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Homoclinic tangencies for hyperbolic sets of large Hausdorff dimension

by

JACOB PALIS and JEAN-CHRISTOPHE YOCCOZ

Instituto de Matemática Pura e Aplicada
Rio de Janeiro, Brazil

Université de Paris-Sud
Orsay, France

Introduction

A fundamental concept in dynamics of a nongradient character is that of a homoclinic orbit, introduced by Poincaré in 1890 [P]: an orbit of intersection at large of the stable and unstable manifolds of a periodic saddle point. It is well known that when such an orbit is *transversal*, it must be accumulated by periodic saddles of the same index (dimension of the stable manifold) as the original saddle with respect to which the homoclinic orbit is doubly asymptotic, as shown by Birkhoff in two dimensions and Smale in general [Bi], [S]. In fact, in this last reference it was proved that transversal homoclinic orbits are always part of a hyperbolic Cantor set, a *horseshoe*, in which the periodic points are dense.

More recently, it has been realized that the creation and unfolding of a *homoclinic tangency*, say for a locally dissipative surface diffeomorphism, gives rise to a striking number of intricate and highly relevant dynamic phenomena: cascades of period doubling bifurcations [YA], infinitely many sinks [N], [R], [PT3], strange attractors of Hénon type [BC], [MV], and hyperbolic Cantor sets combined or not with the previous elements [NP], [PT1], [PT2]. Also, surface diffeomorphisms exhibiting a homoclinic tangency are certainly quite common among nonhyperbolic maps, i.e. maps whose limit set is not hyperbolic. Conjecturally, these homoclinically bifurcating diffeomorphisms may even be dense in the interior of the nonhyperbolic ones, which has turned out to be the case for C^∞ surface diffeomorphisms but in C^1 topology [AM].

Therefore, it seems to us that an important task in dynamics is to unfold the diffeomorphisms exhibiting a homoclinic tangency through k-parameter families and to inquire which of the above or other phenomena are more *common* or *prevalent* in terms of the Lebesgue measure in the parameter space. The main result in the present paper represents a contribution to such a program. Let us first explain it in a more informal way.

A homoclinic tangency may be associated to a single periodic orbit or more generally to a (hyperbolic) basic set. Recall that a basic set for a diffeomorphism is a compact, invariant, hyperbolic and transitive subset of the ambient manifold, which is the maximal invariant subset in some neighbourhood of it; moreover, periodic points are dense in it. We say that a basic set is nontrivial if it does not consist of a periodic orbit. It was proved in previous papers [NP], [PT1], [PT2] that *for a generic one-parameter unfolding if the Hausdorff dimension of the associated basic set is smaller than one, then the initial map exhibiting a homoclinic tangency is a Lebesgue density point of hyperbolic dynamics.* Here we prove a converse to the above statement: *if the Hausdorff dimension of the basic set is bigger than one, then for almost all one-parameter families of diffeomorphims the initial map is not a density point of hyperbolicity.*

To be somewhat more precise, let f be a surface diffeomorphism exhibiting a quadratic homoclinic tangency q between the stable and unstable manifolds of a periodic saddle point p, p being part of a basic set K with Hausdorff dimension $\mathrm{HD}(K)$ bigger than one. Amongst the germs of smooth families $(f_{s,t})$, $|s|<\eta$ and $|t|<\eta$, such that $f_{0,0}=f$, we consider those which unfold the homoclinic tangency at q with positive speed. After a local diffeomorphism in parameter space, we may assume that the homoclinic tangency happens along $t=0$. We then require that the relative variations with respect to s of the logarithms of the stable and unstable eigenvalues at $p_{s,t}$ on one hand, and of the stable and unstable Hausdorff dimensions of $K_{s,t}$ on the other, should not vanish at $s=t=0$. (Here, $p_{s,t}$ and $K_{s,t}$ indicate the continuations of p and K for $|s|$ and $|t|$ small.) These three transversality conditions define an open and dense subset $\mathcal{V}$ in the space of germs of smooth families $(f_{s,t})$, $f_{0,0}=f$; see §1. Let $\mathcal{F}^s(K_{s,t})$ and $\mathcal{F}^u(K_{s,t})$ be the stable and unstable foliations of $K_{s,t}$ and define for $\varepsilon>0$ small, $T_{s,\varepsilon}=\{t\in(-\varepsilon,\varepsilon):$ some leaf of $\mathcal{F}^u(K_{s,t})$ is tangent near q to some leaf of $\mathcal{F}^s(K_{s,t})\}$. We observe that often such orbits of tangency are still called homoclinic, and in fact in our case we even call them primary homoclinic tangencies, since they occur between pieces of leaves of $\mathcal{F}^u(K_{s,t})$ and $\mathcal{F}^s(K_{s,t})$ near the curves in $W^u(p)$ and $W^s(p)$ whose extreme points are p and q.

With the above notations and assumptions our result can be stated as follows.

THEOREM. *For each $f_{s,t}\in\mathcal{V}$, there is $c>0$ such that, for almost all $s\in(-\eta,\eta)$, we have*

$$\limsup_{\varepsilon\to 0}\frac{m(T_{s,\varepsilon})}{\varepsilon}>c$$

where $m(\cdot)$ indicates the Lebesgue measure of the set.

That such a statement could be true as well as its proof was much inspired by the remarkable result of Marstrand [Mar] concerning the positiveness of the Lebesgue measure of almost all linear projections of plane sets of Hausdorff dimension bigger than one. In our case, however, the situation is considerably more delicate due to the *lack of linearity and even smoothness* of the "projections" that we have to consider.

This paper is divided into four sections. The first one contains the precise setting of the problem and a more detailed statement of our result than the one presented above. It contains, moreover an indication of how the proof proceeds in the next three sections, each of them having a different character: *analytic, combinatorial* and *geometric*, respectively. The analytical part of the proof is inspired by Marstrand's theorem (§2). The most important objective in §3 is to establish a combinatorial lemma in the context of symbolic dynamics, which is one of the main new ingredients with respect to Kaufman's proof of Marstrand's theorem in [F]. Finally, in §4, we present geometric estimations on the first and second order variations with respect to parameters of the distance between stable manifolds of nearby points in a basic set.

1. The setting of the problem and statement of the result

1.1. Let M be a smooth surface and f a smooth diffeomorphism of M. Let Λ_1, Λ_2 be two (not necessarily distinct) basic sets of f, nontrivial, topologically mixing and of saddle type.

For $i=1,2$, let $p_i \in \Lambda_i$ be a periodic point. We assume that $W^s(p_1)$ and $W^u(p_2)$ have, at a point $q\in M$, a non-degenerate (i.e. quadratic) tangency.

1.2. We embed f in a smooth 2-parameter family $(f_{s,t})$ of smooth diffeomorphisms of M, with $f_{0,0}=f$. We will only be interested in the family for small s,t, i.e. $|s|,|t|<\eta$ with $\eta>0$ small enough. For small η, Λ_1 and Λ_2 have hyperbolic continuations $\Lambda_1(s,t), \Lambda_2(s,t)$ in $(-\eta,\eta)^2$. For $i=1,2$, $z\in\Lambda_i$, we denote by $z(s,t)$ the point in $\Lambda_i(s,t)$ associated to z, and write $W^s(z,s,t), W^u(z,s,t)$ for the stable and unstable manifolds of $z(s,t)$ relative to $f_{s,t}$.

Let us fix a small number $\varepsilon>0$ and local coordinates $(x,y)\in[-\varepsilon,\varepsilon]^2$ in a neighbourhood V of q such that:

— q has coordinates (0,0);

— the equation of the connected component of q in $W^u(p_2,0,0)\cap V$ is $\{y=0\}$;

— the equation of the connected component of q in $W^s(p_1,0,0)\cap V$ is $\{y=g_1(x)\}$, where $g_1\in C^\infty([-\varepsilon,\varepsilon],[-\varepsilon,\varepsilon])$ satisfies $g_1(0)=g_1'(0)=0$, $1\leqslant g_1''(x)\leqslant 2$ for $x\in[-\varepsilon,\varepsilon]$.

For small η, we can follow these connected components through $(-\eta,\eta)^2$. They will respectively have for equation

$$y = g_2(x,s,t) \quad (\text{with } g_2(x,0,0) \equiv 0),$$
$$y = g_1(x,s,t) \quad (\text{with } g_1(x,0,0) \equiv g_1(x)),$$

for smooth maps $g_1, g_2 \in C^\infty([-\varepsilon,\varepsilon]\times(-\eta,\eta)^2, [-\varepsilon,\varepsilon])$.

1.3. We will make three transversality hypotheses on the family $(f_{s,t})$. The first one is that the quadratic tangency of $W^s(p_1,f)$ and $W^u(p_2,f)$ unfolds generically. Using the implicit function theorem, this means that (with η small enough) we may assume that the coordinates s,t in parameter space are such that:

— for $t<0$ and all $s\in(-\eta,\eta)$, the function $x\mapsto g_1(x,s,t)-g_2(x,s,t)$ is strictly positive in $[-\varepsilon,\varepsilon]$;

— for $t=0$, the function $x\to g_1(x,s,t)-g_2(x,s,t)$ is positive and has a single zero in $[-\varepsilon,\varepsilon]$;

— for all $(x,s,t)\in[-\varepsilon,\varepsilon]\times(-\eta,\eta)^2$, we have

$$\partial_t(g_2-g_1)(x,s,t) \geqslant c > 0,$$

for some constant c (we take ε smaller if necessary).

On the other hand, there exists a neighbourhood U_1 of p_1 in $W^u_{\text{loc}}(p_1)\cap\Lambda_1$, a neighbourhood U_2 of p_2 in $W^s_{\text{loc}}(p_2)\cap\Lambda_2$, and, for $i=1,2$, a continuous map:

$$G_i: U_i \to C^\infty([-\varepsilon,\varepsilon]\times(-\eta,\eta)^2, [-\varepsilon,\varepsilon])$$

with the following properties:

— $G_i(p_i)=g_i$;

— for $(s,t)\in(-\eta,\eta)^2$ and $z\in U_1$ (resp. U_2), $\{y=G_i(z)(x,s,t)\}$ is the equation of the connected component of $W^s(z,s,t)\cap V$ (resp. $W^u(z,s,t)\cap V$) which corresponds (in an obvious meaning) to the component of $W^s(p_1,s,t)\cap V$ (resp. $W^u(p_2,s,t)\cap V$) considered above.

The continuity of G_1, G_2 guarantees that (restricting U_1, U_2 if necessary) we have a continuous map:

$$U_1\times U_2 \xrightarrow{T} C^\infty((-\eta,\eta),(-\eta,\eta))$$

with the following properties, for all $(z_1,z_2)\in U_1\times U_2$:

— if $t<T(z_1,z_2)(s)$, the function $x\to G_1(z_1)(x,s,t)-G_2(z_2)(x,s,t)$ is strictly positive in $[-\varepsilon,\varepsilon]$;

— if $t=T(z_1,z_2)(s)$, the same function is positive with a single zero in $[-\varepsilon,\varepsilon]$.

We can also assume that, for any $z_1 \in U_1$, $z_2 \in U_2$, $x \in [-\varepsilon, \varepsilon]$, $s, t \in (-\eta, \eta)$ we have:

$$\partial_t (G_2(z_2) - G_1(z_1))(x, s, t) \geqslant c > 0.$$

1.4. We now come to our two other transversality hypotheses.

For $s, t \in (-\eta, \eta)$, we define the unstable dimension $\Delta_1(s,t)$ of the basic set $\Lambda_1(s,t)$ of $f_{s,t}$ to be the Hausdorff dimension of $W^u_{\mathrm{loc}}(p_1, s, t) \cap \Lambda_1(s,t)$. Similarly, the stable dimension $\Delta_2(s,t)$ of $\Lambda_2(s,t)$ is the Hausdorff dimension of $W^s_{\mathrm{loc}}(p_2, s, t) \cap \Lambda_2(s,t)$.

We assume that

$$\Delta_1(0,0) + \Delta_2(0,0) > 1.$$

If the two basic sets Λ_1, Λ_2 coincide, $\Delta_1(0,0) + \Delta_2(0,0)$ is just the Hausdorff dimension of $\Lambda = \Lambda_1 = \Lambda_2$.

We also assume that

$$\partial_s \frac{\Delta_1(s,t)}{\Delta_2(s,t)}\bigg|_{(s,t)=(0,0)} \neq 0.$$

(It is known [Man] that Δ_1, Δ_2 are smooth functions of s, t.)

Let n_i, for $i=1,2$, be the period of the periodic point p_i of f. For $(s,t) \in (-\eta, \eta)^2$, let $\bar{\lambda}_1(s,t)$ be the logarithm of the modulus of the unstable eigenvalue of the fixed point $p_1(s,t)$ of $f^{n_1}_{s,t}$. Similarly, let $\bar{\lambda}_2(s,t)$ be the logarithm of the modulus of the stable eigenvalue of the fixed point $p_2(s,t)$ of $f^{n_2}_{s,t}$. We assume that

$$\partial_s \frac{\bar{\lambda}_1(s,t)}{\bar{\lambda}_2(s,t)}\bigg|_{(s,t)=(0,0)} \neq 0.$$

1.5. Before stating our main result, we introduce the following notations. Fix some Riemannian metric on M and denote by d_0 the associated distance.

For $i=1,2$ and $|s| < \eta$, let d_s be the distance on U_i defined by

$$d_s(z, z') = d_0(z(s,0), z'(s,0)).$$

For r small enough, let $B^i_s(r)$ be the d_s-ball in U_i of center p_i and radius r. Let $T_s(r)$ be the image of $B^1_s(r) \times B^2_s(r)$ by the map $(z_1, z_2) \mapsto T(z_1, z_2)(s)$.

THEOREM. *Under the hypotheses above, there are constants $r_1 > 0$, $c_1 > 0$ such that, if $0 < r < r_1$ and $s_0 \in (-\eta, \eta)$ the set*

$$\{(s,t) : |s - s_0| < |\log r|^{-1},\ t \in T_s(r)\}$$

has 2-dimensional Lebesgue measure bigger than $c_1 r |\log r|^{-1}$.

Remark. It is easy to see, and we will prove it later, that we have $T_s(r) \subset [-cr, cr]$, for some fixed $c > 0$. *Therefore, the conclusion of the theorem means that* $\{(s,t) : |s - s_0| <$

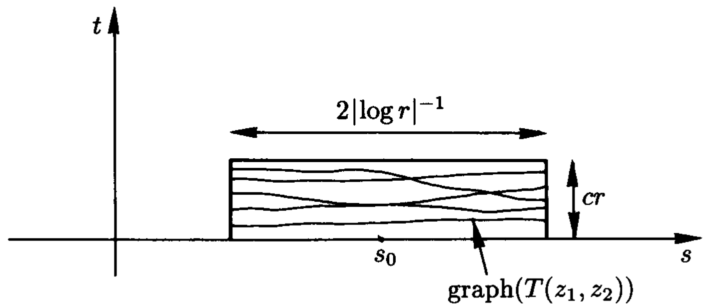

Fig. 1

$|\log r|^{-1}$, $t \in T_s(r)\}$ *has in the rectangle* $[s_0 - |\log r|^{-1}, s_0 + |\log r|^{-1}] \times [-cr, cr]$ *a relative Lebesgue measure bounded from below (independent of* s_0, r *if they are small enough).* See Figure 1.

COROLLARY. *There is a constant* $c_2 > 0$ *such that* for almost all $s \in (-\eta, \eta)$ *we have*:

$$\limsup_{r \to 0} \frac{m(T_s(r))}{r} > c_2$$

(*where* m *is* 1-*dimensional Lebesgue measure*).

Proof of the corollary. Take c_2 small enough. If the corollary is false, there exist $r_2 < r_1$ and a set $A \subset (-\eta, \eta)$ of positive measure such that, for $s \in A$ and $r < r_2$:

$$m(T_s(r)) < 2c_2 r.$$

Let $s_0 \in A$ and $r_3 < r_2$ be such that

$$m([s_0 - |\log r_3|^{-1}, s_0 + |\log r_3|^{-1}] \cap A) > (1 - c_2) 2 |\log r_3|^{-1}.$$

Using the remark which follows the theorem (that $m(T_s(r)) \leqslant cr$ for all $r < r_1$, $s \in (-\eta, \eta)$), we contradict the theorem if c_2 is small enough. □

1.6. The remaining part of the paper is devoted to the proof of the theorem. We give here a short account of the ideas underlying this proof.

