Acta Math., 181 (1998), 109–158

Good points and constructive resolution of singularities

by

S. ENCINAS and O. VILLAMAYOR

Universidad de Valladolid
Valladolid, Spain

Universidad Autonoma de Madrid
Madrid, Spain

To the memory of Professor Manfred Herrmann

Introduction

In this paper we present a new algorithm of resolution of singularities over fields of characteristic zero, making use of invariants that come from Abhyankar's good point theory [Ab1]. We also prove new properties on constructive (or algorithmic) desingularization.

Let us explain what we mean by an algorithm of resolution. Consider a pair (X, W) where W is a regular variety over a base field k (of characteristic zero) not necessarily irreducible (i.e. W smooth over k), and $X \subseteq W$ is a closed non-empty subscheme. Call $\mathcal{C}$ the class of all such pairs (over different base fields k). Natural maps $(\varphi^{-1}(X), W_1) \xrightarrow{\varphi} (X, W)$ arise within this class, for instance if $\varphi: W_1 \to W$ is a smooth map over a fixed field k, or if $\varphi: W_1 \to W$ arises from an arbitrary change of base field.

Fix now a totally ordered set $(I, \leqslant)$ and suppose assigned, for each pair $\mathcal{P}=(X, W)$ of $\mathcal{C}$, a function $\psi_{\mathcal{P}}: X \to I$ which is upper-semi-continuous and takes only finitely many values, say $\{\alpha_1, ..., \alpha_r\} \subseteq I$. Let $\max \psi_{\mathcal{P}}$ be the biggest α_i and set

$$\underline{\operatorname{Max}}\, \psi_{\mathcal{P}} = \{\xi \in X \mid \psi_{\mathcal{P}}(\xi) = \max \psi_{\mathcal{P}}\}.$$

We first require that the assigned function $\psi_{\mathcal{P}}$ be such that $\underline{\operatorname{Max}}\, \psi_{\mathcal{P}}$ $(\subseteq X)$ is regular and closed in W.

Note first that I is independent of $\mathcal{P}=(X, W)$. Roughly speaking, at each point $\xi \in X$, the value $\psi_{\mathcal{P}}(\xi)$ is to quantify how bad ξ is as a singular point, so now the worst

Partially supported by DFG, HE1279/8-1.

singularities define the closed and regular stratum $\underline{\mathrm{Max}}\,\psi_{\mathcal{P}}$. The property of ψ is that singularities "improve" after blowing up $\underline{\mathrm{Max}}\,\psi$ and that desingularization will be achieved by repeating this procedure.

Let $W \xleftarrow{\varphi} W_1$ denote the monoidal transformation with center $\underline{\mathrm{Max}}\,\psi_{\mathcal{P}}$ and set $X_1\,(\subseteq W_1)$ as the strict transform of X. The subscheme X_1 is empty if and only if $\underline{\mathrm{Max}}\,\psi_{\mathcal{P}}=X$, in which case red$(X)$ (X with reduced structure) is regular; if not, $\mathcal{P}_1=(X_1,W_1)$ is also a pair in $\mathcal{C}$ and the exceptional locus of φ in W_1 is a regular hypersurface, say H_1. Now we want to assign a function to $\mathcal{P}_1$, more generally:

Fix $\mathcal{P}_0=(X_0,W_0)$ a pair in $\mathcal{C}$, and suppose that for some index $s\geqslant 0$ we have defined blowing-ups

$$W_0 \xleftarrow{\varphi_1} W_1 \xleftarrow{\varphi_2} \dots \xleftarrow{\varphi_s} W_s$$

in smooth closed centers $C_i\subseteq X_i$, pairs $\mathcal{P}_i=(X_i,W_i)$, X_i the strict transform of X_{i-1}, $i=1,\dots,s$, and that functions

$$\psi_{\mathcal{P}_i}\colon X_i \to (I,\leqslant),\quad i=0,\dots,s-1,$$

have been assigned and $\underline{\mathrm{Max}}\,\psi_{\mathcal{P}_i}=C_i$, $i=0,\dots,s-1$.

($\mathrm{A1}_s$) (Requirement.) We require that the exceptional locus of $W_s\to W_0$ be $E_s=\{H_1,\dots,H_s\}$, a union of regular hypersurfaces having only normal crossings.

($\mathrm{A2}_s$) (Assignment.) If $X_s\neq\varnothing$, an assignment of a function $\psi_{\mathcal{P}_s}\colon X_s\to(I,\leqslant)$, upper-semi-continuous and taking only finitely different values, such that $\underline{\mathrm{Max}}\,\psi_{\mathcal{P}_s}\,(\subseteq X_s)$ is regular and closed in W_s and has only normal crossings with E_s.

(B) For each pair (X_0,W_0) there is an index $s\geqslant 0$ so that $\underline{\mathrm{Max}}\,\psi_{\mathcal{P}_s}=X_s$.

(C) With the setting of (B), if X_0 is reduced, then X_s is regular (and has normal crossings with E_s by ($\mathrm{A2}_s$)).

The last condition (C) is that of a so-called "embedded" desingularization of $X\subseteq W$. Note that ($\mathrm{A1}_s$) is vacuous if $s=0$, and for $s>0$ guaranteed by ($\mathrm{A2}_{s-1}$).

This is an algorithm of resolution (with values at $(I,\leqslant)$), namely an assignment with the conditions (A), (B) and (C). An algorithm was introduced in [V1] to give a constructive proof of desingularization, as opposed to the existential proof in [Hi1]. Constructive resolution allows us to avoid the web of inductive arguments in Hironaka's monumental work and also presents desingularization as an active tool rather than an existential result. The search of applications leads to the study of natural properties as:

(P1) If $(\varphi^{-1}(X),W_1)\to(X,W)$ is defined by $\varphi\colon W_1\to W$, either smooth or an arbitrary change of base field, then the desingularization of $(\varphi^{-1}(X),W_1)$ defined by the algorithm is the fiber product (via φ) of that of (X,W).

(P2) (Equivariance.) If a group acting on W induces an action on a pair (X,W), the action naturally lifts to the desingularization of the pair defined by the algorithm.

(P3) For any (X, W), if $\operatorname{Im}\psi=\{\alpha_1, \ldots, \alpha_r\}$ then $X=\bigcup_{i=1}^{r}\psi^{-1}(\alpha_i)$ is a stratification of X, each stratum $\psi^{-1}(\alpha_i)$ being locally closed, regular and of pure dimension.

Property (P3) is a consequence of:

(P4) The regular and closed subschemes $\underline{\mathrm{Max}}\,\psi_{\mathcal{P}_s}$ (see $(\mathrm{A}2_s)$) are pure dimensional and $\dim \underline{\mathrm{Max}}\,\psi_{\mathcal{P}_s}$ is given by the value $\max \psi_{\mathcal{P}_s}$.

Properties (P4) and (P1) (for an arbitrary change of base field) were initially motivated by the study of stratification of families of schemes (e.g. Hilbert schemes) defined in terms of "algorithmic equiresolution", which we hope to address elsewhere (see also [E]).

Our program of study of canonical properties grows from [V2]. There (P1) and (P2) were proved for the algorithm introduced in [V1] and examples were included to exemplify how the algorithm works (e.g. on the Whitney umbrella) and how group actions lift. We also refer to [V4] and particularly to [EV] for a simple introduction to constructive desingularization. In this work we present a new algorithm and we prove the properties mentioned above. In Remark 6.22 we show how new and old algorithms relate. Proofs are organized to show that these properties hold for both algorithms.

The old algorithm (in [V1], [V2], [V4]) relied on the two main "inductive invariants": w-ord, n (see Definition 4.20 and 6.17). These two invariants were the clue for the inductive argument on the dimension of the ambient space. Together with the two main inductive invariants, there was finally a third invariant involved in the first algorithm, which is simple, non-inductive, and only plays a role when the two main invariants are exhausted (see the monomial case in §2).

Recently an important contribution with another approach to constructive desingularization has appeared in [BM2]. We also refer to [AJ], [AW] and [BP] for short and nice non-constructive proofs of desingularization.

Our new algorithm grows from a fourth invariant: Ab (see (6.18.1), (6.18.2)) which enables us to desingularize taking into account the notion of "good point" introduced by Abhyankar [Ab1]. We refer to [V3] for examples which illustrate that, in general, the new algorithm leads to desingularization in less steps (less monoidal transformations) than the old one.

An important improvement of this presentation with respect to that in [V1] and [V2] is the notion of "assignment of chains and functions" introduced in Definition 6.3, which clarifies the global behavior of the algorithm, avoiding the notion of idealistic exponents. Algorithmic aspects of the proofs are developed in the second half of the last section (§6).

1. Basic objects. Transformations

1.1. Let Z be a Zariski space (i.e. a Noetherian topological space such that each irreducible subset has a unique generic point, cf. [Ha, p. 93]), and $(I, \leqslant)$ a totally ordered set. In what follows, a mapping $f: Z \to (I, \leqslant)$ is said to be a *function* if and only if

(1) $f(Z)=\{\alpha_1, ..., \alpha_s\} \subseteq I$ (f takes only finitely many different values),

(2) for each $\alpha \in I$ the subset $\{\xi \in Z \mid f(\xi) \geqslant \alpha\}$ is closed in Z (i.e. f is upper-semi-continuous).

In our context Z will be the underlying topological space of a scheme of finite type over a field ([Ha, p. 84]), hence a mapping will be a function if and only if both (1) and (2) hold locally at any point of Z.

For f a function as above, we define

- $\max f = \max\{\alpha_1, ..., \alpha_s\}$, the maximal value achieved by f,
- $\underline{\mathrm{Max}}\, f = \{\xi \in Z \mid f(\xi) = \max f\}$, a closed subset of Z.

Example. If $F \subseteq Z$ is closed and $\xi \in F$, let $\operatorname{cod}_\xi(F)$ denote the codimension of F in Z locally at ξ. The map $-\operatorname{cod}: F \to \mathbf{Z}$, $-\operatorname{cod}(\xi) = (-1)\operatorname{cod}_\xi(F)$, is an important example of a function as defined above. Note that the local dimension, say $\dim: F \to \mathbf{Z}$ is also a function.

Definition 1.2. A basic object consists of data $(W, (J, b), E)$ where

(1) W is smooth and pure dimensional over a field k of characteristic zero ([Ha, p. 268]),

(2) J is a coherent sheaf of ideals of $\mathcal{O}_W$, such that $J_\xi \neq 0$ for all $\xi \in W$,

(3) $b \in \mathbf{N}$,

(4) $E = \{H_1, ..., H_r\}$ is a finite set of smooth hypersurfaces of W having only normal crossings.

The dimension of $(W, (J, b), E)$ will be the dimension W and to each such basic object we assign a reduced closed subscheme of W:

$$\operatorname{Sing}(J, b) = \{\xi \in W \mid \nu_\xi(J) \geqslant b\}$$

where $\nu_\xi(J)$ denotes the order of J_ξ at the local regular ring $\mathcal{O}_{W,\xi}$.

1.3. There is an ideal describing the closed set $\operatorname{Sing}(J, b)$. If n is the dimension of the basic object, $\Omega^1_{W/k}$ is locally free of rank n, and so is the dual sheaf $\operatorname{Der}(W/k)$. Define an operator Δ on coherent ideals in $\mathcal{O}_W$ by setting

$$\Delta(J)_\xi = \langle f, D(f) \mid f \in J_\xi,\, D \in \operatorname{Der}(W/k)_\xi \rangle \quad \forall \xi \in W.$$

We claim that $\operatorname{Sing}(J, b) = V(\Delta^{b-1}(J))$ (the closed subset defined by $\Delta^{b-1}(J)$). This can be checked from the fact that, if ξ is a closed point and $x_1, ..., x_n$ is a regular system of parameters of $\mathcal{O}_{W,\xi}$, then

$$\Delta(J)_\xi = \langle f, \partial f/\partial x_i \mid f \in J_\xi,\ i=1,\dots,n\rangle, \tag{1.3.1}$$

so $\nu_\xi(\Delta(J))=\max\{0,\nu_\xi(J)-1\}$. Note also that if J_ξ is generated by equations $f_1,\dots,f_r$ then

$$\Delta(J)_\xi = \langle f_j, \partial f_j/\partial x_i \mid j=1,\dots,r,\ i=1,\dots,n\rangle. \tag{1.3.2}$$

Definition 1.4. A *center* of a basic object $(W,(J,b),E)$ will be a closed and smooth subscheme of $\mathrm{Sing}(J,b)$ which has normal crossings with $E=\{H_1,\dots,H_r\}$.

Let $\varphi: W_1\to W$ denote the monoidal transformation with center C, H_{r+1} the exceptional locus of φ (a smooth hypersurface) and

$$E_1=\{H_1',\dots,H_r'\}\cup\{H_{r+1}\}$$

where H_i' is the strict transform of H_i. There is a unique coherent sheaf of ideals $J_1\subseteq\mathcal{O}_{W_1}$ so that

$$J\mathcal{O}_{W_1}=J_1 I(H_{r+1})^b$$

($I(H_{r+1})$ is the ideal defining $H_{r+1}=\varphi^{-1}(C)$). Now $(W_1,(J_1,b),E_1)$ is also a basic object and

$$(W,(J,b),E)\xleftarrow{\varphi}(W_1,(J_1,b),E_1)$$

will be called the *transformation* of basic objects defined by the center C.

1.5. Note that $\varphi: W_1\to W$ induces a proper map

$$\bar{\varphi}: \mathrm{Sing}(J_1,b)\to\mathrm{Sing}(J,b)$$

which is an isomorphism over $\mathrm{Sing}(J,b)\setminus C$. In particular, $\mathrm{Sing}(J_1,b)$ contains the strict transform of $\mathrm{Sing}(J,b)$.

1.6. *Trivial basic objects.* Note that if $J=I(V)$ where V is any smooth closed subscheme in W, then $\mathrm{Sing}(J,1)=V$, and if

$$\varphi: (W_1,(J_1,1),E_1)\to(W,(J,1),E)$$

is a transformation of basic objects with center C, then $J_1=I(V_1)$ where V_1 is the strict transform of V. $(W,(J,1),E)$ will be called a trivial basic object. So in this case the strict transform of $\mathrm{Sing}(J,b)$ is $\mathrm{Sing}(J_1,b)$.

Definition 1.7. A *resolution* of a basic object $(W,(J,b),E)$ will be a sequence of transformations

$$(W,(J,b),E)=(W_0,(J_0,b),E_0)\leftarrow(W_1,(J_1,b),E_1)\leftarrow\ldots\leftarrow(W_N,(J_N,b),E_N)$$

such that $\mathrm{Sing}(J_N,b)=\varnothing$.

1.8. Let $(W,(J,b),E)$ be a basic object and $\varphi\colon W_1\to W$ a smooth map (of pure relative dimension, [Ha, p. 268]). Set $J_1=J\mathcal{O}_{W_1}$ and

$$E_1=\{\varphi^{-1}(H_1),\ldots,\varphi^{-1}(H_r)\}.$$

Then $(W_1,(J_1,b),E_1)$ is also a basic object and

$$\mathrm{Sing}(J_1,b)=\varphi^{-1}(\mathrm{Sing}(J,b)).$$

This setting (where φ is smooth) will be denoted

$$\varphi\colon(W_1,(J_1,b),E_1)\to(W,(J,b),E)$$

and called the *restriction* defined by φ. Of particular interest is the case where φ is an open immersion or an étale morphism.

1.9. Let $\alpha\colon W'\to W$ be either a restriction or a change of the base field k (k as in Definition 1.2 (1) and W_1' the fiber product):

$$\begin{tikzcd}
(W',(J',b),E') \arrow[d, "\alpha"'] & (W_1',(J_1',b),E_1') \arrow[l, "\varphi'"'] \arrow[d, "\alpha'"] \\
(W,(J,b),E) & (W_1,(J_1,b),E_1). \arrow[l, "\varphi"']
\end{tikzcd}$$

(1) If φ is the transformation with center C, then φ' is the transformation with center $\alpha^{-1}(C)$ and α' is a restriction (resp. a change of base field) (we agree that a transformation on the empty center is the identity).

(2) If φ is a restriction then φ' is a restriction.

In particular a resolution of $(W,(J,b),E)$ induces a resolution of any restriction and of an arbitrary change of base field.

2. The monomial case

The interesting thing with the notion of resolution of basic objects, Definition 1.7, and its link to resolution of singularities will be clarified in the development. For the time being let us say that the clue to constructive desingularization [V1] was to define an assignment: to each basic object $\mathcal{B}=(W,(J,b),E)$ a smooth subscheme $C(\mathcal{B})\subseteq \mathrm{Sing}(J,b)$ having normal crossings with E. Set

$$\mathcal{B}=(W,(J,b),E)\leftarrow (W_1,(J_1,b),E_1)=\mathcal{B}_1$$

as the transformation with center $C(\mathcal{B})$. Now look at the transformation with center $C(\mathcal{B}_1)$, so ultimately such an assignment induces over each basic object $\mathcal{B}$ a sequence of transformations (Definition 1.4):

$$\mathcal{B}\leftarrow \mathcal{B}_1\leftarrow \dots \leftarrow \mathcal{B}_N\leftarrow \dots .$$

We also require that for some N (set $\mathcal{B}_N=(W_N,(J_N,b),E_N)$), $\mathrm{Sing}(J_N,b)$ be empty (as in Definition 1.7).

Here we treat a very special case inspired by [Hi1, p. 312]; but the treatment will illustrate the general strategy. Set $\mathcal{B}=(W,(J,b),E)$, $E=\{H_1,\dots,H_r\}$, and assume that

$$J=I(H_1)^{\alpha_1}\dots I(H_r)^{\alpha_r}, \tag{2.0.1}$$

$\alpha_i\colon W\to \mathbf{Z}\geqslant 0$, $\alpha_i(\xi)=0$ if $\xi\notin H_i$, and α_i locally constant along points of H_i (i.e. constant on each irreducible component of H_i). In this case we call $\mathcal{B}$ a *monomial* basic object.

First note that the singular locus can be expressed in terms of the exponents $\alpha_1,\dots,\alpha_r$ and the hypersurfaces $H_1,\dots,H_r$, namely:

$$\mathrm{Sing}(J,b)=\{\xi\in W\mid \exists i_1,\dots,i_p,\ \alpha_{i_1}(\xi)+\dots+\alpha_{i_p}(\xi)\geqslant b,\ \xi\in H_{i_1}\cap\dots\cap H_{i_p}\}.$$

One can easily check that, locally at any point, $\mathrm{Sing}(J,b)$ is a union of irreducible components with normal crossings, and we wish to select one of them as a center of transformation.

We define a function which depends on (J,b):

$$\begin{gathered}\Gamma(\mathcal{B})\colon \mathrm{Sing}(J,b)\to I_M=\mathbf{Z}\times\mathbf{Q}\times\mathbf{N}^{\mathbf{N}},\\ \Gamma(\mathcal{B})(\xi)=(-\Gamma_1(\xi),\Gamma_2(\xi),\Gamma_3(\xi)),\end{gathered} \tag{2.0.2}$$

where I_M is totally ordered with the usual lexicographic ordering.

Define for $\xi\in\mathrm{Sing}(J,b)$:

$$\Gamma_1(\xi)=\min\{p\mid \exists i_1,\dots,i_p,\ \alpha_{i_1}(\xi)+\dots+\alpha_{i_p}(\xi)\geqslant b,\ \xi\in H_{i_1}\cap\dots\cap H_{i_p}\},$$

$$\Gamma_2(\xi)=\max\left\{\frac{\alpha_{i_1}(\xi)+\dots+\alpha_{i_p}(\xi)}{b}\;\middle|\; p=\Gamma_1(\xi),\ \alpha_{i_1}(\xi)+\dots+\alpha_{i_p}(\xi)\geqslant b,\ \xi\in H_{i_1}\cap\dots\cap H_{i_p}\right\},$$

$$\Gamma_3(\xi)=\max\left\{(i_1,\dots,i_p,0,\dots)\;\middle|\;\Gamma_2(\xi)=\frac{\alpha_{i_1}(\xi)+\dots+\alpha_{i_p}(\xi)}{b},\ \xi\in H_{i_1}\cap\dots\cap H_{i_p}\right\}.$$

$\Gamma_1(\xi)$ is the minimal codimension of the components of $\mathrm{Sing}(J,b)$ locally at ξ. $\Gamma_2(\xi)=b'/b$, where b' is the maximum of $\nu_y(J)$, y being the generic point of a component of $\mathrm{Sing}(J,b)$ containing ξ and of codimension $\Gamma_1(\xi)$.

Set $\max\Gamma(\mathcal{B})=(-p,\omega,\underline{a})$ and $n=\dim W$. One can check that, locally at ξ, $\underline{\mathrm{Max}}\,\Gamma(\mathcal{B})$ is one of the highest-dimensional components of $\mathrm{Sing}(J,b)$ (dimension $n-p$). In particular, if some $\alpha_i(\xi)\geqslant b$ then $p=1$ and $\underline{\mathrm{Max}}\,\Gamma$ is a hypersurface at ξ.

In general, $\underline{\mathrm{Max}}\,\Gamma(\mathcal{B})$ is a union of connected components of the pure dimensional and regular scheme $H_{i_1}\cap\ldots\cap H_{i_p}$ where $\underline{a}=(i_1,\ldots,i_p,0,\ldots)$ (recall that $\max\Gamma=(-p,\omega,\underline{a})$).

Note that $\underline{\mathrm{Max}}\,\Gamma$ has normal crossings with E. Setting $\underline{\mathrm{Max}}\,\Gamma(\mathcal{B})$ as center of the transformation

$$\mathcal{B}=(W,(J,b),E)\leftarrow\mathcal{B}_1=(W_1,(J_1,b),E_1)$$

we naturally obtain an expression

$$J_1=I(H_1')^{\alpha_1}\ldots I(H_r')^{\alpha_r}I(H_{r+1})^{\alpha_{r+1}} \tag{2.0.3}$$

by setting for $\xi_1\in H_i'$ mapping to $\xi\in H_i$, $\alpha_i(\xi_1)=\alpha_i(\xi)$, and if $\xi_1\in H_{r+1}$:

$$\alpha_{r+1}(\xi_1)=(\alpha_{i_1}(\xi)+\ldots+\alpha_{i_p}(\xi))-b=(\omega-1)b.$$

So $\mathcal{B}_1$ is monomial, $\Gamma(\mathcal{B}_1)$ can be defined as above and one can check that $\max\Gamma(\mathcal{B})>\max\Gamma(\mathcal{B}_1)$ and that repeating this construction again and again, we finally come to a resolution of the basic object $\mathcal{B}$ (Definition 1.7).

