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On discrete Möbius groups in all dimensions:
A generalization of Jørgensen's inequality

by

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Introduction

In this paper we present a generalization to all dimensions of the following sharp inequality of Jørgensen [Jø] concerning discrete nonelementary groups:

THEOREM (Jørgensen's inequality). *Let f and g be Möbius transformations of the Riemann sphere. If f and g together generate a discrete nonelementary group, then*

$$|\mathrm{tr}^2(f)-4|+|\mathrm{tr}[f,g]-2|\geq 1.$$

Here $[f,g]=fgf^{-1}g^{-1}$ is the multiplicative commutator and we are identifying f and g with their matrix representatives in $\mathcal{SL}_2\mathbf{C}$. Nonelementary in this setting means not virtually Abelian.

It is the following principle that makes Jørgensen's inequality such a valuable tool.

If two Möbius transformations f and g generate a nonelementary discrete group, then given f, g cannot be too close to the identity.

In higher dimensions the trace seems not to be such a good invariant and moreover we must work in the Lie group $\mathcal{O}^+(1,n)$. Here is the generalization we propose:

THEOREM. *Let f and g be Möbius transformations of* $\mathbf{S}^n$. *If f and g together generate a discrete nonelementary group, then*

$$\max\{\|g^ifg^{-i}-\mathrm{Id}\|: i=0,1,2,..,n\}\geq 2-\sqrt{3}.$$

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Here we are identifying f and g with their matrix representatives in $\mathcal{O}^+(1, n+1)$, Id denotes the identity matrix and $\|\cdot\|$ denotes the Hilbert–Schmidt norm.

We will actually produce a slightly stronger inequality in that if f is not elliptic, then we need only consider the maximum of the terms with $i=0, 1$ (see Corollary 4.3). However the important feature to note from the inequality is that the principle espoused above remains valid in all dimensions, and thus provides the necessary tool to develop certain aspects of the theory of Kleinian groups to higher dimensions. Notice too that in the case of Fuchsian groups we are considering the group $\mathcal{O}^+(1,2)$ and there are exactly two terms in the inequality, corresponding closely to those of Jørgensen's inequality.

There is the drawback that the formulation is not conjugacy invariant (as Jørgensen's inequality is). However this can easily be overcome (see Corollary 4.7). Furthermore, the inequality seems not to be sharp, although it can be improved by considering norms other than the Hilbert–Schmidt norm (see Proposition 4.11). For instance the operator norm (maximum eigenvalue) can be used, this essentially frees both sides of the inequality from dimension and is especially useful in certain circumstances [Mar].

As applications, following Chuchrow [Ch], Jørgensen [Jø], Weilenberg [We] and others [J.K.], [J.M.], we will consider the algebraic limits of discrete nonelementary Möbius groups and using these new inequalities show that under mild (and necessary) restrictions these limits are again discrete and nonelementary, Proposition 5.7. A simple consequence of our results is a proof in all dimensions of the well known fact in dimension two:

A Möbius group is discrete if and only if all its two generator subgroups are.

We then apply these results to the deformation theory of discrete subgroups of the Möbius group validating the following principle:

One cannot continuously deform a discrete nonelementary group, through discrete groups, out of its isomorphism class.

From this we obtain what might be considered a generalization of Weil rigidity to geometrically finite discrete subgroups of the Möbius group, Corollary 7.1. The results along this line are not quite straightforward generalizations of the two dimensional theory. A key ingredient in the low dimensional theory is that finitely generated Kleinian groups are finitely presented [Se] and [Sc]. This is at present unknown for discrete Möbius groups in higher dimensions.

In a companion paper [Mar] we have used the results herein to obtain new lower bounds for the volume of all hyperbolic n-manifolds and in particular a lower bound on the size of an inscribed ball in the Dirichlet region of any discrete group of hyperbolic isometries. The existence of a lower bound is due to Kazdan and Margulis [K.M.].

Our methods for the most part are quite geometric and the inequality we develop and its consequences will hold valid for nonsolvable discrete subgroups of rank one Lie groups and (with some necessary modifications) more generally for isometry groups of negatively curved visibility manifolds. There are two key steps in this generalization. Firstly discrete subgroups of the isometry group of such a space act as so called convergence groups [G.M.1] on the sphere at infinity. These groups are the natural generalizations of uniformly quasiconformal groups and are very much like conformal groups (at least topologically). This observation enables most of the geometric ideas used in the hyperbolic setting (at described herein) to go over to the more general case. Secondly we need a general version of the existence of Zassenhaus neighbourhoods for Lie groups. Such generalizations have been found by Buser and Karcher [B.K.] in their work on Gromov's almost flat manifolds and also by Ballman, Gromov and Schroeder [B.G.S.] in their study of manifolds of nonpositive curvature and, in particular, the Kazdan–Margulis phenomena. The details in this more general setting are rather longer and will appear elsewhere.

The classical theory of discrete subgroups of Lie groups follows the lines laid down by Zassenhaus in his formulation of Bieberbach's solution to Hilbert's problem 18 concerning the finiteness of the isomorphism types of crystallographic groups. They noted that in a Lie group the iterated commutator is contracting in a neighbourhood of the identity and used this fact together with Jordan's lemma and discreteness to find conditions to imply nilpotency. Indeed, even Jørgensen's argument follows this line of approach. We will have to generalize these ideas a little to the case that the group is virtually Abelian as opposed to nilpotent. It is therefore no surprise that we are led to consider carefully the existence of Zassenhaus neighbourhoods.

I would like to acknowledge that I was led to consider this problem in relation to ongoing research with F. W. Gehring concerning other generalizations of Jørgensen's inequality for Kleinian groups, [G.M.2,3]. Also, as should be apparent, our exposition and results here owe a great deal to Jørgensen's original paper [Jø].

§ 1. The Möbius group, Möb(n)

We denote by $\mathbf{B}^n$ the closed unit ball in n-dimensional Euclidean space $\mathbf{R}^n$. The boundary of this ball is the $(n-1)$-sphere $\mathbf{S}^{n-1}$. The Riemannian metric

$ds^2=|dx|^2/(1-|x|^2)^2$ has constant curvature equal to -1 and gives rise to the complete *hyperbolic metric* of int($\mathbf{B}^n$). The geometry of this space is well known, however we note here that the geodesics of this metric are those subarcs of circles or lines lying in $\mathbf{B}^n$ and orthogonal to $\mathbf{S}^{n-1}$. More generally, the *affine* or *totally geodesic* subspaces are the intersections of codimension one spheres or hyperplanes orthogonal to $\mathbf{S}^{n-1}$, with $\mathbf{B}^n$.

There is another useful metric on $\mathbf{B}^n$, namely the restriction of the Euclidean metric to the ball. This metric extends to the boundary sphere $\mathbf{S}^{n-1}$ and there we will call it the *chordal distance*. For two points x and y of $\mathbf{S}^{n-1}$ we denote by $|x-y|$ the chordal distance between them.

By a *Möbius transformation* we mean a conformal self mapping of $\mathbf{B}^n$. Each Möbius transformation is the finite composition of reflections in codimension one spheres orthogonal to the boundary of $\mathbf{B}^n$. The group of all Möbius transformations of the ball, which we denote by Möb(n), is precisely the group of all hyperbolic isometries of $\mathbf{B}^n$. A Möbius transformation which stabilizes the origin is an orthogonal transformation and so an isometry in the chordal metric. If g is a Möbius transformation, then $g(x)=Q\cdot\sigma(x)$, where σ is a reflection in a sphere orthogonal to $\mathbf{S}^{n-1}$ and Q is an orthogonal transformation. Thus every Möbius transformation is naturally identified with a conformal mapping of the boundary $\mathbf{S}^{n-1}$. The classical Poincaré extension provides a converse to this. For proofs of these and other relevant facts we might use concerning Möbius transformations we refer the reader to the books by L. V. Ahlfors [Ah] and A. Beardon [Be]. For completeness we recall here some details that will be paticularly important to us:

Another model for hyperbolic geometry comes from the hyperboloid model. Let

$$\mathcal{Q}=\{(x_0,x_1,\ldots,x_n)\in\mathbf{R}^{n+1}\colon q(x,x)=1,\ x_0>0\},$$

where $q(x,y)=x_0y_0-x_1y_1-x_2y_2-\ldots-x_ny_n$ is the usual quadratic Lorentz form. Then $\mathcal{Q}$ is one sheet of the hyperboloid of two sheets and the map

$$(x_0,x_1,\ldots,x_n)\to\frac{1}{(1+x_0)}(x_1,x_2,\ldots,x_n)$$

is an isometry between the metric induced on $\mathcal{Q}$ from the quadratic form, that is

$$ds^2=dx_1^2+dx_2^2+\ldots+dx_n^2-dx_0^2,$$

and the hyperbolic metric of the ball. The isometries of $\mathcal{Q}$ are precisely the $(n+1)\times(n+1)$ matrices which preserve the quadratic form $q(x,x)$ and the upper half space $x_0>0$. The group of all such matrices is the rank one Lie subgroup $\mathcal{O}^+(1,n)$ of the general linear group $\mathcal{GL}(n+1,\mathbf{R})$. Thus, $\mathcal{O}^+(1,n)$ is the group of all matrices $A=(a_{ij})$, $i,j=0,1,2,\dots,n$, such that $q(Ax,Ax)=q(x,x)$ and the top left entry of A, $a_{00}>0$. If J is the matrix

$$J=\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & -\mathrm{Id}_n & \\ 0 & 0 & \dots & \end{pmatrix}$$

then for all matrices A in $\mathcal{O}^+(1,n)$,

$$AJA^{t}=J \quad \text{and} \quad A^{-1}=JA^{t}J. \tag{2.1}$$

The formula for the correspondence between a matrix $A=(a_{ij})$ and the Möbius transformation g is developed on page 51 of [Be]. It will be quite important to us. If $x\in\mathbf{B}^n$, then $y=g(x)$ has coordinates

$$y_j=\frac{(1+|x|^2)\,a_{0j}+2(x_1\,a_{1j}+\dots+x_n\,a_{nj})}{|x|^2(a_{00}-1)+2(x_1\,a_{10}+\dots+x_n\,a_{n0})+(a_{00}+1)}. \tag{2.2}$$

