &\leqslant c\delta^\gamma \left(\frac{d_{j-1}}{d_j} \right)^\gamma + c d_j^{1-p} r_j^{p-n} \mu(B_j), \end{aligned}$$

and (4.9) follows.

Next we show that

$$a_{j+1} - a_j \leqslant \tfrac{1}{2}(a_j - a_{j-1}) + c\left(\frac{\mu(B_j)}{r_j^{n-p}} \right)^{1/(p-1)}. \tag{4.12}$$

If $a_{j+1}-a_j \leqslant \frac{1}{2}(a_j - a_{j-1})$, the estimate (4.12) is trivial. If $a_j - a_{j-1} < 2(a_{j+1}-a_j)$, then (4.9) implies that

$$\delta^{p\gamma/q} \leqslant c\delta^\gamma + c(a_{j+1}-a_j)^{1-p} \frac{\mu(B_j)}{r_j^{n-p}}.$$

Now choosing $0<\delta=\delta(n,p,\alpha,\beta,\gamma)\leqslant 1$ small enough we obtain

$$\delta^{p\gamma/q} > 2c\delta^\gamma$$

so that

$$(a_{j+1}-a_j)^{p-1} \leqslant c\,\frac{\mu(B_j)}{r_j^{n-p}};$$

hence (4.12) holds also in this case.

Now we are ready to conclude the proof. First we deduce from (4.12) that

$$\begin{aligned} a_k - a_1 \leqslant a_{k+1} - a_1 &= \sum_{j=1}^{k} (a_{j+1} - a_j) \\ &\leqslant \frac{1}{2} \sum_{j=1}^{k} (a_j - a_{j-1}) + c \sum_{j=1}^{k} \left(\frac{\mu(B_j)}{r_j^{n-p}} \right)^{1/(p-1)} \\ &= \frac{1}{2} a_k + c \sum_{j=1}^{k} \left(\frac{\mu(B_j)}{r_j^{n-p}} \right)^{1/(p-1)} \end{aligned}$$

and hence

$$\lim_{k\to\infty} a_k \leqslant 2a_1 + c\sum_{j=1}^{\infty}\left(\frac{\mu(B_j)}{r_j^{n-p}}\right)^{1/(p-1)} \leqslant c\left(\fint_{B_1} u^\gamma\right)^{1/\gamma} + c\mathbf{W}_{1,p}^{\mu}(x_0;2r).$$

Now the theorem follows because by (4.11)

$$\inf_{B_j} u \leqslant a_j$$

for $j=1,2,\dots$, and (for u is lower semicontinuous) we conclude that

$$u(x_0) \leqslant \lim_{j\to\infty} \inf_{B_j} u \leqslant \liminf_{j\to\infty} a_j.$$

Proof of Theorem 1.6. The first inequality was established in [14]. The second inequality follows from Theorem 4.8 because by the weak Harnack inequality in Lemma 2.9 we may pick $\gamma=\gamma(n,p)>p-1$ such that

$$\left(\fint_{B(x_0,r)} u^\gamma\,dx\right)^{1/\gamma} \leqslant c\left(\fint_{B(x_0,2r)} u^\gamma\,dx\right)^{1/\gamma} \leqslant c \inf_{B(x_0,r)} u.$$

COROLLARY 4.13. *Let u be an $\mathcal{A}$-superharmonic function in $\mathbf{R}^n$ with $\inf_{\mathbf{R}^n} u=0$. If $\mu=Tu$, then*

$$c_1\mathbf{W}_{1,p}^{\mu}(x_0;\infty) \leqslant u(x_0) \leqslant c_2\mathbf{W}_{1,p}^{\mu}(x_0;\infty),$$

where c_1 and c_2 are positive constants, depending only on n, p, and the structural constants α and β.

Remark 4.14. Because $\mathcal{A}$-superharmonic functions are lower semicontinuous and satisfy the minimum principle, we can replace $\inf_{B(x_0,r)} u$ in Theorem 1.6 by $\inf_{\partial B(x_0,r)} u$.

4.15. *Harnack's inequality.* As the first application of Theorem 1.6 we establish a Harnack inequality for equations $Tu=\mu$.