In order to estimate the Lebesgue measure of the image of the map $T_s : (z_1, z_2) \mapsto T(z_1, z_2)(s)$, we equip $B^1_{s_0}(r) \times B^2_{s_0}(r)$ with a Radon measure μ and consider, for each s, its image ν_s under T_s. We study ν_s via its Fourier transform $\hat{\nu}_s$ (§2), and want to show that

$$\int \|\hat{\nu}_s\|^2_{L^2}\, ds < c < +\infty.$$

This would indeed easily imply that the support of ν_s (contained in the image of T_s) has positive Lebesgue measure for almost every s, and give an estimate from below for the mean value (with respect to s) of this measure.

This idea was used by Kaufman to give an elegant proof of Marstrand's theorem. The parameter s plays here the role of the angle in Marstrand's theorem, and the map T_s the role of the projection.

For this idea to work, the measure ν_s has to be absolutely continuous with respect to Lebesgue measure for almost all s. In Marstrand's theorem, this is a consequence of an energy estimate on the measure μ and of the variation of the angle. In our case, the energy estimate is essentially the same, but the map T_s depends in a much more complicated way on the parameter.

More precisely, there are various "angles" involved: at a not too small scale, the variation of the "angle" is assured by the relative variation of the logarithms of the eigenvalues (second transversality hypothesis, Proposition 1 in §3.6); at a very small scale, the variation of the angle is assured by the relative variation of the Hausdorff dimensions of Λ_1, Λ_2 (first transversality hypothesis).

The main problem arises from intermediate scales, which might create a singular part of the measures ν_s. In order to avoid this phenomenon, we have to delete, using a stopping time argument, part of the set $B^1_{s_0}(r)\times B^2_{s_0}(r)$, keeping a subset L supporting a positive proportion of μ but for which these intermediate scales do not occur (Proposition 2 in §3.6).

All these considerations rely on a quite good control of the map T_s (Proposition 3.10): approximate formulas for T_s and its first derivative with respect to the parameter s, and bounds for the second derivative. These estimates are themselves consequences of approximate formulas and bounds for the distance between stable manifolds of nearby points in a basic set, measured along these manifolds, and the variation of this distance with parameters (§4).

2. Proof of the theorem: the analysis

2.1. In this section we will give the analytical part of the proof of the theorem. As mentioned before, it is inspired by Kaufman's proof of a theorem of Marstrand presented in [F]. It requires geometrical estimates and a selection lemma that will be proved in later sections. We only state them here.

In the sequel, $c>0$, $r_1>0$, $\beta>1$ are constant independent of later choices.

Let $s_0\in(-\eta,\eta)$, $0<r<r_1$ and I be the interval $[s_0-|\log r|^{-1}, s_0+|\log r|^{-1}]$. Let

$z_1, z_1' \in B^1_{s_0}(r)$, $z_2, z_2' \in B^2_{s_0}(r)$ and $\Delta_i = \Delta_i(s_0, 0)$ for $i=1,2$. We write

$$d = \sup(d_{s_0}(z_1, z_1'), d_{s_0}(z_2, z_2')),$$
$$T(s) = T(z_1, z_2)(s) - T(z_1', z_2')(s), \quad s \in I.$$

It is assumed that there exist a measure μ_i on $B^i_{s_0}(r)$ (for $i=1,2$) and a compact set L in $B^1_{s_0}(r) \times B^2_{s_0}(r)$ such that the following properties hold:

(i) For any ball B_i of d_{s_0}-radius $\varrho \in (0, r)$ contained in $B^i_{s_0}(r)$, we have

$$c^{-1}\varrho^{\Delta_i} < \mu_i(B_i) < c\varrho^{\Delta_i}, \quad i = 1, 2;$$

(ii) For any $s \in I$, $L \subset B^1_s(r) \times B^2_s(r)$;

(iii) $\mu_1 \times \mu_2(L) > c^{-1} r^{\Delta_1 + \Delta_2}$;

(iv) Suppose that $d > r^\beta$; then the set

$$J = \{s \in I : |T(s)| < c^{-1} d\}$$

is empty or is an interval; in the last case we have, for $s \in J$:

$$|T'(s)| \geqslant c^{-1} d |\log r|,$$
$$|T''(s)| \leqslant cd |\log r|^2;$$

(v) Suppose that $d \leqslant r^\beta$, and that $(z_1, z_2) \in L$, $(z_1', z_2') \in L$; then the set

$$J = \{s \in I : |T(s)| < c^{-1} d^{1+c|\log r|^{-1}}\}$$

is empty or is an interval; in the last case, we have, for $s \in J$:

$$|T'(s)| \geqslant c^{-1} d^{1+c|\log r|^{-1}},$$
$$|T''(s)| \leqslant cd^{1-c|\log r|^{-1}} |\log d|^2.$$

2.2. Under these assumptions we will now prove the theorem. Let μ be the restriction of $\mu_1 \times \mu_2$ to L. Let $s_0 \in (-\eta, \eta)$ and $0 < r < r_1$. For $s \in I$, let ν_s be the image of μ by the map $(z_1, z_2) \to T(z_1, z_2)(s)$. We will prove that for almost all $s \in I$, ν_s is absolutely continuous with respect to Lebesgue measure, and will obtain a bound from below for the Lebesgue measure of the support of ν_s. Because this support is contained in $T_s(r)$ (see (ii)), it will prove the theorem.

Let $\hat{\nu}_s$ be the Fourier transform of ν_s. For $p \in \mathbf{R}$, we have

$$|\hat{\nu}_s(p)|^2 = \iint_{L \times L} e^{2\pi i p T_{z,z'}(s)}\, d\mu(z)\, d\mu(z'),$$

with $z=(z_1,z_2)$, $z'=(z'_1,z'_2)$ and

$$T_{z,z'}(s)=T(z_1,z_2)(s)-T(z'_1,z'_2)(s).$$

For $p_0>0$, let

$$\mathcal{I}(p_0)=\int_I\int_{-p_0}^{+p_0}|\hat{\nu}_s(p)|^2\,dp\,ds=\iint_{L\times L}\mathcal{I}_{z,z'}(p_0)\,d\mu(z)\,d\mu(z'),$$

with

$$\mathcal{I}_{z,z'}(p_0)=\int_I\int_{-p_0}^{p_0}e^{2\pi ip}T_{z,z'}(s)\,dp\,ds=\frac{1}{\pi}\int_I\frac{\sin 2\pi p_0 T_{z,z'}(s)}{T_{z,z'}(s)}\,ds.$$

2.3. Fix $z,z'\in L$ and just write T for $T_{z,z'}$. We estimate $\mathcal{I}_{z,z'}(p_0)=\mathcal{I}(p_0)$ in various cases. The letter d has the same meaning as in §2.1.

Case 1: $d>r^{\beta}$. With J as in §2.1 (iv), we have

$$\left|\int_{I-J}\frac{\sin 2\pi p_0T(s)}{T(s)}\,ds\right|\leqslant cd^{-1}|\log r|^{-1}.$$

On I, we use the following (classical) lemma.

LEMMA. *Let T be a C^2 monotonous function on an interval J. For any $p_0>0$:*

$$\left|\int_J\frac{\sin 2\pi pT(s)}{T(s)}\,ds\right|\leqslant c\left(\left(\inf_J|T'|\right)^{-1}+\frac{\sup_J|T''|(\sup_J T-\inf_J T)}{(\inf_J|T'|)^3}\right).$$

Proof. Let $u=T(s)$, $u_1=\sup_J T$, $u_0=\inf_J T$, $\varphi(u)=(T'\circ T^{-1}(u))^{-1}$.
One has, for $u\in[u_0,u_1]$,

$$|\varphi(u)|\leqslant\left(\inf_J|T'|\right)^{-1},$$
$$|\varphi'(u)|\leqslant\sup_J|T''|\left(\inf_J|T'|\right)^{-3}.$$

Also

$$\left|\int_J\frac{\sin 2\pi p_0T(s)}{T(s)}\,ds\right|=\left|\int_{u_0}^{u_1}\frac{\sin 2\pi p_0u}{u}\varphi(u)\,du\right|.$$

Let $\varphi_0=\varphi(0)$ if $0\in[u_0,u_1]$, $\varphi_0=\varphi(u_0)$ if $u_0>0$, $\varphi_0=\varphi(u_1)$ if $u_1<0$. We have

$$|\varphi(u)-\varphi_0|\leqslant\sup_J|T''|\left(\inf_J|T'|\right)^{-3}|u|,$$

hence

$$\int_{u_0}^{u_1}\frac{|\sin 2\pi p_0u|}{|u|}|\varphi(u)-\varphi_0|\,du\leqslant\sup_J|T''|\left(\inf_J|T'|\right)^{-3}(u_1-u_0);$$

On the other hand

$$\left|\int_{u_0}^{u_1} \frac{\sin 2\pi p_0 u}{u}\varphi_0\, du\right| = |\varphi_0|\left|\int_{2\pi p_0 u_0}^{2\pi p_1 u_1} \frac{\sin u}{u}\, du\right| \leqslant c|\varphi_0|,$$

which gives the lemma.

Using the lemma, the definition of J and §2.1 (iv), we get in Case 1:

$$|I_{z,z'}(p_0)| \leqslant cd^{-1}|\log r|^{-1}.$$

Case 2: $d<r^\beta$. Again we have, with the definition of J in §2.1 (v):

$$\left|\int_{I-J} \frac{\sin 2\pi p_0 T(s)}{T(s)}\, ds\right| \leqslant cd^{-1-c|\log r|^{-1}}|\log r|^{-1}.$$

On J, we use the lemma with the estimates of §2.1 (v):

$$\left|\int_{J} \frac{\sin 2\pi p_0 T(s)}{T(s)}\, ds\right| \leqslant cd^{-1-3c|\log r|^{-1}}|\log d|^2,$$

and we conclude that

$$|\mathcal{I}_{z,z'}(p_0)| \leqslant cd^{-1-3c|\log r|^{-1}}|\log d|^2.$$

2.4. Let $0<\varrho<2r$. By §2.1 (i), the set of (z,z') in $L\times L$ for which

$$d=\sup(d_{s_0}(z_1,z_1'), d_{s_0}(z_2,z_2'))<\varrho$$

has $\mu\times\mu$-measure at most $c(r\varrho)^{\Delta_1+\Delta_2}$.

Consequently:

$$\iint_{2^{-n}r\leqslant d<2^{1-n}r} \mathcal{I}_{z,z'}(p_0)\, d\mu(z)\, d\mu(z') \leqslant cr^{2(\Delta_1+\Delta_2)}2^{-n(\Delta_1+\Delta_2)}A_n,$$

with

$$A_n=\begin{cases} r^{-1}2^n|\log r|^{-1} & \text{if } 2^{-n}r>r^\beta, \\ (r/2^n)^{-1-3c|\log r|^{-1}}|\log(r/2^n)|^2 & \text{if } 2^{-n}r\leqslant r^\beta. \end{cases}$$

We take η small enough to have $\Delta_1+\Delta_2>c>1$ and r small enough to have

$$3c|\log r|^{-1}<\tfrac{1}{2}(\Delta_1+\Delta_2)\cdot\frac{\beta_{-1}}{\beta}.$$

Then we get

$$\sum_{n\geqslant 0} 2^{-n(\Delta_1+\Delta_2)}A_n \leqslant cr^{-1}|\log r|^{-1},$$

$$\mathcal{I}(p_0)\leqslant cr^{2\Delta_1+2\Delta_2-1}|\log r|^{-1}.$$

Letting p_0 go to ∞, we conclude that for almost all $s\in I$, ν_s has an L^2-density χ_s with respect to Lebesgue measure and that

$$\int_I \|\chi_s\|_{L^2}^2\,ds \leqslant c r^{2\Delta_1+2\Delta_2-1}|\log r|^{-1}.$$

Therefore there is a set $A\subset I$ of Lebesgue measure $\geqslant c^{-1}|\log r|^{-1}$ such that, for $s\in A$:

$$\|\chi_s\|_{L^2}^2 \leqslant c r^{2\Delta_1+2\Delta_2-1}.$$

On the other hand, the total mass of ν_s satisfies:

$$\|\chi_s\|_{L^1} = \nu_s(\mathbf{R}) = \mu(L) > c^{-1} r^{\Delta_1+\Delta_2}.$$

By the Cauchy–Schwarz inequality, for $s\in A$, the support of ν_s has Lebesgue measure at least $c^{-1}r$. The theorem is proved. □

3. The selection lemma

Our goal in this section is twofold: after recalling some basic material on subshifts of finite type, we express the transversality hypothesis on Hausdorff dimensions in the theorem in a convenient form; we then proceed to construct a set L satisfying the assumptions of §2.1.

More precisely, the contents of this section are as follows. In §§3.1–3.5, we recall the basic facts that we need concerning subshifts of finite type and Gibbs measures. In §3.6, we translate our transversality hypotheses in the symbolic dynamics setting. The end of the section is then devoted to check, from the geometrical estimates given in Proposition 3 in §3.13, the conditions (i)–(v) of §2.1, for an appropriate set L. An outline is first given in §3.7.

3.1. Consider an integer $r\geqslant 2$, and a subshift of finite type Σ^+ of the unilateral full shift on r symbols $\{1,\ldots,r\}$. Let σ be the shift map, and $A=(a_{ij})_{1\leqslant i,j\leqslant r}$ be the transition matrix determining Σ^+. We assume in the sequel that (Σ^+,σ) is topologically mixing.

For $\underline{x}=(x(l))_{l\geqslant 0}$ and $\underline{y}=(y(l))_{l\geqslant 0}$ in Σ^+, define:

$$v(\underline{x},\underline{y}) = \inf\{l\geqslant 0 : x(l)\neq y(l)\},$$
$$d(\underline{x},\underline{y}) = \exp(-v(\underline{x},\underline{y})).$$

Then (Σ^+,d) is a compact ultrametric space, whose balls of positive radius are called cylinders. For a cylinder C, we denote by $v(C)$ the integer such that the diameter of C is $\exp(-v(C))$.

For any continuous function φ on Σ^+, we write $S_n\varphi=\sum_{l=0}^{n-1}\varphi\circ\sigma^l$ for $n\geqslant 0$. For $k,n\geqslant 0$ and $m\geqslant k+n$, $x\in\Sigma^+$, we have:

$$\min_{v(x,y)\geqslant m} S_{k+n}\varphi(y)\geqslant \min_{v(x,y)\geqslant m} S_n\varphi(y)+\min_{v(\sigma^n x,z)\geqslant m-n} S_k\varphi(z),$$
$$\min_{v(x,y)\geqslant m} S_{k+n}\varphi(y)\leqslant \min_{v(x,y)\geqslant m} S_n\varphi(y)+\max_{v(\sigma^n x,z)\geqslant m-n} S_k\varphi(z),$$
$$\min_{v(x,y)\geqslant m} S_{k+n}\varphi(y)\leqslant \max_{v(x,y)\geqslant m} S_n\varphi(y)+\min_{v(\sigma^n x,z)\geqslant m-n} S_k\varphi(z).$$

3.2. Let φ be a strictly positive continuous function on Σ^+. For distinct $x,y\in\Sigma^+$, define, with $m=v(x,y)$:

$$d_\varphi(x,y)=\exp\big(-\min_{v(x,z)\geqslant m} S_m\varphi(z)\big).$$

Putting also $d_\varphi(x,x)=0$, it follows from the inequalities above that d_φ is an ultrametric distance on Σ^+. Moreover, for distinct x,y and $0\leqslant n\leqslant v(x,y)=m$, we have:

$$\min_{v(x,z)\geqslant m} S_n\varphi(z)\leqslant \log\frac{d_\varphi(\sigma^n x,\sigma^n y)}{d_\varphi(x,y)}\leqslant \max_{v(z,x)\geqslant m} S_n\varphi(z).$$

We set $d=d_1$. The identity maps: $(\Sigma^+,d)\to(\Sigma^+,d_\varphi)$, $(\Sigma^+,d_\varphi)\to(\Sigma^+,d)$ are Hölder continuous. The balls of positive radius for d_φ are the cylinders; we write $|B|_\varphi$ for the d_φ-diameter of a subset B of Σ^+.