For any $n\times n$ matrix $B=(b_{ij})$ we define the Hilbert–Schmidt norm as

$$\|B\|^2=\sum_{i,j=0}^{n} b_{ij}^2. \tag{2.3}$$

The Möbius group with the topology of uniform convergence in the chordal metric of $\mathbf{B}^n$ is a geometric realization of $\mathcal{O}^+(1,n)$ with the topology induced by the norm $\|\cdot\|$. That is $\mathcal{O}^+(1,n)$ and $\mathrm{Möb}(n)$ are topologically isomorphic groups. This is simply because the hyperbolic ball is a model for the symmetric space of the Lie group $\mathcal{O}^+(1,n)$.

Given two Möbius transformations g and h of $\mathbf{B}^n$ we denote the distance between them as

$$d(g,h)=\sup\{|g(x)-h(x)|\colon x\in\mathbf{S}^{n-1}\}. \tag{2.4}$$

In order to geometrically relate the inequalities we develop between the matrix representatives and the Möbius transformations themselves, we wish to relate the two quantities $\|A-\mathrm{Id}\|$ and $d(g,\mathrm{Id})$, where A is the matrix in $\mathcal{O}^+(1,n)$ corresponding to the Möbius transformation g.

LEMMA 2.5. *Let $A=(a_{ij})$ correspond to the Möbius transformation g. Then*

(1) $$4(a_{00}^2-1)-2\operatorname{tr}A+2(n+1)=\|A-\mathrm{Id}\|^2$$

(2) $$|g(0)|^2=\frac{a_{00}-1}{a_{00}+1}$$

(3) $$|g(e_i)-e_i|^2=2\left(1-\frac{a_{0i}+a_{ii}}{a_{00}+a_{i0}}\right).$$

Proof. From (2.1) we see that the matrix A has the form

$$A=\begin{pmatrix}a_{00} & u\\ v & B\end{pmatrix} \quad\text{and}\quad A^{-1}=\begin{pmatrix}a_{00} & -v\\ -u & B^t\end{pmatrix}.$$

Hence

$$a_{00}^2-1=a_{10}^2+a_{20}^2+\ldots+a_{n0}^2,$$

and

$$a_{i0}^2+1=a_{i1}^2+a_{i2}^2+\ldots+a_{in}^2.$$

Consequently

$$\|A-\mathrm{Id}\|^2=\operatorname{tr}(A-\mathrm{Id})^t(A-\mathrm{Id})=\operatorname{tr}A^tA-2\operatorname{tr}A+(n+1).$$

Also

$$\begin{aligned}\operatorname{tr}A^tA&=\sum_{i,j=0}^{n}a_{ij}^2=a_{00}^2+\sum_{i=1}^{n}a_{i0}^2+\sum_{j=1}^{n}a_{0j}^2+\sum_{i,j=1}^{n}a_{ij}^2\\&=3a_{00}^2-2+\sum_{i=0}^{n}(a_{i0}^2+1)\\&=4a_{00}^2+(n-3)=4(a_{00}^2-1)+(n+1).\end{aligned}$$

Next, the formula (2.2) yields

$$g(0)_j=\frac{a_{0i}}{a_{00}+1}$$

so that

$$|g(0)|^2=\sum_{j=0}^{n}\frac{a_{0j}^2}{(a_{00}+1)^2}=\frac{a_{00}^2-1}{(a_{00}+1)^2}=\frac{a_{00}-1}{a_{00}+1}.$$

And finally

$$\begin{aligned}|g(e_i)-e_i|^2 &= |g(e_i)|^2+|e_i|^2-g(e_i)_i = 2(1-g(e_i)_i)\\ &= 2\left(1-\frac{a_{0i}+a_{ii}}{a_{00}+a_{i0}}\right).\end{aligned}$$

Henceforth, by a *Möbius group* we will mean subgroup of the group of all Möbius transformations of $\mathbf{B}^n$. The topology of such a subgroup will be that induced from the inclusion. It is well known that a discrete Möbius group acts properly discontinuously in $\operatorname{int}(\mathbf{B}^n)$. The *limit set* $L(G)$ of a Möbius group G is the set of all accumulation points of the *orbit* $G(x)=\{g(x): g\in G\}$ of any $x\in\operatorname{int}(\mathbf{B}^n)$. The *ordinary set* of G is $O(G)=\mathbf{S}^{n-1}-L(G)$. We note here that $L(G)$ consists of either 0, 1, 2 points or is an uncountable perfect set and in the latter case $L(G)$ is the smallest closed G-invariant set.

We will need to define what it means for a group to be nonelementary in higher dimensions and we need to recall the classification of the elementary discrete Möbius groups. Given a discrete Möbius group G, there are only three types of elements that occur. Namely, if $g\in G$, then either g is

elliptic: there is an integer m such that $g^m=\mathrm{Id}$;
parabolic: there is a unique fixed point $x_0\in\mathbf{S}^{n-1}$ such that

$$g^{\pm n}\to x_0 \text{ locally uniformly in } \mathbf{B}^n-\{x_0\} \text{ as } n\to\infty;$$

loxodromic: there are distinct fixed points x_0 and y_0 such that

$$g^{+n}\to x_0 \text{ locally uniformly in } \mathbf{B}^n-\{y_0\} \text{ as } n\to\infty$$

and

$$g^{-n}\to y_0 \text{ locally uniformly in } \mathbf{B}^n-\{x_0\} \text{ as } n\to\infty.$$

In the last case we say that x_0 is the attracting fixed point of g and y_0 is the repulsive fixed point.

If g is a Möbius transformation such that the group $\langle g\rangle=\{g^m: m\in\mathbf{Z}\}$ is not discrete, then we will call g an *irrational rotation*. Notice that simply by definition, if g is an irrational rotation, then there is a sequence of distinct integers $m(j)\to\infty$ such that $g^{m(j)}\to\mathrm{Id}$ as $j\to\infty$. The convergence is of course uniform on $\mathbf{B}^n$.

We say that a Möbius group is *nonelementary* if G contains two elements of infinite order with distinct fixed points and which are not irrational rotations (there is no assumption of discreteness). Otherwise G is *elementary*. It is relevant to note that the usual definition [Jø] of an elementary group being one for which every two elements of infinite order have a common fixed point, fails to be useful in higher dimensions. It is not difficult to construct a pair of irrational rotations with no common fixed points but which both invert the same line. Such a group is nonelementary (in the classical sense) but has an index two subgroup stabilizing a line and so the group itself contains no loxodromics or parabolics.

If G is a discrete nonelementary group, then the limit set of G is perfect and the fixed point pairs of loxodromic elements are pairwise dense in the limit set. Moreover, from the classical Schottky construction, we see that sufficiently high iterates of two loxodromic elements with distinct fixed points will generate a discrete subgroup isomorphic to the free group on two generators. The converse is also true; if G is a discrete Möbius group containing a subgroup isomorphic to a free group of rank two or more, then G is nonelementary. Firstly we need the following well known result which is an easy consequence of the fact that the fixed point set of an isometry is totally geodesic and that compact subgroups stabilize a common point.

PROPOSITION 2.6. *Let g be an elliptic or irrational rotation of* $\mathbf{B}^n$. *Then the fixed point set of g is a nonempty, complete affine subspace.*

Here is the classification of the elementary groups:

PROPOSITION 2.7. *Let G be a discrete elementary subgroup of* Möb(n), *the Möbius group of* $\mathbf{B}^n$. *Then G is virtually Abelian (contains an Abelian subgroup of finite index). Moreover, there is a number $\beta(n)$ such that G contains a solvable subgroup of index less than $\beta(n)$.*

If $L(G)=\emptyset$, then G is finite and G is conjugate into the orthogonal group.

If $L(G)=\{x_0\}$, then every element of infinite order in G is parabolic and G contains at least one such element. Furthermore G is conjugate into the Euclidean group and so the rank of a maximal torsion free Abelian subgroup is no more than n.

If $L(G)=\{x_0, y_0\}$, for distinct x_0 and y_0, then G contains an infinite cyclic subgroup of finite index and every element of G which is of infinite order is loxodromic, furthermore G contains one such element and is conjugate into the similarity group.

Proof. Most of this can already be found in Theorem 2.1 of [Tu] in Tukia's analysis of point stabilizers. We need only make a couple of observations. Firstly, Jordan's theorem states that there is a $\beta(n)$ such that any finite subgroup of $\mathcal{O}(n)$, the orthogonal group, has a normal Abelian subgroup of index less than $\beta(n)$. This together with the fact that the group is virtually Abelian implies the second claim. Secondly, there are the only three possibilities for the limit set, since this set is invariant, the cases $L(G)=\emptyset$ and $L(G)=\{x_0\}$ are covered, while in the last case the stabilizer of the points x_0 and y_0 has index either one or two in G.