THEOREM 4.16. *Suppose that u is a nonnegative $\mathcal{A}$-superharmonic function in $B(x_0,7r)$ and let $\mu=Tu$. If there are $\varepsilon>0$ and $M>0$ such that*

$$\mu(B(x,\varrho)) \leqslant M\varrho^{n-p+\varepsilon}$$

whenever $x\in B(x_0,r)$ and $0<\varrho<4r$, then

$$\sup_{B(x_0,r)} u \leqslant c_1 \inf_{B(x_0,r)} u + c_2 r^{\varepsilon/(p-1)},$$

where $c_1=c_1(n,p,\alpha,\beta)$ *and* $c_2=c_2(n,p,\alpha,\beta,M,\varepsilon)$ *are positive constants.*

Proof. This is a direct consequence of Theorem 1.6 because

$$\begin{aligned}\mathbf{W}^{\mu}_{1,p}(x_0;4r) &= \int_0^{4r}\left(\frac{\mu(B(x_0,\varrho))}{\varrho^{n-p}}\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\\ &\leqslant M^{1/(p-1)}\int_0^{4r}\varrho^{\varepsilon/(p-1)}\frac{d\varrho}{\varrho}=cr^{\varepsilon/(p-1)}.\end{aligned}$$

By a standard iteration (cf. [10, Chapter 6], [19, p. 1441] or [21, p. 269]) it follows from Harnack's inequality in Theorem 4.16 that certain $\mathcal{A}$-superharmonic functions are Hölder continuous.

COROLLARY 4.17. *Suppose that* u *is* $\mathcal{A}$*-superharmonic in* Ω *and* $Tu=\mu$. *If there are positive constants* M *and* ε *such that*

$$\mu(B(x,r))\leqslant Mr^{n-p+\varepsilon}$$

whenever $B(x,2r)\subset\Omega$, *then there is* $\gamma=\gamma(n,p,\alpha,\beta,\varepsilon)>0$ *such that for each compact subset* K *of* Ω *there is a constant* $C>0$ *with*

$$|u(x)-u(y)|\leqslant C|x-y|^{\gamma}$$

whenever $x,y\in K$.

We next show that the restriction for the measure μ in the above theorems is essential; cf. [20].

THEOREM 4.18. *Suppose that* u *is* $\mathcal{A}$*-superharmonic in* $B(x_0,r)$. *If there are positive constants* C *and* γ *such that*

$$|u(x)-u(y)|\leqslant C|x-y|^{\gamma}$$

for every x *and* y *in* $B(x_0,r)$, *then*

$$\mu(B(x_0,\varrho))\leqslant cC^{p-1}\varrho^{n-p+\gamma(p-1)}$$

whenever $0<\varrho<\frac{1}{3}r$; *here* $c=c(n,p,\alpha,\beta)>0$.

Proof. We apply the estimate in Theorem 1.6 to the $\mathcal{A}$-superharmonic function $u-\inf_{B(x_0,3\varrho)}u$ and obtain

$$\begin{aligned}\left(\frac{\mu(B(x_0,\varrho))}{\varrho^{n-p}}\right)^{1/(p-1)} &\leqslant c\int_{\varrho}^{2\varrho}\left(\frac{\mu(B(x_0,t))}{t^{n-p}}\right)^{1/(p-1)}\frac{dt}{t}\\ &\leqslant c\big(u(x_0)-\inf_{B(x_0,3\varrho)}u\big)\leqslant cC\varrho^{\gamma},\end{aligned}$$

and the theorem follows.

4.19. *Continuity of* $\mathcal{A}$*-superharmonic functions.* Next we characterize the continuity of $\mathcal{A}$-superharmonic functions in terms of their Wolff potentials.

THEOREM 4.20. *Suppose that u is $\mathcal{A}$-superharmonic in Ω and $Tu=\mu$. Then u is real valued and continuous at x_0 if and only if for each $\varepsilon>0$ there is $r>0$ such that*

$$\mathbf{W}^{\mu}_{1,p}(x,r)<\varepsilon$$

whenever $x\in B(x_0,r)$.