3.3. Let δ_φ be the Hausdorff dimension of (Σ^+,d_φ). For $1\leqslant j\leqslant r$, let

$$\Sigma_j^+=\{\underline{x}=(x(l))_{l\geqslant 0}\in\Sigma^+ : x(0)=j\}.$$

For $n\geqslant 0$, denote by Σ_j^n the set of cylinders C satisfying $C\subset\Sigma_j^+$, $v(C)\geqslant n+1$ which are maximal with these properties. They form a finite partition of Σ_j^+.

PROPOSITION. *Let $n\geqslant 0$. We have:*

$$\max_{1\leqslant j\leqslant r}\sum_{C\in\Sigma_j^n}\exp\big(-\delta_\varphi\min_C S_n\varphi\big)\geqslant 1,$$
$$\min_{1\leqslant j\leqslant r}\sum_{C\in\Sigma_j^n}\exp\big(-\delta_\varphi\max_C S_n\varphi\big)\leqslant 1.$$

Proof. For $\delta>0$ and a finite family $\mathcal{B}=(B_1,\dots,B_s)$ of cylinders, let

$$H_\delta(\mathcal{B})=\sum|B_i|_\varphi^\delta.$$

First, let $\delta<\delta_\varphi$; consider, for each $1\leqslant j\leqslant r$, a finite covering $\mathcal{B}_j$ of Σ_j^+ by cylinders. The proposition is trivial for $n=0$, so let $n\geqslant 1$. Let $1\leqslant j\leqslant r$, $C\in\Sigma_j^n$; let $1\leqslant k\leqslant r$ be such that $\sigma^n(C)=\Sigma_k^+$, and $\mathfrak{X}B_C$ be the covering of C whose image under σ^n is $\mathcal{B}_k$; by §3.2, we have:

$$H_\delta(\mathcal{B}_C)\leqslant \exp\bigl(-\delta\min_C S_n\varphi\bigr)H_\delta(\mathcal{B}_k).$$

Therefore, if $\mathcal{C}_j$ is the covering of Σ_j^+ given by the various $\mathcal{B}_C$, $C\in\Sigma_j^n$, we have:

$$\max_{1\leqslant j\leqslant r}H_\delta(\mathcal{C}_j)\leqslant D\max_{1\leqslant k\leqslant r}H_\delta(\mathcal{B}_k),$$
$$D=\max_{1\leqslant j\leqslant r}\sum_{C\in\Sigma_j^n}\exp\bigl(-\delta\min_C S_n\varphi\bigr).$$

If we had $D\leqslant 1$, we would get arbitrarily fine coverings with bounded H_δ, contradicting $\delta<\delta_\varphi$. This gives the first inequality in the proposition.

Let now $\delta>\delta_\varphi$, $n\geqslant 1$. Let, for some $1\leqslant j\leqslant r$, $\mathcal{B}$ be a finite covering of Σ_j^+ by cylinders B with $v(B)\geqslant n+1$. For $C\in\Sigma_j^n$, let $\mathfrak{X}B_C$ be the covering of C by those elements of $\mathcal{B}$ which meet C; if $\sigma^n(C)=\Sigma_k^+$, let $\mathcal{C}_C$ be the covering of Σ_k^+ image of $\mathcal{B}_C$ under σ^n. We have:

$$H_\delta(\mathcal{B})=\sum_{C\in\Sigma_j^n}H_\delta(\mathcal{B}_C),$$

and, by §3.2:

$$H_\delta(\mathcal{B}_C)\geqslant \exp\bigl(-\delta\max_C S_n\varphi\bigr)H_\delta(\mathcal{C}_C),$$

hence

$$H_\delta(\mathcal{B})\geqslant D'\inf_{C\in\Sigma_j^n}H_\delta(\mathcal{C}_C),$$
$$D'=\min_{1\leqslant j\leqslant r}\sum_{C\in\Sigma_j^n}\exp\bigl(-\delta\max_C S_n\varphi\bigr).$$

As $\mathcal{C}_c$ has fewer elements than $\mathcal{B}$ and $\delta>\delta_\varphi$, we must have $D'\leqslant 1$, proving the second inequality of the proposition. □

COROLLARY ([MM], [PV]). *The map $\varphi\mapsto\delta_\varphi$, defined on the strictly positive continuous functions on Σ^+, is continuous.*

Proof. Let $C_+(\Sigma^+)$ be the space of strictly positive continuous functions on Σ^+. For $\varphi\in C_+(\Sigma^+)$ and $n\geqslant 1$, define $\delta_n^\pm(\varphi)$ by:

$$\max_{1\leqslant j\leqslant r}\sum_{C\in\Sigma_j^n}\exp\bigl(-\delta_n^+(\varphi)\min_C S_n\varphi\bigr)=1,$$
$$\min_{1\leqslant j\leqslant r}\sum_{C\in\Sigma_j^n}\exp\bigl(-\delta_n^-(\varphi)\max_C S_n\varphi\bigr)=1;$$

we have

$$\delta_n^-(\varphi) \leqslant \delta_\varphi \leqslant \delta_n^+(\varphi)$$

by the proposition. On the other hand, the maps δ_n^+, δ_n^-, for $n \geqslant 1$, form an equicontinuous family on $C_+(\Sigma^+)$, and the sequence $(\delta_n^+ - \delta_n^-)_{n \geqslant 1}$ converge uniformly to 0 on compact subsets of $C_+(\Sigma^+)$. The corollary follows. □

3.4. Let $\gamma > 0$, and $C^\gamma(\Sigma^+)$ be the Banach algebra of Hölder continuous functions of exponent γ on (Σ^+, d). Here we simplify the notation, indicating an element $\underline{x} \in \Sigma^+$ by $x \in \Sigma^+$. For $\varphi \in C^\gamma(\Sigma^+)$ and $x, y \in \Sigma^+$, we have

$$|S_n\varphi(x) - S_n\varphi(y)| \leqslant C(\varphi),$$

for $n \leqslant v(x, y)$ and some constant $C(\varphi)$, v being defined as in (3.1) above. This is called *the bounded oscillation property of Birkhoff sums.*

For $\psi \in C^\gamma(\Sigma^+)$, the Perron–Frobenius operator $L_\psi : C^\gamma(\Sigma^+) \to C^\gamma(\Sigma^+)$ is defined by:

$$L_\psi(\chi)(x) = \sum_{\sigma y = x} \chi(y) \exp(-\psi(y)).$$

We recall Ruelle's theorem, and the relation to Hausdorff dimension ([Bo1], [Bo2], [Man]).

The spectrum of L_ψ is formed by a simple eigenvalue $\varrho_\psi > 0$ and a compact set contained in $\{|z| < \varrho_\psi\}$.

The eigenfunction h_ψ associated to ϱ_ψ is strictly positive; the complementary invariant hyperplane is the kernel of a probability measure ν_ψ on Σ^+, satisfying $L_\psi^*(\nu_\psi) = \varrho_\psi \nu_\psi$.

Normalizing h_ψ by $\nu_\psi(h_\psi) = 1$, the probability measure $\mu_\psi = h_\psi \nu_\psi$ is invariant under σ and ergodic.

3.5. The map $L: \psi \mapsto L_\psi$ from $C^\gamma(\Sigma^+)$ to $\mathcal{L}(C^\gamma(\Sigma^+))$ is analytic, with differential given by:

$$D_\psi L(\Delta\psi)(\chi) = L_\psi(\chi \Delta\psi).$$

The map $\varrho: \psi \mapsto \varrho_\psi$ from $C^\gamma(\Sigma^+)$ to $\mathbf{R}$ is analytic, with:

$$D_\psi \varrho(\Delta\psi) = \varrho_\psi \int \Delta\psi \, d\mu_\psi.$$

Let $\varphi \in C^\gamma(\Sigma^+)$, $\varphi > 0$. The Hausdorff dimension δ_φ of (Σ_A^+, d_φ) is the unique $\delta > 0$ such that $\varrho(\delta\varphi) = 1$. The map $\varphi \mapsto \delta_\varphi$ is analytic on $C_+^\gamma(\Sigma^+)$, with:

$$D_\varphi \delta(\Delta\varphi) = -\frac{\int \Delta\varphi \, d\mu_\psi}{\int \varphi \, d\mu_\psi} \delta_\varphi,$$

where $\psi=\delta_\varphi\varphi$.

Finally, let B be a d_φ-ball of radius r, $n=v(B)$, and A a measurable subset of B; we have:

$$c^{-1}r^\delta \leqslant \mu(B) \leqslant cr^\delta,$$
$$c^{-1}\frac{\mu(A)}{\mu(B)} \leqslant \mu(\sigma^n(A)) \leqslant c\frac{\mu(A)}{\mu(B)},$$

where $\mu=\mu_\psi$, $\delta=\delta_\varphi$ and c depend only on γ, $\|\varphi\|_\gamma$ and $\|\varphi^{-1}\|_0$.

3.6. Let us now translate in the setting of symbolic dynamics our geometrical transversality hypotheses.

Using a Markov partition for Λ_1, we choose a subshift of finite type Σ_1 of the full bilateral left-shift on symbols $\{1,...,r_1\}$, and a homeomorphism $h_1:\Sigma_1\to\Lambda_1$ such that:

$$h_1\circ\sigma = f\circ h_1.$$

Similarly, we choose a subshift of finite type Σ_2 of the full bilateral left-shift on symbols $\{1,...,r_2\}$, and a homeomorphism $h_2:\Sigma_2\to\Lambda_2$ such that

$$h_2\circ\sigma = f^{-1}\circ h_2.$$

Replacing if necessary f by some iterate, we assume that both subshifts are topologically mixing.

For $i\in\{1,2\}$, let Σ_i^+ be the one-sided shifts on symbols $\{1,...,r_i\}$ and $\pi_i:\Sigma_i\to\Sigma_i^+$ be the canonical projections.

We recall that there are continuous linear operators $\Pi_i: C^\gamma(\Sigma_i)\to C^\gamma(\Sigma_i^+)$ and Θ_i: $C^\gamma(\Sigma_i)\to C^\gamma(\Sigma_i)$ such that, for $\psi\in C^\gamma(\Sigma_i)$ $(i=1,2)$:

$$\Pi_i(\psi)\circ\pi_i = \psi+\Theta_i(\psi)-\Theta_i(\psi)\circ\sigma,$$

where $\pi_i:\Sigma_i\to\Sigma_i^+$ is the canonical projection.

We have fixed some Riemannian metric on M. For $z\in\Sigma_1$ and $s,t\in(-\eta,\eta)$, let:

$$\lambda_1(z,s,t) = \log\|T_{h_1(z)(s,t)}f_{s,t}|_{E^u}\|,$$

where E^u is the unstable subspace of $f_{s,t}$ at the point $h_1(z)(s,t)$ of the basic set. Similarly, for $z\in\Sigma_2$, $s,t\in(-\eta,\eta)$ let

$$\lambda_2(z,s,t) = \log\|T_{h_2(z)(s,t)}f_{s,t}^{-1}|_{E^s}\|.$$

We may assume that the Riemannian metrics is such that

$$\lambda_i(z,s,t) \geqslant c > 0, \quad i=1,2,$$

for $z \in \Sigma_i$, $s,t \in (-\eta,\eta)$.

For $s,t \in (-\eta,\eta)$, $i=1,2$, let $\lambda_i(s,t)$ be the map $z \mapsto \lambda_i(z,s,t)$ from Σ_i to $\mathbf{R}$. There exists $\gamma>0$ such that $\lambda_i: (s,t) \to \lambda_i(s,t)$ is a smooth map from $(-\eta,\eta)^2$ to $C^\gamma(\Sigma_i)$.

Let $\varphi_i(s,t)=\Pi_i(\lambda_i(s,t))$; then $(s,t) \mapsto \varphi_i(s,t)$ is a smooth map from $(-\eta,\eta)^2$ to $C^\gamma(\Sigma_i^+)$.

Let $(s,t) \in (-\eta,\eta)$. It is well-known (and we will prove in §4) that the composition

$$W^u_{\mathrm{loc}}(p_1,s,t) \cap \Lambda_1(s,t) \xrightarrow{h^{-1}_{1,s,t}} \Sigma_1 \xrightarrow{\pi_1} (\Sigma_1^+, d_{\varphi_1(s,t)})$$

is a biLipschitz homeomorphism on a neighbourhood of a_1 in Σ_1^+. Therefore the Hausdorff dimension $\Delta_1(s,t)$ of $W^u_{\mathrm{loc}}(p_1,s,t) \cap \Lambda_1(s,t)$ is the same as the Hausdorff dimension $\delta_1(s,t)$ of $(\Sigma_1^+, d_{\varphi_1(s,t)})$.

Similarly, the Hausdorff dimension $\Delta_2(s,t)$ of $W^s_{\mathrm{loc}}(p_2,s,t) \cap \Lambda_2(s,t)$ is the same as the Hausdorff dimension $\delta_2(s,t)$ of $(\Sigma_2^+, d_{\varphi_2(s,t)})$.

For $(s,t) \in (-\eta,\eta)^2$, let

$$\Delta\varphi_i(s,t) = \frac{\partial}{\partial s}(\varphi_i(s,t)) = \Pi_i\left(\frac{\partial}{\partial s}\lambda_i(s,t)\right).$$

Let $\mu_{i,s,t}=\mu_{\psi_i(s,t)}$, with $\psi_i(s,t)=\delta_i(s,t)\varphi_i(s,t)$.

From §3.5 we have that

$$\frac{\partial}{\partial s}\log \delta_i(s,t) = -\frac{\mu_{i,s,t}(\Delta\varphi_i(s,t))}{\mu_{i,s,t}(\varphi_i(s,t))}$$

(and a similar formula holds with $\partial/\partial t$).

Taking η small enough, the transversality hypothesis on the Hausdorff dimensions is therefore equivalent to:

$$|\mu_{1,s,t}(\varphi_1(s,t))\mu_{2,s,t}(\Delta\varphi_2(s,t)) - \mu_{2,s,t}(\varphi_2(s,t))\mu_{1,s,t}(\Delta\varphi_1(s,t))| \geqslant c > 0.$$

For the eigenvalues of the periodic orbit, we have, for $i=1,2$:

$$\bar{\lambda}_i(s,t) = \sum_{j=0}^{n_i-1} \lambda_i(\sigma^j(a_i),s,t) = \sum_{j=0}^{n_i-1} \varphi_i(s,t)(\sigma^j a_i).$$

Hence the transversality hypothesis on the eigenvalues in §1.5 means that (taking η small enough) we have:

$$|S_{n_1}\varphi_1(s,t)(a_1)S_{n_2}\Delta\varphi_2(s,t)(a_2) - S_{n_2}\varphi_2(s,t)(a_2)S_{n_1}\Delta\varphi_1(s,t)a_1)| \geqslant c > 0.$$

3.7. Let us go back to the setting and conditions of §2.1. We identify $W^u_{\mathrm{loc}}(p_1,s,t)\cap\Lambda_1(s,t)$ with a neighbourhood of a_1 in Σ_1^+ and $W^s_{\mathrm{loc}}(p_2,s,t)\cap\Lambda_2(s,t)$ with a neighbourhood of a_2 in Σ_2^+.