3. The good points

3.1. Suppose now that the ideal J is not necessarily monomial but that there exists a monomial part together with another factor, say

$$J=I(H_1)^{\alpha_1}\ldots I(H_r)^{\alpha_r}\mathcal{A}. \tag{3.1.1}$$

A point $\xi\in\mathrm{Sing}(J,b)$ is called *exceptional and good* if

$$\nu_\xi(I(H_1)^{\bar{\alpha}_1}\ldots I(H_r)^{\bar{\alpha}_r}\mathcal{A})<b \tag{3.1.2}$$

where $\bar{\alpha}_i(\xi)$ denotes the remainder of $\alpha_i(\xi)$ modulo b for any point ξ. Clearly $\bar{\alpha}_i$ is locally constant.

If all points in $\mathrm{Sing}(J,b)$ are exceptional and good, one can check that, locally at ξ,

$$\mathrm{Sing}(J,b)=\mathrm{Sing}(I(H_1)^{\alpha_1(\xi)}\ldots I(H_r)^{\alpha_r(\xi)},b)=\bigcup_{\alpha_i(\xi)\geqslant b}H_i.$$

Let $\Gamma(\mathcal{B})\colon \mathrm{Sing}(J,b)\to I_M$ be the function defined in (2.0.2) applied now only to the monomial part of J in the expression (3.1.1). So $\Gamma(\mathcal{B})$ is defined by neglecting the non-monomial part $\mathcal{A}$.

If any point is exceptional and good then $\Gamma_1(\xi)=1$ at any point $\xi\in\mathrm{Sing}(J,b)$ and $\underline{\mathrm{Max}}\,\Gamma$ is a union of components of a hypersurface H_i. In particular, $\underline{\mathrm{Max}}\,\Gamma$ is a hypersurface. The transformation with center $\underline{\mathrm{Max}}\,\Gamma$ is an isomorphism on W, but the transform of $\mathcal{B}=(W,(J,b),E)$ is $\mathcal{B}_1=(W_1,(J_1,b),E_1)$ where

$$J_1=I(H_1)^{\alpha_1}\dots I(H_{i_1-1})^{\alpha_{i_1-1}}I(H_{i_1})^{\alpha'_{i_1}}I(H_{\alpha_{i_1+1}})^{\alpha_{i_1+1}}\dots I(H_r)^{\alpha_r}\mathcal{A},\tag{3.1.3}$$

$\max\Gamma=(-1,\omega,(i_1,0,\dots))$ and $\alpha'_{i_1}=b(\omega-1)$. Again, all points of $\mathrm{Sing}(J_1,b)$ are exceptional and good, and it is easy to check that the sequence of transformations

$$\mathcal{B}\leftarrow\mathcal{B}_1\leftarrow\dots\leftarrow\mathcal{B}_N,$$

defined by the functions $\Gamma(\mathcal{B}_i)$ as in the monomial case, is a resolution of the basic object $\mathcal{B}$ (Definition 1.7) obtained by monoidal transformations, all centers being hypersurfaces (so that all $W_i=W$ in this case).

3.2. Within the setting of (3.1.1) we present a slightly more general situation which is particularly *good*. Define a point $\xi\in\mathrm{Sing}(J,b)$ to be *locally good* if either ξ is exceptional and good ((3.1.2)), or the point is locally monomial, namely if

$$\mathcal{A}_\xi=\mathcal{O}_{W,\xi}.\tag{3.2.1}$$

If each point $\xi\in\mathrm{Sing}(J,b)$ is locally good, then $\mathrm{Sing}(J,b)=\mathrm{Sing}(I(H_1)^{\alpha_1}\dots I(H_r)^{\alpha_r},b)$. Now we neglect $\mathcal{A}$ in (3.1.1) and define Γ as in §2, in terms of $I(H_1)^{\alpha_1}\dots I(H_r)^{\alpha_r}$. In this case, the sequence of transformations consists first in some monoidal transformations at hypersurfaces, say N steps, and then $W_N=W$ and $J_N=I(H_1)^{\bar\alpha_1}\dots I(H_r)^{\bar\alpha_r}\mathcal{A}$ ((3.1.2)). Now all points of $\mathrm{Sing}(J_N,b)$ are locally monomial ((3.2.1)) and then again the procedure in §2, defined in terms of Γ, extends the sequence of transformations to define a resolution.

This shows that the function in §2 defines a unique resolution of $\mathcal{B}$ in case all points of $\mathrm{Sing}(J,b)$ are locally good.

Remark 3.3. (1) $\underline{\mathrm{Max}}\,\Gamma$ is locally defined as an intersection of hypersurfaces in E, hence $\underline{\mathrm{Max}}\,\Gamma$ has normal crossings with E (with the union of hypersurfaces in E).

(2) If $\underline{\mathrm{Max}}\,\Gamma$ is a hypersurface, then $\underline{\mathrm{Max}}\,\Gamma$ is a union of components of some $H_{i_1}\in E$. Assume that $\underline{\mathrm{Max}}\,\Gamma=H_{i_1}$ and fix notation as in Definition 1.4; then $W=W_1$, $H'_{i_1}=\varnothing$ and $H_{r+1}=H_{i_1}$. In particular, we must replace i_1 by $r+1$ in (3.1.3).

4. Codimension and basic objects

4.1. Set $W=\operatorname{Spec} k[X_1,\dots,X_n]$ and let $J\subseteq k[X_1,\dots,X_n]$ be a homogeneous ideal generated by homogeneous elements of degree b. Under these conditions we claim that $\operatorname{Sing}(J,b)$ is a linear variety.

In fact, recall that $\operatorname{Sing}(J,b)=V(\Delta^{b-1}(J))$ (1.3), Euler's formula for homogeneous polynomials

$$bf=\sum_{i=1}^{n} X_i\frac{\partial f}{\partial X_i}$$

(k being of characteristic zero) asserts that $\Delta^{b-1}(J)$ is generated by homogeneous elements of degree one.

4.2. Let R be a regular local ring with maximal ideal $\mathfrak{m}$. If J is an ideal of R, $\nu(J)$ denotes the order of J. If $J\subseteq\mathfrak{m}^b$ (i.e. $\nu(J)\geqslant b$) then we denote by $\operatorname{In}_b J$ the initial part of degree b of J in the graded polynomial ring $\operatorname{Gr}(R)$:

$$\operatorname{In}_b J=J+\mathfrak{m}^{b+1}/\mathfrak{m}^{b+1}\subseteq\mathfrak{m}^b/\mathfrak{m}^{b+1}\subseteq\operatorname{Gr}(R)=\bigoplus_{n\geqslant 0}\mathfrak{m}^n/\mathfrak{m}^{n+1}.$$

If $\nu(J)=b$, then $\operatorname{In}_b J$ is generated by homogeneous polynomials of degree b.

Definition 4.3. Let ξ be a closed point of $\operatorname{Sing}(J,b)$. We define $\tau(\xi)=\tau(J,b)(\xi)$ to be the codimension of the linear variety $\operatorname{Sing}(\operatorname{In}_b J_\xi,b)$ in $\operatorname{Spec}(\operatorname{Gr}(\mathcal{O}_{W,\xi}))$.

Note that $\tau(\xi)\geqslant 0$ and that $\tau(\xi)=0$ if and only if $\nu_\xi(J)>b$.

4.4. It follows easily from (1.3.1) that at a closed point $\xi\in\operatorname{Sing}(J,b)$,

$$\Delta^{b-1}_{\operatorname{Gr}(\mathcal{O}_{W,\xi})}(\operatorname{In}_b J_\xi)=\operatorname{In}_1(\Delta^{b-1}_W(J)_\xi)$$

as ideals in $\operatorname{Gr}(\mathcal{O}_{W,\xi})$. In particular, in case $\nu_\xi(J)=b$, $\tau=\tau(\xi)>0$ and there exists a regular system of parameters $x_1,\dots,x_n$ of $\mathcal{O}_{W,\xi}$ such that

$$\Delta^{b-1}(J)_\xi=\langle x_1,\dots,x_\tau\rangle+I \tag{4.4.1}$$

where $I\subseteq\mathfrak{m}_\xi^2$.

LEMMA 4.5 (Giraud). *Consider a transformation of basic objects:*

$$(W,(J,b),E)\xleftarrow{\varphi}(W_1,(J_1,b),E_1)$$

with center $C\subseteq\operatorname{Sing}(J,b)$ *and denote by* H *the exceptional divisor* (*Definition* 1.4).

Then for all $i\in\{0,...,b\}$, $\Delta^{b-i}(J)\mathcal{O}_{W_1}\subseteq I(H)^i$ *and*

$$\frac{1}{I(H)^i}\Delta^{b-i}(J)\subseteq\Delta^{b-i}(J_1)\,(\subseteq\mathcal{O}_{W_1}).$$

Proof. If $i=b$, $\Delta^0(J)=J$, $\Delta^0(J_1)=J_1$ and the claim is trivial. We argue by decreasing induction on i, so assume that the inclusion holds for some $i>0$. Let $\xi'\in H$ be any closed point, $\xi=\varphi(\xi')$ and choose $x\in\mathcal{O}_{W,\xi}$ such that $I(H)_\xi=(x)$.

It suffices to show that for generators f of $\Delta^{b-(i-1)}(J)$, $f/x^{i-1}\in\Delta^{b-(i-1)}(J_1)$. If $f\in\Delta^{b-i}(J)$ $(\subseteq\Delta^{b-(i-1)}(J))$ then the assertion follows by induction. Therefore, by (1.3.1), it suffices to treat the case $f=D(g)$, for $g\in\Delta^{b-i}(J)_\xi$ and $D\in\mathrm{Der}(W/k)_\xi$. By induction we have

$$\frac{g}{x^i}\in\frac{1}{I(H)^i}\Delta^{b-i}(J)_\xi\subseteq\Delta^{b-i}(J_1)_{\xi'}\,(\subseteq\Delta^{b-(i-1)}(J_1)_{\xi'}).$$

Set $D'=xD$. It can be checked that D' is a derivation (with no poles) locally at $\xi'\in W'$, so $D'(g/x^i)\in\Delta^{b-(i-1)}(J_1)_{\xi'}$. Finally

$$D'\Big(\frac{g}{x^i}\Big)=\frac{D(g)}{x^{i-1}}-iD(x)\frac{g}{x^i},$$

and hence

$$\frac{f}{x^{i-1}}=\frac{D(g)}{x^{i-1}}=D'\Big(\frac{g}{x^i}\Big)+iD(x)\frac{g}{x^i}$$

belongs to $\Delta^{b-(i-1)}(J_1)_{\xi'}$.

COROLLARY 4.6. *Let* $(W,(J,b),E)$ *be a basic object and assume that there is a closed regular subscheme* $Z\subseteq W$ *such that* $I(Z)\subseteq\Delta^{b-1}(J)$.

For any transformation $(W,(J,b),E)\leftarrow(W_1,(J_1,b),E_1)$,

$$I(Z_1)\subseteq\Delta^{b-1}(J_1)$$

where Z_1 *is the strict transform of* Z.

This follows from the property of transformations of trivial objects (1.6) together with Lemma 4.5 applied for $i=1$.

Remark. Note that $I(Z)\subseteq\Delta^{b-1}(J)$ implies $\nu_\xi(J)=b$ for all $\xi\in\mathrm{Sing}(J,b)$ ((1.3.1)); in particular, in this setting it also follows that $\nu_{\xi_1}(J_1)=b$ for $\xi_1\in\mathrm{Sing}(J_1,b)$.

COROLLARY 4.7. *Let* ξ *be a closed point in* $\mathrm{Sing}(J,b)$ *and assume* $\nu_\xi(J)=b$.

(1) *If* $\{x_1,...,x_r\}$ *are as in 4.4, after restriction to a suitable neighborhood of* ξ, *we may assume*:

$$V=V(\langle x_1,...,x_r\rangle)\ \textit{is closed and regular, and}\ I(V)\subseteq\Delta^{b-1}(J).\qquad(4.7.1)$$

In particular, $\mathrm{Sing}(J,b)\subseteq V$ *and* $\mathrm{cod}_\xi(\mathrm{Sing}(J,b))\geqslant\tau=\tau(\xi)$*, where* cod_ξ *denotes the codimension in* W*, locally at* ξ *as in the example in* 1.1.

(2) *Set* $(W,(J,b),E)\xleftarrow{\varphi}(W_1,(J_1,b),E_1)$ *a transformation* (*Definition* 1.4) *and* $\xi_1\in\mathrm{Sing}(J_1,b)$ *such that* $\varphi(\xi_1)=\xi$*. Then* $\nu_{\xi_1}(J_1)=b$ *and* $\tau(\xi)\leqslant\tau(\xi_1)$*. If furthermore the setting is as in* (1)*, then* $\mathrm{Sing}(J_1,b)\subseteq V_1$*, where* V_1 *is the strict transform of* V*.*

(3) *For any transformation* $(W,(J,b),E)\leftarrow(W_1,(J_1,b),E_1)$ *and for any point* $\xi_1\in W_1$ *mapping to* $\xi\in\mathrm{Sing}(J,b)$:

$$b=\nu_\xi(J)\geqslant\nu_{\xi_1}(J_1).$$

Proof. (1) is clear since $\Delta^{b-1}(J)$ is coherent and $\mathrm{Sing}(J,b)=V(\Delta^{b-1}(J))$ (1.3). (2) follows from Corollary 4.6, and (3) from the fact that $\nu_{\xi_1}(\Delta^{b-1}(J_1))\leqslant\nu_{\xi_1}(I(V_1))\leqslant1$ (see Corollary 4.6).

COROLLARY 4.8. *For* $\xi\in\mathrm{Sing}(J,b)$ *as in Corollary* 4.7 *and* $\tau=\tau(\xi)$:

(1) $\mathrm{cod}_\xi(\mathrm{Sing}(J,b))=\tau$ *if and only if* $\Delta^{b-1}(J)_\xi=\langle x_1,...,x_\tau\rangle$ (*in the setting of* (1) *of Corollary* 4.7*, if and only if* $\mathrm{Sing}(J,b)=V$).

(2) *Fix a transformation* $(W,(J,b),E)\xleftarrow{\varphi}(W_1,(J_1,b),E_1)$ (*Definition* 1.4) *and* $\xi_1\in\mathrm{Sing}(J_1,b)$ *so that* $\varphi(\xi_1)=\xi$*. Then* $\mathrm{cod}_{\xi_1}(\mathrm{Sing}(J_1,b))=\tau$ *if and only if* $\mathrm{cod}_\xi(\mathrm{Sing}(J,b))=\tau$*.*

Proof. Replacing W by a suitable open neighborhood of ξ we may assume that the setting is as in (4.7.1), where in addition V is *irreducible*, so

(i) $\mathrm{Sing}(J,b)\subseteq V$,

(ii) $\mathrm{Sing}(J_1,b)\subseteq V_1$,

where V_1 is the strict transform of V, both irreducible, smooth and of pure codimension τ.

(1) is a simple consequence of (i).

(2) We shall prove that equality holds at (i) if and only if it holds at (ii).

If $\mathrm{cod}_\xi(\mathrm{Sing}(J,b))=\tau$ then $\mathrm{Sing}(J,b)=V$ (by (i)). 1.5 asserts that the strict transform of $\mathrm{Sing}(J,b)$, namely V_1, is contained in $\mathrm{Sing}(J_1,b)$, which together with (ii) implies that $\mathrm{Sing}(J_1,b)=V_1$, so $\mathrm{cod}_{\xi_1}(\mathrm{Sing}(J_1,b))=\tau$.

Conversely, if $\mathrm{cod}_{\xi_1}(\mathrm{Sing}(J_1,b))=\tau$, then $\mathrm{Sing}(J_1,b)=V_1$ (by (ii)), which maps surjectively to V. Since $\varphi(\mathrm{Sing}(J_1,b))\subseteq\mathrm{Sing}(J,b)$ (1.5), it follows that $\mathrm{Sing}(J,b)=V$, so $\mathrm{cod}_\xi(\mathrm{Sing}(J,b))=\tau$.

4.9. Let $\varphi:(W',(J',b),E')\to(W,(J,b),E)$ be either a restriction of basic objects (1.8) or an arbitrary change of the base field k (Definition 1.2 (1)), and ξ' a closed point in $\mathrm{Sing}(J',b)=\varphi^{-1}(\mathrm{Sing}(J,b))$ mapping to $\xi\in\mathrm{Sing}(J,b)$. Then $\mathrm{cod}_\xi(\mathrm{Sing}(J,b))=\mathrm{cod}_{\xi'}(\mathrm{Sing}(J',b))$. A regular system of parameters at $\mathcal{O}_{W,\xi}$, say $x_1,...,x_n$, can be extended to a regular system of parameters $x_1,...,x_n,x_{n+1},...,x_m$ at $\mathcal{O}_{W',\xi'}$.

If $J_\xi=\langle f_1,\dots,f_r\rangle$ then $J'_{\xi'}=\langle f_1,\dots,f_r\rangle$ and $\partial f_i/\partial x_j=0$ if $j>n$. It follows from (1.3.2) that

$$\Delta_W^{b-1}(J)\mathcal{O}_{W'}=\Delta_{W'}^{b-1}(J').$$

In particular, the setting of (4.4.1) is preserved (for the same τ) and so is Corollary 4.7 (1) and Corollary 4.8 (1).

PROPOSITION 4.10. *Fix $e\geqslant 0$ and let $(W,(J,b),E)$ be a basic object such that $\tau(J,b)(\xi)\geqslant e$ (Definition 4.3) at any closed point $\xi\in\mathrm{Sing}(J,b)$. Note that we have $\mathrm{cod}_\xi(\mathrm{Sing}(J,b))\geqslant e$ at any ξ (Corollary 4.7 (1)).*

Set $F^{(e)}=\{\xi\in\mathrm{Sing}(J,b)\mid \mathrm{cod}_\xi(\mathrm{Sing}(J,b))=e\}$ (possibly empty). Then:

(1) $F^{(e)}$ is smooth of pure codimension e and is open and closed in $\mathrm{Sing}(J,b)$ (i.e. a union of connected components).

(2) Suppose that $F^{(e)}$ has normal crossings with E and set

$$(W,(J,b),E)\xleftarrow{\ \varphi\ }(W_1,(J_1,b),E_1)$$

as the transformation with center $F^{(e)}$. Then $F_1^{(e)}=\{\xi\in\mathrm{Sing}(J_1,b)\mid \mathrm{cod}_\xi(\mathrm{Sing}(J_1,b))=e\}$ is empty (i.e. $\mathrm{cod}(\mathrm{Sing}(J_1,b))>e$ at any point) and $\mathrm{Sing}(J_1,b)$ can be identified with $\mathrm{Sing}(J,b)\backslash F^{(e)}$.

Proof. (1) Note that at any closed point $\xi\in F^{(e)}$, $\Delta^{b-1}(J)_\xi=\langle x_1,\dots,x_e\rangle$ in (4.4.1), so $\mathrm{Sing}(J,b)=F^{(e)}=V=V(\langle x_1,\dots,x_\tau\rangle)$ locally at ξ, which shows that $F^{(e)}$ is open in $\mathrm{Sing}(J,b)$. From Corollary 4.7 (1), it also follows that, if $F^{(e)}\neq\varnothing$, $F^{(e)}$ consists of points of $\mathrm{Sing}(J,b)$ of maximal dimension, and therefore $F^{(e)}$ is closed by the example in 1.1.

(2) With the setting and notation as in the proof of Corollary 4.8, here $\mathrm{Sing}(J,b)=F^{(e)}=V$ and therefore $\mathrm{Sing}(J_1,b)=V_1$ (strict transform of V). Since the center is V, $V_1(=\mathrm{Sing}(J_1,b))=\varnothing$.

We summarize the previous results as follows:

COROLLARY 4.11. *Let ξ be a closed point in $\mathrm{Sing}(J,b)$ such that $\nu_\xi(J)=b$. Set e such that $\tau(\xi)\geqslant e>0$. There is a restriction to an open neighborhood of ξ,*

$$(W',(J',b),E')\to(W,(J,b),E),$$

and there is a smooth closed subscheme V^e of W' such that:

(1) $\xi\in V^e$ and V^e has pure codimension e.

(2) $I(V^e)\subseteq\Delta^{b-1}(J')$ (so $\mathrm{Sing}(J',b)\subseteq V^e$).

(3) $\tau(\xi')\geqslant e$ and $\mathrm{cod}_{\xi'}(\mathrm{Sing}(J',b))\geqslant e$ for any $\xi'\in\mathrm{Sing}(J',b)$.

(4) *Any sequence of transformations*

$$(W',(J',b),E')\leftarrow(W'_1,(J'_1,b),E'_1)\leftarrow\ldots\leftarrow(W'_N,(J'_N,b),E'_N)$$

induces a sequence of transformations of trivial basic objects (1.6),

$$(W',(I(V^e),1),E')\leftarrow(W'_1,(I(V_1^e),1),E'_1)\leftarrow\ldots\leftarrow(W'_N,(I(V_N^e),1),E'_N),$$

with each V_i^e *the strict transform of* V_{i-1}^e, *and for each index* i,

$$I(V_i^e)\subseteq\Delta^{b-1}(J'_i,b);$$

in particular, $\mathrm{Sing}(J'_i,b)\subseteq V_i^e=\mathrm{Sing}(I(V_i^e),1)$ *and* $\mathrm{cod}_{\xi_i}(\mathrm{Sing}(J'_i,b))\geqslant e$.