Proof. Suppose first that $u(x_0)<\infty$ and that u is continuous at x_0. Fix $\varepsilon>0$ and choose $r>0$ such that

$$|u-u(x_0)|<\tfrac{1}{2}\varepsilon$$

in $\overline{B}(x_0,4r)\subset\Omega$. Then for $x\in B(x_0,r)$ we have by Theorem 1.6 that

$$c\mathbf{W}^{\mu}_{1,p}(x,r)\leqslant u(x)-\inf_{B(x,3r)}u=u(x)-u(x_0)+u(x_0)-\inf_{B(x,3r)}u\leqslant\varepsilon$$

as desired.

For the converse, we may assume that $u(x_0)=0$, for $u(x_0)<\infty$ by Theorem 1.6. Because u is lower semicontinuous we may choose $r_0>0$ such that $u>-\varepsilon$ in $B(x_0,4r_0)$. If $r<r_0$, we now have for all $x\in B(x_0,r)$ that

$$u(x)\leqslant c_2\inf_{B(x,r)}u+(c_2-1)\varepsilon+c_3\mathbf{W}^{\mu}_{1,p}(x;2r)\leqslant c\varepsilon,$$

and the assertion follows.

4.21. *Specific order principle.* The property of the next proposition was called the *specific order principle* in [13], where it was established for $p>n-1$. Now we prove it for all $p>1$.

PROPOSITION 4.22. *Suppose that u and w are $\mathcal{A}$-superharmonic in Ω such that $0\leqslant u,w\leqslant 1$ and $Tu\leqslant Tw$. If $x_j,x_0\in\Omega$ are such that $\lim_{j\to\infty}x_j=x_0$ and*

$$\lim_{j\to\infty}w(x_j)=w(x_0),$$

then

$$\lim_{j\to\infty}u(x_j)=u(x_0).$$

Proof. Fix $\varepsilon>0$ and choose $r_0>0$ such that

$$w\geqslant w(x_0)-\varepsilon$$

on $B(x_0,6r_0)\subset\Omega$. Let $v=u-u(x_0)$ and for $r>0$ write

$$m(x,r)=\inf_{B(x,r)}v.$$

If $r_j=|x_j-x_0|<r_0$, then $m(x_j,r_j)\leqslant 0$ and we have by the potential estimate in Theorem 1.6 that

$$\begin{aligned} v(x_j)-m(x_0,4r_j) &\leqslant c \inf_{B(x_j,r_j)} (v-m(x_0,4r_j))+c\mathbf{W}^{Tu}_{1,p}(x_j;2r_j) \\ &\leqslant cm(x_j,r_j)-cm(x_0,4r_j)+c\mathbf{W}^{Tw}_{1,p}(x_j;2r_j) \\ &\leqslant -cm(x_0,4r_j)+c(w(x_j)-w(x_0)+\varepsilon). \end{aligned}$$

Since $m(x_0,4r_j)\to 0$ and $w(x_j)\to w(x_0)$, we obtain

$$\limsup_{j\to\infty} u(x_j)-u(x_0)=\limsup_{j\to\infty} v(x_j)\leqslant c\varepsilon,$$

as desired.

5. Thin sets and regular points

In this section we apply Theorem 1.6 and show that sets that are thin in the sense of the Wiener integral are also thin in the sense of $\mathcal{A}$-superharmonic functions, that is, we establish Theorem 1.3. As a corollary we obtain a characterization of the p-fine topology in terms of the Cartan property (Theorem 5.2). We also treat the boundary regularity problem and prove Theorem 1.1. Finally, obstacle problems are briefly discussed.

Proof of Theorem 1.3. The sufficiency part was established in [8, Section 4]. We are going to prove the necessity. Let E be p-thin at $x_0\notin E$. We may assume that E is open [8]. Write $B_j=B(x_0,2^{-j})$, $r_j=2^{-j}$, and $E_j=E\cap B_j$. Let $k\geqslant 2$ be an integer, to be specified later. Let $u=\widehat{R}^1_{E_k}(B_{k-2};\mathcal{A})$ be the $\mathcal{A}$-potential of E_k in B_{k-2} and $\mu=Tu$. Then $u\geqslant 1$ on E_k and it remains to prove that (for some k) $u(x_0)<1$. If $\lambda=\inf_{B_k}u$, we have by Lemma 3.9 that

$$\lambda^{p-1}r_k^{n-p}\leqslant c\lambda^{p-1}\operatorname{cap}_p(\{u>\lambda\},B_{k-2})\leqslant c\mu(B_{k-2})=c\mu(B_{k-1}),$$

and so

$$\inf_{B_k} u\leqslant c\left(\frac{\mu(B_{k-1})}{r_{k-1}^{n-p}}\right)^{1/(p-1)}. \tag{5.1}$$