Let us fix $s_0\in(-\eta,\eta)$ and set, for $i=1,2$:

$$\begin{aligned}
&\varphi_i=\varphi_i(s_0,0), &&\Delta\varphi_i=\Delta\varphi_i(s_0,0),\\
&\mu_i=\mu_{i,s_0,0}, &&\\
&S_i=S_{n_i}\varphi_i(a_i), &&\Delta S_i=S_{n_i}\Delta\varphi_i(a_i),\\
&J_i=\int\varphi_i\,d\mu_i, &&\Delta J_i=\int\Delta\varphi_i\,d\mu_i.
\end{aligned}$$

We use d_{φ_i} as distance on Σ_i^+ and denote by $B_i(r)$ the d_{φ_i}-ball of center a_i, radius r. Our transversality hypotheses mean that:

$$S_1\Delta S_2\neq S_2\Delta S_1,\quad J_1\Delta J_2\neq J_2\Delta J_1.$$

Let $z_1,z_1'\in B_1(r)$, $z_2,z_2'\in B_2(r)$ as in §2.1, and

$$T(s)=T(z_1,z_2)(s)-T(z_1',z_2')(s).$$

The main point of properties (iv), (v) in §2.1 is that $|T(s)|$ and $|T'(s)|$ should not *both* be too small at the same time.

Let $\nu_1=v(z_1,z_1')$, $\nu_2=v(z_2,z_2')$.

It will be a consequence of the geometrical estimates of Proposition 3 below that if both $|T(s)|$ and $|T'(s)|$ are small, then

$$\begin{aligned}
D_1&=|S_{\nu_1}\varphi_1(z_1)-S_{\nu_2}\varphi_2(z_2)|,\\
D_2&=|S_{\nu_1}\Delta\varphi_1(z_1)-S_{\nu_2}\Delta\varphi_2(z_2)|,
\end{aligned}$$

are both bounded.

That this cannot happen when ν_1,ν_2 are not too large follow from the hypothesis $S_1\Delta S_2\neq S_2\Delta S_1$: this is the content of Proposition 1 below and will imply conditions (iv) of §2.1.

For large ν_1,ν_2, the Birkhoff sums above are related at most points, through Birkhoff's ergodic theorem, to the mean values of the functions considered.

Roughly speaking, we would like to have, for most z_1,z_2:

$$D_1\text{ bounded}\implies\nu_1/\nu_2\approx J_2/J_1;$$

hence

$$D_2 \approx \nu_2 |J_2/J_1 \Delta J_1 - \Delta J_2| \geqslant c^{-1} \nu_2$$

as $J_2 \Delta J_1 \neq J_1 \Delta J_2$.

The set L in condition (v) of §2.1 is thus constructed by first deleting exceptional points for Birkhoff's ergodic theorem.

But we have also to take the intermediate values of ν_1, ν_2 into account, which are covered neither by Proposition 1 nor by Birkhoff's theorem. This is done using a stopping-time argument. The precise construction of L is done in Proposition 2 below.

The rest of this section is as follows:

— in §3.8, we state and prove Proposition 1;

— in §3.9, we state Proposition 2, which is then proved in §§3.10–3.12;

— in §3.13, we state Proposition 3, to be proven in §4;

— in §§3.14 and 3.15 we finally deduce conditions (i)–(v) of §2.1 from Propositions 1, 2, 3.

3.8. PROPOSITION 1. *Assume that* $S_1 \Delta S_2 \neq S_2 \Delta S_1$. *Then there exist constants* $c_0 > 0$, $r_1 > 0$, $\beta_0 > 1$ *such that the following property holds: let* $0 < r < r_1$, $m_i = v(B_i(r))$, $\nu_i \in \mathbf{N}$ *(for* $i = 1, 2$*); if*

$$1 \leqslant \inf\left(\frac{\nu_1}{m_1}, \frac{\nu_2}{m_2}\right) \leqslant \beta_0,$$

then, for all $z_1 \in B_1(r)$, $z_2 \in B_2(r)$, *we have:*

$$\max(|S_{\nu_1}\varphi_1(z_1) - S_{\nu_2}\varphi_2(z_2)|, |S_{\nu_1}\Delta\varphi_1(z_1) - S_{\nu_2}\Delta\varphi_2(z_2)|) \geqslant c_0^{-1} |\log r|.$$

Proof. In the following, we write c for various positive constants depending only on z_i, r_i, φ_i, $\Delta\varphi_i$. Moreover, the dependence on φ_i, $\Delta\varphi_i$ is only through $\|\varphi_i\|_\gamma$, $\|\Delta\varphi_i\|_\gamma$, $\|\varphi_i^{-1}\|_0$, S_i, ΔS_i, J_i, ΔJ_i. We use repeatedly the bounded oscillation property for the Birkhoff sums of φ_i, $\Delta\varphi_i$, which follows from the Hölder continuity.

With $\varepsilon > 0$ small enough, assume that

$$|S_{\nu_1}\varphi_1(z_1) - S_{\nu_2}\varphi_2(z_2)| < \varepsilon |\log r|,$$

and for instance $m_1 \leqslant \nu_1 \leqslant m_1(1+\varepsilon)$, $m_2 \leqslant \nu_2$.

We have, for $i = 1, 2$:

$$\left| S_{m_i}\varphi_i(z_i) - \frac{m_i}{n_i} S_i \right| < c,$$

$$\left| \frac{m_i}{n_i} S_i - |\log r| \right| < c;$$

as φ_i is positive, we get:

$$c^{-1}(\nu_i-m_i)-c<S_{\nu_i}\varphi_i(z_i)-\frac{m_i}{n_i}S_i<c(\nu_i-m_i)+c,$$
$$-\varepsilon|\log r|<S_{\nu_1}\varphi_1(z_1)-S_{\nu_2}\varphi_2(z_2)<c+c(\nu_1-m_1)-c^{-1}(\nu_2-m_2).$$

For r_1 small enough, this implies:

$$\nu_2\leqslant m_2(1+c\varepsilon),$$
$$\left|S_{\nu_i}\varphi_i(z_i)-\frac{m_i}{n_i}S_i\right|<c\varepsilon|\log r|.$$

We obtain also in a similar way:

$$\left|S_{\nu_i}\Delta\varphi_i(z_i)-\frac{m_i}{n_i}\Delta S_i\right|<c\varepsilon|\log r|.$$

If we had also:

$$|S_{\nu_1}\Delta\varphi_1(z_1)-S_{\nu_2}\Delta\varphi_2(z_2)|<\varepsilon|\log r|,$$

we would get, as $c|\log r|\geqslant m_i\geqslant c^{-1}|\log r|$:

$$|S_1\Delta S_2-S_2\Delta S_1|\leqslant\frac{n_2}{m_2}\left(\left|S_1\left(\frac{m_2}{n_2}\Delta S_2-\frac{m_1}{n_1}\Delta S_1\right)\right|+\left|\Delta S_1\left(\frac{m_1}{n_1}S_1-\frac{m_2}{n_2}S_2\right)\right|\right)\leqslant c\varepsilon,$$

a contradiction for ε small enough. We take $\beta_0=1+\varepsilon$ and $c_0=\varepsilon^{-1}$. □

3.9. PROPOSITION 2. *Assume that* $J_1\Delta J_2\neq J_2\Delta J_1$.

There exist constants $c_1,c_2>0$ *and, for any* $M>0$, *constants* $r(M)>0$, $\varepsilon(M)>0$ *such that, for any* $0<r<r(M)$, *we can find a compact subset* L *of* $B_1(r)\times B_2(r)$ *with the following properties*:

(i) $\mu_1\times\mu_2(L)>\varepsilon(M)\mu_1(B_1(r))\mu_2(B_2(r))$;

(ii) *for any distinct* $y_1,z_1\in B_1(r)$, $y_2,z_2\in B_2(r)$ *such that* $(y_1,y_2),(z_1,z_2)\in L$, *we have*:

$$\sup(|S_{\nu_1}\varphi_1(z_1)-S_{\nu_2}\varphi_2(z_2)|,|S_{\nu_1}\Delta\varphi_1(z_1)-S_{\nu_2}\Delta\varphi_2(z_2)|)\geqslant\sup(M,c_1(\nu_1+\nu_2-c_2|\log r|))$$

where $\nu_i=v(y_i,z_i)$ *for* $i=1,2$.

3.10. *Proof of Proposition* 2. *We may for instance assume that* $J_1\Delta J_2>J_2\Delta J_1$. We will write μ for $\mu_1\times\mu_2$.

Let $\eta>0$ be a small positive constant, to be chosen later, independent of M. As μ_i is ergodic for $i=1,2$, we can find a compact subset $K_i\subset\Sigma_i^+$ with $\mu_i(K_i)>1-\eta$ and an integer n_0 such that, for $n\geqslant n_0$ and $z_i\in K_i$, we have:

$$|S_n\varphi_i(z_i)-nJ_i|<\eta n,$$
$$|S_n\Delta\varphi_i(z_i)-n\Delta J_i|<\eta n.$$

For a cylinder $C\subset\Sigma_i^+$, we define

$$K_i(C)=\sigma^{-m}(K_i)\cap C, \quad m=v(C);$$

if η is small enough (independently of C), we have:

$$\mu_i(K_i(C))\geqslant\tfrac{1}{2}\mu_i(C) \quad \text{(see §3.5)}.$$

LEMMA 1. *Assume that $\eta<c^{-1}$. For $i=1,2$, let $z_i\in\Sigma_i^+$ and q_i,ν_i be integers such that*

$$\bar\nu_i=\nu_i-q_i\geqslant n_0, \quad \sigma^{q_i}(z_i)\in K_i.$$

Then we have

$$(S_{\nu_2}\Delta\varphi_2(z_2)-S_{\nu_1}\Delta\varphi_1(z_1))+|S_{\nu_2}\varphi_2(z_2)-S_{\nu_1}\varphi_1(z_1)|$$
$$\geqslant(S_{q_2}\Delta\varphi_2(z_2)-S_{q_1}\Delta\varphi_1(z_1))-|S_{q_2}\varphi_2(z_2)-S_{q_1}\varphi_1(z_1)|+c(\bar\nu_1+\bar\nu_2).$$

Proof. Writing

$$S_{\nu_i}\varphi_i=S_{q_i}\varphi_i+S_{\bar\nu_i}\varphi_i\circ\sigma^{q_i},$$

we get, as $\bar\nu_i\geqslant n_0$ and $\sigma^{q_i}(z_i)\in K_i$:

$$|S_{\nu_i}\varphi_i(z_i)-S_{q_i}\varphi_i(z_i)-\bar\nu_iJ_i|<\eta\bar\nu_i,$$
$$|S_{\nu_i}\Delta\varphi_i(z_i)-S_{q_i}\Delta\varphi_i(z_i)-\bar\nu_i\Delta J_i|<\eta\bar\nu_i.$$

It then follows that:

$$|S_{\nu_2}\varphi_2(z_2)-S_{\nu_1}\varphi_1(z_1)|\geqslant|\bar\nu_2J_2-\bar\nu_1J_1|-|S_{q_2}\varphi_2(z_2)-S_{q_1}\varphi_1(z_1)|-\eta(\bar\nu_1+\bar\nu_2);$$
$$S_{\nu_2}\Delta\varphi_2(z_2)-S_{\nu_1}\Delta\varphi_1(z_1)\geqslant\bar\nu_2\Delta J_2-\bar\nu_1\Delta J_1+S_{q_2}\Delta\varphi_2(z_2)-S_{q_1}\Delta\varphi_1(z_1)-\eta(\bar\nu_1+\bar\nu_2).$$

It remains to see that:

$$|\bar\nu_2J_2-\bar\nu_1J_1|+\bar\nu_2\Delta J_2-\bar\nu_1\Delta J_1\geqslant c(\bar\nu_1+\bar\nu_2).$$

But we have:

$$\bar{\nu}_2(J_1\Delta J_2 - J_2\Delta J_1) = J_1(\bar{\nu}_2\Delta J_2 - \bar{\nu}_1\Delta J_1) + \Delta J_1(\bar{\nu}_1 J_1 - \bar{\nu}_2 J_2),$$
$$\bar{\nu}_1(J_1\Delta J_2 - J_2\Delta J_1) = J_2(\bar{\nu}_2\Delta J_2 - \bar{\nu}_1\Delta J_1) + \Delta J_2(\bar{\nu}_1 J_1 - \bar{\nu}_2 J_2),$$

which implies the last inequality since $J_1>0$, $J_2>0$, $J_1\Delta J_2 > J_2\Delta J_1$. □

3.11. We now proceed to the construction of L. Fix $M>0$, which we may assume big, and $0<r<r(M)$, with $r(M)$ small, to be determined later. For $i=1,2$, let $m_i=v(B_i(r))$. For $z_i\in B_i(r)$, we have:

$$|S_{m_i}\varphi_i(z_i) - |\log r|\,| \leqslant c,$$
$$c^{-1}|\log r| \leqslant m_i \leqslant c|\log r|.$$

We distinguish two cases. (Recall that we are assuming that $J_1\Delta J_2 > J_2\Delta J_1$.)

Case 1: $S_{m_2}\Delta\varphi_2(a_2) \geqslant S_{m_1}\Delta\varphi_1(a_1)$.

We have

$$\mu_i(K_i(B_i(r))) \geqslant \tfrac{1}{2}\mu_i(B_i(r)).$$

With an integer $m_0=m_0(M)\geqslant n_0$ to be chosen later, pick cylinders $C_i\subset B_i(r)$ satisfying:

$$m_i+m_0+c \geqslant v(C_i) \geqslant m_i+m_0,$$
$$\mu_i(K_i(B_i(r))\cap C_i) \geqslant \tfrac{1}{2}\mu_i(C_i),$$

and define

$$L = (C_1\cap K_1(B_1(r)))\times(C_2\cap K_2(B_2(r))).$$

We have

$$\mu(L) \geqslant \tfrac{1}{4}\mu_1(C_1)\mu_2(C_2) \geqslant \tfrac{1}{4}e^{-c(m_0+c)}\mu_1(B_1(r))\mu_2(B_2(r)),$$

hence condition (i) in Proposition 2 is satisfied if $\varepsilon(M)<\frac{1}{4}e^{-c(m_0+c)}$. Let $y_1, y_2, z_1, z_2, \nu_1, \nu_2$ be as in condition (ii) of Proposition 2.

We have

$$|S_{m_1}\varphi_1(z_1) - S_{m_2}\varphi_2(z_2)| \leqslant c$$

and, by the hypothesis of Case 1:

$$S_{m_2}\Delta\varphi_2(z_2) - S_{m_1}\Delta\varphi_1(z_1) \geqslant -c.$$

Also, $\sigma^{m_i}(z_i)\in K_i$ and $\nu_i - m_i = \bar{\nu}_i \geqslant m_0 \geqslant n_0$; with $m_i=a_i$, we then get from Lemma 1:

$$\begin{aligned}(S_{\nu_2}\Delta\varphi(z_2) - S_{\nu_1}\Delta\varphi_1(z_1)) + |S_{\nu_2}\varphi_2(z_2) - S_{\nu_1}\varphi_1(z_1)| &\geqslant c((\nu_1+\nu_2)-(m_1+m_2)) - c' \\ &\geqslant cm_0 - c'.\end{aligned}$$

With $m_0 > cM$ and c_1, c_2^{-1} small enough, this implies the inequality in condition (ii) of Proposition 2.

3.12. *Case* 2: $S_{m_2}\Delta\varphi_2(a_2) < S_{m_1}\Delta\varphi_1(a_1)$.

The construction of L is more intricate, involving a stopping time argument.

Consider the family $\mathcal{F}$ of products $C = C_1 \times C_2$, where C_i is a cylinder contained in $B_i(r)$, such that, with $q_i = v(C_i)$, there exist points $u_1 \in C_1$, $u_2 \in C_2$ with:

$$|S_{q_1}\varphi_1(u_1) - S_{q_2}\varphi_2(u_2)| < M,$$
$$|S_{q_1}\Delta\varphi_1(u_1) - S_{q_2}\Delta\varphi_2(u_2)| < M.$$

We order $\mathcal{F}$ by inclusion, and denote by $\mathcal{F}_0$ the subfamily of maximal elements of $\mathcal{F}$. For $B > 0$, let $\mathcal{F}_B$ be the subfamily of $\mathcal{F}_0$ formed by the products $C = C_1 \times C_2 \in \mathcal{F}_0$ with:

$$v(C_i) \leqslant B|\log r|, \quad i = 1, 2.$$

LEMMA 2. *For $M, B > c$, we have*:

$$K_1(B_1(r)) \times K_2(B_2(r)) \subset \bigcup_{\mathcal{F}_B} C.$$

Proof. Let $u_i \in K_i(B_i(r))$, for $i = 1, 2$. For all $m \geqslant m_1$, select an integer $\tau(m) \geqslant m_2$ such that

$$|S_m\varphi_1(u_1) - S_{\tau(m)}\varphi_2(u_2)| \leqslant c,$$
$$\tau(m_1) = m_2, \quad \tau(m+1) \geqslant \tau(m).$$

(this is possible because $\varphi_1, \varphi_2 > 0$).