4.12. Set $R=k[[Z_1,\ldots,Z_e,X_1,\ldots,X_n]]$ and $\bar{R}=R/(Z_1,\ldots,Z_e)=k[[X_1,\ldots,X_n]]$. Let ν denote the order in R and $\bar{\nu}$ the order in $\bar{R}$. If $f\in R$, we denote $f\bar{R}$ the class of f in $\bar{R}$. One can check that

$$\nu(f)\geqslant b\quad\Leftrightarrow\quad\bar{\nu}\left(\frac{\partial^i f}{\partial Z_1^{i_1}\ldots\partial Z_e^{i_e}}\bar{R}\right)\geqslant b-i,\quad i=0,\ldots,b-1,\ i_1+\ldots+i_e=i.$$

In fact, setting $f=\sum a_\alpha z^\alpha$, $a_\alpha\in k[[X_1,\ldots,X_n]]$, $\alpha=(\alpha_1,\ldots,\alpha_n)$, $|\alpha|=\alpha_1+\ldots+\alpha_n$, the equivalence can be rephrased as

$$\nu(f)\geqslant b\quad\Leftrightarrow\quad\bar{\nu}(a_\alpha)\geqslant b-|\alpha|\quad\forall\alpha,\ |\alpha|<b.$$

COROLLARY 4.13. *Let* $(W,(J,b),E)$ *be a basic object and* $V\,(\subseteq W)$ *be a regular subvariety of codimension* e. *If* $\xi\in V$ *then*

$$\nu_{W,\xi}(J)(\xi)\geqslant b\quad\Leftrightarrow\quad\nu_{V,\xi}(\Delta^i(J)\mathcal{O}_V)\geqslant b-i,\quad i=0,\ldots,b-1,$$

where $\nu_{W,\xi}$ *(resp.* $\nu_{V,\xi}$*) denotes the order at the local ring* $\mathcal{O}_{W,\xi}$ *(resp. at* $\mathcal{O}_{V,\xi}$*).*

Definition 4.14. Let $\mathcal{B}=(W,(J,b),E)$ be a basic object such that $b=\max\{\nu_\xi(J)\mid\xi\in\mathrm{Sing}(J,b)\}$. Let V^e be a smooth closed subscheme of codimension e and assume that $I(V^e)\subseteq\Delta^{b-1}(J)$ (so $\mathrm{Sing}(J,b)\subseteq V^e$). We define the *coefficient ideal* of $\mathcal{B}$ on V^e:

$$C(J)=\sum_{i=0}^{b-1}(\Delta^i(J))^{b!/(b-i)}\mathcal{O}_{V^e}.$$

It follows from Corollary 4.13 that $\mathrm{Sing}(J,b)=\mathrm{Sing}(C(J),b!)$. Note that $C(J)_\xi=0$ if and only if $\mathrm{Sing}(J,b)=V^e$ locally at ξ. In particular, if $F^{(e)}=\varnothing$ then $C(J)_\xi\neq0$ for any ξ, so $(V^e,(C(J),b!),\varnothing)$ is a basic object and $\mathrm{Sing}(J,b)=\mathrm{Sing}(C(J),b!)\,(\subseteq V^e)$ as closed subsets in W.

PROPOSITION 4.15. *Let* $(W_0,(J_0,b),E_0)$ *be a basic object and* V_0^e *be a smooth closed subscheme of pure codimension e such that:*

(1) $I(V_0^e)\subseteq\Delta^{b-1}(J)$ *as in Definition* 4.14; *note that in that case* $\tau(\xi)\geqslant e$ *for any* $\xi\in\mathrm{Sing}(J,b)$.

(2) V_0^e *has normal crossings with* E_0 *and* $V_0^e\not\subset H$ *for any* $H\in E_0$.

Assume that $F^{(e)}$ *is empty (see Proposition* 4.10) *and set the basic object as* $(V_0^e,(C(J_0),b!),\overline{E}_0)$, *where* $\overline{E}_0=\{H\cap V_0^e\mid H\in E_0\}$.

Any sequence of transformations of basic objects,

$$(W_0,(J_0,b),E_0)\leftarrow(W_1,(J_1,b),E_1)\leftarrow\ldots\leftarrow(W_s,(J_s,b),E_s),$$

induces a sequence of transformations with the same center,

$$(V_0^e,(C(J_0),b!),\overline{E}_0)\leftarrow(V_1^e,(C(J_0)_1,b!),\overline{E}_1)\leftarrow\ldots\leftarrow(V_s,(C(J_0)_s,b!),\overline{E}_s),$$

and $\mathrm{Sing}(J_s,b)=\mathrm{Sing}(C(J_0)_s,b!)$ $(\subseteq V_s^e\subseteq W_s)$.

Proof. Let $H_k\subseteq W_k$ be the exceptional hypersurface corresponding to the transformation $(W_{k-1},(J_{k-1},b),E_{k-1})\leftarrow(W_k,(J_k,b),E_k)$.

We set for $k>0$,

$$[\Delta^{b-i}(J_0)]_k=\frac{1}{I(H_k)^i}[\Delta^{b-i}(J_0)]_{k-1}\mathcal{O}_{W_k},$$

so that

$$C(J_0)_k=\sum_{i=0}^{b-1}[\Delta^{b-i}(J_0)]_k^{b!/i}\mathcal{O}_{V_k^e}.$$

We begin by formulating a claim, say:

Claim(s). For any index $k=0,\ldots,s$:

(1) $[\Delta^{b-i}(J_0)]_k\subseteq\Delta^{b-i}(J_k)$.

(2) At any closed point $\xi_k\in\mathrm{Sing}(C(J_0)_k,b!)$ there is a regular system of parameters $z_{k,1},\ldots,z_{k,e},x_{k,1},\ldots,x_{k,n-e}$ such that:

(a) $I(V_k^e)_{\xi_k}=\langle z_{k,1},\ldots,z_{k,e}\rangle$.

(b) Setting $R_k=\widehat{\mathcal{O}}_{W_k,\xi_k}$, $\overline{R}_k=\widehat{\mathcal{O}}_{V_k^e,\xi_k}$, there is a set of generators $\{f_k^{(\lambda)}\}$ of J_kR_k,

$$f_k^{(\lambda)}=\sum_\alpha a_{k,\alpha}^{(\lambda)}Z_k^\alpha,\quad a_{k,\alpha}^{(\lambda)}\in k'[[X]]=\overline{R}_k,$$

so that

$$(a_{k,\alpha}^{(\lambda)})^{b!/(b-|\alpha|)}\in C(J_0)_k\overline{R}_k \tag{4.15.1}$$

for all α with $|\alpha|<b$.

Before we proceed with the proof of our claim, let us point out that if (1) holds then $C(J_0)_k \subseteq C(J_k)$ and in particular,

$$(\mathrm{Sing}(J_k, b) =) \, \mathrm{Sing}(C(J_k), b!) \subseteq \mathrm{Sing}(C(J_0)_k, b!).$$

On the other hand, if (2) holds at any $\xi_k \in \mathrm{Sing}(C(J_0)_k, b!)$, it follows from (4.15.1) that $\xi_k \in \mathrm{Sing}(J_k, b)$, so

$$\mathrm{Sing}(C(J_0)_k, b!) \subseteq \mathrm{Sing}(J_k, b).$$

As for Claim(0), (1) is trivial and (2) follows from the fact that $a_{k,\alpha}^{(\lambda)} \in \Delta^{b-|\alpha|}(J_0)\bar{R}_0$ if $|\alpha| < b$.

We now assume Claim(s) and consider a sequence of transformations of length $s+1$. Since $[\Delta^{b-i}(J_0)]_s \subseteq \Delta^{b-i}(J_s)$, we have

$$[\Delta^{b-i}(J_0)]_{s+1} = \frac{1}{I(H_{s+1})^i} [\Delta^{b-i}(J_0)]_s \subseteq \frac{1}{I(H_{s+1})^i} \Delta^{b-i}(J_s) \subseteq \Delta^{b-i}(J_{s+1}).$$

See Lemma 4.5 for the last inclusion.

Let $\xi_{s+1} \in \mathrm{Sing}(C(J_0)_{s+1}, b!)$ be a closed point, $\xi_s \in \mathrm{Sing}(C(J_0)_s, b!)$ the image in W_s. After a finite extension of the base field and a linear change involving only the variables $x_{s,j}$ in R_s, we may assume at $R_{s+1} = \widehat{\mathcal{O}}_{W_{s+1}, \xi_{s+1}}$ a regular system of parameters $z_{s+1,1}, \ldots, z_{s+1,e}, x_{s+1,1}, \ldots, x_{s+1,n-e}$ with

$$I(H_{s+1})_{\xi_{s+1}} = \langle x_{s+1,1} \rangle, \quad x_{s+1,1} = x_{s,1},$$
$$I(V_{s+1}^e)_{\xi_{s+1}} = \langle z_{s+1,1}, \ldots, z_{s+1,e} \rangle, \quad z_{s+1,j} = z_{s,j}/x_{s,1},$$

and define

$$f_{s+1}^{(\lambda)} = \frac{f_s^{(\lambda)}}{x_{s,1}^b} = \sum_\alpha a_{s+1,\alpha}^{(\lambda)} Z_{s+1}^\alpha$$

so that $a_{s+1,\alpha}^{(\lambda)} = a_{s,\alpha}^{(\lambda)} / x_{s,1}^{b-|\alpha|}$. In particular,

$$(a_{s+1,\alpha}^{(\lambda)})^{b!/(b-|\alpha|)} \in C(J_0)_{s+1} \bar{R}_{s+1}.$$

This proves Claim($s+1$) and Proposition 4.15.

Definition 4.16. Let $(W, (J, b), E)$ be a basic object and assume that $\tau(J, b)(\xi) \geqslant e$ for any $\xi \in \mathrm{Sing}(J, b)$. Define

$$\mathrm{ord}_e(J, b) : \mathrm{Sing}(J, b) \to \mathbf{Q} \cup \{\infty\},$$

$\mathrm{ord}_e(J, b)(\xi) = \infty$ if and only if $\mathrm{cod}_\xi(\mathrm{Sing}(J, b)) = e$. If $\mathrm{cod}_\xi(\mathrm{Sing}(J, b)) > e$, setting V^e in a suitable open restriction so that $I(V^e) \subseteq \Delta^{b-1}(J)$ (Corollary 4.11), and $C(J) \subseteq \mathcal{O}_{V^e}$ as in Definition 4.14, then

$$\mathrm{ord}_e(J, b)(\xi) = \frac{\nu_\xi(C(J))}{b!} \in \mathbf{Q}$$

where ν_ξ denotes the order at $\mathcal{O}_{V^e, \xi}$.

PROPOSITION 4.17. *The function defined above* (*Definition* 4.16) *is independent of the choice of* V^e.

Proof. The statement is clear if $\mathrm{ord}_e(J,b)(\xi)=\infty$, so assume that $\mathrm{cod}_\xi(\mathrm{Sing}(J,b))>e$ and, after suitable open restriction, that furthermore $F^{(e)}=\varnothing$.

Since $\mathrm{ord}_e(J,b)$ is clearly a function as in 1.1, it suffices to assume that $\xi\in\mathrm{Sing}(J,b)$ is a closed point. Multiplying by $\mathbf{A}^1_k$: $W_0=W\times\mathbf{A}^1_k$, $V^e_0=V^e\times\mathbf{A}^1_k$, $J_0=J\mathcal{O}_{W_0}$, we get a restriction

$$(W,(J,b),E)\leftarrow(W_0,(J_0,b),E_0),$$

so $I(V^e_0)\subseteq\Delta^{b-1}(J_0)$ (4.9) and $C(J_0)=C(J)\mathcal{O}_{V^e_0}$.

Set $\xi_0=(\xi,0)$, $L_0=\{\xi\}\times\mathbf{A}^1_k$. If $b'=\nu_\xi(C(J))$, then $\nu_{\xi'}(C(J_0))=b'$ for any $\xi'\in L_0$ (4.9).

Consider the transformation with center ξ_0, say

$$(W_0,(J_0,b),E)\rightarrow(W_1,(J_1,b),E_1).$$

Let H_1 be the exceptional divisor, L_1 the strict transform of L_0, V^e_1 the strict transform of V^e_0 and $\xi_1=L_1\cap H_1$. Set $(V^e_1,(C(J_0)_1,b!),\overline{E}_1)$ as the transform of $(V^e_0,(C(J_0),b!),\varnothing)$ and $(V^e_1,(\mathcal{A}_1,b'),\overline{E}_1)$ as the transform of $(V^e_0,(C(J_0),b'),\varnothing)$. Note that the order of $\mathcal{A}_1$ at points in $L_1\setminus\{\xi_1\}$ is b', so $\nu_{\xi_1}(\mathcal{A}_1)=b'$ (see Corollary 4.7 (3)), and

$$C(J_0)_1=I(\overline{H}_1)^{b'-b!}\mathcal{A}_1$$

where $\overline{H}_1=H_1\cap V^e_1$. Note that $b'-b!$ is the highest power of $I(\overline{H}_1)$ that one can factor out.

Suppose that we have defined inductively $(W_k,(J_k,b),E_k)$, L_k, V^e_k, H_k, $\overline{H}_k=H_k\cap V^e_k$, $\mathcal{A}_k\subseteq\mathcal{O}_{V^e_k}$ and $\xi_k=L_k\cap H_k$ such that

$$C(J_0)_k=I(\overline{H}_k)^{k(b'-b!)}\mathcal{A}_k,\quad b'=\nu_{\xi_k}(\mathcal{A}_k).$$

Consider the transformation with center at ξ_k: $(W_k,(J_k,b),E_k)\leftarrow(W_{k+1},(J_{k+1},b),E_{k+1})$. Let H_{k+1} be the exceptional divisor, L_{k+1} be the strict transform of L_k, V^e_{k+1} be the strict transform of V^e_k and $\xi_{k+1}=L_{k+1}\cap H_{k+1}$ (closed point). Set $(V^e_{k+1},(C(J_0)_{k+1},b!),\overline{E}_{k+1})$ as the transform of $(V^e_k,(C(J_0)_k,b!),\overline{E}_k)$ and $(V^e_{k+1},(\mathcal{A}_{k+1},b'),\overline{E}_{k+1})$ as the transform of $(V^e_k,(\mathcal{A}_k,b'),\overline{E}_k)$. By Corollary 4.7 (3), $\nu_{\xi_1}(\mathcal{A}_{k+1})=b'$ and

$$C(J_0)_{k+1}=I(\overline{H}_{k+1})^{(k+1)(b'-b!)}\mathcal{A}_{k+1}.$$

In this way, for any natural number N we have defined a sequence of transformations

$$(W_0,(J_0,b),E_0)\leftarrow\ldots\leftarrow(W_N,(J_N,b),E_N)$$

and $\mathrm{cod}_{\xi_N}(\mathrm{Sing}(J_N,b))\geqslant e+1$; in particular, the local codimension in W_N is

$$\mathrm{cod}_{\xi_N}(\mathrm{Sing}(J_N,b)\cap H_N)\geqslant e+1.$$

Now it is clear by Proposition 4.15 that

$$\begin{aligned}\mathrm{cod}_{\xi_N}(\mathrm{Sing}(J_N,b)\cap H_N)=e+1 \quad&\Leftrightarrow\quad \mathrm{cod}_{\xi_N}(\mathrm{Sing}(C(J_0)_N,b!)\cap\overline{H}_N)=1\\ &\Leftrightarrow\quad N(b'-b!)\geqslant b!\end{aligned}\tag{4.17.1}$$

where the second codimension is considered in V^e_N. Moreover, in this case,

$$\mathrm{Sing}(J_N,b)\cap H_N=\overline{H}_N$$

is a permissible center.

Suppose $N(b'-b!)\geqslant b!$ and set $C_0=\mathrm{Sing}(J_N,b)\cap H_N$. Consider $(W_N,(J_N,b),E_N)\leftarrow(W'_1,(J'_1,b),E'_1)$ as the transformation with center C_0, and let H'_1 be the exceptional divisor. Note that this transformation in V^e_N is the identity map, $\overline{H}'_1=H'_1\cap V'^e_1=\overline{H}_N$,

$$C(J'_1)=I(\overline{H}'_1)^{N(b'-b!)-b!}\mathcal{A}'_1,$$

and it is therefore clear that

$$\mathrm{cod}_{\xi'_1}(\mathrm{Sing}(J'_1,b)\cap H'_1)=e+1 \quad\Leftrightarrow\quad N(b'-b!)-b!\geqslant b!\,.\tag{4.17.2}$$

We can iterate this process of transformations (isomorphisms in V^e_N) at centers of codimension $e+1$ exactly l_N times, for

$$l_N=\left[\frac{N(b'-b!)}{b!}\right]$$

where the brackets denote the integer part.

Equations (4.17.1) and (4.17.2) show that the number l_N depends on the codimension of the singular locus and not on the choice of V^e. Finally note that

$$\mathrm{ord}_e(J,b)(\xi)-1=\frac{b'}{b!}-1=\lim_{N\to\infty}\frac{l_N}{N}=\lim_{N\to\infty}\frac{[N(b'-b!)/b!]}{N}.\tag{4.17.3}$$

4.18. In the particular case of $e=0$ ($V^e=W$) the map defined in Definition 4.16 is

$$\mathrm{ord}_0(J,b)(\xi)=\frac{\nu_\xi(J)}{b}.$$

4.19. Let $(W,(J,b),E)$ be a basic object such that $\tau(\xi)\geqslant e$ for any $\xi\in\mathrm{Sing}(J,b)$. Let $(W,(J,b),E)\leftarrow(W_1,(J_1,b),E_1)$ be a transformation with center C, and let H_1 denote the exceptional divisor. Let $\xi \in \mathrm{Sing}(J,b)$, V^e and $C(J) \subseteq \mathcal{O}_{V^e}$ be as in Definition 4.14 (in a suitable neighborhood). Set V_1^e as the strict transform of V^e. Recall that the number $\mathrm{ord}_e(J,b)$ does not depend on the choice of V^e, and by Proposition 4.15 that

$$\mathrm{Sing}(J,b) = \mathrm{Sing}(C(J), b!)\ (\subseteq V^e), \quad \mathrm{Sing}(J_1,b) = \mathrm{Sing}(C(J)_1, b!)\ (\subseteq V_1^e).$$

Define $b'\colon W_1 \to \mathbf{Z}\geqslant 0$ as follows: if $\xi_1 \in W_1 \setminus H_1$, set $b'(\xi_1)=0$, and for $\xi_1 \in H_1$ with image say $\xi \in C$, set $b'(\xi_1) = \nu_{C'}(C(J))$, the order of the ideal $C(J)$ at the generic point of the unique irreducible component C' of C which contains ξ.

Note that

$$C(J)_1 = I(\overline{H}_1)^{b'-b!}\mathcal{A}_1$$

where $\overline{H}_1 = H_1 \cap V_1^e$ and $b'\ (\geqslant b!)$ is a locally constant function on $\overline{H}_1$. The exponent which appears in this expression above divided by $b!$ is independent of V^e, since

$$\frac{b'(\xi_1)-b!}{b!} = \mathrm{ord}_e(J,b)(C')-1.$$

On the other hand, if $\xi_1 \in \mathrm{Sing}(J_1,b)$ then

$$\mathrm{ord}_e(J_1,b)(\xi_1) = \frac{b'(\xi_1)-b!}{b!} + \frac{\nu_{\xi_1}(\mathcal{A}_1)}{b!}.$$

We now attach a new map $\text{w-ord}_e(J_1,b)$ to the transformation, setting $\text{w-ord}_e(J_1,b)(\xi_1) = \nu_{\xi_1}(\mathcal{A}_1)/b!$.

Definition 4.20. Let $(W_0,(J_0,b),E_0)$ be a basic object such that $\tau(J,b)(\xi) \geqslant e$, $\forall \xi \in \mathrm{Sing}(J,b)$. Given a sequence of transformations, with centers C_i, $i=0,\dots,k-1$,

$$(W_0,(J_0,b),E_0) \leftarrow (W_1,(J_1,b),E_1) \leftarrow \dots \leftarrow (W_k,(J_k,b),E_k),$$

we define a map $\text{w-ord}_e(J_k,b)\colon \mathrm{Sing}(J_k,b) \to \mathbf{Q}$ by induction on k.

If $k=0$, define $\text{w-ord}_e(J_0,b) = \mathrm{ord}_e(J_0,b)$ (Definition 4.16). Suppose for $k>0$ the existence of maps

$$\text{w-ord}_e(J_r,b)\colon \mathrm{Sing}(J_r,b) \to \mathbf{Q}, \quad r=0,\dots,k-1,$$

and assume, for any point $\xi \in \mathrm{Sing}(J_0,b)$, the choice of a regular variety V_0^e of codimension e, as in Corollary 4.11; assume also expressions of the form

$$C(J_0)_{k-1} = I(\overline{H}_1)^{\alpha_1} \dots I(\overline{H}_{k-1})^{\alpha_{k-1}}\mathcal{A}_{k-1}$$

where V_{k-1}^e is the strict transform of V_0^e and the exponents $\alpha_1,\dots,\alpha_{k-1}$ are locally constant functions on H_i independent of the choice of V^e.

For the index k we have the following expression in $\mathcal{O}_{V_k^e}$ (with V_k^e the strict transform of V_{k-1}^e):

$$C(J_0)_k = I(\overline{H}_1^{\alpha_1}) \dots I(\overline{H}_{k-1})^{\alpha_{k-1}} I(\overline{H}_k)^{\alpha_k} \mathcal{A}_k$$

where α_k is a locally constant function on $\overline{H}_k$; for $\xi \in \overline{H}_k$ set

$$\beta_k(\xi) = \frac{\alpha_k(\xi)}{b!} = \operatorname{ord}_e(J_{k-1}, b)(C') - 1 \tag{4.20.1}$$

where C' is the unique component of C_{k-1} which contains the image of ξ in W_{k-1}.

Now we define for each $\overline{H}_i$ a function $\beta_i \colon W_k \to \mathbf{Q}$, $\beta_i(\xi) = \alpha_i(\xi)/b!$, so for any $\xi \in \operatorname{Sing}(J_k, b)$,

$$\begin{aligned} \text{w-ord}_e(J_k, b)(\xi) = \frac{\nu_\xi(\mathcal{A}_k)}{b!} &= \operatorname{ord}_e(J_k, b)(\xi) - \sum_{\xi \in \overline{H}_i} \frac{\alpha_i(\xi)}{b!} \\ &= \operatorname{ord}_e(J_k, b)(\xi) - \sum_{\xi \in \overline{H}_i} \beta_i(\xi). \end{aligned} \tag{4.20.2}$$

Note that the definition of the map $\text{w-ord}_e(J_k, b)$ depends only on the map $\operatorname{ord}_e(J_k, b)$ and the given sequence of transformations.

Remark 4.21. The maps ord_e and w-ord_e are functions as in 1.1, and if

$$(W, (J, b), E) \leftarrow (W', (J', b), E')$$

is a restriction (1.8) or arises from an arbitrary change of the base field k in Definition 1.2 (1), and $\xi' \in \operatorname{Sing}(J', b)$ maps to $\xi \in \operatorname{Sing}(J, b)$, then $\operatorname{ord}_e(J, b)(\xi) = \operatorname{ord}(J', b)(\xi')$ and the same holds for w-ord_e.