Moreover, it follows from Lemma 3.7 that for $j>k-2$

$$\mu(B_j)\leqslant c\operatorname{cap}_p(E_j,B_{k-2})\leqslant c\operatorname{cap}_p(E_j,B_{j-1}).$$

Hence, keeping (5.1) in mind, we obtain from Theorem 1.6 that

$$\begin{aligned} u(x_0) &\leqslant c\inf_{B_k} u+c\mathbf{W}^{\mu}_{1,p}(x_0,r_{k-1}) \\ &\leqslant c\sum_{j=k-1}^{\infty}\left(\frac{\operatorname{cap}_p(E_j,B_{j-1})}{r_j^{n-p}}\right)^{1/(p-1)}\leqslant\frac{1}{2}, \end{aligned}$$

where $c=c(n,p,\alpha,\beta)>0$ and the last inequality follows by choosing k large enough. This completes the proof.

Using Theorem 1.3 we have that the Cartan property characterizes fine topologies in nonlinear potential theory. Recall that the $\mathcal{A}$-fine topology is the coarsest topology in $\mathbf{R}^n$ that makes all $\mathcal{A}$-superharmonic functions in $\mathbf{R}^n$ continuous.

THEOREM 5.2. *Suppose that* $E\subset\mathbf{R}^n$ *and* $x_0\in\overline{E}$. *Then the following are equivalent:*

(i) x_0 *is not an* $\mathcal{A}$*-fine limit point of* $E\setminus\{x_0\}$.

(ii) E *is p-thin at* x_0.

(iii) (Cartan property) *There is an* $\mathcal{A}$*-superharmonic function* u *in a neighborhood of* x_0 *such that*

$$\liminf_{\substack{x\to x_0\\ x\in E\setminus\{x_0\}}} u(x)>u(x_0).$$

(iv) *There are open neighborhoods* U *and* V *of* x_0 *such that*

$$\widehat{R}^1_{E\cap U}(V;\mathcal{A})(x_0)<1.$$

Proof. It was shown in [9] that (i) and (ii) are equivalent. In [8] it was established that (iii) implies (iv) and that (iv) implies (ii). The missing link that (ii) yields (iii) follows from Theorem 1.3.

5.3. *Boundary regularity.* Next we show that regular boundary points can be characterized by the Wiener test, that is, we prove Theorem 1.1.

We begin with recalling the Perron process. Let $f:\partial\Omega\to[-\infty,\infty]$ be a function. Here we make the convention that if Ω is unbounded, the boundary $\partial\Omega$ is taken with respect to the one point compactification $\mathbf{R}^n\cup\{\infty\}$ of $\mathbf{R}^n$. Hence $\partial\Omega$ is always compact. Define the upper Perron solution $\overline{H}_f$ in Ω to be the function

$$\overline{H}_f=\inf\{u:u\in\mathcal{U}_f\},$$

where $\mathcal{U}_f$ consists of all $\mathcal{A}$-superharmonic functions u in Ω such that u is bounded below and that $\liminf_{x\to y}u(x)\geqslant f(y)$ for all $y\in\partial\Omega$.

The lower Perron solution $\underline{H}_f$ is defined analogously via $\mathcal{A}$-subharmonic functions so that

$$\overline{H}_f=-\underline{H}_{-f}.$$

It is fundamental that in each component of Ω, $\overline{H}_f$ is either $\mathcal{A}$-harmonic, or $\overline{H}_f\equiv\infty$ or $\overline{H}_f\equiv-\infty$ [11].