Let $\sigma\colon \mathbf{B}^n \to \mathbf{B}^n$ be a Möbius transformation. We will say that σ *inverts a line* if there is a hyperbolic line ℓ which σ setwise fixes and whose endpoints are interchanged by σ. It follows from the classification of the elements in a Möbius group that σ inverts a line only if σ is either elliptic or an irrational rotation.

The following lemma is quite useful.

LEMMA 2.8. *Suppose that σ inverts a line. If $A \in \mathcal{O}^+(1,n)$ is the matrix representing σ, then*

$$\|A-\mathrm{Id}\| \geq 1/2.$$

Proof. Suppose σ inverts the line ℓ. Since σ is either elliptic or an irrational rotation, we see that σ has a unique fixed point x_0 on ℓ. Let V be the hyperbolically affine subspace of $\mathbf{B}^n$ perpendicular to ℓ and passing through x_0. Then V is invariant under σ, and separating in $\mathbf{B}^n$. Consequently σ interchanges the components of $\mathbf{B}^n - V$. Let z_0 be the most distant point of $\mathbf{S}^{n-1}$ from V in the chordal meric. Considering the tangent plane to V at x_0 and the fact that σ interchanges components, one easily finds that $|\sigma(z_0)-z_0|\sqrt{2}$. It is easy to see that we may assume by conjugating with an appropriate orthogonal transformation, that $z_0=e_1=(1,0,\ldots,0)$. The above inequality then yields $\sigma(e_1)_1<0$. From our formula (2.2) we obtain (as $a_{00}+a_{10}>0$)

$$a_{01}+a_{11}<0.$$

If $a_{11} \leq 1/2$, then we are done as $\|A-\mathrm{Id}\| > |a_{11}-1|$. Otherwise, $a_{01} < -1/2$ and we are again done as $\|A-\mathrm{Id}\| \geq |a_{01}|$.

Finally in this section we wish to establish the following theorem which will give us a geometric characterization of our later results, much in the same vein as those obtained in [G.M.3]. The geometric content of the result is contained in the existence of the topological isomorphism between the Möbius group and $\mathcal{O}^+(1,n)$.

THEOREM 2.9. *Let g be a Möbius transformation and $A=(a_{ij})$ be the corresponding representative in $\mathcal{O}^{+}(1,n)$. If $d(g,\mathrm{Id})<\delta$, then $\|A-\mathrm{Id}\|<2-\sqrt{3}$, where*

$$\delta=\frac{1}{2\sqrt{16+n}}.$$

Proof. Firstly, if $d(g,\mathrm{Id})<\delta$, then $|g(0)|^2<\delta^2$, and so from Lemma 2.5

$$\frac{a_{00}-1}{a_{00}+1}<\delta^2.$$

Thus

$$a_{00}^2-1<4\frac{\delta^2}{(1-\delta^2)^2}.$$

Next we have

$$|g(e_i)-e_i|^2<\delta^2$$

and so from Lemma 2.5

$$2\left(1-\frac{a_{0i}+a_{ii}}{a_{00}+a_{i0}}\right)<\delta^2.$$

This yields

$$\left(1-\frac{\delta^2}{2}\right)(a_{00}+a_{i0})<a_{ii}+a_{0i}. \tag{2.10}$$

We now observe that $d(g,\mathrm{Id})=d(g^{-1},\mathrm{Id})$. Considering the matrix A^{-1} we find the appropriate version of (2.10) (see the proof of Lemma 2.5 for the form of A^{-1}),

$$\left(1-\frac{\delta^2}{2}\right)(a_{00}-a_{i0})<a_{ii}-a_{i0}. \tag{2.11}$$

Adding (2.10) and (2.11) yields

$$\left(1-\frac{\delta^2}{2}\right)a_{00}<a_{ii}.$$

Thus

$$1-a_{ii}<\frac{\delta^2}{2}a_{00}$$

as $1-a_{00}\leqslant 0$. Summing gives

$$\sum_{i=0}^{n}(1-a_{ij})<n\frac{\delta^2}{2}a_{00}. \tag{2.12}$$

Combining this with our estimates for a_{00}^2-1 yields

$$\begin{aligned}4(a_{00}^2-1)+2\sum_{i=0}^{n}(1-a_{ii}) &< 16\frac{\delta^2}{(1-\delta^2)^2}+n\delta^2\frac{1+\delta^2}{1-\delta^2}\\ &< (16+n)\frac{\delta^2}{(1-\delta^2)^2}.\end{aligned}$$

Finally observe that the left hand side is $\|A-\mathrm{Id}\|^2$ by Lemma 2.5, and by our choice of δ the right hand side is less than $2-\sqrt{3}$.

§ 3. Zassenhaus neighbourhoods

In this section we introduce the key concept of Zassenhaus neighbourhoods for Lie groups. We will want to calculate the size of such a neighbourhood for the group $\mathcal{O}^+(1,n)$.

Definition 3.1. Let U be a neighbourhood of the *identity* e of a Lie group $\mathcal{G}$. We say that U is a *Zassenhaus neighbourhood* for $\mathcal{G}$ if it has the following property: Let Γ be any discrete subgroup of $\mathcal{G}$. Then $\Gamma\cap U$ lies in a connected nilpotent Lie subgroup of $\mathcal{G}$.

It is Zassenhaus' theorem that such neighbourhoods exist in arbitrary Lie groups. For more information on these neighbourhoods and their applications one should consult Chapter VIII of M. Raghunathan's book [Ra]. We will need to work through some parts of the chapter to get the estimates we need.

We recall that for two matrices A and B

$$\|A+B\|\leqslant\|A\|+\|B\| \quad\text{and}\quad \|AB\|\leqslant\|A\|\,\|B\|.$$

If $A=\mathrm{Id}+\xi$ with $\|\xi\|<1$, then the series

$$\mathrm{Id}+\sum_{n=1}^{\infty}(-1)^n\xi^n$$

converges uniformly to A^{-1}. Thus for any matrix X

$$\|A^{-1}X\| = \left\|X+\sum_{n=1}^{\infty}(-1)^n\xi^n X\right\| \leqslant \|X\|+\sum_{n=1}^{\infty}\|\xi^n\|\,\|X\| = \|X\|\left(1+\frac{\|\xi\|}{1-\|\xi\|}\right) = \frac{\|X\|}{1-\|\xi\|}.$$

Similarly

$$\|XA^{-1}\| \leqslant \frac{\|X\|}{1-\|\xi\|}.$$

Suppose now that $A=\mathrm{Id}+\xi$ and $B=\mathrm{Id}+\eta$, where $\max\{\|\xi\|,\|\eta\|\}<1$. Then

$$[A,B]-\mathrm{Id} = ABA^{-1}B^{-1}-\mathrm{Id} = (AB-BA)A^{-1}B^{-1} = (\xi\eta-\eta\xi)A^{-1}B^{-1}.$$

It follows by applying the above estimates twice, that

$$\begin{aligned}\|[A,B]-\mathrm{Id}\| &\leqslant \frac{\|\xi\eta-\eta\xi\|}{(1-\|\eta\|)(1-\|\xi\|)}\\ &\leqslant 2\frac{\|\xi\|\,\|\eta\|}{(1-\|\eta\|)(1-\|\xi\|)}.\end{aligned}$$

Consequently, if $\max\{\|\xi\|,\|\eta\|\}<r<2-\sqrt{3}$, then

$$\|[A,B]-\mathrm{Id}| \leqslant c\min(\|A-\mathrm{Id}\|,\|B-\mathrm{Id}\|\}, \tag{3.2}$$

where $c=r/(1-r)^2<1$.

THEOREM 3.3. *The set* $\Omega=\{A\in\mathcal{GL}(n,R)\colon\ \|A-\mathrm{Id}\|<2-\sqrt{3}\}$ *is a Zassenhaus neighbourhood for* $\mathcal{GL}(n,R)$.

Proof. Consider the exponential mapping $\exp\colon \mathcal{M}(n,R)\to\mathcal{GL}(n,R)$ defined by the series

$$x=\exp X=\sum_{k=0}^{\infty}\frac{X^k}{k!}.$$

This series converges absolutely and if $\|X\|<\ln 2$, then

$$\|\exp X-\mathrm{Id}\| \leqslant \sum_{k=1}^{\infty}\frac{\|X\|^k}{k!} \leqslant \exp\|X\|-1<1.$$

And if $\|x-\mathrm{Id}\|<1$, then the logarithmic series

$$X=\ln x=\sum_{k=1}^{\infty}\frac{(x-\mathrm{Id})^k(-1)^{k+1}}{k}$$

also converges absolutely. Consequently if we set

$$U=\{X\in\mathfrak{M}(n,R)\colon\ \|X\|<\ln 2\},$$

then one easily finds by rearranging the terms of the absolutely convergent series that $\ln\circ\exp|U$ is the identity mapping. Since both are continuous, both are homeomorphisms onto their images.