PROPOSITION 4.22. *Let $(W, (J, b), E)$ be a basic object such that $\tau(J, b)(\xi) \geqslant e$, $\forall \xi \in \operatorname{Sing}(J, b)$. Consider the sequence of transformations*

$$(W, (J, b), E) = (W_0, (J_0, b), E_0) \leftarrow (W_1, (J_1, b), E_1) \leftarrow \dots \leftarrow (W_k, (J_k, b), E_k)$$

and the function $\text{w-ord}_e(J_k, b) \colon \operatorname{Sing}(J_k, b) \to \mathbf{Q}$. A smooth closed subscheme C_k of $\operatorname{Sing}(J_k, b)$ having normal crossings with E_k defines an enlargement of the sequence of transformations

$$\begin{aligned} &(W_0, (J_0, b), E_0) \leftarrow (W_1, (J_1, b), E_1) \leftarrow \dots \\ &\qquad \dots \leftarrow (W_k, (J_k, b), E_k) \xleftarrow{\varphi_{k+1}} (W_{k+1}, (J_{k+1}, b), E_{k+1}). \end{aligned}$$

If $\text{w-ord}_e(J_k, b)$ is locally constant along C_k (constant on each irreducible component) then

$$\text{w-ord}_e(J_k, b)(\varphi(\xi)) \geqslant \text{w-ord}_e(J_{k+1}, b)(\xi) \quad \forall \xi \in \operatorname{Sing}(J_{k+1}, b).$$

Proof. To study the inequality we may assume that $\varphi(\xi)\in C_k$, and after suitable open restriction, that C_k is irreducible with generic point y. We assume here that $\text{w-ord}_e(J_k,b)(\varphi(\xi))=\text{w-ord}_e(J_k,b)(y)=b''/b!$. It can be checked that:

(1) $\nu_{\varphi(\xi)}(\mathcal{A}_k)=\nu_y(\mathcal{A}_k)=b''$.

(2) The basic object $(V^e_{k+1},(\mathcal{A}_{k+1},b''),\overline{E}_{k+1})$ is the transform of $(V^e_k,(\mathcal{A}_k,b''),\overline{E}_k)$ at the permissible center C_k.

Finally the inequality follows from Corollary 4.7 (3).

5. Idealistic closed sets

Definition 5.1. Let W be smooth of pure dimension n over a field k of characteristic zero, and $E=\{H_1,...,H_r\}$ a finite set of smooth hypersurfaces of W having only normal crossings. A *weak idealistic closed set* is

$$(W,F,E,\{U^{(i)}\xrightarrow{\alpha_i}W\}_{i\in I},\{(J^{(i)},b_i)\}_{i\in I})$$

where

(1) I is a finite set,

(2) F is a closed subset of W,

(3) for each $i\in I$, $\alpha_i\colon U^{(i)}\to W$ is smooth and $W=\bigcup_{i\in I}\operatorname{Im}\alpha_i$,

(4) if $E^{(i)}=\{\alpha_i^{-1}(H)\mid H\in E,\ \alpha_i^{-1}(H)\neq\varnothing\}$ then for each $i\in I$, $(U^{(i)},(J^{(i)},b_i),E^{(i)})$ is a basic object and

$$\alpha_i^{-1}(F)=\operatorname{Sing}(J^{(i)},b_i). \tag{5.1.1}$$

Weak idealistic closed sets will be denoted by gothic letters,

$$\mathfrak{F}=(W,F,E,\{U^{(i)}\xrightarrow{\alpha_i}W\}_{i\in I},\{(J^{(i)},b_i)\}_{i\in I}),$$

and F will be called the *singular locus* of $\mathfrak{F}$,

$$\operatorname{Sing}(\mathfrak{F})=F.$$

We say that $\mathfrak{F}$ is an *n-dimensional weak idealistic closed* set if in addition $\dim U^{(i)}=\dim W=n$ for any i, in which case the α_i are open immersions or étale morphisms.

5.2. If $\alpha\colon W'\to W$ is either a restriction (1.8) or a change of the base field k, then $\mathfrak{F}$ induces by fiber products a weak idealistic closed set $\mathfrak{F}'$ on W' (see 1.9) denoted

$$\mathfrak{F}\xleftarrow{\alpha}\mathfrak{F}',$$

which we shall call a *restriction* of $\mathfrak{F}$ when α is smooth.

5.3. Let C be a closed smooth subscheme of W having normal crossings with $E=\{H_1, ..., H_r\}$. Set $\varphi: W_1 \to W$ as the monoidal transformation with center C. We can argue as in 1.9 to show that, by taking fiber products, there is a finite covering $\{\beta_i: U_1^{(i)} \to W_1\}_{i\in I}$ of W_1 by smooth morphisms, and basic objects $(U_1^{(i)}, (J_1^{(i)}, b_i), E_1^{(i)})$ (transforms of original basic objects) for each index of the same finite set I.

However it is not clear, and in general not true, that there exists a closed set $F_1 \subseteq W_1$ such that

$$\beta_i^{-1}(F_1) = \mathrm{Sing}(J_1^{(i)}, b_i).$$

Definition 5.4. A closed smooth center $C \subseteq F = \mathrm{Sing}(\mathfrak{F})$ having normal crossings with E is said to be *permissible* if the latter condition holds. In such case the morphism $\mathfrak{F} \leftarrow \mathfrak{F}_1$ is called a *transformation* of weak idealistic sets, where

$$\mathfrak{F}_1 = (W_1, F_1, E_1, \{U_1^{(i)} \xrightarrow{\beta_i} W_1\}_{i\in I}, \{(J_1^{(i)}, b_i)\}_{i\in I}).$$

Definition 5.5. An n-dimensional weak idealistic closed set $\mathfrak{F}$ is said to be an n-dimensional *idealistic closed set* if, in addition, for any sequence

$$\mathfrak{F} = \mathfrak{F}_0 \xleftarrow{\varphi_1} \mathfrak{F}_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_k} \mathfrak{F}_k,$$

with each φ_i either a restriction or a transformation at a permissible center, then any closed and smooth subscheme of $\mathrm{Sing}(\mathfrak{F}_k)$ having normal crossings with E_k is permissible.

5.6. Any n-dimensional basic object $(W, (J, b), E)$ defines naturally an n-dimensional idealistic closed set.

A finite open covering $\{U^{(i)}\}$ of W (or an étale covering) defines by restrictions of $(W, (J, b), E)$ (1.8) also a structure of idealistic closed set where, naturally, $F = \mathrm{Sing}(J, b)$.

Other examples will show up in Theorem 6.6.

Definition 5.7. Let $\mathfrak{F}$ be an n-dimensional idealistic closed set. A *resolution* of $\mathfrak{F}$ is a sequence of transformations

$$\mathfrak{F} = \mathfrak{F}_0 \xleftarrow{\varphi_1} \mathfrak{F}_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_k} \mathfrak{F}_k$$

such that $\mathrm{Sing}(\mathfrak{F}_k)$ is empty.

5.8. Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two idealistic closed sets on the same regular scheme W. $\mathfrak{F}$ is defined in terms of E (hypersurfaces with normal crossings) and $\mathfrak{F}'$ in terms of E'. Assume that $\mathrm{Sing}(\mathfrak{F}') \subseteq \mathrm{Sing}(\mathfrak{F})$ and suppose that if C is any smooth closed subscheme of $\mathrm{Sing}(\mathfrak{F}')$ having normal crossings with E' then C has normal crossings with E (for instance, if $E = \varnothing$ or $E = E'$).

It is clear that any transformation

$$\mathfrak{F}' \leftarrow \mathfrak{F}'_1$$

induces a transformation

$$\mathfrak{F} \leftarrow \mathfrak{F}_1,$$

and the same holds for restrictions or change of base field (1.9).

Definition 5.9. Given $\mathfrak{F}$ and $\mathfrak{F}'$ as above, we shall say that $\mathfrak{F}' \subseteq \mathfrak{F}$ if the following properties hold:

(1) $\mathrm{Sing}(\mathfrak{F}') \subseteq \mathrm{Sing}(\mathfrak{F})$.

(2) Any sequence of transformations and restrictions over $\mathfrak{F}'$, say

$$\mathfrak{F}' = \mathfrak{F}'_0 \leftarrow \dots \leftarrow \mathfrak{F}'_k,$$

induces a sequence over $\mathfrak{F}$, say

$$\mathfrak{F} = \mathfrak{F}_0 \leftarrow \dots \leftarrow \mathfrak{F}_k,$$

so that $\mathrm{Sing}(\mathfrak{F}'_i) \subseteq \mathrm{Sing}(\mathfrak{F}_i)$, $i=0, \dots, k$, and furthermore, if $C \subseteq \mathrm{Sing}(\mathfrak{F}'_k)$ is closed and regular and has normal crossings with E'_k then C has also normal crossings with E_k.

Example 5.10. Let $(W, (J, b), E)$ be a basic object, and $V^e \subseteq W$ a closed and smooth subscheme of codimension e. Assume $I(V^e) \subseteq \Delta^{b-1}(J)$. Setting $\mathfrak{F}'$ and $\mathfrak{F}$ as the idealistic closed sets defined by the basic objects $(W, (J, b), E)$ and $(W, (I(V^e), 1), E)$, then $\mathfrak{F}' \subseteq \mathfrak{F}$.

Definition 5.11. Set $\mathfrak{F}$, $\mathfrak{F}'$, W, E and E' as in 5.8. We say that $\mathfrak{F}$ is equivalent to $\mathfrak{F}'$ ($\mathfrak{F} \sim \mathfrak{F}'$) if $\mathfrak{F} \subseteq \mathfrak{F}'$, $\mathfrak{F}' \subseteq \mathfrak{F}$ and $E=E'$.

Example 5.12. (1) The basic objects $(W, (J, b), E)$ and $(W, (J^2, 2b), E)$ define equivalent idealistic closed sets.

(2) Fix $\mathfrak{F}$ an idealistic closed set and notation as in Definition 5.1. If for each $i \in I$ there is a finite set I_i and smooth morphisms

$$\beta_{i,j}: U^{(i,j)} \to U^{(i)}, \quad j \in I_i,$$

such that $U^{(i)} = \bigcup_{j \in I_i} \mathrm{Im}\, \beta_{i,j}$, then

$$\mathfrak{F}' = (W, F, E, \{U^{(i,j)} \xrightarrow{\alpha_i \circ \beta_{i,j}} W\}_{i \in I, j \in I_i}, \{(J^{(i,j)}, b_i)\})$$

is an idealistic closed set equivalent to $\mathfrak{F}$.

Remark 5.13. (1) Set W, $E=E'$ and $\mathfrak{F}\sim\mathfrak{F}'$ as in Definition 5.11. Then $\mathrm{Sing}(\mathfrak{F})=\mathrm{Sing}(\mathfrak{F}')$ in W and if

$$\mathfrak{F}=\mathfrak{F}_0\leftarrow\ldots\leftarrow\mathfrak{F}_k$$

is a resolution of $\mathfrak{F}$ (Definition 5.7), it induces naturally a resolution

$$\mathfrak{F}'=\mathfrak{F}'_0\leftarrow\ldots\leftarrow\mathfrak{F}'_k$$

and equalities

$$\mathrm{Sing}(\mathfrak{F}_i)=\mathrm{Sing}(\mathfrak{F}'_i),\quad i=0,1,\ldots,k. \tag{5.13.1}$$

(2) Let $\mathfrak{F}\xleftarrow{\alpha}\mathfrak{F}'$ be a restriction of $\mathfrak{F}$ (resp. a change of base field). Then a resolution of $\mathfrak{F}$,

$$\mathfrak{F}=\mathfrak{F}_0\leftarrow\ldots\leftarrow\mathfrak{F}_k,$$

induces (by pullback) a resolution of $\mathfrak{F}'$,

$$\mathfrak{F}'=\mathfrak{F}'_0\leftarrow\ldots\leftarrow\mathfrak{F}'_k,$$

and restrictions (resp. changes of base field) $\alpha_i:\mathfrak{F}'_i\to\mathfrak{F}_i$ inducing morphisms

$$\alpha_i:\mathrm{Sing}(\mathfrak{F}'_i)\to\mathrm{Sing}(\mathfrak{F}_i). \tag{5.13.2}$$

Definition 5.14. An n-dimensional idealistic closed set $\mathfrak{F}$ (Definition 5.5) is said to be of *codimension* $\geqslant e$ if for each $i\in I$ there is a closed smooth subscheme $V^{e,i}\subseteq U^{(i)}$ of pure codimension e, such that:

(1) $I(V^{e,i})\subseteq\Delta^{b_i-1}(J^{(i)})$ (in particular, $\mathrm{Sing}(J^{(i)},b_i)\subseteq V^{e,i}$).

(2) At any $\xi\in\mathrm{Sing}(J^{(i)},b_i)$ there is a regular system of parameters $x_1,\ldots,x_d$ of $\mathcal{O}_{U^{(i)},\xi}$ with the following conditions:

(a) $I(V^{e,i})_\xi=\langle x_1,\ldots,x_e\rangle$.

(b) If $E^{(i)}_\xi=\{H\in E^{(i)}\mid\xi\in H\}$ then for any $H\in E^{(i)}_\xi$ there exists an index $i_H>e$ so that $I(H)_\xi=\langle x_{i_H}\rangle$.

Also if $\mathfrak{F}$ is a weak idealistic closed set (Definition 5.1), we say that $\mathfrak{F}$ has codimension $\geqslant e$ if these conditions hold for $\mathfrak{F}$.

Note that to any basic object $(U^{(i)},(J^{(i)},b_i),E^{(i)})$ we associate a basic object $(V^{e,i},(C(J^{(i)}),b_i!),\overline{E}^{(i)})$ such that $\mathrm{Sing}(J^{(i)},b_i)=\mathrm{Sing}(C(J^{(i)}),b_i!)$, and this equality holds after any sequence of transformations (see Proposition 4.15).

Example 5.15. (1) Let $\mathfrak{F}$ be an idealistic closed set. Then $\mathfrak{F}$ is of codimension $\geqslant 0$ ($V^{e,i}=U^{(i)}$). In fact, for $e=0$ all conditions in Definition 5.14 are vacuous.

(2) Let $(W,(J,b),E)$ be a basic object where $E=\varnothing$, and let $e>0$. If $\tau(J^{(i)},b_i)(\xi)\geqslant e$, $\forall \xi\in \operatorname{Sing}(J^{(i)},b_i)$, then Corollary 4.11 asserts that there is a finite set I and an open covering $\{U^{(i)}\}_{i\in I}$ so that the restrictions of $(W,(J,b),E)$ to the different $U^{(i)}$ define a structure of n-dimensional idealistic closed set of codimension $\geqslant e$.

5.16. Let $\mathfrak{F}$ be an n-dimensional idealistic closed set of codimension $\geqslant e$. Condition (1) of Definition 5.14 implies that $\tau(J^{(i)},b_i)(\xi)\geqslant e$ for each $\xi\in\operatorname{Sing}(J^{(i)},b_i)$, $i\in I$. In particular, functions

$$\operatorname{ord}_e(J^{(i)},b_i)\colon \operatorname{Sing}(J^{(i)},b_i)\to \mathbf{Q}\cup\{\infty\}$$

are defined so that whenever $\xi_i\in\operatorname{Sing}(J^{(i)},b_i)$ and $\xi_j\in\operatorname{Sing}(J^{(j)},b_j)$, $i,j\in I$, map to the same point $\xi\in\operatorname{Sing}(\mathfrak{F})$ then $\operatorname{ord}_e(J^{(i)},b_i)(\xi_i)=\operatorname{ord}_e(J^{(j)},b_j)(\xi_j)$ (see Definition 4.16). We say that the functions $\operatorname{ord}_e(J^{(i)},b_i)$ patch so as to define a function $\operatorname{Sing}(\mathfrak{F})\to\mathbf{Q}\cup\{\infty\}$, namely:

LEMMA 5.17. *With the setting as above, where $\mathfrak{F}$ is an n-dimensional idealistic closed set of codimension $\geqslant e$, the different functions*

$$\operatorname{ord}_e(J^{(i),b})\colon \operatorname{Sing}(J^{(i)},b_i)\to \mathbf{Q}\cup\{\infty\},\quad i\in I,$$

patch and define a function (1.1)

$$\operatorname{ord}_e(\mathfrak{F})\colon \operatorname{Sing}(\mathfrak{F})\to \mathbf{Q}\cup\{\infty\}.$$

Proof. This follows from formula (4.17.3), where the value of the function is expressed in terms of the codimension of the singular locus.

5.18. One can check that if $\mathfrak{F}$ is an n-dimensional idealistic closed set and $\mathfrak{F}\leftarrow\mathfrak{F}_1$ is a transformation, then naturally $\mathfrak{F}_1$ is an n-dimensional idealistic closed set. Furthermore, if $\mathfrak{F}$ has codimension $\geqslant e$ then $\mathfrak{F}_1$ has also codimension $\geqslant e$. The same holds for restrictions or change of base fields (5.2).

A sequence of transformations

$$\mathfrak{F}=\mathfrak{F}_0\leftarrow\mathfrak{F}_1\leftarrow\ldots\leftarrow\mathfrak{F}_k$$

induces for each $i\in I$ (Definition 5.1) a sequence of transformations of basic objects,

$$\begin{aligned}(U^{(i)},(J^{(i)},b_i),E^{(i)})=(U_0^{(i)},(J_0^{(i)},b_i),E_0^{(i)})\leftarrow(U_1^{(i)},(J_1^{(i)},b_i),E_1^{(i)})\leftarrow\ldots\\ \ldots\leftarrow(U_k^{(i)},(J_k^{(i)},b_i),E_k^{(i)}),\end{aligned}$$

and we have attached functions in Definition 4.20 to such a sequence, say

$$\text{w-ord}_e(J_k^{(i)}, b_i) \colon \operatorname{Sing}(J_k^{(i)}, b_i) \to \frac{1}{b_i!}\mathbf{Z} \cup \{\infty\} \subseteq \mathbf{Q} \cup \{\infty\}, \tag{5.18.1}$$

and for each hypersurface $H_j \in E_k^{(i)}$ a function

$$\beta_{k,j}^{(i)} \colon U_k^{(i)} \to \mathbf{Q}, \quad \beta_{k,j}^{(i)}(\xi) = \frac{\alpha_{k,j}^{(i)}(\xi)}{b_i!}$$

(see (4.20.1) and (4.20.2)).

COROLLARY 5.19. *Let $\mathfrak{F}$ be an n-dimensional idealistic closed set of codimension $\geqslant e$, fix a sequence of transformations*

$$\mathfrak{F} = \mathfrak{F}_0 \leftarrow \mathfrak{F}_1 \leftarrow \ldots \leftarrow \mathfrak{F}_k$$

and set for $s=0, \ldots, k$, $\mathfrak{F}_s = (W_s, F_s, E_s, \{U_s^{(i)} \to W_s\}_{i \in I}, \{(J_s^{(i)}, b_i)\}_{i \in I})$ as in Definition 5.1. The functions $\text{w-ord}_e(J_k^{(i)}, b_i)$ (5.18) patch and define a function

$$\text{w-ord}_e(\mathfrak{F}_k) \colon \operatorname{Sing}(\mathfrak{F}_k) \to \mathbf{Q} \cup \{\infty\},$$

and $\text{w-ord}_e(\mathfrak{F}_0) = \operatorname{ord}_e(\mathfrak{F}_0)$. For each $H_j \in E_k$, the functions $\beta_{k,j}^{(i)}$ (5.18) patch to define functions

$$\beta_j(\mathfrak{F}_k) \colon W_k \to \mathbf{Q}$$

which are locally constant along $H_j \in E_k$ (see Lemma 5.17 and formulas (4.20.1) and (4.20.2).

COROLLARY 5.20. *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be equivalent idealistic closed sets (Definition 5.11) of codimension $\geqslant e$. Then*

$$\operatorname{ord}_e(\mathfrak{F}) = \operatorname{ord}_e(\mathfrak{F}'),$$

and given a sequence of transformations

$$\mathfrak{F} = \mathfrak{F}_0 \leftarrow \ldots \leftarrow \mathfrak{F}_k,$$
$$\mathfrak{F}' = \mathfrak{F}'_0 \leftarrow \ldots \leftarrow \mathfrak{F}'_k$$

as in Definition 5.11, we have

$$\text{w-ord}_e(\mathfrak{F}_k) = \text{w-ord}_e(\mathfrak{F}'_k)$$

and for each $H_j \in E_k$,

$$\beta_j(\mathfrak{F}_k) = \beta_j(\mathfrak{F}'_k).$$

This follows directly from (4.17.3), (4.20.1) and (4.20.2).

Remark 5.21. It is clear from Corollary 4.7 and Corollary 4.11 that if $\mathfrak{F}$ is an idealistic closed set of codimension $\geqslant e$, then $\operatorname{cod}_\xi F \geqslant e$ (codimension of F in W locally at ξ) at any point $\xi \in F$.

THEOREM 5.22. *Let* $\mathfrak{F}$ *be an idealistic closed set of codimension* $\geqslant e$. *Set* $F^e=\{\xi\in\operatorname{Sing}(\mathfrak{F})\,|\,\operatorname{cod}_\xi \operatorname{Sing}(\mathfrak{F})=e\}$.

(1) $\xi\in F^e$ *if and only if* $\operatorname{ord}_e(\mathfrak{F})(\xi)=\infty$.

(2) F^e *is an open and closed set in* $\operatorname{Sing}(\mathfrak{F})$ *(it is a union of connected components).*

(3) F^e *is smooth of pure codimension* e *and has normal crossings with* E.

(4) *Let* $\mathfrak{F}\leftarrow\mathfrak{F}_1$ *be a transformation and set* $F_1^e=\{\xi\in\operatorname{Sing}(\mathfrak{F}_1)\,|\,\operatorname{cod}_\xi \operatorname{Sing}(\mathfrak{F}_1)=e\}$. *Then* F_1^e *is the strict transform of* F^e. *In particular, if the center of the transformation is* $C=F^e$ *then* F_1^e *is empty and* $\operatorname{Sing}(\mathfrak{F}_1)$ *can be identified with* $\operatorname{Sing}(\mathfrak{F})\setminus F^e$.

Proof. (1) follows from Definition 4.16; (2) from Proposition 4.10; (3) from Definition 5.14 (2); and finally, (4) follows from Corollary 4.8.

6. Algorithms of resolution

The proof of desingularization of an embedded variety $X\subset W$ is closely related to that of principalization of ideals: given a sheaf of ideals in a regular variety, say $I\subset\mathcal{O}_W$, define a morphism of regular varieties $W\leftarrow W'$ so that $I'=I\mathcal{O}_{W'}$ is locally principal and $V(I')$ (algebraic subset defined by I') is a union of hypersurfaces having only normal crossings.

In fact, both results undergo the same general scheme of proof. The following development will state both problems in a unified frame.