Moreover, it was shown in [11] that if $f:\partial\Omega\to\mathbf{R}$ is continuous, then $\overline{H}_f=\underline{H}_f$ in Ω and $\overline{H}_f$ is $\mathcal{A}$-harmonic there; if $p=n$ one must assume in addition that $\mathbf{R}^n\setminus\Omega$ is not of n-capacity zero.

We call a boundary point $x_0\in\partial\Omega$ *regular* if

$$\lim_{x\to x_0}\overline{H}_f(x)=f(x)$$

whenever $f:\partial\Omega\to\mathbf{R}$ is continuous. If Ω is bounded, this definition results in the same regularity concept as it was mentioned in the Introduction. Indeed, if $f\in W^{1,p}(\Omega)$ is continuous on $\bar{\Omega}$, then $\overline{H}_f-f\in W_0^{1,p}(\Omega)$ (see [11, 6.2] or [10, 9.29]). Hence by the uniqueness we have that a point x_0, which is regular in the sense of Perron solutions, is also regular in the sense of the Introduction. Conversely, let x_0 be regular according to the definition in the Introduction and approximate a continuous function f uniformly on $\partial\Omega$ by functions $f_j\in C^\infty(\mathbf{R}^n)$. Then

$$\lim_{x\to x_0}\overline{H}_f(x)=f(x_0)$$

because $\overline{H}_{f_j}-f_j\in W_0^{1,p}(\Omega)$ and $\overline{H}_{f_j}$ converges to $\overline{H}_f$ uniformly in Ω [10, 9.30].

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. That the divergence of the Wiener integral $W_p(\mathbf{R}^n\setminus\Omega,x_0)$ implies the regularity of x_0 was proved by Maz'ya [18] if Ω is bounded; the general case was treated in [11]. See also [10, 6.16 and Chapter 9], where a somewhat simpler proof for Maz'ya's estimate is given.

For the converse, suppose that

$$W_p(\mathbf{R}^n\setminus\Omega,x_0)<\infty.$$

If x_0 is an isolated boundary point, it never is regular as easily follows by using the maximum principle and the removability of singletons for bounded $\mathcal{A}$-harmonic functions (cf. [7], [11]). Hence we are free to assume that x_0 is an accumulation point of $E=\mathbf{R}^n\setminus\Omega$. Because E is p-thin at x_0, we now infer from Theorem 1.3 that there are balls $B_i=B(x_0,r_i)$, $i=1,2$, such that $r_1<r_2$ and an $\mathcal{A}$-superharmonic function u in B_2 such that $0\leqslant u\leqslant 1$, $u=1$ in $B_2\cap E\setminus\{x_0\}$ and $u(x_0)\leqslant\frac{1}{2}$. Next, choose a function $\varphi\in C^\infty(\mathbf{R}^n)$ such that $\varphi\leqslant u$ in $E\cap\overline{B}_1\setminus\{x_0\}$ and that $\varphi=1$ in a neighborhood of x_0. Consider the upper Perron solution $\overline{H}_\varphi$ taken in the open set $B_1\cap\Omega$. Because the set of the irregular boundary points of $B_1\cap\Omega$ is of p-capacity zero [11, 5.6] and because $\overline{H}_\varphi\in W^{1,p}(B_1\cap\Omega)$ (see [11, 6.2] or [10, 9.29]) it follows from the generalized comparison principle [13, 3.3] that

$$\overline{H}_\varphi\leqslant u$$

in $B_1 \cap \Omega$. In particular,

$$\liminf_{x \to x_0} \overline{H}_\varphi(x) \leqslant \liminf_{\substack{x \to x_0 \\ x \in \Omega}} u(x) = u(x_0) \leqslant \tfrac{1}{2} < 1 = \varphi(x_0).$$

Hence x_0 is not regular boundary point of $B_1 \setminus \Omega$. Since the barrier characterization for regularity [11] implies that the regularity is a local property, it follows that x_0 is not a regular boundary point of Ω. Theorem 1.1 is proved.

In [13] we termed a boundary point x_0 of a bounded Ω *exposed*, if there is a continuous function $h: \overline{\Omega} \to \mathbf{R}$, $\mathcal{A}$-harmonic in Ω, such that $h(x_0)=0$ and $h>0$ on $\overline{\Omega} \setminus \{x_0\}$.