Let $\Delta_m = S_{\tau(m)}\Delta\varphi_2(u_2) - S_m\Delta\varphi_1(u_1)$ for $m \geqslant m_1$. We have $\tau(m+1) \leqslant \tau(m) + c$, hence:

$$|\Delta_m - \Delta_{m+1}| \leqslant c,$$

and, by the hypothesis of Lemma 2:

$$-c|\log r| < \Delta_{m_1} < c.$$

We also clearly have

$$c^{-1}m \leqslant \tau(m) \leqslant cm.$$

With $r(M)$ small enough, apply Lemma 1, taking $q_i = m_i$ and $B'|\log r| < \nu_i < B|\log r|$, $\nu_2 = \tau(\nu_1)$, $B' > c$; we get:

$$\Delta_{\nu_1} > \Delta_{m_1} - c' + c(\nu_1 + \nu_2 - m_1 - m_2) > 0,$$

hence there exists $m_1 \leqslant m \leqslant \nu_1$ such that

$$|S_m\varphi_1(u_1) - S_{\tau(m)}\varphi_2(u_2)| \leqslant c,$$
$$|\Delta_m| = |S_m\Delta\varphi_1(u_1) - S_{\tau(m)}\Delta\varphi_2(u_2)| \leqslant c.$$

Let C_1 be the smallest cylinder with $v(C_1) < m$ containing u_1, and C_2 be the smallest cylinder with $v(C_2) < \tau(m)$ containing u_2. We have

$$m - c \leqslant v(C_1) < m \leqslant \nu_1 < B|\log r|,$$
$$\tau(m) - c \leqslant v(C_2) < \tau(m) \leqslant \nu_2 < B|\log r|,$$

hence $C = C_1 \times C_2$ belongs to $\mathcal{F}$; this proves Lemma 2. □

For $C \in \mathcal{F}_0$, define:

$$W_1(C) = \bigcup_{\substack{C' \in \mathcal{F}_0 \\ C \cap C' \neq \varnothing}} C',$$
$$W_2(C) = \bigcup_{\substack{C' \in \mathcal{F}_0 \\ C \cap C' \neq \varnothing}} W_1(C').$$

Let also $U = \bigcup_{C \in \mathcal{F}_B} C$.

Choose elements $C^1, \ldots, C^N$ of $\mathcal{F}_B$, with N maximal ($\mathcal{F}_B$ is finite), such that:

$$C^{i+1} \not\subset \bigcup_{j=0}^{i} W_2(C^j), \quad 1 \leqslant i < N.$$

We then have

$$U \subset \bigcup_{j=1}^{N} W_2(C^j).$$

With an integer $m_0 = m_0(M) \geqslant n_0$ to be chosen later, select as in Case 1, for $1 \leqslant j \leqslant N$, a product of cylinders $\widehat{C}^j = \widehat{C}_1^j \times \widehat{C}_2^j \subset C^j = C_1^j \times C_2^j$ such that, for $i = 1, 2$:

$$m_0 + v(C_i^j) \leqslant v(\widehat{C}_i^j) \leqslant m_0 + c + v(C_i^j),$$
$$\mu_i(\widehat{C}_i^j \cap K_i(C_i^j)) \geqslant \tfrac{1}{2}\mu_i(\widehat{C}_i^j).$$

See Figure 2.

Finally, define

$$L = \bigcup_{j=1}^{N} (\widehat{C}_1^j \cap K_1(C_1^j)) \times (\widehat{C}_2^j \cap K_2(C_2^j)).$$

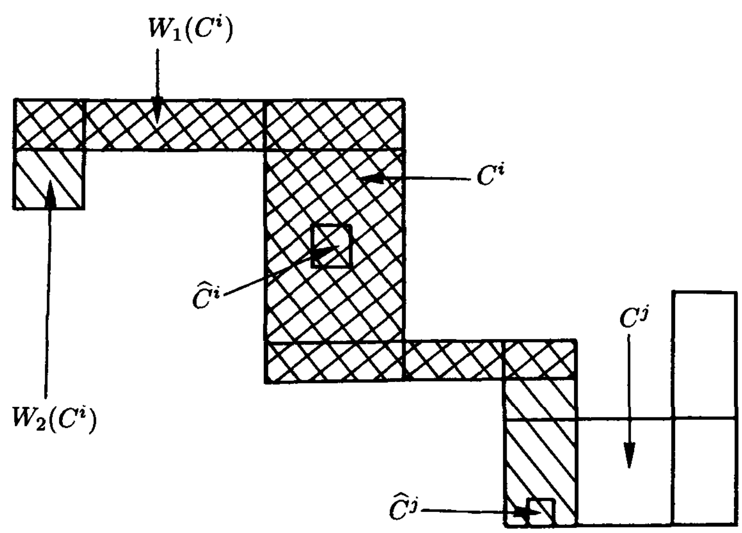

Fig. 2

We first check condition (i) of Proposition 2.

The C^j being disjoint by construction, we have:

$$\mu(L) \geqslant \tfrac{1}{4}\Sigma\mu(\widehat{C}^j) \geqslant \tfrac{1}{4}e^{-c(m_0+c)}\Sigma\mu(C^j).$$

On the other hand, if $C, C' \in \mathcal{F}_0$ satisfy $C \cap C' \neq \varnothing$, we must have for instance $C_1 \subset C'_1$, $C_2 \supset C'_2$; because φ_1, φ_2 are positive, this implies:

$$|v(C_i) - v(C'_i)| \leqslant cM.$$

From this, it follows, for $C \in \mathcal{F}_0$:

$$\mu(W_1(C)) \leqslant ce^{cM}\mu(C),$$
$$\mu(W_2(C)) \leqslant ce^{cM}\mu(C).$$

By Lemma 2 we have:

$$U \supset K_1(B_1(r)) \times K_2(B_2(r)).$$

Therefore, we obtain

$$\tfrac{1}{4}\mu(B_1(r) \times B_2(r)) \leqslant \mu(U) \leqslant \Sigma\mu(W_2(C^j)) \leqslant ce^{cM}\Sigma\mu(C^j),$$
$$\mu(L) \geqslant \tfrac{1}{16}c^{-1}e^{-c(M+m_0+c)}\mu(B_1(r) \times B_2(r)),$$

and condition (i) is satisfied provided

$$\varepsilon(M) < \tfrac{1}{16}c^{-1}e^{-c(M+m_0+c)}.$$

Let $y_1, y_2, z_1, z_2, \nu_1, \nu_2$ be as in condition (ii) of Proposition 2. Let $1 \leqslant j, k \leqslant N$ be such that $(y_1, y_2) \in \widehat{C}^j$, $(z_1, z_2) \in \widehat{C}^k$.

First assume that $j<k$. Then, for $i=1,2$:

$$\nu_i \leqslant v(C_i^j) \leqslant B|\log r|,$$

hence, with c_2^{-1} small enough

$$\max(M, c_1(\nu_1+\nu_2-c_2|\log r|)) = M;$$

if the inequality in Proposition 2 was not valid, the minimal product $\widetilde{C}=\widetilde{C}_1 \times \widetilde{C}_2$ containing (y_1, z_1) and (y_2, z_2) (with $v(\widetilde{C}_i)=\nu_i$) would belong to $\mathcal{F}$. But then $\widetilde{C} \subset W_1(C^j)$ and $C^k \subset W_2(C^j)$, contradicting the choice of the C^l.

Assume now that $j=k$. With $q_i = v(C_i^j)$, we apply Lemma 1 to get:

$$\begin{aligned}|S_{\nu_2}\Delta\varphi_2(z_2) - S_{\nu_1}\Delta\varphi_1(z_1)| + |S_{\nu_2}\varphi_2(z_2) - S_{\nu_1}\varphi_1(z_1)| \\ \geqslant (S_{q_2}\Delta\varphi_2(z_2) - S_{q_1}\Delta\varphi_1(z_1)) - |S_{q_2}\varphi_2(z_2) - S_{q_1}\varphi_1(z_1)| + c(\nu_1+\nu_2-q_1-q_2).\end{aligned}$$

From the definition of $\mathcal{F}, \mathcal{F}_B$, we have:

$$\begin{gathered}|q_i| \leqslant B|\log r|, \\ |S_{q_2}\varphi_2(z_2) - S_{q_1}\varphi_1(z_1)| \leqslant c+M, \\ |S_{q_2}\Delta\varphi_2(z_2) - S_{q_1}\Delta\varphi_1(z_1)| \leqslant c+M,\end{gathered}$$

and the inequality of the proposition follows, provided c_1, c_2^{-1} are small enough and $m_0 > cM$ (recall that $\nu_i - q_i \geqslant m_0$).

The proof of Proposition 2 is complete. □

3.13. We now state the geometrical estimates proved in §4, and deduce from them the assumptions in §2.1.

PROPOSITION 3. *Let η be small enough. There exist constants $r_2>0$, $c>0$ such that, if $z_1, z_1' \in \Sigma_1^+$, $z_2, z_2' \in \Sigma_2^+$ satisfy:*

$$\begin{aligned} d_0(z_i, a_i) &< r_2 \quad \textit{for } i=1,2, \\ d_0(z_i', a_i) &< r_2 \quad \textit{for } i=1,2,\end{aligned}$$

then, writing

$$\begin{aligned}\tau_1(s) &= \log|T(z_1, z_2)(s) - T(z_1', z_2)(s)|, \\ \tau_2(s) &= \log|T(z_1', z_2)(s) - T(z_1', z_2')(s)|, \\ \nu_i &= v(z_i, z_i') \quad \textit{for } i=1,2,\end{aligned}$$

the following estimates, for $i=1,2$, hold:

$$\left|\tau_i(s)+\sum_{j=0}^{\nu_i-1}\lambda_i(\sigma^j z_i, s, T(z_1,z_2)(s))\right|\leqslant c,$$

$$\left|\frac{d}{ds}\left(\tau_i(s)+\sum_{j=0}^{\nu_i-1}\lambda_i(\sigma^j z_i, s, T(z_1,z_2)(s))\right)\right|\leqslant c,$$

$$\left|\frac{d^2}{ds^2}\tau_i(s)\right|\leqslant c\nu_i^2,$$

$$|T(z_1,z_2)(s)|\leqslant c\sup[d_s(z_1,z_1),d_s(z_2,a_2)]=cd_s,$$

$$\left|\frac{d}{ds}T(z_1,z_2)(s)\right|\leqslant cd_s|\log d_s|.$$

3.14. In the context of §2.1, let $r_1>0$, $\beta>1$ to be determined later.

Let $s_0\in(-\eta,\eta)$, $0<r<r_1$, $I=[s_0-|\log r|^{-1}, s_0+|\log r|^{-1}]$. Let $z_1, z_1'\in B^1_{s_0}(r)$, $z_2, z_2'\in B^2_{s_0}(r)$. Let $\Delta_i=\Delta_i(s_0,0)$,

$$\begin{aligned} d&=\sup(d_{s_0}(z_1,z_1'), d_{s_0}(z_2,z_2')),\\ T(s)&=T(z_1,z_2)(s)-T(z_1',z_2')(s)\\ &=[T(z_1,z_2)(s)-T(z_1',z_2)(s)]+[T(z_1',z_2)(s)-T(z_1',z_2')(s)]\\ &=T_1(s)-T_2(s),\quad s\in I,\\ &\qquad |T_i(s)|=\exp\tau_i(s).\end{aligned}$$

We assume r_1 small enough to have, for any $s\in(-\eta,\eta)$,

$$B^i_s(r_1)\subset B^i_0(r_2)$$

which means that we are in the domain of validity of Proposition 3.

Let $\nu_i=v(z_i,z_i')$ for $i=1,2$.

Let $\varphi_i=\varphi_i(s_0,0)$, $\Delta\varphi_i=\Delta\varphi_i(s_0,0)$, $\mu_i=\mu_{i,s_0,0}$ (cf. §3.9).

Property (i) of §2.1 follows from §3.5 and Proposition 3 (the metrics d_{s_0} and d_{φ_i} being equivalent).

We now check property (iv), using Proposition 1. We have seen in §3.7 that the hypothesis of Proposition 1 is satisfied ($S_1\Delta S_2\neq S_2\Delta S_1$).

According to Proposition 3 above, we have:

$$|T(z_1,z_2)(s)|=c\sup[d_s(z_1,a_1),d_s(z_2,a_2)]\leqslant cr^u,\quad 0<u<1.$$

Therefore:

$$\left|\sum_{j=0}^{\nu_i-1}\lambda_i(\sigma^j z_i,s,T(z_1,z_2)(s))-\sum_{j=0}^{\nu_i-1}\lambda_i(\sigma^j z_i,s_0,0)\right|<c\nu_i|\log r|^{-1},\quad s\in I.$$

Also

$$\left|\sum_{j=0}^{\nu_i-1}\lambda_i(\sigma^j z_i,s_0,0)-S_{\nu_i}\varphi_i(z_i)\right|<c,$$

hence $|\tau_i(s)+S_{\nu_i}\varphi_i(z_i)|<c\nu_i|\log r|^{-1}$ (clearly $\nu_i\geqslant c^{-1}|\log r|$).

We have:

$$\begin{aligned}\frac{d}{ds}\sum_{j=0}^{\nu_i-1}\lambda_i(\sigma^j z_i,s,T(z_1,z_2)(s))&=\sum_{j=0}^{\nu_i-1}\frac{\partial}{\partial s}\lambda_i(\sigma^j z_i,s,T(z_1,z_2)(s))\\&\quad+\frac{d}{ds}T(z_1,z_2)(s)\sum_{j=0}^{\nu_i-1}\frac{\partial}{\partial t}\lambda_i(\sigma^j z_i,s,T(z_1,z_2)(s),\end{aligned}$$

with, as above:

$$\left|\sum_{j=0}^{\nu_i-1}\frac{\partial}{\partial s}\lambda_i(\sigma^j z_i,s,T(z_1,z_2)(s))-S_{\nu_i}\Delta\varphi_i(z_i)\right|<c\nu_i|\log r|^{-1}$$

and $|dT(z_1,z_2)(s)/ds|\leqslant r^u$. Therefore

$$\left|\frac{d}{ds}\tau_i(s)+S_{\nu_i}\Delta\varphi_i(z_i)\right|<c\nu_i|\log r|^{-1}.$$

Let $1<\beta<\beta_0$, with β_0 as in Proposition 1. We apply Proposition 1 with balls B_1,B_2 of radius cr containing $B^1_{s_0}(r),B^2_{s_0}(r)$ (the respective distances are equivalent). We assume for instance that

$$d=d_{s_0}(z_1,z_1')>r^\beta,$$

which implies, with the notations of Proposition 1:

$$m_1\leqslant\nu_1\leqslant\beta_0 m_1,$$

if r_1 is small enough.