Definition 6.1. Let an *algebraic class* $\mathcal{G}$ be a class of objects $\operatorname{Ob}(\mathcal{G})$ and arrows

$$\mathfrak{F}\leftarrow\mathfrak{F}_1,\quad \mathfrak{F},\mathfrak{F}_1\in\operatorname{Ob}(\mathcal{G}),$$

called *transformations*, subject to the following conditions:

(1) To each $\mathfrak{F}\in\operatorname{Ob}(\mathcal{G})$ there is an assigned scheme $\operatorname{Sing}(\mathfrak{F})$.

(2) A transformation $\mathfrak{F}\xleftarrow{\varphi}\mathfrak{F}_1$ can be identified with a closed and regular subscheme C_φ of $\operatorname{Sing}(\mathfrak{F})$, called the *center* of φ, and each such transformation induces a morphism of schemes

$$\bar{\varphi}\colon \operatorname{Sing}(\mathfrak{F}_1)\to\operatorname{Sing}(\mathfrak{F})$$

which in terms defines an isomorphism

$$\operatorname{Sing}(\mathfrak{F}_1)\setminus\bar{\varphi}^{-1}(C_\varphi)\cong\operatorname{Sing}(\mathfrak{F})\setminus C_\varphi.$$

Example 6.2. (1) Define $\mathcal{S}$ by setting $X\in\operatorname{Ob}(\mathcal{S})$ if and only if X is a scheme which is separated and locally finite over some field of characteristic zero. Define $\operatorname{Sing}(X)$ as the singular locus with reduced structure, and a transformation

$$X\leftarrow X_1$$

to be a monoidal transformation on a closed regular subscheme $C_\varphi \subseteq \mathrm{Sing}(X)$.

(2) Let $\mathcal{C}$ be the class of idealistic closed sets (Definition 5.5), with $\mathrm{Sing}(\mathfrak{F})$ as in Definition 5.1 and transformations as in Definition 5.4.

(3) We can define $\mathcal{D}$ by setting $\mathfrak{F} \in \mathrm{Ob}(\mathcal{D})$ where $\mathfrak{F}=(W,I)$ consists of a non-zero sheaf of ideals $I \subset \mathcal{O}_W$, W a regular variety, and

$$\mathrm{Sing}(\mathfrak{F}) = \{\xi \in W \mid I_\xi \in \mathcal{O}_{W,\xi} \text{ is not (locally) principal}\},$$

and if C is closed and regular in $\mathrm{Sing}(\mathfrak{F}) \subset W$, it defines the monoidal transformation $W \leftarrow W'$. Then set $\mathfrak{F} \leftarrow \mathfrak{F}'$ where $\mathfrak{F}'=(W',I')$, $I'=I\mathcal{O}_{W'}$.

Definition 6.3. Let $\mathcal{G}$ be an algebraic class and $(I,\leqslant)$ be a totally ordered set. An *assignment of chains and functions* from $\mathcal{G}$ to $(I,\leqslant)$ will be a set

$$\mathrm{CF}(\mathcal{G},I)$$

where an element consists of data, say

$$\mathcal{L}_k = \begin{cases} \mathfrak{F}_0 \xleftarrow{\varphi_1} \dots \xleftarrow{\varphi_k} \mathfrak{F}_k, \quad \mathfrak{F}_i \in \mathrm{Ob}(\mathcal{G}), \\ g_i = g_i(\mathfrak{F}_i) \colon \mathrm{Sing}(\mathfrak{F}_i) \to I, \quad i=0,\dots,k, \end{cases}$$

each g_i being a function (1.1) and each φ_i a transformation in the algebraic class $\mathcal{G}$.

A sequence of transformations in $\mathcal{G}$ together with functions as in (6.3.1) will be called a *chain of length* k. We require that the set $\mathrm{CF}(\mathcal{G},I)$ satisfy the following properties:

(A_0) For each $\mathfrak{F} \in \mathrm{Ob}(\mathcal{G})$ there is a unique chain of length zero with $\mathfrak{F}_0=\mathfrak{F}$, say

$$\mathcal{L}_0 = \begin{cases} \mathfrak{F}_0, \\ g_0 \colon \mathrm{Sing}(\mathfrak{F}_0) \to I. \end{cases}$$

(B_k) For each chain $\mathcal{L}_k$ of length k ((6.3.1)) there is a set $C(\mathcal{L}_k)$ and each element of this set is a closed and regular subscheme in $\underline{\mathrm{Max}}\, g_k$. This set $C(\mathcal{L}_k)$ will be called the *criterion of choice* of the centers for the given chain.

(C_k) For each chain $\mathcal{L}_k$ of length k ((6.3.1)) and for each $C \in C(\mathcal{L}_k)$ there are a transformation $\mathfrak{F}_k \leftarrow \mathfrak{F}_{k+1}$ and an enlargement of $\mathcal{L}_k$ to

$$\mathcal{L}_{k+1} = \begin{cases} \mathfrak{F}_0 \xleftarrow{\varphi_1} \dots \xleftarrow{\varphi_k} \mathfrak{F}_k \xleftarrow{\varphi_{k+1}} \mathfrak{F}_{k+1}, \quad \mathfrak{F}_i \in \mathrm{Ob}(\mathcal{G}), \\ g_i \colon \mathrm{Sing}(\mathfrak{F}_i) \to I, \quad i=0,\dots,k,k+1, \end{cases} \tag{6.3.2}$$

with $g_0,\dots,g_k$ as in (6.3.1), and

$$g_k(\bar{\varphi}_{k+1}(\xi)) \geqslant g_{k+1}(\xi) \quad \forall \xi \in \mathrm{Sing}(\mathfrak{F}_{k+1}) \tag{6.3.3}$$

with equality if $\bar{\varphi}_{k+1}(\xi) \notin C_{\varphi_{k+1}}$, $\bar{\varphi}_{k+1} \colon \mathrm{Sing}(\mathfrak{F}_{k+1}) \to \mathrm{Sing}(\mathfrak{F}_k)$ as in Definition 6.1.

(D) Any chain of length $k+1$ arises from one of length k as in (C_k).

Remark 6.4. From the definition above one deduces easily that for any chain of length k as in (6.3.1):

$$g_j(\bar{\varphi}_{j+1}(\xi)) \geqslant g_{j+1}(\xi) \quad \forall \xi \in \mathrm{Sing}(\mathfrak{F}_{j+1}),\ j=0,1,\dots,k-1,$$

and, in particular,

$$\max g_0 \geqslant \max g_1 \geqslant \dots \geqslant \max g_k.$$

Example 6.5. Set $\mathcal{S}$ as in Example 6.2 (1), and $I=\mathbf{N}^{\mathbf{N}}$ ordered lexicographically. One can adapt the Hilbert–Samuel function so as to define, for each $X \in \mathrm{Ob}(\mathcal{S})$,

$$H(X)\colon \mathrm{Sing}(X) \to (I, \leqslant)$$

as an upper-semi-continuous function (1.1). So (A_0) holds.

Condition (D) says that chains are to be constructed by induction on the length. Set the criterion of choice (B_k) as $C \in C(\mathcal{L}_k)$ if and only if C is closed, regular and contained in $\underline{\mathrm{Max}}\, g_k$, and set $g_k = H(X_k)$.

Fix a chain $\mathcal{L}_k$. For any j, $\underline{\mathrm{Max}}\, g_j = \underline{\mathrm{Max}}\, H(X_j)$ is known as the Hilbert–Samuel stratum of the scheme X_j. Now, $C \in C(\mathcal{L}_j)$ if and only if C is permissible in the sense of Hironaka ([Hi2, p. 71]), and the inequalities (6.3.3) are known as Bennett's theorem [Ben]. So this defines an assignment of chains and functions from $\mathcal{S}$ to $\mathbf{N}^{\mathbf{N}}$, say $\mathcal{H}(\mathcal{S}, \mathbf{N}^{\mathbf{N}})$.

THEOREM 6.6 (Aroca [Hi2, Theorem 1, p. 100], [G2, Theorem 3.12]; see also [G3, p. 233] for characteristic p). *Let $\mathcal{S}$ be the algebraic class and $\mathcal{H}(\mathcal{S}, \mathbf{N}^{\mathbf{N}})$ the assignment of chains and functions defined above* (*Example* 6.5). *Fix $X \in \mathrm{Ob}(\mathcal{S})$ and assume that X is a closed subscheme of W, where W is smooth and of pure dimension n over a field of characteristic zero. There exists an n-dimensional idealistic closed set $\mathfrak{F}$ such that:*

If $\mathcal{L}_k \in \mathcal{H}(\mathcal{S}, \mathbf{N}^{\mathbf{N}})$ is a chain over X, say

$$\mathcal{L}_k = \begin{cases} X_0 \leftarrow \dots \leftarrow X_k, & X_i \in \mathrm{Ob}(\mathcal{S}),\ X_0 = X, \\ H(X_i)\colon \mathrm{Sing}(X_i) \to \mathbf{N}^{\mathbf{N}}, & i=0,\dots,k, \end{cases}$$

with $\max H(X_0) = \dots = \max H(X_k)$, *then there is a sequence of transformations* (*Definition* 5.4)

$$\mathfrak{F} = \mathfrak{F}_0 \leftarrow \dots \leftarrow \mathfrak{F}_k$$

such that $\mathrm{Sing}(\mathfrak{F}_i) = \underline{\mathrm{Max}}\, H(X_i)$, $i=0,\dots,k$. *And if we consider an enlargement of the chain $\mathcal{L}_k$:*

$$\mathcal{L}_{k+1} = \begin{cases} X_0 \leftarrow \dots \leftarrow X_k \leftarrow X_{k+1}, & X_i \in \mathrm{Ob}(\mathcal{S}), \\ H(X_i)\colon \mathrm{Sing}(X_i) \to \mathbf{N}^{\mathbf{N}}, & i=0,\dots,k,k+1, \end{cases}$$

and the corresponding sequence of transformations

$$\mathfrak{F}=\mathfrak{F}_0 \leftarrow \dots \leftarrow \mathfrak{F}_k \leftarrow \mathfrak{F}_{k+1},$$

then either

$$\max H(X_k) > \max H(X_{k+1}) \quad \textit{and} \quad \mathrm{Sing}(\mathfrak{F}_{k+1}) = \varnothing$$

or

$$\max H(X_k) = \max H(X_{k+1}) \quad \textit{and} \quad \mathrm{Sing}(\mathfrak{F}_{k+1}) = \underline{\mathrm{Max}}\, H(X_{k+1}).$$

Definition 6.7. Fix an algebraic class $\mathcal{G}$ and a totally ordered set $(I, \leqslant)$. An *algorithm of resolution* of $\mathcal{G}$ with values at I will be an assignment of chains and functions $\mathrm{CF}(\mathcal{G}, I)$ (Definition 6.3) with the following properties:

(1) For any chain of length k ((6.3.1)), say

$$\mathcal{L}_k = \begin{cases} \mathfrak{F}_0 \xleftarrow{\varphi_1} \dots \xleftarrow{\varphi_k} \mathfrak{F}_k, & \mathfrak{F}_i \in \mathrm{Ob}(\mathcal{G}), \\ f_i \colon \mathrm{Sing}(\mathfrak{F}_i) \to I, & i = 0, \dots, k, \end{cases}$$

we have $C(\mathcal{L}_k)=\{\underline{\mathrm{Max}}\, f_k\}$ (B_k in Definition 6.3), and in (6.3.2) either $\mathrm{Sing}(\mathfrak{F}_k)=\varnothing$ or $\max f_k > \max f_{k+1}$. In particular, this means that $\underline{\mathrm{Max}}\, f_k$ is regular and the criterion of choices (B_k) reduces to $\underline{\mathrm{Max}}\, f_k$.

This already says that, fixing $\mathfrak{F} \in \mathrm{Ob}(\mathcal{G})$, if there is a chain of length k so that $\mathfrak{F}=\mathfrak{F}_0$ then the chain is unique and $\max f_0 > \dots > \max f_k$.

(2) For each $\mathfrak{F} \in \mathrm{Ob}(\mathcal{G})$ there is an index k and a chain of length k such that $\mathfrak{F}=\mathfrak{F}_0$ and $\mathrm{Sing}(\mathfrak{F}_k)$ is empty.

Note that the chain in (2) is a resolution of $\mathfrak{F}$ (Definition 5.7) which is uniquely determined.

6.8. Let $\mathcal{C}$ be as in Example 6.2 (2), and let $\mathcal{C}(n)$ consist of those $\mathfrak{F} \in \mathrm{Ob}(\mathcal{C})$ which are n-dimensional idealistic closed sets (Definition 5.5). Given $\mathfrak{F} \in \mathrm{Ob}(\mathcal{C}(n))$ and a transformation $\mathfrak{F} \leftarrow \mathfrak{F}_1$ in $\mathcal{C}$, it is clear that also $\mathfrak{F}_1 \in \mathrm{Ob}(\mathcal{C}(n))$. So $\mathcal{C}(n)$ is also an algebraic class.

In the same way we can define for $\mathcal{S}$ as in Example 6.2 (1) the subclass $\mathcal{S}(n)$ of schemes which admit a closed embedding in a smooth n-dimensional W (smooth over some field k of characteristic zero).

An algorithm of resolution of $\mathcal{C}(n)$ with values at $(I_n, \leqslant)$ together with Theorem 6.6 would provide for any $X \in \mathrm{Ob}(\mathcal{S}(n))$ a unique sequence of transformations

$$X = X_0 \leftarrow X_1 \leftarrow \dots \leftarrow X_k$$

on centers $C_i \subseteq \underline{\mathrm{Max}}\, H(X_i)$ (the Hilbert–Samuel stratum) such that

$$\max H(X_0) = \dots = \max H(X_{k-1}) > \max H(X_k) \quad \text{in } \mathbf{N}^{\mathbf{N}}.$$

A result of Hironaka [Hi2, p. 71] states that if

$$X_0 \leftarrow X_1 \leftarrow \dots \leftarrow X_n \leftarrow \dots$$

is a sequence of monoidal transformations at centers C_i contained in the Samuel stratum of X_i, then for some index m,

$$\max H(X_m) = \max H(X_{m'}) \quad \forall m' \geqslant m.$$

One can finally check that an algorithm of resolution on $\mathcal{S}(n)$ can be defined with values at $\mathbf{N}^{\mathbf{N}} \times I_n$, ordered lexicographically, which essentially means that an algorithm of desingularization can be achieved from an algorithm of resolution of $\mathcal{C}(n)$. Furthermore, one can also check that an algorithm of principalization of ideals (Example 6.2 (3)) will also follow from an algorithm of resolution of $\mathcal{C}(n)$.

If $\mathfrak{F} \in \mathrm{Ob}(\mathcal{C}(n))$ is actually an idealistic closed set of codimension $\geqslant e$ (Definition 5.14) and $\mathfrak{F} \leftarrow \mathfrak{F}_1$ is a transformation in $\mathcal{C}(n)$, then $\mathfrak{F}_1$ is also of codimension $\geqslant e$. So set $\mathcal{C}(n,e)$ as the algebraic class consisting of those objects, and naturally

$$\mathcal{C}(n,n) \subseteq \mathcal{C}(n,n-1) \subseteq \dots \subseteq \mathcal{C}(n,1) \subseteq \mathcal{C}(n,0) = \mathcal{C}(n).$$

Since both desingularization and principalization follow from an algorithm of resolution of $\mathcal{C}(n)$, our main goal is to define I_n and an algorithm of resolution on $\mathcal{C}(n)$. But we will first argue by decreasing induction on e, defining totally ordered sets $(I_{n,e}, \leqslant)$ and an algorithm of resolution on $\mathcal{C}(n,e)$, and finally setting $I_n = I_{n,0}$. This inductive procedure will be clarified in the proof of Theorem 6.13.

6.9. We shall construct an algorithm of resolution on $\mathcal{C}(n)$ (values at I_n) (Definition 6.7) with the following additional properties:

(1) Compatibility with equivalence: Note that if $\mathfrak{F}$ and $\mathfrak{F}'$ are equivalent (Definition 5.11), then a resolution of one induces a resolution of the other (Definition 5.7). With the setting as in Remark 5.13 (1) we will show that the assigned functions are equal, namely

$$f_k(\mathfrak{F}_k) = f_k(\mathfrak{F}'_k),$$

as functions on $\mathrm{Sing}(\mathfrak{F}_k) = \mathrm{Sing}(\mathfrak{F}'_k)$ ((5.13.1)). In particular, both $\mathfrak{F}$ and $\mathfrak{F}'$ undergo the same resolution via the algorithm (see Definition 6.7 (1)).

(2) The center $\underline{\mathrm{Max}}\, f_k$ (Definition 6.7) is of pure dimension and its codimension in W_k (notation as in Definition 5.1) is given by the value $\max f_k$.

(3) Fix $\mathfrak{F}\in\mathrm{Ob}(\mathcal{C}(n))$ and $\alpha\colon\mathfrak{F}'\to\mathfrak{F}$, an étale restriction or a change of base field. Fix notation as in Remark 15.3 (2). We will show that

$$f_i(\mathfrak{F}'_i)=f_i(\mathfrak{F}_i)\circ\alpha_i$$

for $\alpha_i\colon\mathrm{Sing}(\mathfrak{F}'_i)\to\mathrm{Sing}(\mathfrak{F}_i)$ as in (5.13.2). In particular, the algorithmic resolution of $\mathfrak{F}'$ is obtained from that of $\mathfrak{F}$.

Note that both $\mathcal{C}(n)$, $\mathcal{C}(n,e)$ and also $\mathcal{S}(n)$ (6.8) are closed by étale restriction and by arbitrary change of base field.

6.10. Consider a set with two elements $\{G,B\}$ (G=good, B=bad) ordered by $G<B$.

Given totally ordered sets $(I_1,\leqslant)$, $(I_2,\leqslant)$, we shall always consider $(I_1\times I_2,\leqslant)$ to be ordered lexicographically and set

$$\mathrm{pr}_1\colon I_1\times I_2\to I_1,\quad \mathrm{pr}_2\colon I_1\times I_2\to I_2$$

as the usual projections. An element α of $(I,\leqslant)$ will be denoted by $\infty(I)$ if $\alpha>\beta$ for any $\beta\in I$, $\beta\neq\alpha$. If such an element exists, it is clearly unique. If not we will sometimes enlarge I to $I\cup\{\infty(I)\}$ so as to add such an element.

6.11. *Claim*(n,e). There is an ordered set $(\bar{I}_e,\leqslant)$ with $\infty(\bar{I}_e)\in\bar{I}_e$, and an assignment of chains and functions (Definition 6.3)

$$\mathrm{CF}(\mathcal{C}(n,e),I_e)$$

where $I_e=\{G,B\}\times\bar{I}_e$, with the following conditions and properties:

(1) For any $\mathfrak{F}\in\mathcal{C}(n,e)$ let

$$\mathcal{L}_0=\begin{cases}\mathfrak{F}=\mathfrak{F}_0,\\ g_0^e\colon\mathrm{Sing}(\mathfrak{F}_0)\to I_e\end{cases}$$

be the assigned chain of length zero ((A_0) in Definition 6.3). Then

(a) $\mathrm{pr}_1(g_0^e(\xi))=B$ for any $\xi\in\mathrm{Sing}(\mathfrak{F})=F$ (i.e. the first coordinate of g_0^e is always B),

(b) $g_0^e(\xi)=(B,\infty(\bar{I}_e))$ if and only if $\xi\in F^{(e)}$ ($F^{(e)}=\{\xi\in F\mid\mathrm{cod}_\xi(F)=e\}$ as in Theorem 5.22).

Note that $(B,\infty(\bar{I}_e))$ is the biggest element of I_e (here $\infty(I_e)=(B,\infty(\bar{I}_e))$), so $\underline{\mathrm{Max}}\, g_0^e=F^{(e)}$ if $F^{(e)}\neq\varnothing$.

(2) If $F^{(e)}$ is not empty then $C(\mathcal{L}_0)=\{F^{(e)}\}$. The criterion of choice of centers for chains of length zero ((B_0) in Definition 6.3) reduces to $F^{(e)}$ if it is not empty.

(3) Let $\mathcal{L}_k$ be a chain of length k in $\mathrm{CF}(\mathcal{C}(n,e),I_e)$. Assume that either $k=0$ and $F^{(e)}=\varnothing$ or that $k>0$ and $\mathrm{pr}_1(\max g_k^e)=B$. Set i_0 as the smallest index such that

$$(\max g_{i_0-1}^e>)\max g_{i_0}^e=\ldots=\max g_k^e$$

(see Remark 6.4). Consider the chain $\mathcal{L}_{i_0}$, the truncation of $\mathcal{L}_k$ at level i_0:

$$\mathcal{L}_{i_0}=\begin{cases}\mathfrak{F}_0\xleftarrow{\varphi_1}\ldots\xleftarrow{\varphi_k}\mathfrak{F}_{i_0}, & \mathfrak{F}_i\in\mathrm{Ob}(\mathcal{G}),\\ g_i^e\colon \mathrm{Sing}(\mathfrak{F}_i)\to I, & i=0,\ldots,i_0.\end{cases}$$

Note that $\mathrm{pr}_1(\max g_j^e)=B$ for $j=0,\ldots,k$ by Remark 6.4, and that $\max g_k^e<(B,\infty(\bar{I}_e))$ by (1) and (2) above and (4) of Theorem 5.22. We require:

Case $k=i_0$. There is an idealistic closed set $\mathfrak{F}'_{i_0}$ of codimension $\geqslant e+1$ (with $\mathfrak{F}_{i_0}\in\mathrm{Ob}(\mathcal{C}(n,e+1))$) and $\mathfrak{F}'_{i_0}\subseteq\mathfrak{F}_{i_0}$ (so any sequence of transformations and restrictions over $\mathfrak{F}'_{i_0}$ induces the same sequence over $\mathfrak{F}_{i_0}$ (Definition 5.9)) such that:

- $\mathrm{Sing}(\mathfrak{F}'_{i_0})=\underline{\mathrm{Max}}\,g_{i_0}^e(\mathfrak{F}_{i_0})$.
- $C\in C(\mathcal{L}_{i_0})$ if and only if C is a permissible center for $\mathfrak{F}'_{i_0}$ (Definition 5.4).

Case $i_0<k$. Note that if $i_0=0$ then $F^e=\varnothing$ by (1) and (2), otherwise $\max g_0^e=(B,\infty(\bar{I}_e))>\max g_1^e\geqslant\max g_k^e$. So we define the chain $\mathcal{L}_{i_0}$ as above and consider $\mathfrak{F}'_{i_0}$ as in the case $k=i_0$.