By the barrier characterization of the regularity ([10, 9.8], [11]) an exposed boundary point is always regular. In [13, 4.1] it was proved that also the converse is true provided that the operator T obeys the specific order principle 4.21. By Proposition 4.22 this is always the case so that we have:

THEOREM 5.4. *A boundary point x_0 of a bounded open set Ω is regular if and only if it is exposed.*

5.5. *Obstacle problems.* There are several other problems in nonlinear potential theory that have been solved for $p>n-1$ only, but rather straightforwardly follow once Theorem 1.3 is established for each $p \in (1, n]$. In this respect we mention here obstacle problems.

Let $\psi: \mathbf{R}^n \to \mathbf{R}$ be a bounded function. A function u is said to be a local solution to the obstacle problem at the point x_0 if there is an open neighborhood Ω of x_0 such that

$$\begin{cases} u \in W^{1,p}(\Omega), \\ u \geqslant \psi \ \ p\text{-quasieverywhere}, \\ \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx \geqslant 0 \ \text{ whenever } \varphi \in W_0^{1,p}(\Omega) \\ \text{ is such that } u + \varphi \geqslant \psi \ \ p\text{-quasieverywhere.} \end{cases} \tag{5.6}$$

Let

$$\bar{\psi}(x_0) = \inf_{r>0} \ p\text{-}\operatorname*{ess\,sup}_{B(x_0,r)} \psi,$$

where

$$p\text{-}\operatorname*{ess\,sup}_{B(x_0,r)} \psi = \inf\{t : \psi \leqslant t \ \ p\text{-quasieverywhere in } B(x_0,r)\}.$$

Write for $\varepsilon>0$

$$E_\varepsilon = \{x : \psi(x) \geqslant \bar{\psi}(x_0) - \varepsilon\}.$$

Now we have:

THEOREM 5.7. *If no E_ε is p-thin at x_0, $\varepsilon>0$, then each local solution to the obstacle problem* (5.6) *is continuous at x_0.*

Conversely, if E_ε is p-thin for some $\varepsilon>0$, there is a local solution to (5.6) *that cannot be made continuous at x_0.*

Proof. The first assertion is well known and it was proved by Michael and Ziemer [19]; see also [8].

The second assertion was established in [8] under the additional restriction that $p>n-1$. Next we show that it follows from Theorem 1.3 for all $p\in(1,n]$. To this end, suppose that there is $\varepsilon>0$ such that E_ε is p-thin at x_0. Appealing to Theorem 1.3 we infer that there is a bounded $\mathcal{A}$-superharmonic function u in a ball $B=B(x_0,r_0)$ such that $u=\sup\psi$ on $E_\varepsilon\setminus\{x_0\}$ and

$$u(x_0)=\bar{\psi}(x_0)-\tfrac{1}{2}\varepsilon.$$

Because u is lower semicontinuous, we may assume that

$$u>\bar{\psi}(x_0)-\varepsilon$$

in B. Also there is no loss of generality in assuming that $u\in W^{1,p}(B)$ (see Proposition 2.7). Let v be the $\mathcal{A}$-superharmonic solution to the obstacle problem (5.6) with $\Omega=B$ so that $u-v\in W_0^{1,p}(B)$. Since $u\geqslant\psi$ p-quasieverywhere, we have that $v\leqslant u$ in B (cf. [7, 2.8]). It follows that v cannot be continuous at x_0, since

$$\begin{aligned}\liminf_{x\to x_0} v(x) &\leqslant \operatorname{ess}\liminf_{x\to x_0} u(x)=u(x_0)\\ &<\bar{\psi}(x_0)=\inf_{r>0}\ p\text{-}\operatorname{ess}\sup_{B(x_0,r)}\psi\\ &\leqslant \limsup_{x\to x_0} v(x),\end{aligned}$$

for $v\geqslant\psi$ p-quasieverywhere.

Remark 5.8. A similar problem for double obstacle problems was studied in [15, Theorem 5.2]. The sufficiency part of the Wiener test was established there for all p's but the necessity was proved only if $p>n-1$ or if the operator in question is linear. By using Theorem 5.7 one can use the argument in [15, Theorem 5.2] and easily establish the necessity part without any restriction for the type p of the operator.