If the set J of §2.1, property (iv), is empty, there is nothing to prove. Assume that J contains a point $s_1\in I$. We have

$$|\log d+S_{\nu_1}\varphi_1(z_1)|\leqslant c,$$

hence
$$|\tau_1(s)-\log d|\leqslant c,\quad s\in I,$$
and also
$$||T_1(s_1)|-|T_2(s_1)||<c^{-1}d.$$
Therefore
$$|\tau_1(s_1)-\tau_2(s_1)|\leqslant c.$$
But
$$|\tau_2(s_1)+S_{\nu_2}\varphi_2(z_2)|\leqslant c\nu_2|\log r|^{-1}$$
and
$$|\tau_2(s_1)-\log d|\leqslant c$$
imply $\nu_2\leqslant c|\log r|$. We then get:
$$|S_{\nu_1}\varphi_1(z_1)-S_{\nu_2}\varphi_2(z_2)|\leqslant c.$$
Then, by Proposition 1 and the estimates above for $d\tau_i(s)/ds$, we get
$$\left|\frac{d}{ds}(\tau_1(s)-\tau_2(s))\right|>c^{-1}|\log r|,\quad s\in I.$$
Also
$$\left|\frac{d^2}{ds^2}(\tau_1(s)-\tau_2(s))\right|<c|\log r|^2.$$
This shows that J is an interval and gives the estimates of property (iv) in §2.1.

3.15. For $s\in I$, there is a constant c such that
$$B_{s_0}(c^{-1}r)\subset B_s(r)\subset B_{s_0}(cr).$$

Indeed, the distance d_s is equivalent to $d_{i,s,0}$, the distance d_{s_0} to the distance $d_{i,s_0,0}$, and the property is clearly true for the balls relative to these distances.

The hypothesis $J_1\Delta J_2\neq J_2\Delta J_1$ of Proposition 2 is satisfied, as we have seen in §3.10. With M to be determined later, we apply Proposition 2 in balls $B_i(c^{-1}r)$, in order to satisfy property (ii) of §2.1. Property (iii) of §2.1 (with L as in Proposition 2) follows from the conclusion (i) of Proposition 2 and §3.5.

We now check property (v). We therefore assume that $(z_1,z_2)\in L$, $(z_1',z_2')\in L$ and $d\leqslant r^\beta$ (with $\beta>1$ as above). Again there is nothing to prove when J is empty, hence we assume that there exists $s_1\in I$ with
$$||T_1(s_1)|-|T_2(s_1)||<c^{-1}d^{1+c|\log r|^{-1}}.$$

Assume for instance that $d=d_{s_0}(z_1, z_1')$. We have:

$$|\log d+S_{\nu_1}\varphi_1(z_1)| \leqslant c,$$
$$|\tau_1(s)+S_{\nu_1}\varphi_1(z_1)| \leqslant c\nu_1|\log r|^{-1},$$

and therefore

$$|\tau_2(s_1)-\tau_1(s_1)| < c|\log r|^{-1}|\log d|,$$
$$|S_{\nu_1}\varphi_1(z_1)-S_{\nu_2}\varphi_2(z_2)| < c(\nu_1+\nu_2+|\log d|)|\log r|^{-1}.$$

But, $|\log d+S_{\nu_1}\varphi_1(z_1)| \leqslant c$ implies

$$c^{-1}|\log d| < \nu_1 < c|\log d|,$$

and for the last inequality to hold we must then also have $c^{-1}|\log d|<\nu_2<c|\log d|$. Therefore

$$|S_{\nu_1}\varphi_1(z_1)-S_{\nu_2}\varphi_2(z_2)| < c|\log d|\,|\log r|^{-1}.$$

For M big enough (and r_1 small enough), we have

$$\sup(M, c_1(\nu_1+\nu_2-c_2|\log r|)) > c|\log d|\,|\log r|^{-1},$$

hence, by Proposition 2:

$$|S_{\nu_1}\Delta\varphi_1(z_1)-S_{\nu_2}\Delta\varphi_2(z_2)| > \sup(M, c_1\{(\nu_1+\nu_2)-c_2|\log r|\}).$$

Then we will have, for $s\in I$, r_1 small enough, M big enough:

$$\left|\frac{d}{ds}(\tau_1(s)-\tau_2(s))\right| > \tfrac{1}{2}\sup(M, c_1(\nu_1+\nu_2-c_2|\log r|)),$$
$$\left|\frac{d^2}{ds^2}(\tau_1(s)-\tau_2(s))\right| < c|\log d|^2,$$

from which we deduce easily that J is an interval and that the estimates of property (v) of §2.1 hold in J.

We have thus reduced the proof of the theorem to the proof of Proposition 3.

4. The geometrical estimates

The aim of this section is to prove Proposition 3 and thus finish the proof of our main result.

In §4.1, using the smooth dependence on parameters of the stable and unstable foliations of the basic set Λ_1, we introduce appropriate local charts around each point of Λ_1.

These charts are then used in §§4.2–4.4 to obtain estimates of the distance between two nearby stable manifolds of Λ_1 along these manifolds and its variation with parameters. The main part of the calculation actually takes place in some group of jets; we therefore adopt a slightly more abstract setting to make this apparent.

In §4.5, we do some preparatory work in order to finally obtain in §4.6 the estimates in Proposition 3. The calculation is quite long but straightforward; it consists essentially in translating the estimates on the distances between stable manifolds (§4.4), via the implicit function theorem, to estimates on the parameter intervals corresponding to various tangencies as in Proposition 3.

4.1. We start from constants $\varepsilon>0$, $c_0>1$, $c_1>0$ and a continuous map L:

$$\Sigma_1 \to C^\infty([-c_0,c_0]^2\times(-\eta,\eta)^2, M)$$

with the following properties.

(i) Let $z_1\in\Sigma_1$, $s,t\in(-\eta,\eta)$; the map $L_{s,t}(z_1)$:

$$(x,y)\to L(z_1)(x,y,s,t)$$

is an embedding of $[-c_0,c_0]^2$ into M whose image $U_{s,t}(z_1)$ contains an ε-neighbourhood of $h_{1,s,t}(z_1)$; we have

$$\begin{aligned}
&L_{s,t}(z_1)(0,0)=h_{1,s,t}(z_1),\\
&L_{s,t}(z_1)(x,0)\subset W^s(z_1,s,t), \quad x\in[-c_0,c_0],\\
&L_{s,t}(z_1)(0,y)\subset W^u(z_1,s,t), \quad y\in[-c_0,c_0].
\end{aligned}$$

(ii) Let $z_1\in\Sigma_1$, $s,t\in(-\eta,\eta)$; the map $F_{s,t}(z_1)$:

$$F_{s,t}(z_1)=(L_{s,t}(\sigma z_1))^{-1}\circ f_{s,t}\circ L_{s,t}(z_1)$$

is defined on $[-1,1]^2$; it may be written in this square under the form

$$F_{s,t}(z_1)(x,y)=(\pm x\exp H_{s,t}(z_1)(x,y), \pm y\exp K_{s,t}(z_1)(x,y)),$$

where the $+$ or $-$ signs depend (continuously) only on z_1. The maps $F_{s,t}(z_1)$ together define a continuous map F:

$$\Sigma_1\to C^\infty([-1,1]^2\times(-\eta,\eta)^2, [-c_0,c_0]^2),$$

and similarly for $H_{s,t}(z_1)$, $K_{s,t}(z_1)$.

We have $K_{s,t}(z_1)(0,0)=\lambda_1(z_1,s,t)$ (cf. §3.6) and define $\varrho_1(z_1,s,t)=-H_{s,t}(z_1)(0,0)$; for $z_1\in\Sigma_1$, $s,t\in(-\eta,\eta)^2$, we have:

$$\lambda_1(z_1,s,t)\geqslant c_1>0,$$
$$\varrho_1(z_1,s,t)\geqslant c_1>0.$$

(iii) Let $x_1\in\Sigma_1(j)$, $z_1'\in\Sigma_1(k)$, with $j\neq k$ (i.e. they belong to distinct elements of the Markov partition). Then

$$U_{s,t}(z_1)\cap U_{s,t}(z_1')=\varnothing,\quad \text{for all } (s,t)\in(-\eta,\eta)^2.$$

4.2. We consider now the following slightly more general situation.

Let P be an open set in a parameter space $\mathbf{R}^d$. For $i\geqslant 0$, let $F_i\colon[-1,1]^2\times P\to\mathbf{R}^2$ be a map which may be written in the form:

$$F_i(x,y,p)=(x\exp H_i(x,y,p),y\exp K_i(x,y,p))$$

with smooth maps $H_i,K_i\colon[-1,1]^2\times P\to\mathbf{R}$.

Define

$$\varrho_i(p)=-H_i(0,0,p),$$
$$\lambda_i(p)=K_i(0,0,p),$$

and assume that, for some constant $c_1>0$, all $p\in P$, $i\geqslant 0$:

$$\varrho_i(p)\geqslant c_1,$$
$$\lambda_i(p)\geqslant c_1.$$

For $n\geqslant 0$, define

$$\varrho^{(n)}(p)=\sum_{i=0}^{n-1}\varrho_i(p),$$
$$\lambda^{(n)}(p)=\sum_{i=0}^{n-1}\lambda_i(p).$$

Assume that we have, for some constant $c_2>0$:

$$|\partial_x H_i(x,y,p)|\leqslant c_2,$$
$$|\partial_y H_i(x,y,p)|\leqslant c_2,$$
$$|\partial_x K_i(x,y,p)|\leqslant c_2,$$
$$|\partial_y K_i(x,y,p)|\leqslant c_2.$$

Let $\theta \in (0,1]$. Consider a smooth map

$$p \to (x_0(p), y_0(p))$$

with values in $(0,\theta]^2$. As long as $x_i(p) \in (0,\theta]$, $y_i(p) \in (0,\theta]$, we define:

$$(x_{i+1}(p), y_{i+1}(p)) = F_i(x_i(p), y_i(p), p).$$

Define

$$X_i(p) = \log x_i(p), \quad Y_i(p) = \log y_i(p);$$

then we have

$$\begin{aligned} |X_{i+1}(p) - X_i(p) + \varrho_i(p)| &\leqslant c_2(x_i(p) + y_i(p)), \\ |Y_{i+1}(p) - Y_i(p) - \lambda_i(p)| &\leqslant c_2(x_i(p) + y_i(p)). \end{aligned}$$

From now on, we adopt the following convention: We denote by $c_1, c_2, c_3, \ldots$ constants which depend only on:

— bounds on F_i, H_i, K_i, uniform in i;

— bounds on the map g introduced in §4.4;

— bounds on the map $p \mapsto X_0(p)$.

We also use the letter c for such (unspecified) constants.

We assume that

$$\theta < \tfrac{1}{4} c_1 c_2^{-1} e^{-c_1/2} (1 - e^{-c_1})^{-1}.$$

LEMMA 1. *Assume that for some $n \geqslant 0$, we have*

$$\begin{aligned} &0 < x_0(p) < \theta, \\ &0 < y_0(p) < \theta \exp(-\lambda^{(n)}(p)). \end{aligned}$$

Then $(x_i(p), y_i(p))$ is defined for $0 \leqslant i \leqslant n$ and we have:

$$\begin{aligned} |X_i(p) - X_0(p) + \varrho^{(i)}(p)| &< \tfrac{1}{2} c_1, \\ |Y_i(p) - Y_0(p) - \lambda^{(i)}(p)| &< \tfrac{1}{2} c_1. \end{aligned}$$

Proof. This is clear if $i=0$; assume that it is true for $0 \leqslant i \leqslant j < n$. Then, for $0 \leqslant i \leqslant j$, we have:

$$\begin{aligned} &0 < x_i(p) < \theta e^{c_1/2} e^{-ic_1} \quad (< \theta \text{ if } i > 0), \\ &0 < y_i(p) < \theta e^{c_1/2} e^{(i-n)c_1} \quad (< \theta), \end{aligned}$$

and therefore

$$c_2 \sum_{i=0}^{j} (x_i(p)+y_i(p)) < \frac{2\theta c_2 e^{c_1/2}}{1-e^{-c_1}} < \tfrac{1}{2}c_1$$

which proves the statement of the lemma for $j+1$. □

We recall, for further purposes the estimates

$$0 < x_i(p) < \theta e^{c_1/2} e^{-ic_1},$$
$$0 < y_i(p) < \theta e^{c_1/2} e^{(i-n)c_1},$$

under the hypotheses of the lemma.

Let us now study partial derivatives with respect to the parameters. We write $p=(p_1, ..., p_d)$ and ∂_j for $\partial/\partial p_j$.

Let $1 \leqslant j \leqslant d$. For $0 \leqslant i \leqslant n$, define

$$J_i^{(j)} = \begin{pmatrix} \partial_j X_i \\ \partial_j Y_i \end{pmatrix}, \quad V_i^{(j)} = \begin{pmatrix} \partial_j H_i(x_i, y_i, p) \\ \partial_j K_i(x_i, y_i, p) \end{pmatrix},$$
$$M_i = \begin{pmatrix} 1 + x_i \partial_x H_i(x_i, y_i, p) & y_i \partial_y H_i(x_i, y_i, p) \\ x_i \partial_x K_i(x_i, y_i, p) & 1 + y_i \partial_y K_i(x_i, y_i, p) \end{pmatrix}.$$

Then, for $0 \leqslant i < n$, we have

$$J_{i+1}^{(j)} = M_i J_i^{(j)} + V_i^{(j)}.$$

For $0 \leqslant i \leqslant n$, define

$$M^{(i)} = M_{i-1} \ldots M_0 = \begin{pmatrix} A^{(i)} & B^{(i)} \\ C^{(i)} & D^{(i)} \end{pmatrix},$$
$$M_{(i)} = M_{n-1} \ldots M_i = \begin{pmatrix} A_{(i)} & B_{(i)} \\ C_{(i)} & D_{(i)} \end{pmatrix},$$
$$M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix},$$

with estimates (under the hypothesis of Lemma 1):

$$|a_i - 1| < c_2 e^{c_1/2} \theta e^{-ic_1},$$
$$|c_i| < c_2 e^{c_1/2} \theta e^{-ic_i},$$
$$|b_i| < c_2 e^{c_1/2} \theta e^{(i-n)c_i},$$
$$|d_i - 1| < c_2 e^{c_1/2} \theta e^{(i-n)c_i}.$$

Using the recursion formulas for the $A^{(i)}, A_{(i)}, \ldots$ it is easy to prove that there exist constants $c_3, c_4 > 0$ depending only on c_1, c_2 such that if we assume that

$$\theta < c_3$$

then we will have, for $0 \leqslant i \leqslant n$:

$$\begin{aligned}
|A^{(i)}-1| &< c_4\theta, \\
|B^{(i)}| &< c_4\theta e^{(i-n)c_1}, \\
|C^{(i)}| &< c_4\theta, \\
|D^{(i)}-1| &< c_4\theta e^{(i-n)c_1}, \\
|A_{(i)}-1| &< c_4\theta e^{-ic_1}, \\
|B_{(i)}| &< c_4\theta, \\
|C_{(i)}-1| &< c_4\theta e^{-ic_1}, \\
|D_{(i)}-1| &< c_4\theta.
\end{aligned}$$

We also have

$$M_{(0)} = M_{(i)} M^{(i)},$$

and therefore

$$\begin{aligned}
B_{(0)} &= A_{(i)} B^{(i)} + B_{(i)} D^{(i)}, \\
D_{(0)} &= C_{(i)} B^{(i)} + D_{(i)} D^{(i)}, \\
|B_{(0)} - B_{(i)}| &< c\theta e^{(i-n)c_1}, \\
|D_{(0)} - D_{(i)}| &< c\theta e^{(i-n)c_1}.
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
J_n^{(j)} &= M_{(0)} J_0^{(j)} + \sum_{i=0}^{n-1} M_{(i+1)} V_i^{(j)}, \\
\partial_j X_n &= A_{(0)} \partial_j X_0 + B_{(0)} \partial_j Y_0 + \sum_{i=0}^{n-1} (A_{(i+1)} \partial_j H_i + B_{(i+1)} \partial_j K_i), \\
\partial_j Y_n &= C_{(0)} \partial_j X_0 + D_{(0)} \partial_j Y_0 + \sum_{i=0}^{n-1} (C_{(i+1)} \partial_j H_i + D_{(i+1)} \partial_j K_i).
\end{aligned}$$

Suppose that on our domain we have

$$\begin{aligned}
|\partial_j \partial_x H_i| &\leqslant c, \\
|\partial_j \partial_y H_i| &\leqslant c, \\
|\partial_j \partial_x K_i| &\leqslant c, \\
|\partial_j \partial_y K_i| &\leqslant c.
\end{aligned}$$

The following lemma is then immediate.