Now we require that there be a sequence of $k-i_0$ transformations over $\mathfrak{F}'_{i_0}$:

$$\begin{tikzcd}\mathcal{L}_k\colon \mathfrak{F}_0 & \cdots\arrow[l] & \mathfrak{F}_{i_0}\arrow[l] & \cdots\arrow[l] & \mathfrak{F}_k\arrow[l]\\ & & \mathfrak{F}'_{i_0}\arrow[u] & \cdots\arrow[l] & \mathfrak{F}'_k\arrow[l]\arrow[u]\end{tikzcd}$$

such that:

- $\mathrm{Sing}(\mathfrak{F}'_j)=\underline{\mathrm{Max}}\,g_j^e$, $j=i_0,\ldots,k$.
- $C\in C(\mathcal{L}_j)$ ((C$_j$) in Definition 6.3) if and only if C is a permissible center for $\mathfrak{F}'_j$ (Definition 5.4), where $\mathcal{L}_j$ is the truncation of $\mathcal{L}_k$ at level j, and $j=i_0,\ldots,k$.

Induction. Set

$$\begin{tikzcd}\mathcal{L}_{k+1}\colon \mathfrak{F}_0 & \cdots\arrow[l] & \mathfrak{F}_{i_0}\arrow[l] & \cdots\arrow[l] & \mathfrak{F}_k\arrow[l] & \mathfrak{F}_{k+1}\arrow[l]\\ & & \mathfrak{F}'_{i_0}\arrow[u] & \cdots\arrow[l] & \mathfrak{F}'_k\arrow[l]\arrow[u] & \mathfrak{F}'_{k+1}\arrow[l]\arrow[u]\end{tikzcd}$$

where $\mathfrak{F}'_{i_0}$ (of codimension $\geqslant e+1$) is as in the case $i_0<k$ and $\mathcal{L}_{k+1}$ is an enlargement of $\mathcal{L}_k$ defined by a choice of a permissible center $C\in C(\mathcal{L}_k)$. Then either

$$\max g_k^e>\max g_{k+1}^e\quad\text{and}\quad \mathrm{Sing}(\mathfrak{F}'_{k+1})=\varnothing$$

or

$$\max g_k^e = \max g_{k+1}^e \quad \text{and} \quad \underline{\mathrm{Max}}\, g_{k+1}^e = \mathrm{Sing}(\mathfrak{F}'_{k+1}).$$

(4) If $k>1$ and $\mathrm{pr}_1(\max g_k^e)=G$ then:

(a) $\underline{\mathrm{Max}}\, g_k^e$ is regular and $C(\mathcal{L}_k)=\{\underline{\mathrm{Max}}\, g_k^e\}$ (see (C_k) in Definition 6.3). So if $\mathrm{pr}_1(\max g_k^e)=G$, there is a unique choice of center in (B_k) in Definition 6.3 (see Remark 3.3 (1)).

(b) If

$$\mathfrak{F}_0 \leftarrow \dots \leftarrow \mathfrak{F}_k \leftarrow \mathfrak{F}_{k+1}$$

is the unique enlargement defined in (4)(a) above, then either $\mathrm{Sing}(\mathfrak{F}_{k+1})=\varnothing$ or $\max g_k^e > \max g_{k+1}^e$ (in which case $\mathrm{pr}_1(\max g_{k+1}^e)=G$).

(c) The case $\mathrm{Sing}(\mathfrak{F}_{k+1})=\varnothing$ holds for some k after finitely many transformations.

(5) Given

$$\mathfrak{F}_0 \leftarrow \dots \leftarrow \mathfrak{F}_n \leftarrow \dots,$$

an infinite sequence in $\mathcal{C}(n,e)$, such that for any k,

$$\mathfrak{F}_0 \leftarrow \dots \leftarrow \mathfrak{F}_k$$

is as in (6.3.1) for some chain in $\mathrm{CF}(\mathcal{C}(n,e), I_e)$, then there is an index s such that

$$\max g_s^e = \max g_{s+1}^e = \dots .$$

Note that for an infinite sequence as above $\mathrm{pr}_1(\max g_r^e)=B$ for all $r\geqslant 1$, by (4)(c).

(6) (a) If $\mathfrak{F}$ and $\mathfrak{F}'$ are equivalent in $\mathcal{C}(n,e)$ (Definition 5.11), then any chain $\mathcal{L}_k$ in $\mathrm{CF}(\mathcal{C}(n,e), I_e)$ with $\mathfrak{F}_0=\mathfrak{F}$ induces a chain $\mathcal{L}'_k$ with $\mathfrak{F}'_0=\mathfrak{F}'$ so that $\mathrm{Sing}(\mathfrak{F}_i)=\mathrm{Sing}(\mathfrak{F}'_i)$ ((5.13.1)) and

$$g_i^e(\mathfrak{F}_i) = g_i^e(\mathfrak{F}'_i), \quad i=0,\dots,k,$$

as functions on $\mathrm{Sing}(\mathfrak{F}_i)=\mathrm{Sing}(\mathfrak{F}'_i)$.

(b) If $\mathrm{pr}_1(\max g_k^e)=G$ then the centers $C_k=\underline{\mathrm{Max}}\, g_k^e$ are of pure dimension and the codimension is determined by the value $\max g_k^e$.

(c) If $\mathfrak{F}'$ is an étale restriction of $\mathfrak{F}$ or if $\mathfrak{F}'$ is obtained from $\mathfrak{F}$ by a change of the base field, then any chain $\mathcal{L}_k$ in $\mathrm{CF}(\mathcal{C}(n,e), I_e)$ with $\mathfrak{F}_0=\mathfrak{F}$ induces, by étale restriction or change of base field, a chain $\mathcal{L}'_k$ in $\mathrm{CF}(\mathcal{C}(n,e), I_e)$ with $\mathfrak{F}'_0=\mathfrak{F}'$ and $g_k^e(\mathfrak{F}'_k)=g_k^e(\mathfrak{F}_k)\circ\alpha_k$ (α_k as in (5.13.2)).

6.12. *Proof of Claim*(n,n). Set $\bar{I}_n=\{\infty\}$ so $I_n=\{G,B\}\times\{\infty\}$ as in 6.11. For any $\mathfrak{F}\in\mathrm{Ob}(\mathcal{C}(n,n))$, $\mathrm{Sing}(\mathfrak{F})$ is either empty or $\mathrm{Sing}(\mathfrak{F})=F^{(n)}$ consists of finitely many closed points.

We set $g_0^n(\xi)=(B,\infty)$ for any $\xi\in F^{(n)}=\mathrm{Sing}(\mathfrak{F})$. Now $\max g_0^n=(B,\infty)$ and $\underline{\mathrm{Max}}\, g_0^n=\mathrm{Sing}(\mathfrak{F})$. We declare $\underline{\mathrm{Max}}\, g_0^n$ to be the unique center as our criterion $(C(\mathcal{L}_0)=\{\underline{\mathrm{Max}}\, g_0^n\})$, and set

$$\mathfrak{F}_0 \leftarrow \mathfrak{F}_1$$

as the transformation at such a center. Theorem 5.22 asserts that $\mathrm{Sing}(\mathfrak{F}_1)$ is empty.

Clearly all conditions of $\mathrm{Claim}(n,n)$ are fulfilled. Furthermore, what we obtain is an algorithm of resolution of the algebraic class $\mathcal{C}(n,n)$, as defined in Definition 6.7, that clearly fulfills all properties of 6.9.

We shall address $\mathrm{Claim}(n,e)$ for $e<n$ in 6.14. The following theorem is to show how an algorithm of resolution of $\mathcal{C}(n,e)$, $e\leqslant n$, can be achieved from $\mathrm{Claim}(n,e)$. Recall that we are ultimately interested in an algorithm of resolution of $\mathcal{C}(n,0)=\mathcal{C}(n)$ (6.8).

THEOREM 6.13. *Fix (n,e) and assume the following two hypotheses:*

(H1) *The existence of an algorithm of resolution on $\mathcal{C}(n,e+1)$ with values at an ordered set $\Lambda_{n,e+1}$ (Definition 6.7) and such that the properties in 6.9 hold.*

(H2) *The existence of I_e so that Claim(n,e) holds.*

Then there is an algorithm of resolution of $\mathcal{C}(n,e)$, with values at $\Lambda_{n,e}$, satisfying the properties of 6.9, where

$$\Lambda_{n,e}=I_e\times(\Lambda_{n,e+1}\cup\{\infty\}).$$

We shall organize the proof of Theorem 6.13 as follows:

Step 1. We attach to each $\mathfrak{F}\in\mathrm{Ob}(\mathcal{C}(n,e))$ a unique sequence

$$\mathfrak{F}=\mathfrak{F}_0\leftarrow\mathfrak{F}_1\leftarrow\ldots\leftarrow\mathfrak{F}_k \tag{6.13.1}$$

such that

$$\mathrm{Sing}(\mathfrak{F}_k)=\varnothing. \tag{6.13.2}$$

This sequence (6.13.1) will be constructed in such a way that there is a chain of functions in $\mathrm{CF}(\mathcal{C}(n,e),I_e)$ associated to (6.13.1).

Step 2. We define functions

$$f_i^e\colon \mathrm{Sing}(\mathfrak{F}_i)\to\Lambda_{n,e},\quad i=0,\ldots,k,$$

so that, for all $\mathfrak{F}\in\mathrm{Ob}(\mathcal{C}(n,e))$, the sequence (6.13.1) together with these functions f_i^e define an algorithm of resolution (Definition 6.7).

Step 3. We show that the algorithm of resolution constructed in Steps 1 and 2 fulfills the properties of 6.9.

Step 1. Fix $\mathfrak{F}\in\mathrm{Ob}(\mathcal{C}(n,e))$. If $F^{(e)}\subseteq\mathrm{Sing}(\mathfrak{F})$ is not empty, set

$$\mathfrak{F}=\mathfrak{F}_0\leftarrow\mathfrak{F}_1$$

as noted in 6.11 (2): the transformation with center $F^{(e)}$.

Suppose that (6.13.1) has been defined so as to induce a chain of functions of length k in $\mathrm{CF}(\mathcal{F}(n,e),I_e)$. If $k=0$ and $F^{(e)}\neq\varnothing$, this is done by 6.11 (2). So we are left with two possibilities:

(1) Either $k=0$ and $F^{(e)}=\varnothing$ or $k>0$ and $\mathrm{pr}_1(\max g_k^e)=B$.

(2) $\mathrm{pr}_1(\max g_k^e)=G$.

(1) If either $k=0$ and $F^{(e)}=\varnothing$ or $k>0$ and $\mathrm{pr}_1(\max g_k^e)=B$, set i_0 as in 6.11 (3). Now we can assume that the sequence

$$\mathfrak{F}'_{i_0}\leftarrow\ldots\leftarrow\mathfrak{F}'_k$$

of 6.11 (3)(c) consists of the first steps of the resolution of $\mathfrak{F}'_{i_0}\in\mathrm{Ob}(\mathcal{C}(n,e+1))$. This assumption can be made because we are constructing (6.13.1) inductively on k and we also assume that there is an algorithm of resolution of $\mathcal{C}(n,e+1)$ with values at $\Lambda_{n,e+1}$.

Set now the next transformation of (6.13.1) by choosing C_k to be the center assigned in $\mathrm{Sing}(\mathfrak{F}'_k)$ by the algorithm of the resolution mentioned above.

(2) If $\mathrm{pr}_1(\max g_k^e)=G$, then set $C_k=\underline{\mathrm{Max}}\, g_k^e$ and apply 6.11 (4)(c) to come to the case $\mathrm{Sing}(\mathfrak{F}_k)=\varnothing$.

Note that the hypotheses (H1) and (H2) together with (3) and (5) of 6.11 assert that $\mathrm{pr}_1(\max g_k^e)=G$ will hold for some k big enough. Finally Step 1 follows from 6.11 (4). In this way we attach to each $\mathfrak{F}\in\mathrm{Ob}(\mathcal{C}(n,e))$ a unique sequence (6.13.1) in $\mathrm{CF}(\mathcal{C}(n,e),I_e)$ so that $\mathrm{Sing}(\mathfrak{F}_k)=\varnothing$ ((6.13.2)).

Step 2.

Step 2.1. First we define the functions f_i^e only along the closed sets $\underline{\mathrm{Max}}\, g_i^e$.

Step 2.2. We define f_i as a function on all $\mathrm{Sing}(\mathfrak{F}_i)$.

Step 2.3. We show that the functions f_i define an algorithm of resolution (Definition 6.7).

Step 2.1. (1) If $i=0$ and $C_0=F^{(e)}$ (case $F^{(e)}\neq\varnothing$), then set for $\xi\in\underline{\mathrm{Max}}\, g_0^e=F^{(e)}$ (see 6.11 (1)):

$$f_0^e(\xi)=(g_0^e(\xi),\infty(\Lambda_{n,e+1}))=(\max g_0^e,\infty(\Lambda_{n,e+1}))\in\Lambda_{n,e}.$$

(2) If either $i=0$ and $F^{(e)}=\varnothing$ or $i>0$ and $\mathrm{pr}_1(\max g_i^e)=B$, then set for $\xi\in\underline{\mathrm{Max}}\, g_i^e$:

$$f_i^e(\xi)=(g_i^e(\xi),f_i^{e+1}(\xi))=(\max g_i^e,f_i^{e+1}(\xi))\in\Lambda_{n,e}$$

where $f_i^{e+1}\colon \mathrm{Sing}(\mathfrak{F}_i') \to \Lambda_{n,e+1}$ are the functions given by the algorithm of resolution for $\mathfrak{F}_i' \in \mathcal{C}(n,e+1)$ as in 6.11 (3).

(3) If $\mathrm{pr}_1(\max g_i^e) = G$, then set

$$f_i^e(\xi) = (g_i^e(\xi), \infty(\Lambda_{n,e+1})) = (\max g_i^e, \infty(\Lambda_{n,e+1})), \quad \xi \in \underline{\mathrm{Max}}\, g_i^e.$$

One can easily check that all

$$f_i^e\colon \underline{\mathrm{Max}}\, g_i^e \to \Lambda_{n,e}, \quad i=0,\dots,k-1, \tag{6.13.3}$$

are as in 1.1.

Step 2.2. In this step we extend (6.13.3) to define

$$f_i^e\colon \mathrm{Sing}(\mathfrak{F}_i) \to \Lambda_{n,e}, \quad i=0,\dots,k-1.$$

Recall that $\mathfrak{F}_i \leftarrow \mathfrak{F}_{i+1}$ in (6.13.1) is defined as a transformation with center $C_i \subseteq \underline{\mathrm{Max}}\, g_i^e \subseteq \mathrm{Sing}(\mathfrak{F}_i)$; in fact, the construction in Step 1 was done so that (6.13.1) induces a chain in $\mathcal{C}(n,e)$, so the inclusion is given by (B_k) in Definition 6.3. Since $\mathrm{Sing}(\mathfrak{F}_k)$ is empty, it is clear that $\mathrm{Sing}(\mathfrak{F}_{k-1}) = C_{k-1}$ (Definition 6.1); in particular,

$$f_{k-1}^e\colon \mathrm{Sing}(\mathfrak{F}_{k-1}) \to \Lambda_{n,e}$$

is already defined. Assume, by decreasing induction, that

$$f_i^e\colon \mathrm{Sing}(\mathfrak{F}_i) \to \Lambda_{n,e}, \quad i=l, l+1, \dots, k-1,$$

are defined as functions (1.1), and

$$f_i^e(\bar{\varphi}_{i+1}(\xi)) \geqslant f_{i+1}^e(\xi) \quad \forall \xi \in \mathrm{Sing}(\mathfrak{F}_i)$$

with equality if $\bar{\varphi}_{i+1}(\xi) \notin C_i$. Define now

$$f_{l-1}^e\colon \mathrm{Sing}(\mathfrak{F}_{l-1}) \to \Lambda_{n,e}$$

by setting:

(a) For $\xi \in \underline{\mathrm{Max}}\, g_{l-1}^e$, $f_{l-1}^e(\xi)$ as in Step 2.1.

(b) For $\xi \in \mathrm{Sing}(\mathfrak{F}_{l-1}) \setminus \underline{\mathrm{Max}}\, g_{l-1}^e$, we have $\xi \notin C_{l-1}$ and one can identify ξ with $\xi' \in \mathrm{Sing}(\mathfrak{F}_l)$; set

$$f_{l-1}^e(\xi) = f_l^e(\xi').$$

Now we check that:

(1) $f_{l-1}^{e}(\bar{\varphi}_l(\xi))\geqslant f_l^e(\xi)$ for any $\xi\in\mathrm{Sing}(\mathfrak{F}_l)$.

(2) f_{l-1}^{e} is upper-semi-continuous (1.1).

(3) $\underline{\mathrm{Max}}\, f_{l-1}^{e}=C_{l-1}$ and $\max f_{l-1}^{e}>\max f_l^e$.

(1) By construction of f_{l-1}^{e} it suffices to check the inequality only if $\bar{\varphi}_l(\xi)\in C_{l-1}$. Since the first coordinates of $f_l^e(\xi)$ and $f_{l-1}^{e}(\bar{\varphi}_l(\xi))$ are defined by $g_l^e(\xi)$ and $g_{l-1}^{e}(\bar{\varphi}_l(\xi))$, we may also reduce to the case $g_l^e(\xi)=g_{l-1}^{e}(\bar{\varphi}_l(\xi))$ ((6.3.3)). Since now we assume $\bar{\varphi}_l(\xi)\in C_{l-1}$, 6.11 (4)(b) asserts that this equality can only hold if $\mathrm{pr}_1(\max g_l^e)=B$. But then the second coordinates (see (2) in Step 2.1 and Definition 6.7 (1)):

$$f_{l-1}^{e+1}(\bar{\varphi}_l(\xi))>f_l^{e+1}(\xi) \tag{6.13.4}$$

by the assumption (H1) of Theorem 6.13.

(2) Fix $\xi\in\mathrm{Sing}(\mathfrak{F}_{l-1})$ and $\xi'\in\overline{\{\xi\}}$: ξ' is a specialization of ξ. Since the coordinates involved in the definition of f_{l-1}^{e} fulfill the conditions of 1.1, it suffices to show that $f_{l-1}^{e}(\xi')\geqslant f_{l-1}^{e}(\xi)$. If $\xi'\notin\underline{\mathrm{Max}}\, g_{l-1}^{e}$, both ξ and ξ' can be identified with points in $\mathrm{Sing}(\mathfrak{F}_l)$ and the assertion is clear. If $\xi'\in\underline{\mathrm{Max}}\, g_{l-1}^{e}$ then the inequality follows from (6.13.3).

(3) is a case by case treatment. If $\mathrm{pr}_1(\max g_{l-1})=G$, the inequality follows from (4)(b) in 6.11. If $\mathrm{pr}_1(\max g_{l-1})=B$ then it follows from (6.13.4).

Step 2.3. The assertion grows now from the construction and properties (3), (4) and (5) in 6.11.

Step 3. This now follows from our definition of f_i^e; in fact, it follows from part (6)(a), (6)(b) and (6)(c) of 6.11 together with the assumption (H1) in Theorem 6.13. This proves Theorem 6.13.

6.14. *Proof of Claim*(n,e) (6.11). Claim(n,n) was proved in 6.12. We shall prove Claim(n,e) for $e<n$. Recall that for $e=n$ we defined $\bar{I}_n=\{\infty\}$ (6.12). Set now, for $e<n$,

$$\bar{I}_e=((\mathbf{Q}\cup\{\infty\})\times(\mathbf{Z}\cup\{\infty\}))\sqcup I_M$$

where $\sqcup$ denotes the disjoint union, ordered so that if $\beta\in(\mathbf{Q}\cup\{\infty\})\times(\mathbf{Z}\cup\{\infty\})$ and $\alpha\in I_M$ then $\beta>\alpha$, where I_M denotes the totally ordered set defined in (2.0.2).

Set $I_e=\{G,B\}\times\bar{I}_e$. We shall define now an assignment of chains and functions $\mathrm{CF}(\mathcal{C}(n,e),I_e)$ as in 6.11. In particular, functions

$$g_i^e\colon\mathrm{Sing}(\mathfrak{F}_i)\to I_e$$

will be defined, and we shall call $\mathrm{Ab}_i = \mathrm{pr}_1 \circ g_i^e$ (first coordinate) and $\bar{g}_i^e = \mathrm{pr}_2 \circ g_i^e$ (second coordinate):

$$\mathrm{Ab}_i\colon \mathrm{Sing}(\mathfrak{F}_i) \to \{G, B\},$$
$$\bar{g}_i^e\colon \mathrm{Sing}(\mathfrak{F}_i) \to \bar{I}_e.$$

The first coordinate will indicate where the second coordinate lies, namely for $\xi \in \mathrm{Sing}(\mathfrak{F}_i)$,

$$\bar{g}_i^e(\xi) \in (\mathbf{Q} \cup \{\infty\}) \times (\mathbf{Z} \cup \{\infty\}) \quad \Leftrightarrow \quad \mathrm{Ab}_i(\xi) = B,$$

and therefore $\bar{g}_i^e \in I_M$ if and only if $\mathrm{Ab}_i(\xi) = G$. Recall that $B > G$ and that I_e is ordered lexicographically. In particular, $(B, (\infty, \infty))$ is the biggest element of I_e.

Our task in this proof is twofold: on the one hand to define the assignment on I_e as above (i.e. defining chains of length k and proving the conditions in Definition 6.3), on the other hand to show that these chains of length k fulfill the conditions of 6.11. All this will be carried out by induction on k.

We organize the proof of 6.11 by dividing it into the following steps:

Step 1. We begin by defining chains of length zero (of what is to be $\mathrm{CF}(\mathcal{C}(n,e), I_e)$) by setting ($\mathrm{A}_0$) and ($\mathrm{B}_0$) (($\mathrm{B}_k$) for $k=0$) in Definition 6.3, and showing that conditions (1) and (2) of 6.11 hold.

Step 2. We prove condition (6) of 6.11 for chains of length zero.

Step 3. Assume, inductively on k, the definition of chains and functions of length k:

$$\mathcal{L}_k = \begin{cases} \mathfrak{F}_0 \leftarrow \dots \leftarrow \mathfrak{F}_k, \\ g_i^e\colon \mathrm{Sing}(\mathfrak{F}) \to I_e, \quad i = 0, \dots, k, \end{cases} \tag{6.14.1}$$

so that the inequalities of Remark 6.4 hold, and with conditions (1), (2) and (6) of 6.11. At this step we also introduce some additional hypotheses, ($\mathrm{C1}_k$), ($\mathrm{C2}_k$), ($\mathrm{C3}_k$), ($\mathrm{C4}_k$), and prove that ($\mathrm{C1}_0$), ($\mathrm{C2}_0$), ($\mathrm{C3}_0$) and ($\mathrm{C4}_0$) hold for chains of length zero. This will allow us to continue the development with the assumption that also ($\mathrm{C1}_k$), ($\mathrm{C2}_k$), ($\mathrm{C3}_k$) and ($\mathrm{C4}_k$) hold for chains $\mathcal{L}_k$ of length $k \geqslant 0$.