Next set

$$\Omega=\{x\in\mathcal{GL}(n,R)\colon\ \|x-\mathrm{Id}\|<2-\sqrt{3}\}.$$

If $x\in\Omega$, then

$$\|\ln x\|\leq\sum_{k=1}^{\infty}\|x-\mathrm{Id}\|^k\leq\frac{2-\sqrt{3}}{\sqrt{3}-1}<\ln 2.$$

Consequently, if we define

$$V=\{\ln x\colon\ x\in\Omega\},$$

then V is contained in U and $\exp V=\Omega$. Moreover, for all $x\in\Omega$ and $Y\in V$ if $y=\exp Y\in\Omega$, then

$$\begin{aligned}\|\mathrm{Ad}_x(Y)\|=\|\ln xyx^{-1}\|&=\left\|\sum_{k=1}^{\infty}\frac{(xyx^{-1}-\mathrm{Id})^k(-1)^{k+1}}{k}\right\|\\&=\left\|x\sum_{k=1}^{\infty}\frac{(y-\mathrm{Id})^k(-1)^{k+1}}{k}x^{-1}\right\|\\&\leq\|x\|\,\|x^{-1}\|\left\|\sum_{k=1}^{\infty}(y-\mathrm{Id})^k\right\|\\&\leq\frac{(1+2-\sqrt{3})^2(2-\sqrt{3})}{1-(2-\sqrt{3})}<\ln 2.\end{aligned}$$

Thus $\mathrm{Ad}_x(V)$ lies in U. Next, we have seen already in (3.2) that $[\Omega,\Omega]=\{[\omega,\eta]\colon\ \omega,\eta\in\Omega\}$ lies in Ω. That same estimate implies if F is any relatively compact subset of Ω and if we define the recursive sequence

$$F^0=F\quad\text{and}\quad F^{n+1}=[F,F^n],$$

and if $x\in F^{n+1}$, then

$$\|x-\mathrm{Id}\|\leq c^n(2-\sqrt{3}),$$

where c is a constant less than one and depends only on the diameter of F.

Now, the sets U, V and Ω satisfy all the desired properties of Lemma 8.20 in [Ra] and further Ω has the additional desired property for the proof of Theorem 8.16 in [Ra] and so by that theorem Ω is a Zassenhaus neighbourhood of the identity for $\mathcal{GL}(n,R)$.

The following corollaries are not the generalizations of Jørgensen's inequality we seek. We include them for their interest and to point out the usefulness of Zassenhaus neighbourhoods. The proofs are immediate from the definition and the fact that a discrete nonelementary group is not nilpotent.

COROLLARY 3.4. *Let A, B be two elements of $\mathcal{O}^+(1,n)$ generating a discrete group. Then*

$$\max\{\|A-\mathrm{Id}\|, \|B-\mathrm{Id}\|\} \geqslant 2-\sqrt{3},$$

unless $\langle A,B\rangle$ is an elementary nilpotent group.

The geometric formulation of the above is

COROLLARY 3.5. *Let g and h be two Möbius transformations generating a discrete subgroup of* Möb(n). *Then*

$$\max\{d(g,\mathrm{Id}), d(h,\mathrm{Id})\} \geqslant \frac{1}{2\sqrt{16+n}}.$$

unless $\langle g,h\rangle$ is an elementary nilpotent group.

Notice that in Corollary 3.4 the right hand bound is independent of dimension (of course the dimension is intrinsic to the norm on the left hand side), while in the geometric inequality Corollary 3.5 the left hand side is dimension free while the right hand side depends only on $\sqrt{n}$. We ask if the dependence on $\sqrt{n}$ of the left hand side of the inequality of Corollary 3.5 is really necessary?

The most general formulation of Corollary 3.4 (and one which we will have cause to use) in terms of any number k of generators (possibly $k=\infty$) is:

COROLLARY 3.6. *Let $\{A_i: i=1,2,\dots,k\}$ be a collection of elements of $\mathcal{O}^+(1,n)$ generating a discrete group G. Then*

$$\max\{\|A_i-\mathrm{Id}\|: i=1,2,\dots,k\} \geqslant 2-\sqrt{3}$$

unless G is an elementary nilpotent group.

The key to the inequality that we really seek is to replace B by $[A, B]$ in Corollary 3.4 and to get the weaker conclusion that $\langle A, B\rangle$ is virtually Abelian and so elementary. This is the content of the next section.

§ 4. The generalized Jørgensen inequality

Throughout this section we will abuse notation and think of a Möbius transformation g and its matrix representative $A \in \mathcal{O}^+(1, n)$ as being the same. Thus by $\|g-\mathrm{Id}\|$ we will mean $\|A-\mathrm{Id}\|$. Also by a *discrete group* we mean a discrete subgroup of Möb(n). To begin with we need a couple of lemmas.

LEMMA 4.1. *Suppose that f and g are two Möbius transformations generating a discrete group. If f is either parabolic or loxodromic and if the group $\langle f, g^{-1}fg\rangle$ is elementary, then the group $\langle f, g\rangle$ is also elementary.*

Proof. Let us consider the two cases that f is parabolic and loxodromic separately.

(i) Suppose that f is parabolic with parabolic fixed point x_0. As the group $\langle f, g^{-1}fg\rangle$ is elementary, x_0 is the unique point of the limit set and every element of this group stabilizes that point. In particular $g^{-1}fg(x_0)=x_0$. Consequently f fixes $g(x_0)$ and since f is parabolic, this is impossible unless $g(x_0)=x_0$. Thus the group generated by f and g is discrete and stabilizes x_0 and so is elementary.

(ii) Suppose that f is loxodromic with two fixed points x_0 and y_0. As above, since the group $\langle f, g^{-1}fg\rangle$ is elementary every element of this group must stabilize the set $\{x_0, y_0\}$. Thus $g^{-1}fg(\{x_0, y_0\})=\{x_0, y_0\}$ so that f stabilizes the set $g(\{x_0, y_0\})$. Since f is loxodromic we must have $g(\{x_0, y_0\})=\{x_0, y_0\}$. Thus the group generated by f and g stabilizes the set $\{x_0, y_0\}$. This easily implies that the group $\langle f, g\rangle$ is elementary.

LEMMA 4.2. *Let f and g be two Möbius transformations generating a discrete group. Suppose that f is elliptic and p is the dimension of the fixed point set of f. If the group*

$$G=\langle g^{-i}fg^{i}\colon i=0, 1, 2, \ldots, p+1\rangle$$

is elementary then either $\langle f, g\rangle$ is elementary or $\|f-\mathrm{Id}\|>1/2$.

Proof. We introduce some terminology. Given a set of points $\{x_0, x_1, \ldots, x_n\}$ the *span* of the points $\mathrm{sp}(x_0, x_1, \ldots, x_n)$ is the smallest (hyperbolically) affine subspace containing the points. Since we know what all the affine subspaces are, the span of the points $x_0, x_1, \ldots, x_n$ is the intersection of $\mathbf{B}^n$ and the sphere of smallest dimension containing the points and orthogonal to the boundary.

Let $V = \mathrm{fix}(f)$. As remarked, V is a nonempty affine subspace. We break the proof up into three parts.

(1) Suppose that G is finite. Then it is well known that there is an affine subspace F pointwise fixed by every element of G. Clearly F is a subspace of V. Let $x_0 \in F$. Then $x_0 = g^{-i}fg^i(x_0)$ for all $i = 0, 1, 2, \ldots, p+1$. Thus $g^i(x_0) \in V$ for all $i = 0, 1, 2, \ldots, p+1$. Set

$$x_i = g^i(x_0), \quad i = 0, 1, 2, \ldots, p+1.$$

Let j be the smallest integer such that $\mathrm{sp}(x_0, x_1, x_2, \ldots, x_j) = \mathrm{sp}(x_0, x_1, \ldots, x_j, x_{j+1})$. Such a minimal j exists since the $\{x_j\}$ are $p+2$ points (though not necessarily distinct) lying on the p-dimensional affine subspace V. Now, since g is an isometry of the space we have

$$g(\mathrm{sp}(x_0, x_1, x_2, \ldots, x_j) = \mathrm{sp}(g(x_0), g(x_1), g(x_2), \ldots, g(x_j)) = \mathrm{sp}(x_1, x_2, \ldots, x_j, x_{j+1}).$$

But $\mathrm{sp}(x_1, x_2, \ldots, x_j, x_{j+1})$ is a subspace of $\mathrm{sp}(x_0, x_1, x_2, \ldots, x_j)$ by the choice of j. A simple dimension count then implies

$$g(\mathrm{sp}(x_0, x_1, x_2, \ldots, x_j)) = \mathrm{sp}(x_0, x_1, x_2, \ldots, x_j).$$

Since $\mathrm{sp}(x_0, x_1, x_2, \ldots, x_j)$ is an affine space which is mapped to itself by g, we know that g has a fixed point in this space, or possibly in its boundary on $\mathbf{S}^{n-1}$. Since f stabilizes this entire subspace pointwise, f and g have a common fixed point and so the group they generate is elementary, as it is assumed discrete.

(2) Suppose that $L(G) = \{x_0\}$. Since the limit set is G-invariant we must have $f(x_0) = x_0$ and for all $i = 1, 2, \ldots, p+1$, $g^{-i}fg^i(x_0) = x_0$. Thus, as above if we set $x_i = g^i(x_0)$, then $\{x_i\}$ are $p+2$ points lying in the fixed point set of f. The argument given in (1) now applies and we find that $\langle f, g \rangle$ is elementary.

(3) Suppose that $L(G) = \{x_0, y_0\}$. Again as $L(G)$ is G invariant, every element of G must fix or interchange the points x_0 and y_0. If every element of G pointwise fixes x_0 and y_0 then the argument given in (2) suffices to deduce that the group $\langle f, g \rangle$ is elementary. If there is some element of G which does not fix x_0 and y_0 then there must be a generator which does not. Thus for some $i \in \{0, 1, 2, \ldots, p+1\}$, $g^{-i}fg^i$ must interchange the points x_0 and y_0 and thus must invert the hyperbolic line between them. But then f must invert the hyperbolic line joining the points $g(x_0)$ and $g(y_0)$. Consequently, by Lemma 2.8, $\|f - \mathrm{Id}\| > 1/2$. This last observation completes the proof of the lemma.