LEMMA 2. *There are constants* c_3, c_4 *depending only on* c_1, c_2 *such that, if*

$$\begin{aligned}
&0 < x_0(p) < \theta,\\
&0 < y_0(p) < \theta \exp(-\lambda^{(n)}(p)),\\
&\theta < c_3 < \tfrac{1}{4} c_1 c_2^{-1} e^{-c_1/2} (1-e^{-c_2})^{-1},
\end{aligned}$$

then, one has:

$$\left|\left(\partial_j X_n + \sum_{i=0}^{n-1} \partial_j \varrho_i\right) - A_{(0)} \partial_j X_0 - B_{(0)} \left(\partial_j Y_0 + \sum_{i=0}^{n-1} \partial_j \lambda_i\right)\right| \leqslant c\theta,$$
$$\left|\partial_j Y_n - C_{(0)} \partial_j X_0 - D_{(0)} \left(\partial_j Y_0 + \sum_{i=0}^{n-1} \partial_j \lambda_i\right)\right| \leqslant c\theta,$$

where $|A_{(0)}-1| < c_4\theta$, $|B_{(0)}| < c_4\theta$, $|C_{(0)}| < c_4\theta$, $|D_{(0)}-1| < c_4\theta$.

4.3. We will also need estimate for second partial derivatives. Let $1 \leqslant j, k \leqslant d$. Define

$$J_i^{(j,k)} = \begin{pmatrix} \partial_j \partial_k X_i \\ \partial_j \partial_k Y_i \end{pmatrix}.$$

Then we have

$$J_{i+1}^{(j,k)} = M_i J_i^{(j,k)} + S_i (J_i^{(j)} \overset{s}{\otimes} J_j^{(k)}) + S_i^{(j)} J_i^{(k)} + S_i^{(k)} J_i^{(j)} + V_i^{(j,k)},$$

with

$$S_i^{(j)} = \begin{pmatrix} x_i \partial_x \partial_j H_i & y_i \partial_y \partial_j H_i \\ x_i \partial_x \partial_j K_i & y_i \partial_y \partial_j K_i \end{pmatrix}, \quad V_i^{(j,k)} = \begin{pmatrix} \partial_j \partial_k H_i \\ \partial_j \partial_k K_i \end{pmatrix},$$
$$S_i = \begin{pmatrix} x_i \partial_x H_i + x_i^2 \partial_x^2 H_i & x_i y_i \partial_x \partial_y H_i & y_i \partial_y H_i + y_i^2 \partial_y^2 H_i \\ x_i \partial_x K_i + x_i^2 \partial_x^2 K_i & x_i y_i \partial_x \partial_y K_i & y_i \partial_y K_i + y_i^2 \partial_y^2 K_i \end{pmatrix},$$

and we write the symmetric tensor product $J_i^{(j)} \overset{s}{\otimes} J_i^{(k)}$ with coordinates

$$\begin{pmatrix} \partial_j X_i \partial_k X_i \\ \partial_j X_i \partial_k Y_i + \partial_k X_i \partial_j Y_i \\ \partial_j Y_i \partial_k Y_i \end{pmatrix}.$$

We have

$$J_n^{(j,k)} = M_{(0)} J_0^{(j,k)} + \sum_{i=0}^{n-1} M_{(i+1)} R_i^{(j,k)},$$

with $R_i^{(j,k)} = S_i (J_i^{(j)} \overset{s}{\otimes} J_i^{(k)}) + S_i^{(j)} J_i^{(k)} + S_i^{(k)} J_i^{(j)} + V_i^{(j,k)}$.

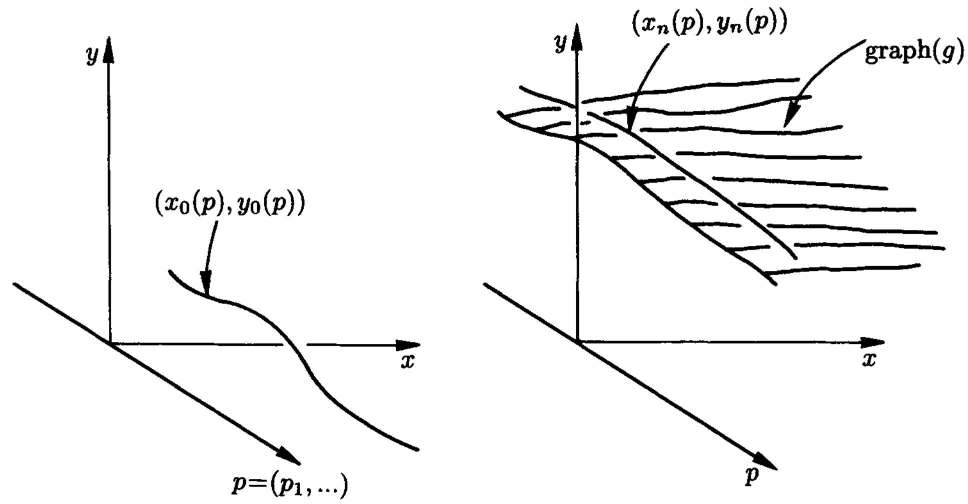

Fig. 3

Assume that there are constants c, C_0 such that all second partial derivatives of H_i, K_i are bounded in our domain by c, and that:

$$\|J_i^{(j)}\| < C_0 n,$$
$$\|J_i^{(k)}\| < C_0 n.$$

Then, for some constant $C_1 = C_1(C_0)$ we have:

$$\sup(\|S_i\|, \|S_i^{(j)}\|, \|S_i^{(k)}\|) \leqslant c\exp(-\inf(i, n-i)c_1),$$
$$\|J_n^{(j,k)} - M_{(0)} J_0^{(j,k)}\| < C_1 n^2.$$

4.4. We now fix θ such that

$$\theta < c_3, \quad \theta < \tfrac{1}{10} c_4^{-1} \quad \text{(cf. Lemma 2).}$$

We add the following new feature: there is a smooth map $g: [0,\theta] \times P \to [0,\theta]$, which satisfies

$$g(x,p) > c > 0, \quad x \in [0,\theta], \ p \in P,$$

and such that:

$$y_n(p) = g(x_n(p), p), \quad p \in P.$$

See Figure 3. We also assume that $p \to (x_0(p), y_0(p))$ satisfies the hypothesis of Lemma 2.

Write

$$\widetilde{\partial_j X_n} = \partial_j X_n + \sum_{i=0}^{n-1} \partial_j \varrho_i,$$
$$\widetilde{\partial_j Y_0} = \partial_j Y_0 + \sum_{i=0}^{n-1} \partial_j \lambda_i.$$

For $1 \leqslant j \leqslant d$, we have:

$$y_n \partial_j Y_n = \partial_j g + x_n \partial_x g \partial_j X_n,$$

with $|x_n \partial_x g| < ce^{-c_1 n}$ and $|\widetilde{\partial_j X_n} - \partial_j X_n| < cn$.

Joining this to the conclusion of Lemma 2, we obtain (as $y_n > c > 0$)

$$|\widetilde{\partial}_j X_n - A_{(0)} \partial_j X_0 - B_{(0)} \widetilde{\partial_j Y_0}| < c,$$
$$|C_{(0)} \partial_j X_0 + D_{(0)} \widetilde{\partial_j Y_0}| < c + ce^{-c_1 n} |\widetilde{\partial_j X_n}|,$$

from which we deduce, as $|\partial_j X_0| \leqslant c$, for $n \geqslant c_5$:

$$|\widetilde{\partial_j X_n}| \leqslant c,$$
$$|\widetilde{\partial_j Y_0}| \leqslant c.$$

For second derivatives, we have, for $1 \leqslant j \leqslant k \leqslant d$:

$$\begin{aligned} y_n(\partial_j Y_n \partial_k Y_n + \partial_j \partial_k Y_n) = {} & \partial_j \partial_k g + x_n \partial_j \partial_x g \partial_k X_n \\ & + x_n \partial_k \partial_x g \partial_j X_n + (x_n \partial_x g + x_n^2 \partial_x^2 g) \partial_j X_n \partial_k X_n \\ & + x_n \partial_x g \partial_j \partial_k X_n. \end{aligned}$$

Then, we see from §4.3 and above that we have for $n \geqslant c_5$:

$$\|J_i^{(j)}\| < cn,$$
$$\|J_n^{(j,k)} - M_{(0)} J_0^{(j,k)}\| < cn^2.$$

We have

$$|\partial_j Y_n| < c, \quad |\partial_k Y_n| < c,$$

and therefore

$$|\partial_j \partial_k Y_n| < c + ce^{-c_1 n} |\partial_j \partial_k X_n|.$$

In the same way as for first derivatives, that allows to conclude that, for $n \geqslant c_6 \geqslant c_5$,

$$|\partial_j \partial_k Y_0| < cn^2.$$

4.5. We now proceed to prove Proposition 3. We use the notations of §4.1. Let $P=(-\eta,\eta)^2$, η small enough.

The maps

$$\Sigma_1 \xrightarrow{H} C^\infty([-1,1]^2\times P,\mathbf{R}),$$
$$\Sigma_1 \xrightarrow{K} C^\infty([-1,1]^2\times P,\mathbf{R})$$

are continuous, hence there is a constant $c_2>0$ such that, for all $z_1\in\Sigma_1$, $s,t\in(-\eta,\eta)^2$, $x,y\in[-1,1]$:

$$|\partial_x H_{s,t}(z_1)(x,y)|\leqslant c_2,$$
$$|\partial_y H_{s,t}(z_1)(x,y)|\leqslant c_2,$$
$$|\partial_x K_{s,t}(z_1)(x,y)|\leqslant c_2,$$
$$|\partial_y K_{s,t}(z_1)(x,y)|\leqslant c_2.$$

We determine then, from c_1 (in §4.1) and c_2 above, constants c_3, c_4 as in Lemma 2, and choose θ with

$$\theta<c_3, \quad \theta<\tfrac{1}{10}c_4^{-1}.$$

There exists $n_0>0$ such that, for all $s,t\in(-\eta,\eta)$, $x_1,x_1'\in\Sigma_1^+$ with $v(z_1,z_1')\geqslant n_0$, the point $h_{1,s,t}(z_1)$ lies in $U_{s,t}(z_1')$ and the equation of $W^s_{\text{loc}}(z_1,s,t)\cap U_{s,t}(z_1')$ is

$$y=g_{z_1/z_1'}(x,s,t)$$

in the coordinate system given by $L_{s,t}(z_1')$.

For each z_1, z_1', the map $g_{z_1/z_1'}$ is smooth, and these maps together give a continuous map:

$$V_i \xrightarrow{G} C^\infty([-c_0,c_0]\times(-\eta,\eta)^2,[-c_0,c_0])$$

where $V_1=\{(z_1,z_1')\in\Sigma_1^+\times\Sigma_1^+ : v(z_1,z_1')\geqslant n_0\}$.

We choose $n_0>0$ big enough such that, for $(z_1,z_1')\in V_1$, the image of $g_{z_1/z_1'}$ is actually contained in $\left[-e^{-c_1/2}\frac{1}{2}\theta, e^{-c_1/2}\frac{1}{2}\theta\right]$.

We choose, once and for all, an integer m_0, multiple of the period of a_1, such that the point $q'=f^{m_0}(q)$ belongs to $U_{0,0}(a_1)$, with coordinates $(\theta_1,0)$, where $0<|\theta_1|\leqslant\frac{1}{2}\theta$.

We will consider tangencies near q' instead of q. The map T' related to q' and the map T related to q satisfy $T'(\sigma^{m_0}z_1,\sigma^{-m_0}z_2)=T(z_1,z_2)$. Once m_0 is fixed, it is clearly equivalent to prove Proposition 3 for T or T'. We will actually prove it for T'. But to keep notations simple, we assume that $m_0=0$, $T=T'$.

Let $0<\varepsilon_1<\frac{1}{2}|\theta_1|$ be a small number such that the component of q' in $W^u(a_2,0,0)\cap [\theta_1-\varepsilon_1,\theta_1+\varepsilon_1]\times\left[-\frac{1}{2}\theta,\frac{1}{2}\theta\right]$ in the coordinate system $L_{0,0}(a_1)$ has equation:

$$y=\varphi(x),\quad |x-\theta_1|\leqslant\varepsilon_1,$$

with $\varphi(\theta_1)=\varphi'(\theta_1)=0$, $|\varphi''(x)|>c>0$.

Let n_1 be an integer $\geqslant n_0$, and let $z_1,z_1'\in\Sigma_1^+$, $z_2\in\Sigma_2^+$ with $v(z_1,a_1)\geqslant n_1$, $v(z_1',a_1)\geqslant n_1$, $v(z_2,a_2)\geqslant n_1$. We claim that if ε_1 is small enough and n_1 is big enough the following properties hold, with constants independent of s, t, z_1, z_1', z_2 (provided η is also small enough).

(i) The connected component of $W^u(z_2,s,t)\cap[\theta_1-\varepsilon_1,\theta_1+\varepsilon_1]\times\left[-\frac{1}{2}\theta,\frac{1}{2}\theta\right]$ we are interested in has equation

$$y=\varphi_{z_2/z_1'}(x,s,t)$$

in the coordinate system $L_{s,t}(z_1')$, for a smooth function $\varphi_{z_2/z_1'}$; all partial derivatives of $\varphi_{z_2/z_1'}$ and $g_{z_1/z_1'}$ of order up to 3 are bounded by a constant c.

(ii) For $s,t\in(-\eta,\eta)$, $|x-\theta_1|<\varepsilon_1$, we have

$$\begin{aligned}|\partial_x^2\varphi_{z_2/z_1'}(x,s,t)|&>c>0,\\|\partial_t\varphi_{z_2/z_1'}(x,s,t)|&>c>0,\\|\partial_t(\varphi_{z_2/z_1'}-g_{z_1/z_1'}(x,s,t))|&>c>0,\\|\partial_x^2(\varphi_{z_2/z_1'}-g_{z_1/z_1'}(x,s,t))|&>c>0.\end{aligned}$$

(iii) For $s,t\in(-\eta,\eta)$, $x,x'\in[\theta_1-\varepsilon_1,\theta_1+\varepsilon_1]$, we have

$$\begin{aligned}|\partial_x g_{z_1/z_1'}(x,s,t)|&\leqslant c|g_{z_1/z_1'}(x',s,t)|,\\|\partial_x^2 g_{z_1/z_1'}(x,s,t)|&\leqslant c|g_{z_1/z_1'}(x',s,t)|.\end{aligned}$$

(iv) For $s,t\in(-\eta,\eta)$ the function $\partial_x\varphi_{z_1/z_1'}$ (resp. $\partial_x(\varphi_{z_2/z_1'}-g_{z_1/z_1'})$) has a (unique) zero $\hat c(s,t)$ (resp. $c(s,t)$) in $[\theta_1-\varepsilon_1,\theta_1+\varepsilon_1]$. We have

$$\begin{aligned}|\partial_s c|\leqslant c_7,&\quad|\partial_t c|\leqslant c_7,\\|\partial_s\hat c|\leqslant c_7,&\quad|\partial_t\hat c|\leqslant c_7.\end{aligned}$$

and second partial derivatives are bounded by c.

(v) For $s\in(-\eta,\eta)$, the points $T(z_1,z_2)(s)$ and $T(z_1',z_2)(s)$ belong to $(-\eta,\eta)$; we write $\bar t=t-T(z_1',z_2)(s)$. In the parameter coordinates $(s,\bar t)$, all estimates above are still valid.