Step 4. Under the assumption of an enlargement $\mathcal{L}_{k+1}$ of $\mathcal{L}_k$ by a transformation on a center $C \in C(\mathcal{L}_k)$, we define the function

$$g_{k+1}^e\colon \mathrm{Sing}(\mathfrak{F}_{k+1}) \to I_e$$

and the criterion of choice of centers $C(\mathcal{L}_{k+1})$ so that ($\mathrm{C1}_{k+1}$), ($\mathrm{C2}_{k+1}$), ($\mathrm{C3}_{k+1}$) and ($\mathrm{C4}_{k+1}$) also hold. Finally we check that we have defined an assignment of chains and functions (Definition 6.3) which, in addition, satisfies hypotheses (C1), (C2), (C3) and (C4).

Step 5. We prove that the assignment of chains and functions satisfies conditions (1), (2) and (6) of 6.11.

Step 6. We prove that condition (3) of 6.11 holds.

Step 7. Finally we prove conditions (4) and (5) of 6.11.

Step 1. Fix I_e defined as above. Set now for each $\mathfrak{F}\in\mathrm{Ob}(\mathcal{C}(n,e))$,

$$g_0^e(\mathfrak{F})=g_0^e\colon \mathrm{Sing}(\mathfrak{F})\to I_e=\{B,G\}\times \bar{I}_e,\quad \mathrm{Ab}_0(\xi)=B,$$
$$g_0^e(\xi)=(B,\text{w-ord}_e(\mathfrak{F})(\xi),n_e(\mathfrak{F})(\xi))$$

for $\text{w-ord}_e(\mathfrak{F})$ defined (for $\mathfrak{F}=\mathfrak{F}_0$) as $\text{w-ord}_e(\mathfrak{F}_0)=\mathrm{ord}_e(\mathfrak{F}_0)$ in Corollary 5.19. Set

$$n_0(\mathfrak{F})(\xi)=\begin{cases}\infty & \text{if w-ord}_e(\mathfrak{F})(\xi)=\infty,\\ \#\{H\in E\mid \xi\in H\} & \text{if w-ord}_e(\mathfrak{F})(\xi)<\infty,\end{cases}$$

$(\text{w-ord}_e(\mathfrak{F})(\xi),n_0(\mathfrak{F})(\xi))\in(\mathbf{Q}\cup\{\infty\})\cup(\mathbf{Z}\cup\{\infty\})\subseteq\bar{I}_e$. Now we set (B_0) (Definition 6.3), namely we fix the criterion of choice of centers $C(\mathcal{L}_0)$:

6.15. It follows easily from the definition that $g_0^e(\mathfrak{F})$ is a function as defined in 1.1. Moreover, Theorem 5.22 asserts that $\xi\in F^{(e)}$ if and only if $g_0^e(\mathfrak{F})(\xi)=(B,\infty,\infty)$. In particular, $F^{(e)}\neq\varnothing$ if and only if $\max g_0^e(\mathfrak{F})=(B,\infty,\infty)$, in which case $\underline{\mathrm{Max}}\, g_0^e(\mathfrak{F})=F^{(e)}$.

Let

$$\mathcal{L}_0=\begin{cases}\mathfrak{F}_0,\\ g_0^e\colon \mathrm{Sing}(\mathfrak{F}_0)\to I_e\end{cases}$$

be a chain of length zero. If $F^e\neq\varnothing$ then we agree to set $C(\mathcal{L}_0)=\{\underline{\mathrm{Max}}\, g_0^e\}$. If $F^e=\varnothing$ then we set $C\in C(\mathcal{L}_0)$ if and only if C is a permissible center for $\mathfrak{F}_0$ (Definition 5.4) contained in $\underline{\mathrm{Max}}\, g_0^e(\mathfrak{F}_0)$. In this way chains of length zero are defined as in Definition 6.3, and conditions (1) and (2) of 6.11 hold.

Step 2. Condition (6)(a) follows from the first assertion in Corollary 5.20 and from the fact that $E=E'$ in our notion of equivalence (Definition 5.11). Condition (6)(c) follows from the formula (4.17.3) which, in turn, is invariant by restrictions or change of base field. Condition (6)(b) is vacuous for chains of length zero, since $\mathrm{pr}_1(\max g_0^e)=B$.

Step 3. Now we assume the definition of chains and functions of length k, say $\mathcal{L}_k$ ((6.3.1)), together with a criterion of choice of centers (B_k), such that the inequalities of Remark 6.4 and the conditions (1), (2) and (6) of 6.11 hold.

6.16. Fix an index j, $0\leqslant j\leqslant k$, and assume $\mathfrak{F}_j$ locally defined by $(U_j^{(i)},(J_j^{(i)},b_i),E_j^{(i)})$ (notation as in 5.18), which we simply denote by $(U_j,(J_j,b),E_j)$.

Recall that in Definition 4.20 and 3.1 we have established the following expressions of products of ideals:

$$\begin{aligned} C(J_j) &= I(\overline{H}_1)^{a_j(1)} \dots I(\overline{H}_j)^{a_j(j)} \mathcal{A}_j, \\ \operatorname{red}(C(J_j)) &= I(\overline{H}_1)^{\overline{a_j(1)}} \dots I(\overline{H}_j)^{\overline{a_j(j)}} \mathcal{A}_j \end{aligned} \tag{6.16.1}$$

in a regular subscheme of codimension e in W_j, for $j=0, \dots, k$ ($\overline{a_s(i)}$ the remainder modulo $b!$ of $a_s(i)$ as in 3.1).

For $\xi_k \in \operatorname{Sing}(\mathfrak{F}_k)$, let $\xi_j \in \operatorname{Sing}(\mathfrak{F}_j)$ denote the image of ξ_k, $j=0, \dots, k$. Since the inequalities of Remark 6.4 hold by the assumption on k, we also know that

$$g^e_{j-1}(\xi_{j-1}) \geqslant g^e_j(\xi_j), \quad \mathrm{Ab}_{j-1}(\xi_{j-1}) \geqslant \mathrm{Ab}_j(\xi_j).$$

6.17. We will also assume for $0 \leqslant j \leqslant k$ the following definitions and conditions linking the functions $g^e_j(\mathfrak{F}_j)$:

($\mathrm{C1}_j$) If $\mathrm{Ab}_{j-1}(\xi_{j-1}) = B$ then $\text{w-ord}_e(\mathfrak{F}_{j-1})(\xi_{j-1}) \geqslant \text{w-ord}_e(\mathfrak{F}_j)(\xi_j)$.

We now define $E^-_j(\xi_j) \subseteq E_j$ inductively:

(D_0) $E^-_0(\xi_0) = \{H \in E_0 \mid \xi_0 \in E_0\}$.

(D_j)

$$E^-_j(\xi_j) = \begin{cases} \{H \in E_j \mid \xi_j \in H\} & \text{if } \text{w-ord}_e(\mathfrak{F}_{j-1})(\xi_{j-1}) > \text{w-ord}_e(\mathfrak{F}_j)(\xi_j), \\ \{H \in [E^-_{j-1}(\xi_{j-1})]_j \mid \xi_j \in H\} & \text{if } \text{w-ord}_e(\mathfrak{F}_{j-1})(\xi_{j-1}) = \text{w-ord}_e(\mathfrak{F}_j)(\xi_j) \end{cases}$$

where $[E^-_{j-1}(\xi_{j-1})]_j$ denotes the strict transform of hypersurfaces in $E^-_{j-1}(\xi_{j-1})$.

Finally set

$$n_j(\xi_j) = \#(E^-_j).$$

Remark. In our development it will be enough to understand the behavior of the function n_j along closed points in $\underline{\mathrm{Max}}\, \text{w-ord}_e(\mathfrak{F}_j)$. Note that ($\mathrm{C1}_j$) implies that

$$\max \text{w-ord}_e(\mathfrak{F}_0) \geqslant \dots \geqslant \max \text{w-ord}_e(\mathfrak{F}_j).$$

Set i'_0 as follows:

- If $\max \text{w-ord}_e(\mathfrak{F}_0) = \max \text{w-ord}_e(\mathfrak{F}_j)$ then $i'_0 = 0$.
- If $\max \text{w-ord}_e(\mathfrak{F}_0) > \max \text{w-ord}_e(\mathfrak{F}_j)$ then set i'_0 so that

$$\max \text{w-ord}_e \mathfrak{F}_{i'_0-1} > \max \text{w-ord}_e \mathfrak{F}_{i'_0} = \max \text{w-ord}_e \mathfrak{F}_j.$$

Finally set $E_j^- \subset E_j$ as the hypersurfaces of E_j which are strict transforms of hypersurfaces in $E_{i_0'}$, and note that for $\xi_j \in \underline{\mathrm{Max}}\,\text{w-ord}_e(\mathfrak{F}_j)$:

$$E_j^-(\xi_j) = \{H \in E_j^- \mid \xi_j \in H\}.$$

($\mathrm{C2}_j$) If $\mathrm{Ab}_j(\xi_j) = G$ then ξ_j is a locally good point of $(C(J_j), b!)$ (3.2), $C(J_j)$ as in (6.16.1).

($\mathrm{C3}_j$)

$$g_j^e(\xi_j) = \begin{cases} (\mathrm{Ab}_j(\xi_j), (\text{w-ord}_e(\mathfrak{F}_j)(\xi_j), n_j(\xi_j))) & \text{if } \mathrm{Ab}_j(\xi_j) = B, \\ (\mathrm{Ab}_j(\xi_j), \Gamma(\xi_j)) & \text{if } \mathrm{Ab}_j(\xi_j) = G \end{cases}$$

for Γ as in (2.0.2). Note that $\Gamma(\xi_j)$ is well defined if $\mathrm{Ab}_j(\xi_j) = G$ by (C2); in fact, Γ is defined for locally good points (3.2).

($\mathrm{C4}_j$) If $\mathrm{pr}_1(\max g_k^e) = G$ then $C(\mathcal{L}_k) = \{\underline{\mathrm{Max}}\, g_k^e\}$.

Condition ($\mathrm{C1}_0$) is vacuous, Step 1 asserts that for $j = 0$ ($\mathrm{C3}_0$) holds, where only the first line applies ($\mathrm{Ab}_0(\xi) = B$). Conditions ($\mathrm{C2}_0$) and ($\mathrm{C4}_0$) are also vacuous.

6.18. *Step* 4. Now we will define the function g_{k+1}^e and the criterion of choice of centers $C(\mathcal{L}_{k+1})$, under the assumption of an enlargement of the chain $\mathcal{L}_k$ to a chain $\mathcal{L}_{k+1}$ obtained by a transformation on a center $C_k \in C(\mathcal{L}_k)$.

Case A. $\mathrm{pr}_1(\max g_k^e) = B$. We want to define a value $g_{k+1}^e(\xi_{k+1})$ for $\xi_{k+1} \in \mathrm{Sing}(\mathfrak{F}_{k+1})$.

As $C_k \subseteq \underline{\mathrm{Max}}\, g_k^e$ then ($\mathrm{C3}_k$) and Proposition 4.22 will guarantee that

$$\text{w-ord}_e(\mathfrak{F}_{k+1})(\xi_{k+1}) \leqslant \text{w-ord}_e(\mathfrak{F}_k)(\xi_k).$$

In fact, $\text{w-ord}_e(\mathfrak{F}_k)$ is actually constant along C_k. We define now n_{k+1} as in (D_{k+1}) (6.17).

Recall from Remark 6.4 that $\max g_0^e \geqslant \ldots \geqslant \max g_k^e$. Let i_0 be the smallest index such that

$$\max g_{i_0-1}^e > \max g_{i_0}^e = \ldots = \max g_k^e.$$

Now set $\mathrm{Ab}_{k+1}(\xi_{k+1})$ as follows:

(1) If

$$\text{w-ord}_e(\mathfrak{F}_k)(\xi_k) > \text{w-ord}_e(\mathfrak{F}_{k+1})(\xi_{k+1})$$

or if

$$\text{w-ord}_e(\mathfrak{F}_k)(\xi_k) = \text{w-ord}_e(\mathfrak{F}_{k+1})(\xi_{k+1}) \quad \text{and} \quad n_k(\xi_k) > n_{k+1}(\xi_{k+1}),$$

then

$$\mathrm{Ab}_{k+1}(\xi_{k+1}) = \begin{cases} B & \text{if } \xi_{k+1} \in \mathrm{Sing}(\mathrm{red}(C(J)_{k+1}), b!) \text{ and } \text{w-ord}_e(\mathfrak{F}_{k+1})(\xi_{k+1}) \neq 0, \\ G & \text{if } \xi_{k+1} \notin \mathrm{Sing}(\mathrm{red}(C(J)_{k+1}), b!) \text{ or } \text{w-ord}_e(\mathfrak{F}_{k+1})(\xi_{k+1}) = 0. \end{cases} \tag{6.18.1}$$

(2) If $\text{w-ord}_e(\mathfrak{F}_k)(\xi_k)=\text{w-ord}_e(\mathfrak{F}_{k+1})(\xi_{k+1})$ and $n_k(\xi_k)=n_{k+1}(\xi_{k+1})$ then

$$\mathrm{Ab}_{k+1}(\xi_{k+1})=$$
$$\begin{cases} B & \text{if } \xi_{k+1}\in \mathrm{Sing}([\mathrm{red}(C(J_{i_0}))]_{k+1},b!) \text{ and } \text{w-ord}_e(\mathfrak{F}_{k+1})(\xi_{k+1})\neq 0, \\ G & \text{if } \xi_{k+1}\notin \mathrm{Sing}([\mathrm{red}(C(J_{i_0}))]_{k+1},b!) \text{ or } \text{w-ord}_e(\mathfrak{F}_{k+1})(\xi_{k+1})=0 \end{cases} \tag{6.18.2}$$

where $(V^e_{k+1},([\mathrm{red}(C(J_{i_0}))]_{k+1},b!),\overline{E}_{k+1})$ here is the transform of the basic object $(V^e_{i_0},(\mathrm{red}(C(J_{i_0})),b!),\varnothing)$ ($\mathrm{red}(C(J_{i_0}))$ and $V^e_{i_0}=V^e$ as in (6.16.1)). It should be noted that if $\xi_{k+1}\notin\mathrm{Sing}([\mathrm{red}(C(J_{i_0}))]_{k+1},b!)$ then ξ_{k+1} is a locally good point of $(C(J_{k+1}),b!)$ (3.2); in fact,

$$\mathrm{Sing}([\mathrm{red}(C(J_{k+1}))],b!)\subseteq \mathrm{Sing}([\mathrm{red}(C(J_{i_0}))]_{k+1},b!)$$

and points of $\mathrm{Sing}(C(J_{k+1}),b!)\backslash\mathrm{Sing}([\mathrm{red}(C(J_{k+1}))],b!)$ are good ((3.1.2)). So condition ($\mathrm{C2}_{k+1}$) holds. Condition ($\mathrm{C1}_{k+1}$) is guaranteed by Proposition 4.22. We now define g^e_{k+1} as in ($\mathrm{C3}_{k+1}$), and finally condition ($\mathrm{C4}_{k+1}$) is vacuous within this case.

We can now check (6.3.3), namely that $g^e_k(\xi_k)\geqslant g^e_{k+1}(\xi_{k+1})$, with equality if $\xi_k\notin C_k$. This reduces to the following cases:

- $\mathrm{Ab}_k(\xi_k)=\mathrm{Ab}_{k+1}(\xi_{k+1})=B$, in which case the inequality follows from

$$\text{w-ord}_e(\mathfrak{F}_k)(\xi_k)\geqslant \text{w-ord}_e(\mathfrak{F}_{k+1})(\xi_{k+1})$$

and our definition of $n_k(\xi_k)$ and $n_{k+1}(\xi_{k+1})$.

- $\mathrm{Ab}_k(\xi_k)=\mathrm{Ab}_{k+1}(\xi_{k+1})=G$, in which case $g^e_k(\xi_k)=g^e_{k+1}(\xi_k)$. In fact, since $C_k\subseteq \underline{\mathrm{Max}}\, g^e_k$ and we are within Case A, we have $\xi_k\notin C_k$.

Now we define the criterion of choice of centers (B_{k+1}): $C(\mathcal{L}_{k+1})$ (notation as in Definition 6.3).

- If $\mathrm{pr}_1(\max g^e_{k+1})=B$ then $C\in C(\mathcal{L}_{k+1})$ if and only if C is a closed smooth subscheme of $\underline{\mathrm{Max}}\, g^e_{k+1}$, permissible for $\mathfrak{F}_{k+1}$.
- If $\mathrm{pr}_1(\max g^e_{k+1})=G$ then $C(\mathcal{L}_{k+1})=\{\underline{\mathrm{Max}}\, g^e_{k+1}\}$.

Case B. $\mathrm{pr}_1(\max g^e_k)=G$. By ($\mathrm{C2}_k$) all the points of $\mathrm{Sing}(\mathfrak{F}_k)$ are locally good points of $(C(J_k),b!)$ (3.2). ($\mathrm{C4}_k$) asserts that there is only one choice of center, namely $C_k=\underline{\mathrm{Max}}\, g^e_k$, so all the points of $\mathrm{Sing}(\mathfrak{F}_{k+1})$ will also be locally good points of $(C(J_{k+1}),b!)$.

Now, for $\xi_{k+1}\in\mathrm{Sing}(\mathfrak{F}_{k+1})$, we set $\mathrm{Ab}_{k+1}(\xi_{k+1})=G$ and

$$g^e_{k+1}(\xi_k)=(\mathrm{Ab}_{k+1}(\xi_{k_1}),\Gamma(\xi_{k+1})),$$

Γ as in (2.0.2).

We fix now the criterion of choice of centers to be $C(\mathcal{L}_{k+1})=\{\underline{\mathrm{Max}}\, g^e_{k+1}\}$. ($\mathrm{C1}_{k+1}$) is vacuous and ($\mathrm{C2}_{k+1}$), ($\mathrm{C3}_{k+1}$) and ($\mathrm{C4}_{k+1}$) will hold.

We can finally check that we have defined an assignment of chains and functions (Definition 6.3) satisfying, in addition, conditions (C1), (C2), (C3) and (C4) introduced in 6.17.

Step 5. Conditions (1) and (2) of 6.11 only apply to chains of length zero and they were shown to hold for this assignment in Step 1. Condition (6)(a) follows from Corollary 5.20, and (6)(c) follows from formulas (4.20.1), (4.20.2), (4.17.3) and the fact that they are all invariant by restrictions or arbitrary change of base field.

Note finally from (2.0.2) that the codimension of $\underline{\operatorname{Max}}\,\Gamma$ is given by the first coordinate of $\max\Gamma$, so (6)(b) follows from (C2) and (C3).

Step 6. Let $\mathcal{L}_k$ be a chain of length k ((6.3.1)) and set the index i_0 as in 6.11 (3).

Step 6.1. We consider the case $k=0$ and $F^e=\varnothing$, and construct a weak idealistic closed set $\mathfrak{F}'_0$ as in condition (3) of 6.11.

Step 6.2. Here we consider $k>0$ and $\mathrm{pr}_1(\max g_k^e)=B$ and construct a weak idealistic closed set $\mathfrak{F}'_{i_0}$ as in condition (3) of 6.11.

Step 6.3. We show that $\mathfrak{F}'_{i_0}$ is in fact an idealistic closed set.

First of all we need a previous lemma:

LEMMA 6.19. *Let W be a regular algebraic variety over k, and let V be a regular subvariety of W of codimension τ at a closed point $\xi\in V$.*

There exists an étale neighborhood U of ξ, $e:(U,\xi)\to(W,\xi)$, so that U admits a retraction on $e^{-1}(V)$.

Let $x_1,\dots,x_n$ be a regular system of parameters of $\mathcal{O}_{W,\xi}$ such that $I(V)_\xi=(x_1,\dots,x_\tau)$. We can define a morphism $f:W\to\mathbf{A}^n_k$ replacing W by an open neighborhood of ξ by setting $f^{\#}(X_i)=x_i$ for $i=1,\dots,n$, where $\mathbf{A}^n_k=\mathrm{Spec}(k[X_1,\dots,X_n])$. So we may assume, after suitable restriction, that f is an étale neighborhood of $(\mathbf{A}^n_k,\underline{0})$ and the subvariety $V=f^{-1}(\{X_1=0,\dots,X_\tau=0\})$. Let $r:\mathbf{A}^n_k\to\mathbf{A}^{n-\tau}_k$ be the natural retraction. The morphism

$$f'=r\circ(f|_V):V\to\mathbf{A}^{n-\tau}_k$$

is étale. Consider now the fiber product U':

$$\begin{tikzcd} W \arrow[r, "r\circ f"] & \mathbf{A}^{n-\tau}_k \\ U' \arrow[u, "e'"] \arrow[r, "r'"] & V. \arrow[u, "f'"'] \end{tikzcd}$$

Let $i:V\to W$ denote the inclusion and note that $f'=(r\circ f)\circ i$. Since U' is a fiber product, i induces a section of r', and we identify $e^{-1}(V)$ with the image of such a section. Finally note that the section defines a retraction.

6.20. A retraction of W on $V\,(\subseteq W)$ will allow us to lift an ideal $I\,(\subseteq \mathcal{O}_V)$ to, say $\tilde{I}\,(\subseteq \mathcal{O}_W)$.

Step 6.1. Now suppose that $k=0$, that F^e is empty, and set $\max g_0^e=(B,\omega,a)$. Note that $\omega\in\mathbf{Q}\geqslant 1$ and $a\in\mathbf{Z}\geqslant 0$. Locally at any closed point $\xi\in\underline{\operatorname{Max}}\, g_0^e$ there is an open set $U^{(i)}$, so that $\mathfrak{F}_0$ is defined (after restriction) by a basic object $(U^{(i)},(J^{(i)},b_i),E^{(i)})$; there is also a regular variety of codimension e, say $V^e\subseteq U^{(i)}$ (Definition 5.14), and a coefficient ideal $C(J^{(i)})$ in $\mathcal{O}_{V^e}$ (Definition 4.14).