It is worthwhile observing at this point that the number p is at most $n-1$, while $p=n-1$, implies f is a reflection and therefore inverts a line, and so $\|f-\mathrm{Id}\|>1/2$. Thus either

$$p \leq n-2 \quad \text{or} \quad \|f-\mathrm{Id}\|>1/2.$$

As corollaries to the above two lemmas, we can formulate preliminary versions of the generalizations of Jørgensen's inequality we seek. We let $\beta(n)$ denote the constant of Proposition 2.7.

COROLLARY 4.3. *Let f and g together generate a discrete group and suppose that f is loxodromic or parabolic. Then*

$$\max\{\|f-\mathrm{Id}\|, \|[f,g]-\mathrm{Id}\|\} \geq 2-\sqrt{3}$$

and

$$\max\{\|f-\mathrm{Id}\|, \|g^{-1}fg-\mathrm{Id}\|\} \geq 2-\sqrt{3},$$

unless $\langle f, g\rangle$ is elementary and contains a solvable subgroup of index no more than $\beta(n)$.

Notice that in Corollary 4.3 both inequalities must hold and neither one implies the other. The formulation of the above in the case that f is elliptic is the following:

COROLLARY 4.4. *Let f and g generate a discrete group and suppose that f is elliptic and $k=\dim(\mathrm{fix}(f))$. Then*

$$\max\{\|f-\mathrm{Id}\|, \|[f,g^i]-\mathrm{Id}\|: i=1,2,\dots,k+1\} \geq 2-\sqrt{3}$$

and

$$\max\{\|g^{-i}fg^i-\mathrm{Id}\|: i=0,1,2,\dots,k+1\} \geq 2-\sqrt{3},$$

unless $\langle f, g\rangle$ is elementary and contains a solvable subgroup of index no more than $\beta(n)$.

Proof of Corollaries 4.3 *and* 4.4. In each case if the given inequality fails to hold, then we find that the group

$$\langle f, [f,g]\rangle = \langle f, gfg^{-1}\rangle$$

in the first case and the group

$$\langle f, [f, g^i]: i=1,2,\ldots,k+1\rangle = \langle g^i f g^{-i}: i=0,1,2,\ldots,k+1\rangle$$

in the second case are nilpotent and consequently elementary, by Corollaries 3.4 and 3.6. Thus the two Lemmas 4.1 and 4.2 imply in each respective case that $\langle f, g\rangle$ is elementary, since $\|f-\mathrm{Id}\|>1/2$ is impossible given that the inequality fails to hold. The remark about the index of a solvable subgroup comes from the structure of the elementary groups, Proposition 2.7.

Putting together the above results into one inequality and recalling that the dimension of $\operatorname{fix}(A)\leq n-2$, unless f is of even order and $\|f-\mathrm{Id}\|>1/2$ we obtain what we consider to be the correct generalization of Jørgensen's inequality to higher dimensions.

THEOREM 4.5 (Generalized Jørgensen inequality). *Let f and g generate a discrete group. Then*

$$\max\{\|g^{-i}fg^i-\mathrm{Id}\|: i=0,1,2,\ldots,n-1\}\geq 2-\sqrt{3}$$

and

$$\max\{\|f-\mathrm{Id}\|, \|[f,g^i]-\mathrm{Id}\|: i=1,2,\ldots,n-1\}\geq 2-\sqrt{3}$$

unless $\langle f, g\rangle$ is an elementary group with a solvable subgroup of index no more than $\beta(n)$.

The appropriate geometric form is

COROLLARY 4.6. *Let f and g be Möbius transformations generating a discrete group. Then*

$$\max\{d(g^{-i}fg^i, \mathrm{Id}): i=0,1,2,\ldots,n-1\}\geq \frac{1}{2\sqrt{16+n}}$$

and

$$\max\{d(f, \mathrm{Id}), d([f,g^i], \mathrm{Id}): i=1,2,\ldots,n-1\}\geq \frac{1}{2\sqrt{16+n}}$$

unless f and g together generate an elementary group with a solvable subgroup of index no more than $\beta(n)$. Moreover, if f is not elliptic, then it suffices to consider only those terms with $i=0$ or 1.

Notice that Jørgensen's inequality is formulated in a conjugacy invariant fashion. The inequality we have found is not. This is actually no problem at all since if G is discrete and nonelementay, then so is any conjugate of G. Thus the inequality is valid for G and all its conjugates. One can then obtain a conjugacy invariant formulation by looking at the adjoint orbit of G and considering the infimum. As an example, a conjugacy invariant formulation of Corollary 4.5 is

COROLLARY 4.7. *Let f and g generate a discrete nonelementary Möbius group and suppose that f is loxodromic or parabolic. Then*

$$\min\{\max\{\|hfh^{-1}-\mathrm{Id}\|, \|h[f,g]h^{-1}-\mathrm{Id}\|\}: h\in \mathrm{Möb}(n)\}\geq 2-\sqrt{3}.$$

Of course similar versions of the other inequalities will hold.

In view of Lemma 2.5 we can formulate Corollary 4.3 in a version involving traces that is close to Jørgensen's original inequality. Let f and g be Möbius transformations with corresponding matrix representatives $A=(a_{ij})$ and $B=(b_{ij})$. Set $C=B^{-1}AB=(c_{ij})$.

COROLLARY 4.8. *If f and g generate a discrete nonelementary Möbius group and if f is not elliptic, then*

$$2\,\mathrm{tr}(\mathrm{Id}-A)+4\max\{a_{00}^2, c_{00}^2\}\geq 11-4\sqrt{3}>4\frac{1}{14}.$$

Here is an interesting application of this inequality. By a *pure translation* we mean a parabolic Möbius transformation conjugate to $x\to x+a$, some $a\in \mathbf{R}^n$.

COROLLARY 4.9. *Suppose that f is a pure translation and that f and g together generate a discrete nonelementary subgroup of* Möb(n). *Then*

$$\max\{|f(0)|, |g^{-1}fg(0)|\}\geq \frac{1}{15}.$$

Proof. Since f is a pure translation $\mathrm{tr}(A)=\mathrm{tr}(\mathrm{Id})$. Thus from Corollary 4.8

$$\max\{a_{00}^2-1, c_{00}^2-1\}\geq \frac{(2-\sqrt{3})^2}{4}.$$

Now by Lemma 2.5

$$|f(0)|^2=\frac{a_{00}^2-1}{(a_{00}+1)^2} \quad\text{and}\quad |g^{-1}fg(0)|^2=\frac{c_{00}^2-1}{(c_{00}+1)^2}.$$

The function $h(t)=(t^2-1)/(1+t)^2$ is increasing in t and therefore one of the terms above must exceed $h(\sqrt{1+((2-\sqrt{3})^2/4)})=0.0044\ldots$. The result then follows.

By applying Corollary 3.4 instead of Corollary 4.8, the same argument shows

COROLLARY 4.10. *Suppose that f and g are pure translations generating a discrete nonelementary Möbius group. Then*

$$\max\{|f(0)|, |g(0)|\} \geqslant \frac{1}{15}.$$

Notice the independence of dimension in both Corollaries 4.9 and 4.10.

With some more work we could go on to develop from our inequalities the appropriate version of the Shimutzu–Leutbecher inequality (the existence of precisely invariant horospheres at the fixed points of pure translations). However the best one could hope for is the result already obtained by N. Weilenberg [We, Proposition 4] and so we do not pursue the matter.

Finally we prove another generalisation of Theorem 4.5 that is particularly useful in certain circumstances, see [Mar]. A function $N(f, g)$ is a *pseudo-distance* on the Möbius group generating the usual topology if $N: \mathrm{Möb}(n)\times\mathrm{Möb}(n)\to\mathbf{R}$ is symmetric and nonnegative, $N(f, g)=0$ if and only if $f\equiv g$, and $N(f_i, g)\to 0$ if and only if $f_i\to g$ uniformly in $\mathbf{B}^n$. Given such a pseudodistance we define $N(f)=N(f, \mathrm{Id})$.

Here is the torsion free version of the general result we seek. The result in the case that there is torsion is a quite straightforward consequence of the argument given here and Theorem 4.5, which we leave for the reader to develop.

PROPOSITION 4.11. *Let N be a pseudo-distance on* Möb(n) *generating the usual topology. Suppose that there are* $\delta>0$ *and* $c<1$ *such that*

$$\max\{N(h_1), N(h_2)\} <\delta \quad\Rightarrow\quad N([h_1, h_2]) <c\delta.$$

If f and g together generate a discrete nonelementary torsion free group, then

$$\max\{N(f), N(g)\} \geqslant \delta.$$

Proof. Given f and g set $h_1=[f, g]$ and recursively define $h_2=[f, h_1], \ldots,$ $h_{n+1}=[f, h_n]$ and so on. If $\max\{N(f), N(g)\}<\delta$, then by hypothesis $N(h_i)\to 0$. Since N generates the usual topology we se that

$$||h_i-\mathrm{Id}||+||[f, h_i]-\mathrm{Id}|| \to 0.$$

Thus the group $G_i=\langle h_i, f\rangle$ is eventually elementary by Theorem 4.5. Since $G_i=\langle f, (h_{i-1})f(h_{i-1})^{-1}\rangle$, Lemma 4.1 implies that $G_{i-1}=\langle f, h_{i-1}\rangle$ is also elementary. Inductively we find that $G_0=\langle f, g\rangle$ is elementary.