References

[1] ADAMS, D. R., L^p potential theory techniques and nonlinear PDE, in *Potential Theory* (M. Kishi, ed.), pp. 1–15. Walter de Gruyter & Co., Berlin, 1992.

[2] ADAMS, D. R. & HEDBERG, L. I., *Function Spaces and Potential Theory*. In preparation.

[3] ADAMS, D. R. & MEYERS, N. G., Thinness and Wiener criteria for non-linear potentials. *Indiana Univ. Math. J.*, 22 (1972), 169–197.

[4] DAL MASO, G., MOSCO, U. & VIVALDI, M. A., A pointwise regularity theory for the two-obstacle problem. *Acta Math.*, 163 (1989), 57–107.

[5] GARIEPY, R. & ZIEMER, W. P., A regularity condition at the boundary for solutions of quasilinear elliptic equations. *Arch. Rational Mech. Anal.*, 67 (1977), 25–39.

[6] HEDBERG, L. I. & WOLFF, TH. H., Thin sets in nonlinear potential theory. *Ann. Inst. Fourier*, 33 (1983), 161–187.

[7] HEINONEN, J. & KILPELÄINEN, T., $\mathcal{A}$-superharmonic functions and supersolutions of degenerate elliptic equations. *Ark. Mat.*, 26 (1988), 87–105.

[8] — On the Wiener criterion and quasilinear obstacle problems. *Trans. Amer. Math. Soc.*, 310 (1988), 239–255.

[9] HEINONEN, J., KILPELÄINEN, T. & MARTIO, O., Fine topology and quasilinear elliptic equations. *Ann. Inst. Fourier*, 39 (1989), 293–318.

[10] — *Nonlinear Potential Theory of Degenerate Elliptic Equations*. Oxford University Press, Oxford, 1993.

[11] KILPELÄINEN, T., Potential theory for supersolutions of degenerate elliptic equations. *Indiana Univ. Math. J.*, 38 (1989), 253–275.

[12] — Nonlinear potential theory and PDEs. To appear in *Potential Analysis*.

[13] KILPELÄINEN, T. & MALÝ, J., Generalized Dirichlet problem in nonlinear potential theory. *Manuscripta Math.*, 66 (1989), 25-44.

[14] — Degenerate elliptic equations with measure data and nonlinear potentials. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19 (1992), 591–613.

[15] KILPELÄINEN, T. & ZIEMER, W. P., Pointwise regularity of solutions to nonlinear double obstacle problems. *Ark. Mat.*, 29 (1991), 83–106.

[16] LINDQVIST, P. & MARTIO, O., Two theorems of N. Wiener for solutions of quasilinear elliptic equations. *Acta Math.*, 155 (1985), 153–171.

[17] LITTMAN, W., STAMPACCHIA, G. & WEINBERGER, H. F., Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat. (3)*, 17 (1963), 43–77.

[18] MAZ'YA, V. G., On the continuity at a boundary point of solutions of quasi-linear elliptic equations. *Vestnik Leningrad Univ. Math.*, 3 (1976), 225–242; English translation of *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.*, 25 (1970), 42–55 (Russian).

[19] MICHAEL, J. H. & ZIEMER, W. P., Interior regularity for solutions to obstacle problems. *Nonlinear Anal.*, 10 (1986), 1427–1448.

[20] RAKOTOSON, J. M. & ZIEMER, W. P., Local behavior of solutions of quasilinear elliptic equations with general structure. *Trans. Amer. Math. Soc.*, 319 (1990), 747–764.

[21] SERRIN, J., Local behavior of solutions of quasi-linear equations. *Acta Math.*, 111 (1964), 247–302.

[22] TRUDINGER, N. S., On Harnack type inequalities and their application to quasilinear elliptic equations. *Comm. Pure Appl. Math.*, 20 (1967), 721–747.

[23] WIENER, N., Certain notions in potential theory. *J. Math. Phys.*, 3 (1924), 24–51; Reprinted in *Norbert Wiener: Collected works*, Vol. 1, pp. 364–391. MIT Press, 1976.

[24] ZIEMER, W. P., *Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation*. Springer-Verlag, New York, 1989.

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