Note that $\nu_\xi(C(J^{(i)}))=\omega b_i!\in\mathbf{Z}\geqslant 0$. At step $k=0$, $\omega b_i!\geqslant b_i!\geqslant 1$, so there exists a regular system of parameters $\bar{y}_{e+1},\dots,\bar{y}_n$ of $\mathcal{O}_{V^e,\xi}$ such that

$$\bar{y}_{e+1}\in\Delta^{\omega b_i!-1}(C(J^{(i)}))$$

(see 4.4). Denote by $\widetilde{C(J^{(i)})}$ the ideal obtained by lifting $C(J^{(i)})$ to $U^{(i)}$, or more exactly, to an étale neighborhood defined in terms of a retraction on V^e (Lemma 6.19), and set $y_j\in\mathcal{O}_{W,\xi}$ by lifting $\bar{y}_j\in\mathcal{O}_{V^e,\xi}$.

Denote $b_i'=\omega b_i!$. Now define at a suitable neighborhood of the closed point ξ the sheaf of ideals $\mathcal{B}_0^{(i)}$ so that

$$(\mathcal{B}_0^{(i)})_\xi=(x_1^{b_i'},\dots,x_e^{b_i'})+\widetilde{C(J^{(i)})}_\xi+(x_i^{b_i'}\mid I(H)_\xi=(x_i),\,\xi\in H\in E^{(i)}) \tag{6.20.1}$$

where $x_1,\dots,x_n$ is a regular system of parameters of $\mathcal{O}_{U^{(i)},\xi}$ as in Definition 5.14.

6.21. Let us note that:

(1) $\operatorname{In} y_{e+1}$ is linearly independent of $\operatorname{In} x_1,\dots,\operatorname{In} x_e$ in $\operatorname{Gr}(\mathcal{O}_{W,\xi})$ (4.4).

(2) $(x_1,\dots,x_e,y_{e+1})\subseteq\Delta^{b_i'-1}(\mathcal{B}_0^{(i)})$.

(3) $\operatorname{Sing}(\mathcal{B}_0,b_i')=\underline{\operatorname{Max}}\, g_0^e(\mathfrak{F})$ (after restriction of $\underline{\operatorname{Max}}\, g_0^e(\mathfrak{F})$ to a suitable open neighborhood of ξ).

These three conditions hold at an étale neighborhood. In particular, there is a weak idealistic closed set, $\mathfrak{F}_0'$, such that:

- $\mathfrak{F}_0'$ is locally defined by the basic objects $(U^{(i)},(\mathcal{B}_0^{(i)},b_i'),\varnothing)$ (Definition 5.1).
- From the definition of g_0^e in Step 1, it can be checked that $\operatorname{Sing}(\mathfrak{F}_0')=\underline{\operatorname{Max}}\, g_0^e(\mathfrak{F})$.
- $\mathfrak{F}_0'$ is of codimension $\geqslant e+1$ by 6.21, and $E_0'=\varnothing$.

If $C\in C(\mathcal{L}_0)$ then $C\subseteq\operatorname{Sing}(\mathfrak{F}_0')$ and has normal crossings with $E_0'=\varnothing$. Conversely, if $C\subseteq\operatorname{Sing}(\mathfrak{F}_0')$ then C has normal crossings with E_0. In fact, locally at any closed point $\xi\in C$, C is contained in the intersection $\bigcap_{\xi\in H\in E_0}H$. So if $\mathfrak{F}_0\leftarrow\mathfrak{F}_1$ is the transformation with center $C\in C(\mathcal{L}_0)$, then C is permissible for $\mathfrak{F}_0'$. Set $(U_1^{(i)},(\mathcal{B}_1^{(i)},b_i'),E_1')$ as the transform of $(U^{(i)},(\mathcal{B}_0^{(i)},b_i'),E_0')$. $\mathcal{B}_1^{(i)}$ can be written as

$$\mathcal{B}_1^{(i)}=\mathcal{X}_1+\widetilde{C(J^{(i)})}_1+\mathcal{X}_1'$$

where $\mathcal{X}_1$ and $\mathcal{X}'_1$ are the transform of the first and the third term of the sum, respectively, in the formula (6.20.1). One can check that either $\max g_0^e > \max g_1^e$, in which case $\mathrm{Sing}(\mathcal{B}_1^{(i)}, b'_i) = \varnothing$, or $\max g_0^e = \max g_1^e$, in which case $\mathrm{Sing}(\mathcal{B}_1^{(i)}, b'_i) = \underline{\mathrm{Max}}\, g_1^e \cap U_1^{(i)}$.

Step 6.2. Suppose $k>0$ and $\max g_k^e = (B, \omega, a)$. Recall that i_0 is defined in terms of k as in 6.11 (3). Locally at any closed point $\xi \in \underline{\mathrm{Max}}\, g_j^e$, $i_0 \leqslant j \leqslant k$, $\mathfrak{F}_j$ is defined by a basic object $(U_j, (J_j, b), E_j)$. Set $b' = \omega b!$ and note that $b' \in \mathbf{Z} \geqslant 0$ but ω can be smaller than 1. Set also $b'' = b' b!$.

Case $k = i_0$. Recall that $g_{i_0}^e = (\mathrm{Ab}_{i_0}, \bar{g}_{i_0}^e)$ and that formula (6.18.1) applies for Ab_{i_0}. We define at a suitable neighborhood of a closed point $\xi \in \underline{\mathrm{Max}}\, g_{i_0}^e$ the sheaf of ideals $\mathcal{B}_{i_0}$:

$$\begin{aligned}(\mathcal{B}_{i_0})_\xi = (x_1^{b''}, \ldots, x_e^{b''}) + (\tilde{\mathcal{A}}_{i_0})_\xi^{b''/b'} + (\widetilde{C(J_{i_0})})_\xi^{b''/b!} \\ + (x_i^{b''} \mid I(H)_\xi = (x_i),\, H \in E_{i_0}^-(\xi)) + \widetilde{\mathrm{red}}(C(J_{i_0}))_\xi^{b''/b!}\end{aligned} \tag{6.21.1}$$

where $x_1, \ldots, x_n$ is a regular system of parameters of $\mathcal{O}_{U_{i_0}, \xi}$ as in Definition 5.14, $\mathcal{A}_{i_0}$ and $\mathrm{red}(C(J_{i_0}))$ are as in (6.16.1), and $E_{i_0}^-(\xi)$ as in (D_{i_0}) (6.17). Note that $\#(E_{i_0}^-(\xi))$ is constant (and equal to a) along a neighborhood of $\xi = \xi_{i_0}$ in $\underline{\mathrm{Max}}\, g_{i_0}^e$. We now want to define a structure of an idealistic closed set of codimension $\geqslant e+1$ (in the setting of Definition 5.14) in terms of (6.21.1). So first set $\xi_{i'_0} \in \mathrm{Sing}(\mathfrak{F}_{i'_0})$ as the image of $\xi_{i_0} \in \mathrm{Sing}(\mathfrak{F}_{i_0})$, where i'_0 is as in the remark in 6.17, so $\max \text{w-ord}_{i'_0} = \ldots = \max \text{w-ord}_{i_0}$. Locally at $\xi_{i'_0}$ we may argue as in Step 6.1 to find a regular system of parameters $x'_1, \ldots, x'_e, y'_{e+1}, \ldots, y'_n$ so that $\bar{y}'_{e+1} \in \Delta^{b'-1}(\mathcal{A}_{i'_0})$ where $\bar{y}'_{e+1}$ denotes the image of y'_{e+1} in $\mathcal{O}_{V^e_{i'_0}, \xi_{i'_0}}$.

Applying Corollary 4.6 to the induced sequence

$$(V_{i'_0}^e, (\mathcal{A}_{i'_0}, b'), \varnothing) \leftarrow \ldots \leftarrow (V_{i_0}^e, (\mathcal{A}_{i_0}, b'), \overline{E}_{i_0}^+)$$

(recall that $\nu_{\xi_{i'_0}}(\mathcal{A}_{i'_0}) = \ldots = \nu_{\xi_{i_0}}(\mathcal{A}_{i_0}) = b'$), and setting $I(V') = (\bar{y}'_{e+1}) \subseteq \Delta^{b'-1}(\mathcal{A}_{i'_0})$, we may assume that there is a regular system of parameters $\{x_1, \ldots, x_e, y_{e+1}, \ldots, y_n\}$ locally at ξ_{i_0} so that $x_1, \ldots, x_e, y_{e+1}$ are strict transforms of $x'_1, \ldots, x'_e, y'_{e+1}$. Setting E_j^- as in the remark in 6.17 and $E_j^+ = E_j \setminus E_j^-$, we may also assume that:

- $\bar{y}_{e+1} \in \Delta^{b'-1}(\mathcal{A}_{i_0})$.
- If $\xi_{i_0} \in H \in E_{i_0}^+$ then $I(H \cap V_{i_0}^e)_{\xi_{i_0}} = (y_{s_H})$ for some $s_H > e+1$.

Now we argue as in Step 6.1 (6.21) to check that there is a weak idealistic closed set $\mathfrak{F}'_{i_0}$ such that:

- $\mathfrak{F}'_{i_0}$ is locally defined by the basic object $(U_{i_0}, (\mathcal{B}_{i_0}, b''), E'_{i_0})$, where now $E'_{i_0} = E_{i_0}^+$.
- $\mathrm{Sing}(\mathfrak{F}'_{i_0}) = \underline{\mathrm{Max}}\, g_{i_0}^e$.
- $\mathfrak{F}'_{i_0}$ is of codimension $\geqslant e+1$ and $E'_{i_0} = E_{i_0}^+$.
- $C \in C(\mathcal{L}_{i_0})$ if and only if C is permissible for $\mathfrak{F}'_{i_0}$.

Case $i_0<k$. In this case formula (6.18.2) applies for Ab_k, where $g_k^e=(\mathrm{Ab}_k,\bar{g}_k^e)$. For $i_0\leqslant j\leqslant k$, $\max g_j^e=(B,\omega,a)$. If $(U_j,(\mathcal{B}_j,b''),E_j')$ is the transform of $(U_{i_0},(\mathcal{B}_{i_0},b''),E_{i_0}')$ and $\xi\in\underline{\mathrm{Max}}\,g_j^e$, then

$$\begin{aligned}(\mathcal{B}_j)_\xi=(x_1^{b''},\dots,x_e^{b''})+(\tilde{A}_j)_\xi^{b''/b'}+(\widetilde{[C(J_{i_0})]_j})_\xi^{b''/b!}\\+(x_i^{b''}\mid I(H)_\xi=(x_i),\,H\in E_j^-(\xi))+\widetilde{\mathrm{red}}([C(J_{i_0})]_j)_\xi^{b''/b!}\end{aligned}\tag{6.21.2}$$

where $x_1,\dots,x_n$ is a regular system of parameters of $\mathcal{O}_{U_j,\xi}$ as in Definition 5.14. The same argument as in Case $k=i_0$ applies to show that $\#(E_j^-(\xi))=a$ for any point of $\underline{\mathrm{Max}}\,g_j^e$ (D(j) in 6.17). It is easy to check that $\mathrm{Sing}(\mathcal{B}_j,b'')=\underline{\mathrm{Max}}\,g_j^e\cap U_j$, so that the different basic objects $(U_j,(\mathcal{B}_j,b''),E_j')$ define a weak idealistic closed set $\mathfrak{F}_j'$ such that $\mathrm{Sing}(\mathfrak{F}_j')=\underline{\mathrm{Max}}\,g_j^e$. In fact, $\mathfrak{F}_j'$ is the transform of $\mathfrak{F}_{i_0}'$. As in the case above, we argue as in the previous case to see that $C\in C(\mathcal{L}_j)$ if and only if C is a permissible center for $\mathfrak{F}_k'$ (Definition 5.4).

Induction. If we consider an enlargement to a chain of length $k+1$ by a choice of a center $C_k\in C(\mathcal{L}_k)$, then formula (6.21.2) holds for $j=k+1$ to define $\mathfrak{F}_{k+1}'$ such that

$$\begin{aligned}\max g_k^e>\max g_{k+1}^e\quad&\Leftrightarrow\quad \mathrm{Sing}(\mathfrak{F}_{k+1}')=\varnothing,\\ \max g_k^e=\max g_{k+1}^e\quad&\Rightarrow\quad \mathrm{Sing}(\mathfrak{F}_{k+1}')=\underline{\mathrm{Max}}\,g_{k+1}^e.\end{aligned}$$

Step 6.3. We have shown that formula (6.21.2) and the equalities $\mathrm{Sing}(\mathfrak{F}_j')=\underline{\mathrm{Max}}\,g_j^e$ hold after transformations. It can be checked that these equalities are preserved also by restrictions. We conclude that $\mathfrak{F}_{i_0}'$ is, in fact, an idealistic closed set of codimension $\geqslant e+1$.

Step 7. Condition 6.11 (4) follows from the definition of g_k^e given in ($\mathrm{C3}_j$) and the process defined in §§ 2 and 3. Condition (5) of 6.11 reduces to the case $\mathrm{pr}_1(\max g_N^e)=B$ for all N by (4)(b) and (4)(c) of 6.11. Note that the definition of $\max g_k^e$ is in terms of the functions w-ord$_e$ and n. Note that w-ord$_e$ takes values in $\mathbf{Z}/m$ for some m big enough; in fact, the index i in (5.18.1) will range in the finite set I (Definition 5.1 (1)). On the other hand, n takes values between $0,1,\dots,\dim W$. So $\max g_N^e$ cannot improve (decrease) infinitely many times; in particular, $\max g_N=\max g_{N+1}=\dots$ for some N big enough in the sequence in 6.11 (5). This proves condition (5) of 6.11.

Remark 6.22. Setting $\mathrm{Ab}_0(\xi)=B$, $\forall\xi\in\mathrm{Sing}(\mathfrak{F}_0)$ (see Step 1 of 6.14), and

$$\mathrm{Ab}(\xi_k)=\begin{cases}B & \text{if w-ord}_e(\xi_k)>0,\\ G & \text{if w-ord}_e(\xi_k)=0\end{cases}$$

(see Step 4 of 6.14), we recover the algorithm introduced in [V1]. In such case the first coordinate of g_k^e is G if and only if $C(J)_k$ is monomial in $\mathcal{O}_{V_k^e}$ ((2.0.1)), and if the first coordinate is B then the last term of the sum should be neglected in (6.21.1) and (6.21.2).

6.23. *Equivariance.* Let $\mathfrak{F}$ be an idealistic closed set of dimension n (Definition 5.5), and set W and $E=\{H_1,\dots,H_r\}$ as in Definition 5.1. In what follows we will consider an isomorphism $\Theta\colon W\to W$ (not necessarily of k-varieties) with the additional condition that $\Theta(H_i)=H_i$ for any hypersurface $H_i\in E$. Such an isomorphism defines naturally a restriction of $\mathfrak{F}$ (5.2), say $\mathfrak{F}^\Theta$, now with the same W and the same $E=\{H_1,\dots,H_r\}$. We shall say that Θ acts on the idealistic closed set $\mathfrak{F}$ if in addition $\mathfrak{F}$ is equivalent to $\mathfrak{F}^\Theta$ (Definition 5.11).

Fix W, E and Θ as before. If $\Theta(X)=X$, where X is a subvariety of W, then one can easily check that Θ acts on the n-dimensional idealistic closed set $\mathfrak{F}$ defined in Theorem 6.6.

The combination of properties (1) and (3) in 6.9 says that, setting

$$\mathfrak{F}_j=(W_j,F_j,E_j,\{U_j^{(i)}\to W_j\}_{i\in I},\{(J_j^{(i)},b_i)\})$$

as the transforms of $\mathfrak{F}=\mathfrak{F}_0$ defined in terms of the resolution, then:

(1) For any $\xi_j\in\mathrm{Sing}(\mathfrak{F}_j)$, $\Theta(\xi_j)\in\mathrm{Sing}(\mathfrak{F}_j)$ (Definition 5.11).

(2) $f(\mathfrak{F}_j^\Theta)(\xi_j)=f(\mathfrak{F}_j)(\Theta(\xi_j))$ (6.9 (3)).

(3) $f(\mathfrak{F}_j)(\xi_j)=f(\mathfrak{F}_j)(\Theta(\xi_j))$ (6.9 (1)).

Finally (3) implies that

(4) $\Theta(\underline{\mathrm{Max}}\, f(\mathfrak{F}_j))=\underline{\mathrm{Max}}\, f(\mathfrak{F}_j)$,

which asserts that any such Θ will lift to the resolution of $\mathfrak{F}$ (i.e. will act on each $\mathfrak{F}_i$), and ultimately that the embedded desingularization defined by the the algorithm is equivariant.

References

[Ab1] ABHYANKAR, S. S., Good points of a hypersurface. *Adv. in Math.*, 68 (1988), 87–256.

[Ab2] — Analytic desingularization in characteristic zero. Preprint, 1996.

[AHV1] AROCA, J. M., HIRONAKA, H. & VICENTE, J. L., *The Theory of Maximal Contact.* Memorias de Matemática del Instituto Jorge Juan, 29. Consejo Superior de Investigaciones Científicas, Madrid, 1975.

[AHV2] — *Desingularization Theorems.* Memorias de Matemática del Instituto Jorge Juan, 30. Consejo Superior de Investigaciones Científicas, Madrid, 1977.

[AJ] ABRAMOVICH, D. & JONG, A. J. DE, Smoothness, semistability and toroidal geometry. *J. Algebraic Geom.*, 6 (1997), 789–801.

[AW] ABRAMOVICH, D. & WANG, J., Equivariant resolution of singularities in characteristic 0. *Math. Res. Lett.*, 4 (1997), 427–433.

[Ben] BENNETT, B. M., On the characteristic function of a local ring. *Ann. of Math.*, 91 (1970), 25–87.

[Ber] BERTHELOT, P., Altérations des variétés algébriques (d'après A. J. de Jong), in *Séminaire Bourbaki*, vol. 1995/96, exp. n° 815. *Astérisque*, 241 (1997), 273–311.

[BM1] BIERSTONE, E. & MILMAN, P., A simple constructive proof of canonical resolution of singularities, in *Effective Methods in Algebraic Geometry* (Castiglioncello, 1990), pp. 11–30. Progr. Math., 94. Birkhäuser, Boston, MA, 1991.

[BM2] — Canonical desingularization in characteristic zero by blowing up the maximal strata of a local invariant. *Invent. Math.*, 128 (1997), 207–302.

[BP] BOGOMOLOV, F. & PANTEV, T., Weak Hironaka theorem. *Math. Res. Lett.*, 3 (1996), 299–307.

[Ca] CANO, F., *Desingularización de Superficies.* Seminario Iberoamericano de Matemáticas (Singularidades en Tordesillas), Fascículo 1. Valladolid, 1995.

[CGO] COSSART, V., GIRAUD, J. & ORBANZ, U., *Resolutions of Surface Singularities.* Lecture Notes in Math., 1101. Springer-Verlag, Berlin–New York, 1984.

[E] ENCINAS, S., Resolución constructiva de singularidades de familias de esquemas. Ph.D. Thesis, Universidad de Valladolid, 1996.

[EV] ENCINAS, S. & VILLAMAYOR, O., Constructive desingularization and equivariance: Introductory notes, in *Working Week on Mountains and Singularities* (Obergurgl, 1997).

[G1] GIRAUD, J., *Étude locale des singularités.* Publ. Math. Orsay, 26. U.E.R. Mathématique, Université Paris XI, Orsay, 1972.

[G2] — Sur la théorie du contact maximal. *Math. Z.*, 137 (1974), 285–310.

[G3] — Contact maximal en caractéristique positive. *Ann. Sci. École Norm. Sup.* (4), 8 (1975), 201–234.

[G4] — Remarks on desingularization problems. *Nova Acta Leopoldina*, 52 (240) (1981), 103–107.

[Ha] HARTSHORNE, R., *Algebraic Geometry.* Graduate Texts in Math., 52. Springer-Verlag, New York, 1983.

[Hi1] HIRONAKA, H., Resolution of singularities of an algebraic variety over a field of characteristic zero, I; II. *Ann. of Math.*, 79 (1964), 109–203; 205–326.

[Hi2] — Idealistic exponents of singularity, in *Algebraic Geometry* (Baltimore, MD, 1976), pp. 52–152. John Hopkins Univ. Press, Baltimore, MD, 1977.

[HIO] HERRMANN, M., IKEDA, S. & ORBANZ, U., *Equimultiplicity and Blowing Up.* Springer-Verlag, Berlin–New York, 1988.

[J] JONG, A. J. DE, Smoothness, semi-stability and alterations. Preprint 916, University of Utrecht, 1995.

[LT1] LEJEUNE-JALABERT, M. & TEISSIER, B., Normal cones and sheaves of relative jets. *Compositio Math.*, 28 (1974), 305–331.

[LT2] — *Quelques calculs utiles pour la résolution des singularités.* Centre de Mathématiques de l'École Polytechnique, 1971.

[M] MOH, T. T., Canonical uniformization of hypersurface singularities of characteristic zero. *Comm. Algebra*, 20 (1992), 3207–3251.

[O] ODA, T., Infinitely very near-singularity points (complex analytic singularities). *Adv. Stud. Pure Math.*, 8 (1986), 363–404.

[S] SPIVAKOVSKY, M., Resolution of singularities, in *Journées singulières et jacobiennes.* Institut Fourier, Grenoble, 1996.

[V1] VILLAMAYOR, O., Constructiveness of Hironaka's resolution. *Ann. Sci. École Norm. Sup.* (*4*), 22 (1989), 1–32.

[V2] — Patching local uniformizations. *Ann. Sci. École Norm. Sup.* (*4*), 25 (1992), 629–677.

[V3] — On good points and a new canonical algorithm of resolution of singularities, in *Real Analytic and Algebraic Geometry* (Trento, 1992), pp. 277–291. de Gruyter, Berlin–New York, 1995.

[V4] — Introduction to the algorithm of resolution, in *Algebraic Geometry and Singularities* (La Rábida, 1991), pp. 123–154. Progr. Math., 134. Birkhäuser, Basel, 1996.

[Z] ZARISKI, O., Local uniformization on algebraic varieties. *Ann. of Math.*, 41 (1940), 852–860.

S. ENCINAS
Departamento de Matemática Aplicada F.
E.T.S. Arquitectura
Universidad de Valladolid
Avda. de Salamanca, s/n
E-47014 Valladolid
Spain
sencinas@cpd.uva.es

O. VILLAMAYOR
Departamento de Matemáticas
Universidad Autonoma de Madrid
Ciudad Universitaria de Canto Blanco
E-28049 Madrid
Spain
villamayor@uam.es

Received November 25, 1996
Received in revised form December 2, 1997