There is another natural pseudo-distance to consider other than that induced by the Hilbert–Schmidt norm and that is the distance induced by the operator norm. If $A\in \mathfrak{gl}(n, \mathbf{R})$, we define

$$\|A\|_{\mathrm{op}}=\max\{\|Av\|: \|v\|=1\}.$$

The usefulness of this norm is evidenced by the fact that it measures only the size of the largest eigenvalue of A^tA and therefore is essentially dimension free, for instance every orthogonal transformation has norm one. The argument immediately following Definition 3.1 shows that the estimate we found for the Hilbert–Schmidt norm also applies in the case of the operator norm. This establishes the following.

PROPOSITION 4.12. *Let f and g generate a discrete nonelementary torsion free Möbius group. Then*

$$\max\{\|f-\mathrm{Id}\|_{\mathrm{op}}, \|[f, g]-\mathrm{Id}\|_{\mathrm{op}}\}\geqslant 2-\sqrt{3}.$$

Again the reader may easily develop the correct form of the above in the case that there is torsion.

§ 5. Algebraic convergence and discreteness

There are many rather nice consequences of Jørgensen's inequality in the classical case of Kleinian groups. One of the more important is that the limit of Kleinian groups is again Kleinian (that is discrete and nonelementary). One of course suspects, since we have a version of the inequality in all dimensions, that this result should also be valid in all dimensions. We will find an easy example to show that this is not in fact the case. The example will motivate the assumption that we will have to make in order to proceed.

There will also be some bother in showing that the limit of discrete nonelementary groups is nonelementary. There is a simple test to decide if two elements fix a common point in a Kleinian group, namely the trace of the commutator of the elements is two. This is no longer true in higher dimensions. This test together with Jørgensen's inequality enables one to readily establish Lemma 9 of [J.K.] showing that if the limit pair $\{g, h\}$ of a sequence $\{g_i, h_i\}$ of pairs of elements generating a discrete group fix a common point, then so too must g_i and h_i for all sufficiently large i. This is used to prove that the limit of nonelementary groups is nonelementary. Another problem is that the geometry is more complicated and to obtain the reduction to the two generator case, as is done in the Kleinian case, we must turn to more algebraic methods. Moreover, any finite group can occur as a discrete subgroup of a Möbius group (since all such groups embed in the orthogonal group). Thus any finite group can be a subgroup of a discrete nonelementary Möbius group. Consequently the classification of the elementary groups does not help nearly as much in the general setting as it does in the classical case. One more confounding issue is that the fixed point set of some power of an elliptic element may be larger than that of the element itself. Consequently it may be that f is elliptic and g is loxodromic with $\langle f, g\rangle$ nonelementary, while $\langle f^m, g\rangle$ is elementary. This cannot happen in dimension three.

EXAMPLE. *For each $n \geq 4$, there is a sequence of finitely generated discrete nonelementary subgroups of* Möb(n), *such that the generators converge uniformly on* $\mathbf{B}^n$ *to Möbius transformations which do not generate a discrete group.*

Proof. Let Γ be a finitely generated nonelementary Fuchsian group. Let Γ_0 be the Poincaré extension of Γ to $\mathbf{B}^n$ (obtained by first extending to $\mathbf{B}^3$ and then to $\mathbf{B}^4$ and so on). Then Γ_0 fixes a codimension two affine subspace. Let Γ_0 denote the group generated by Γ_0 together with the rotation of order n about this affine subspace. This sequence of groups is easily seen to have the desired properties if we are careful enough to arrange that the rotations converge to an irrational rotation.

Another point to this example is that we could arrange that the rotations converge to the identity. Thus the limit group would be discrete but the groups in the sequence are not eventually factors of the limit group (that is there is no eventual homomorphism back). However it is always the case (for finitely generated groups) in dimension three that there is an eventual homomorphism back.

The example actually shows essentially the only way the limit theorem can fail. It suggests two natural paths to follow. Firstly we could assume that no finite index subgroup stabilizes a codimension two or more subspace. This is a geometric restriction that seems rather hard to apply in many circumstances. In the torsion free case it amounts to asking that there is no totally geodesic submanifold of codimension two or more carrying the fundamental group of the quotient. A more natural restriction is that if G_i splits as $G=G_i'\times H_i$, where G_i' is nonelementary and H_i is finite, then there should be a uniform bound on the order of the H_i. We prefer to make a somewhat stronger algebraic restriction which will be easier to apply in the situations we are particularly interested in, namely continuous families of deformations. We will try to point out where with more work one could weaken the hypothesis to either of those above.

Definition 5.1. Let $\{G_i\}_{i\in I}$ be a family of groups. We say that $\{G_i\}$ has *uniformly bounded torsion* if there is an integer N with the following property:

$$\text{if } g \in G_i \text{ for some } i, \text{ then } \operatorname{ord}(g) \leq N \text{ or } \operatorname{ord}(g) = \infty.$$

That is, there is a uniform bound on the order of finite cyclic subgroups. The following lemma is clear.

LEMMA 5.2. *Let G be a finitely generated discrete subgroup of* Möb(n) *with a torsion free subgroup of index M. Then every finite cyclic subgroup has order less than M.*

At this point it is worthwhile recalling Selberg's theorem [Se] which implies that every finitely generated Möbius group contains a normal torsion free subgroup of finite index. Thus the content of the definition is solely in the uniformity of the order of the torsion elements. Notice that such a hypothesis will be trivially satisfied if each group G_i is torsion free, or if each group is the isomorphic image of a fixed finitely generated Möbius group. We also raise here the question of whether or not there is an appropriate converse to Lemma 5.2. The Burnside groups provide a counterexample to the general question, but these groups cannot be linear.

We need another useful lemma which is well known. Indeed the purely topological content of this result is also true and due to Newman [Ne].

LEMMA 5.3. *For each integer $m>1$ there is a positive number $\delta(m)$ such that if g is a periodic Möbius transformation of period less than or equal to m, which is not the identity, then*

$$\|g-\mathrm{Id}\|>\delta(m).$$

The next lemma too is easy.

LEMMA 5.4. *Let $\{g_j\}$ be a sequence of Möbius transformations converging uniformly to another Möbius tranformation g. If* $\operatorname{ord}(g_j)<N$ *for all $j=1,2,\ldots$ then, eventually the order of the g_j is a constant $m<N$ and g is elliptic of order m.*

We are now in a position to obtain the first convergence theorem. It is analogous to Proposition 1 of [Jø].

THEOREM 5.5 *Let G be a nonelementary subgroup of* Möb(n) *and ψ_m a sequence of mappings of G into* Möb(n) *such that for each $m=1,2,\ldots$ the group $\psi_m(G)$ is discrete and the family $\{\psi_m(G)\}$ has uniformly bounded torsion. If for each $g\in G$,*

$$\psi_m(g)\to g \quad \text{as} \quad m\to\infty,$$

then G is discrete.

Proof. Suppose that G is not discrete. Then there is a sequence of elements $\{h_j\}$ in G converging to the identity. Since G is nonelementary it has two elements which are either parabolic or loxodromic and have distinct fixed points. Label these as f_1 and f_2. It follows from continuity that for $i=1,2$ and sufficiently large j and m,

$$\|\psi_m(h_j)-\mathrm{Id}\|+\|[\psi_m(h_j),\psi_m(f_i)]-\mathrm{Id}\|<\min\{\delta(N),2-\sqrt{3}\}$$

where N is the bound on the order of the finite cyclic subgroups and $\delta(N)$ is the number in Lemma 5.2. It follows that $\psi_m(h_j)$ is not elliptic and that the group generated by $\psi_m(h_j)$ and $\psi_m(f_i)$ is elementary by Corollary 4.3. Moreover, as f_i is not elliptic, Lemma 5.3 together with the assumption of bounded torsion implies that for large enough m, $\psi_m(f_i)$ is not elliptic. Consequently $\psi_m(h_j)$ and $\psi_m(f_i)$ both have common fixed points. So then do $\psi_m(f_i)$, $i=1,2$. Consequently in the limit we find that f_1 and f_2 have a common fixed point and this is the desired contraction.

Analogously to Proposition 2 of [Jø] we have

THEOREM 5.6. *A finitely generated nonelementary subgroup of* Möb(n) *is discrete if and only if every two generator subgroup is discrete.*

Proof. Suppose that G is nonelementary and every two generator subgroup is discrete and yet G itself is not discrete. Then there is a sequence $\{h_j\}$ converging to the identity in G. Now G has bounded torsion as it is finitely generated. Thus we may assume that the h_j are not elliptic. Furthermore G has parabolic or loxodromic elements f_1 and f_2 with distinct fixed points by hypothesis. Since $\langle h_j, f_i\rangle$ is discrete for each j and $i=1,2$, the argument given in Theorem 5.4 will now imply that for large enough j, h_j and f_i, $i=1,2$, will have common fixed points. This is a contradiction.

We now seek to establish the appropriate version of the proposition of Jørgensen and Klein on the algebraic convergence of groups, see [J.K.]. We need a definition.