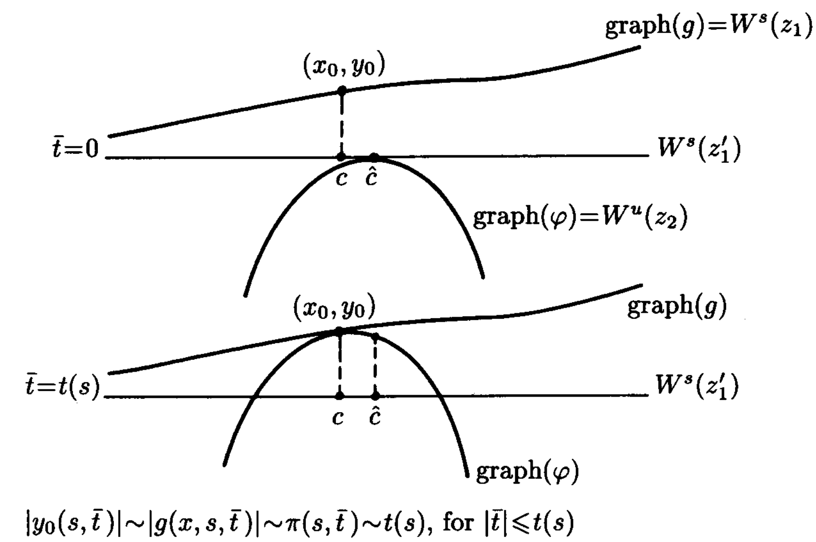

Fig. 4

All estimates and claims are straightforward, taking first ε_1 very small and then n_1 very big; estimate (iii) holds because the stable foliation of the basic set is, uniformly in s, t, of class $C^{1+\alpha}$ for some $\alpha>0$.

4.6. We fix points z_1, z_1', z_2 as above, but we will assume that

$$v(z_1, a_1) \geqslant n_2, \quad v(z_1', a_1) \geqslant n_2, \quad v(z_2, a_2) \geqslant n_2,$$

for an integer $n_2 \geqslant n_1$, still to be chosen.

Let $n = v(z_1, z_1') - n_0$; for $0 \leqslant i < n$, let

$$F_i\colon [-1,1]^2 \times (-\eta, \eta)^2 \to \mathbf{R}^2$$

be the map $(x, y, s, t) \mapsto F_{s,t}(\sigma^i z_1')(x, y)$.

By changing the signs of the coordinates in the coordinate systems $L_{s,t}(\sigma^i z_1')$, $0 \leqslant i \leqslant n$, we may assume that the F_i are exactly of the form considered in §4.2.

We will work with the parameter coordinates $(s, \bar{t})$ considered in §4.5 (v). To keep notations simple, we just write (s, t) again for these new coordinates. We therefore have $T(z_1', z_2)(s) = 0$, which means:

$$\varphi(\hat{c}(s, 0), s, 0) \equiv 0.$$

We are interested in the function $T(z_1, z_2)(s) = t(s)$ which is defined by:

$$\varphi(c(s, t(s)), s, t(s)) = g(c(s, t(s)), s, t(s)).$$

We have written φ for $\varphi_{z_2/z_1'}$ and g for $g_{z_1/z_1'}$. See Figure 4.

Write $\tilde{g}=g_{\sigma^n z_1/\sigma^n z_1'}$. We have, as $v(\sigma^n z_1, \sigma^n z_1')$ is a fixed integer n_0:

$$\tfrac{1}{2}e^{-c_1/2}\theta > |\tilde{g}(x,s,t)| > c > 0.$$

By Lemma 1 in §4.2, we have therefore, for $x\in[\theta_1-\varepsilon_1, \theta_1+\varepsilon_1]$, $s,t\in(-\eta,\eta)$:

$$0 < c < |g(x,s,t)| \exp \lambda^{(n)}(s,t) < \theta.$$

Let us just write $\pi(s,t)$ for $\exp -\lambda^{(n)}(s,t)$. We estimate $|c(s,t)-\hat{c}(s,0)|$. We have

$$|\hat{c}(s,0)-\hat{c}(s,t)| < c_7|t|,$$
$$\partial_x\varphi(\hat{c}(s,t),s,t)=0,$$
$$|\partial_x\varphi(c(s,t),s,t)| = |\partial_x g(c(s,t),s,t)| < c\pi(s,t),$$

hence (as $|\partial_u^2\varphi|>c>0$)

$$|\hat{c}(s,0)-c(s,t)| < c[\pi(s,t)+|t|].$$

But $\varphi(\hat{c}(s,0),s,0)=\partial_x\varphi(\hat{c}(s,0),s,0)=0$. Therefore

$$|\varphi(c(s,t),s,0)| < c[(\pi(s,t))^2+t^2].$$

On the other hand, writing $a(s)=\partial_t\varphi(c(s,t(s)),s,t(s))$, we have:

$$|a(s)| \geqslant c > 0,$$

and, for $|t|\leqslant|t(s)|$,

$$|\partial_t\varphi(c(s,t(s)),s,t)-a(s)| \leqslant c|t(s)|.$$

Summarizing, the function $\chi\colon u\to\varphi(c(s,t(s)),s,u)$ for $|u|\leqslant|t(s)|$ satisfies

$$|\chi(0)| < c[\pi(s,t(s))^2+t(s)^2],$$
$$\chi(t(s)) = g(c(s,t(s)),s,t(s)),$$
$$|a(s)| = |\chi'(t(s))| \geqslant c > 0,$$
$$|\chi'(u)-\chi'(t(s))| < c|t(s)|.$$

Let $y_0(s,t)=g(c(s,t),s,t)$. We have

$$\left|\log\frac{|y_0(s,t)|}{\pi(s,t)}\right| < c,$$

hence $|t(s)-y_0(s,t(s))/a(s)|<c(\pi(s,t(s)))^2$.

In particular, writing $\pi(s)=\pi(s,t(s))$:

$$\left|\log\frac{|t(s)|}{\pi(s)}\right|<c.$$

Let us now estimate $t'(s)$. From the defining relation

$$\psi(s,t(s))=0,$$

with $\psi(s,t)=\varphi(c(s,t),s,t)-g(c(s,t),s,t)$, we get

$$t'(s)=-[\partial_t\psi(s,t(s))]^{-1}\partial_s\psi(s,t(s)),$$

where

$$\partial_s\psi=\partial_s\varphi+\partial_s c\,\partial_x\varphi-\partial_s y_0,$$
$$\partial_t\psi=\partial_t\varphi+\partial_t c\,\partial_x\varphi-\partial_t y_0.$$

We have

$$\partial_t\varphi(c(s,t(s)),s,t(s))=a(s),$$
$$|\partial_x\varphi(c(s,t(s)),s,t(s))|=|\partial_x g(c(s,t(s)),s,t(s))|<c\pi(s)$$

(by §4.5 (iii)),

$$|\partial_s\varphi(c(s,t(s)),s,t(s))-\partial_s\varphi(\hat{c}(s,0),s,0)|<c\pi(s)$$

(because $|t(s)|<c\pi(s)$, $|c(s,t)-\hat{c}(s,0)|<c\pi(s)$),

$$\partial_s\varphi(\hat{c}(s,0),s,0)=0$$

(because $\partial_x\varphi(\hat{c}(s,0),s,0)\equiv\varphi(\hat{c}(s,0),s,0)\equiv 0$).

On the other hand, with $x_0(s,t)=c(s,t)$, we have, in the notation of §4.2:

$$y_n(s,t)=\tilde{g}(x_n(s,t),s,t)$$

with estimates:

$$|\partial_s\log|x_0|\,|<\frac{c_7}{\theta_1-\varepsilon_1},\quad |\partial_t\log|x_0|\,|<\frac{c_7}{\theta_1-\varepsilon_1}.$$

Taking n_2 big enough, the calculations in §4.4 are valid and we get:

$$\left|\frac{\partial_s y_0}{y_0}+\sum_{i=0}^{n-1}\partial_s\lambda_i\right|\leqslant c,$$
$$|\partial_t y_0|<c\pi(s,t)n<c\pi(s,t)|\log\pi(s,t)|.$$

Therefore

$$
\begin{aligned}
|\partial_s\psi(s,t(s))-y_0(s,t(s))\partial_s\lambda^{(n)}(s,t(s))| &< c\pi(s),\\
|\partial_t\psi(s,t(s))-a(s)| &< c\pi(s)|\log\pi(s)|,
\end{aligned}
$$

with $|y_0(s,t(s))-a(s)t(s)|<c(\pi(s))^2$.

We conclude that:

$$
\left|\frac{t'(s)}{t(s)}+\sum_{i=0}^{n-1}\partial_s\lambda_i(s,t(s))\right|\leqslant c.
$$

Let us now estimate from above the second derivative $t''(s)$. We have:

$$
\partial_t\psi t''+\partial_t^2\psi(t')^2+2\partial_t\partial_s\psi t'+\partial_s^2\psi=0,
$$

hence

$$
|t''(s)|<c(|\partial_s^2\psi|+\pi(s)|\log\pi(s)|\,|\partial_s\partial_t\psi|+\pi^2(s)|\log\pi(s)|\,|\partial_t^2\psi|),
$$

with

$$
\begin{aligned}
\partial_s^2\psi &= \partial_s^2c\partial_x\varphi+(\partial_sc)^2\partial_x^2\varphi+2\partial_sc\partial_s\partial_x\varphi+\partial_s^2\varphi-\partial_s^2y_0,\\
\partial_s\partial_t\psi &= \partial_s\partial_tc\partial_x\varphi+\partial_sc\partial_tc\partial_x^2\varphi+\partial_sc\partial_t\partial_x\varphi+\partial_tc\partial_s\partial_x\varphi+\partial_s\partial_t\varphi-\partial_s\partial_ty_0,\\
\partial_t^2\psi &= \partial_t^2c\partial_x\varphi+(\partial_tc)^2\partial_x^2\varphi+2\partial_tc\partial_t\partial_x\varphi+\partial_t^2\varphi-\partial_t^2y_0.
\end{aligned}
$$

According to §4.4, we have, for n_2 big enough:

$$
\begin{aligned}
\partial_s^2\log|y_0| &< cn^2,\\
\partial_s\partial_t\log|y_0| &< cn^2,\\
\partial_t^2\log|y_0| &< cn^2,\\
\partial_s\log|y_0| &< cn,\\
\partial_t\log|y_0| &< cn,
\end{aligned}
$$

therefore

$$
\begin{aligned}
|\partial_s^2y_0| &< cn^2\pi(s),\\
|\partial_s\partial_ty_0| &< cn^2\pi(s),\\
|\partial_t^2y_0| &< cn^2\pi(s).
\end{aligned}
$$

We already obtain:

$$
\begin{aligned}
|\partial_t^2\psi(s,t(s))| &< c,\\
|\partial_s\partial_t\psi(s,t(s))| &< c.
\end{aligned}
$$

In the formula for $\partial_s^2\psi(s,t(s))$ we have:

$$|\partial_x g(c(s,t(s)),s,t(s))| = |\partial_x\varphi(c(s,t(s)),s,t(s))| < c\pi(s).$$

We compare $\partial_s c(s,t(s))$ with $\partial_s\hat{c}(s,0)$. We have

$$\partial_x\partial_s\varphi(\hat{c}(s,t),s,t)+\partial_x^2\varphi(\hat{c}(s,t),s,t)\partial_s\hat{c}(s,t)\equiv 0,$$
$$\partial_x\partial_s(\varphi-g)(c(s,t),s,t)+\partial_x^2(\varphi-g)(c(s,t),s,t)\partial_s c(s,t)\equiv 0,$$

with

$$|\partial_x\partial_s\varphi(\hat{c}(s,0),s,0)-\partial_x\partial_s\varphi(c(s,t(s)),s,t(s))| < c\pi(s),$$
$$|\partial_x^2\varphi(\hat{c}(s,0),s,0)-\partial_x^2(\varphi-g)(c(s,t(s)),s,t(s))| < c\pi(s)$$

(recall §4.5 (iii)), hence

$$|\partial_s c(s,t(s))-\partial_s\hat{c}(s,0)| < c(\pi(s)+|\partial_x\partial_s g(c(s,t(s)),s,t(s))|.$$

Let

$$\begin{cases} \tilde{x}_0(u,s,t)=u, & u\in[\theta_1-\varepsilon_1,\theta_1+\varepsilon_1],\\ \tilde{y}_0(u,s,t)=g(u,s,t), & u\in[\theta_1-\varepsilon_1,\theta_1+\varepsilon_1]. \end{cases}$$

The discussion in §§4.2–4.4 applies to $(\tilde{x}_0,\tilde{y}_0)$ (depending now on three parameters u,s,t) and we get

$$|\partial_x\partial_s g(c(s,t(s)),s,t(s))| = |\partial_u\partial_s\tilde{y}_0| < cn^2\pi(s).$$

Therefore

$$|\partial_s c(s,t(s))-\partial_s\hat{c}(s,0)| < cn^2\pi(s).$$

But we observe that

$$\partial_s^2\varphi(\hat{c}(s,0),s,0)+\partial_s\hat{c}(s,0)\partial_s\partial_x\varphi(\hat{c}(s,0),s,0)=0,$$
$$\partial_s\partial_x\varphi(\hat{c}(s,0),s,0)+\partial_s\hat{c}(s,0)\partial_x^2\varphi(\hat{c}(s,0),s,0)=0,$$

therefore

$$|\partial_s^2\varphi(c(s,t(s)),s,t(s))+\partial_s c(s,t(s))\partial_s\partial_x\varphi(c(s,t(s)),s,t(s))| \leqslant cn^2\pi(s),$$
$$|\partial_s\partial_x\varphi(c(s,t(s)),s,t(s))+\partial_s c(s,t(s))\partial_x^2\varphi(c(s,t(s)),s,t(s))| \leqslant cn^2\pi(s),$$

and we conclude that

$$|\partial_s^2\psi(s,t(s))| < cn^2\pi(s),$$
$$|t''(s)| < cn^2\pi(s),$$
$$\left|\frac{d^2}{ds^2}\log|t(s)|\right| < cn^2.$$

Recapitulating, we have proved, as $\log\pi(s) = -\sum_{i=0}^{n-1}\lambda_i(s,t(s))$:

$$\left|\log|t(s)| + \sum_{i=0}^{n-1}\lambda_i(s,t(s))\right| < c,$$
$$\left|\frac{d}{ds}\log|t(s)| + \sum_{i=0}^{n-1}\partial_s\lambda_i(s,t(s))\right| < c,$$
$$\left|\frac{d^2}{ds^2}\log|t(s)|\right| < cn^2.$$

Let us see that this indeed gives the estimations for τ_1 as in Proposition 3. We have

$$\left|\sum_{i=0}^{n-1}\lambda_i(s,t(s)) - \sum_{i=0}^{n-1}\lambda_i(s,0)\right| < cn\pi(s) < c,$$
$$\left|\sum_{i=0}^{n-1}\partial_s\lambda_i(s,t(s)) - \sum_{i=0}^{n-1}\partial_s\lambda_i(s,0)\right| < cn\pi(s) < c.$$

As our t-variable here is really $\bar{t} = t - T(z_1', z_2)(s)$, we have in fact

$$\partial_s\lambda_i(s,0) = \frac{d}{ds}\lambda_1(\sigma^i z_1', s, T(z_1', z_2)(s))$$

where the notations of §3.10 are used in the right hand term. We have $\tau_1(s) = \log|t(s)|$. The integer ν_1 in Proposition 3 is here $n+n_0+m_0$: more precisely, in the sum $\sum_{j=0}^{n-1}\lambda_1(\sigma^j z_1', s, T(z_1', z_2)(s))$, the sum $\sum_{j=0}^{n-1}\lambda_i(s,0)$ missed the first m_0 terms (when we replaced q by q') and the last n_0 terms. But m_0, n_0 are fixed integers. Therefore we have proved the required estimates for $\tau_1(s)$. The estimates for τ_2 are true for the same reasons. The last two estimates in Proposition 3 are immediate, writing:

$$T(z_1,z_2) = T(z_1,z_2) - T(z_1,a_2) + T(z_1,a_2) - T(a_1,a_2).$$

Proposition 3 is, therefore, proved. □

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Jacob Palis
Instituto de Matemática Pura e Aplicada
Estrada Dona Castorina, 110
Caixa Postal 34021
22460 Rio de Janeiro, RJ
Brazil
E-mail address: jpalis@impa.br

Jean-Christophe Yoccoz
Département de Mathématiques
Université de Paris-Sud
Bâtiment 425
91405 Orsay Cedex
France

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