Definition 5.7. Let $\{G_j\}$ be a sequence of subgroups of Möb(n) each with the same finite number of generators say $\{g_{j,1}, g_{j,2}, \dots, g_{j,m}\}$. If for each $i=1,2,\dots,m$ we have

$$g_{j,i} \to g_i \quad \text{as} \quad j \to \infty,$$

then we say that the groups G_j *converge algebraically* to the Möbius group

$$G = \langle g_1, g_2, \dots, g_m \rangle.$$

Given an element $g \in G - \{\mathrm{Id}\}$, we can express g in a minimal fashion as a product of the generators

$$g = \prod_{i=1}^{N} g_i^{p(i)}$$

where for each i, g_i is some generator. Given such a minimal reduction of g we define the *word length* of g as $w(g)=\Sigma p(i)$. We set $w(\mathrm{Id})=0$.

PROPOSITION 5.8. *Let G be the algebraic limit of a sequence of m generator discrete nonelementary subgroups of* Möb(n) *of uniformly bounded torsion. Then G is discrete and nonelementary.*

Proof. Let the sequence of discrete nonelementary groups converging to G be denoted $\{G_i\}$. Since G and G_i have the same finite number of generators the mappings $g_{i,j} \to g_j$ given by the correspondence of generators will have the property needed in Theorem 5.5. Evidently then G will be discrete if it is nonelementary. This is what we shall show. Firstly we claim:

(1) *G is not finite.* To see this let us suppose that G were finite and let $m=\max\{w(g):g \in G\}<\infty$. As G_j is infinite there are words of every length. Let $\{h_j\}$ be a sequence of words of length $m+1$. Because the word length of h_j is uniformly bounded we may pass to a subsequence so that $h_j \to h \in G$. Now h has a representation as a word of length no more than m. Let the corresponding word in G_j be f_j and set $h_j'=f_j^{-1}h_j$. Then $1<w(h_j')\leq 2m+1$, and $h_j' \to \mathrm{Id}$. By Lemma 5.3 and since $\{G_j\}$ has uniformly bounded torsion, for j sufficiently large, h_j' is not elliptic. Clearly h_j' is not the identity. Next, since G_j is nonelementary at least one of the generators of G_j cannot stabilize (setwise) the fixed point set of h_j'. Let g_j be such a generator. Since there is a

finite set of generators, we may assume by passing to a subsequence that $g_j \to g$, a generator of G. We now easily obtain a contradiction to Corollary 4.3 as $\|h_j' - \mathrm{Id}\| + \|g_j h_j' g_j^{-1} - \mathrm{Id}\| \to 0$, but the group $\langle h_j', g_j \rangle$ is discrete and nonelementary. Thus G is infinite.

Hence we may choose $h \in G$ of infinite order (recall that G is finitely generated and so contains a torsion free subgroup of finite index). Let $\{h_j\}$ be the corresponding words in $\{G_j\}$. Then it is clear $h_j \to h$. Since $\{G_j\}$ has uniformly bounded torsion and since for all fixed p, $h_j^p \to h^p \neq \mathrm{Id}$, for all j sufficiently large h_j is not elliptic. Again choose a generator g_j which does not stabilize the fixed point set of h_j and assume $g_j \to g$ a generator of G. We have now obtained the desired reduction to the two generator case as $\langle g_j, h_j \rangle$ is discrete and nonelementary and the generators are converging to the group $\langle g, h \rangle$.

We claim:

(2) *The limit h is parabolic or loxodromic.* To see this let us suppose, for the purpose of contradiction, that this is not the case. Since h is either elliptic or an irrational rotation there is an integer k such that h^k is so close to the identify (possibly equal) that

$$\|h^k - \mathrm{Id}\| + \|[h^k, g] - \mathrm{Id}\| < 2 - \sqrt{3}.$$

From the uniform convergence, we find that for all sufficiently large j,

$$\|h_j^k - \mathrm{Id}\| + \|[h_j^k, g_j] - \mathrm{Id}\| < 2 - \sqrt{3}.$$

Consequently by Corollary 4.3, the group $\langle g_j, h_j \rangle$ is elementary. This is impossible since h_j^k is loxodromic or parabolic (as h_j is) and g_j was chosen so that it did not setwise stabilize the fixed points.

We next claim:

(3) *The elements g and h do not have a common fixed point unless g is parabolic and h is loxodromic, or g is loxodromic and h is parabolic.* Suppose g and h do have a common fixed point. There are two cases.

Firstly, let us suppose that the group $\langle g, h \rangle$ is discrete. Then since g and h have a common fixed point the group is elementary and so virtually Abelian. Therefore there are integers j and k such that h^p and $gh^q g^{-1}$ commute. This is because both of the elements h and $f = ghg^{-1}$ are of infinite order and so some power lies in any finite index subgroup. Then

$$\|[h^p, f^q] - \mathrm{Id}\| + \|[[h^p, f^q], h^p] - \mathrm{Id}\| = 0.$$

By continuity and Corollary 4.3 we see that for all sufficiently large i, the group generated by the elements h_i^p and $[h_i^p, f_i^q]$ is elementary, where $f_i = g_i^{-1} h_i g_i$. But this is the group generated by h_i^p and $g_i^{-1} h_i^q g_i$ and both of these elements are parabolic or loxodromic with distinct fixed points (since g_i does not fix the fixed points of h_i). This is impossible.

Secondly, if $\langle g, h\rangle$ is not discrete, then there is a sequence f_k converging to the identity in $\langle g, h\rangle$. For sufficiently large k

$$\|f_k-\mathrm{Id}\|+\|[f_k, h]-\mathrm{Id}\|+\|[f_k, ghg^{-1}]-\mathrm{Id}\| \ll 1.$$

The uniformly bounded torsion assumption will now imply that the approximants to f_k are not elliptic and by continuity and Corollary 4.3 the group generated by the approximants to f_k and h and the group generated by the approximants to f_k and ghg^{-1} are elementary. Thus the approximations to h and to ghg^{-1} have common fixed points. This is not the case by construction.

It remains only to observe that h and $g^{-1}hg$ are both parabolic or loxodromic with distinct fixed points (notice that in the loxodromic case that they may have one common fixed point). Thus G is nonelementary and this establishes the proposition.

We are now in a position to assert the following generalization of the theorems of Chuckrow, Marden and Jørgensen, see [Jø]. Weilenberg also has an n-dimensional version of the following theorem in the case that G_0 is torsion free, [We, Theorem 3]. We need the following easy lemma (cf. Lemma 3 [Jø] and notice that the appropriate version of the combination theorem is true in all dimensions, and that the weaker assumption that f is not the identity, with the same conclusion, is false in higher dimensions):

LEMMA 5.9. *Let G be a discrete nonelementary group and $f\in G$ not elliptic. Then there is a loxodromic $g\in G$ such that $\langle f, g\rangle$ is discrete, nonelementary and isomorphic to the free group of rank two.*

THEOREM 5.10. *Let G_0 be a finitely generated discrete nonelementary subgroup of* Möb(n). *For each natural number m, let ψ_m be an isomorphism of G_0 onto a discrete group G_m and suppose that for some choice of generators $\{g_i\colon i=1,2,\ldots,k\}$ the images $\psi_m(g_i)$ converge to a Möbius transformation*

$$\psi_m(g_i)\to\psi(g_i) \quad \text{as} \quad m\to\infty.$$

Then the group G generated by transformations $\{\psi(g_i): i=1,2,\dots,k\}$ is discrete and nonelementary, furthermore ψ is an isomorphism of G_0 onto G.

Proof. It is clear that the mapping $\psi: G_0 \to G$ is a homomorphism onto. Let f be an element of $G_0 - \{\mathrm{Id}\}$. If f is elliptic of order p say, then so too is $\psi_m(f)$ for all m. Consequently, Lemma 5.4 implies that $\|\psi_m(f)-\mathrm{Id}\|$ is bounded below by a number depending only on the dimension. Thus $\psi_m(f)$ does not approach the identity and so $\psi(f) \neq \mathrm{Id}$. If f is not elliptic, then according to Lemma 5.9 choose a loxodromic $g \in G_0$ so that the group $\langle f, g\rangle$ is discrete, nonelementary and isomorphic to the free group of rank two.

The group $\langle \psi_m(f), \psi_m(g)\rangle$ is discrete and is isomorphic to $\langle f, g\rangle$, therefore it is not virtually Abelian and so nonelementary. If $\psi(f)=\mathrm{Id}$, then by continuity for sufficient large m

$$\max\{\|\psi_m(g)^{-i}\psi_m(f)\psi_m(g)^i - \mathrm{Id}\|: i=0,1,2,\dots,n-1\} < 2-\sqrt{3}.$$

And this implies by Theorem 4.5 that $\langle \psi_m(f), \psi_m(g)\rangle$ is elementary. This contradiction implies that $\psi_m(f)$ cannot approach the identity and so $\psi(f) \neq \mathrm{Id}$. Thus the eventual homomorphism ψ is an isomorphism. By Proposition 5.8 the limit group G is discrete and nonelementary since a sequence of isomorphic groups has bounded torsion.

Note how important it was to us that the maps ψ_m were isomorphisms as it enabled us to omit the assumption of uniformly bounded torsion. A. Marden used a version of the above theorem in the Kleinian case to show that every group on the boundary of Schottky space is discrete. Schottky space is more complicated in higher dimensions (for instance the Cantor set limit sets may have topologically distinct complements). However the above suggests that essentially the same result is true. We hope to return to this at another time.

It is worthwhile remarking at this point how one might weaken the hypothesis of bounded torsion. In Theorem 5.5 one will have control on the order of the cyclic groups that lie in the appropriate splittings of the G_m by (a necessary) hypothesis. Then one needs to control the elliptics that really do lie in the nonelementary part of the group. To control these elliptics we need an n-dimensional version of Lemma 2 of [JØ] and Lemma 4 of [J.K.]. The first lemma asserts that if $X_i \to X$, $Y_i \to Y$, with $\langle X_i, Y_i\rangle$ discrete nonelementary, then X elliptic implies X_i is elliptic for i sufficiently large and the orders of the X_i are uniformly bounded. This is false in higher dimensions (even if the limit $\langle X, Y\rangle$ is discrete nonelementary). The second lemma implies the existence of elements of infinite order and word length bounded in terms of the number of generators in a discrete nonelementary group. This seems rather hard to guarantee in higher dimensions. However there is hope that the arguments can be made to work under the geometric assumption that the limit set does not lie in the boundary of any codimension two (hyperbolically) affine subspace. We shall report on this elsewhere.

§ 6. Continuous families of discrete groups

In this section we will replicate what we can from Section 4 of [Jø]. In particular we are aiming to get a n-dimensional version of the main theorem of [J.K.] and the appropriate version of Theorem 3 of [Jø]. Unfortunately we will always have to deal with the troublesome problem of bounded torsion.

Thus motivated we define $\mathcal{D}(n:r)$ to be that subset of

$$\mathcal{O}^+(1,n)\times\mathcal{O}^+(1,n)\times\ldots\times\mathcal{O}^+(1,n) \quad (r \text{ copies})$$

consisting of all points $(g_1, g_2, \ldots, g_r)$ such that $\langle g_1, g_2, \ldots, g_r\rangle$ is a discrete subgroup of the n-dimensional Möbius group, Möb(n). As such it has a presentation with exactly r generators. We give $\mathcal{D}(n:r)$ the topology it inherits as a subspace, this corresponds to the notion of algebraic convergence and is easily seen to be compatible with the natural topology induced by the norm $\|\cdot\|$. Notice that each point of $\mathcal{D}(n:r)$ is a discrete group with a canonical choice of generators and that the same group may represent different points of $\mathcal{D}(n:r)$ depending on how the generators are chosen. $\mathcal{D}(n:r)$ is naturally a subset of $\mathcal{D}(n:r+1)$ as we may add an extra generator with the additional relation that it is the identity. According to Theorem 5.5 the subset of $\mathcal{D}(n:r)$ consisting of those groups whose finite cyclic subgroups have order less than M is closed (one needs to make the additional observation that the limit of such groups also has the bound M on the order of the finite cyclic subgroups, as we will see). In particular the finitely generated torsion free groups form a closed set.

It follows from the compact core theorem for three manifolds of P. Scott [Sc] together with Selberg's theorem [Se], that every subgroup of $\mathcal{D}(2:r)$ is finitely presented. The question of whether every element of $\mathcal{D}(n:r)$ is finitely presented or not is unknown and difficult. It would follow from such a core theorem for n-dimensional hyperbolic manifolds.

Consequently we must make some assumptions about the existence of finite presentations. The following is the appropriate version of Theorem 2 of [J.K.]. It is unfortunately and necessarily more complicated. We go through the details for completeness and because there are a few differences in our situation.

THEOREM 6.1. *Let $\{G_m\}_{m=1}^{\infty}$ be a sequence of r generator discrete nonelementary subgroups of* Möb(n) *converging algebraically to the group G. Suppose that the sequence has uniformly bounded torsion and that G is finitely presented. Then G too is a discrete nonelementary subgroup of* Möb(n) *and the correspondence from the generators of G to their approximants in G_m extends for all sufficiently large m to a homomorphism of G onto G_m.*

Proof. We denote the generators of G_m *by* $g_{m,1}, g_{m,2}, \ldots, g_{m,r}$. By hypothesis we have for each $i=1,2,\ldots,m$,

$$g_{m,i} \to g_i \quad \text{as} \quad m \to \infty.$$

There is only one possibility of extending the natural correspondence of generators (which we denote by ψ_m), namely for each finite sequence $(p(i), q(i))$ we should have

$$\prod_{i=1}^{N} g_{q(i)}^{p(i)} \to \prod_{i=1}^{N} g_{m,q(i)}^{p(i)}.$$

A necessary and sufficient condition for this correspondence to define a mapping of G is that

$$\prod_{i=1}^{N} g_{q(i)}^{p(i)} = \mathrm{Id} \quad \Rightarrow \quad \prod_{i=1}^{N} g_{m,q(i)}^{p(i)} = \mathrm{Id}.$$

Our assumption is that G has a finite presentation. Let $R_1, R_2, \ldots, R_m$ be a basis for the relations in G each R_i being written as a product of powers of g_i and let $R_{m,i}$ denote the same word in the group G_m in terms of the generators for G_m. We need only show that there is a number M such that if $m > M$, then $R_{m,i} = \mathrm{Id}$. It is clear that

$$R_{m,i} \to \mathrm{Id} \quad \text{as} \quad m \to \infty,$$

as for each m, $R_{m,i}$ has the same uniformly bounded word length. Since the groups G_m have uniformly bounded torsion, it must be the case that for all sufficiently large m, $R_{m,i}$ is either the identity, parabolic or loxodromic. We already know from what we have proved above that G is discrete and nonelementary. Thus choose a pair of loxodromics h_1 and h_2 in G generating a free group of rank two. Then the sequence of approximants $h_{m,1}$ and $h_{m,2}$ must be either parabolic or loxodromic. From Corollary 4.3, and the above for sufficiently large m, the group generated by $R_{m,i}$ and $h_{m,j}$, $j=1,2$ is discrete and elementary and both generators have infinite order. The least such m for which the above statement holds true is the M that we seek. For then $R_{m,i}$ has infinite order and has either

(a) four distinct fixed points (both $h_{m,i}$ are loxodromic) or

(b) three distinct fixed points (one $h_{m,i}$ parabolic the other loxodromic) or

(c) two distinct fixed points (both $h_{m,i}$ are parabolic).

It only remains to observe that the last case cannot occur in a discrete group, that is a parabolic and loxodromic cannot share a common fixed point, see for instance Theorem 6.7 of [G.M.1]. This concludes the proof.

We remark here that we could replace the assumption that G has a finite presentation by the assumption that each G_m is finitely presented and the word length of each relation in the presentation is bounded independently of m.

Given two points (groups) G and H in $\mathcal{D}(n\colon r)$ we denote by $d(G,H)$ the *distance* from G to H defined by

$$d(G,H) = \max\{\|g_i - h_i\| \colon i = 1, 2, \ldots r\}$$

where $\{g_i\}$ and $\{h_i\}$ are the canonical set of generators. It is not difficult to see that this metric is compatible with the usual topologies of all spaces concerned.

The following lemma will prove extremely useful. It's proof is inherent, more or less, in the proof of Theorem 6.1 but will go through the details.

LEMMA 6.2. *Let $G \in \mathcal{D}(n\colon r)$ and $\{g_1, g_2, \ldots, g_r\}$ be the generating set for G. Let $\{H_i \colon i \in I\}$ be a family of subgroups of $\mathcal{D}(n\colon r)$ such that each torsion element has order less than M. Then there are positive numbers ε and δ depending only on the choice of generators for G, the number M and the dimension n, such that if $h_i \in H_i$ and*

$$d(G, H_i) < \varepsilon \quad \text{and} \quad \|h_i - \mathrm{Id}\| < \delta,$$

then $h_i = \mathrm{Id}$.

Proof. There is a positive number δ_1 such that for all i and $h \in H_i$, if $\|h - \mathrm{Id}\| < \delta_1$, then h is not elliptic. Choose a pair of loxodromic elements f_1 and f_2 of G generating a discrete nonelementary subgroup isomorphic to a free group of rank 2. By the continuity of the commutator and local compactness of the group, there is a positive number δ_2 such that for all Möbius transformations h, if $\|h - \mathrm{Id}\| < \delta_2$, then $\|[h, f] - \mathrm{Id}\| < 1/8$ for all Möbius transformations f with $\min\{\|f - f_i\| \colon i = 1, 2\} < 1$.

We choose $\delta = \min\{\delta_1, \delta_2, 1/8\}$. Next, choose ε so small that if $d(G, H) < \varepsilon$, then the words h_1 and h_2 in the appropriate generating set for H, corresponding to the elements f_1 and f_2 satisfy $\|h_1-f_1\|+\|h_2-f_2\|<1$, and neither h_1 nor h_2 is elliptic (every loxodromic has a neighbourhood which contains no elliptic elements) and h_1 and h_2 have distinct fixed points.

With these choices of ε and δ we are done, for $\|h-\mathrm{Id}\|<\delta$ implies that for $i=1,2$, $\|[h_i,h]-\mathrm{Id}\|+\|h-\mathrm{Id}\|<2\sqrt{3}$. This implies by Corollary 4.3, that $\langle h_i, h\rangle$ is discrete and elementary, for both i. Since neither h nor h_i is elliptic, we obtain, as before, that h must be equal to the identity.

Here is the main result of this section. It is the n-dimensional version of Theorem 3 [J∅].

THEOREM 6.3. *Let $\mathcal{E}$ be a compact connected subset of $\mathcal{D}(n:r)$. Then $\mathcal{E}$ consists entirely of isomorphic groups.*

Proof. Recall that each point of $\mathcal{E}$ has a canonical choice of generators. Let us first consider those elements of $\mathcal{E}$ for which the order of a finite maximal cyclic subgroup is no more than m. Call this set $\mathcal{E}_m$.

(1) *$\mathcal{E}_m$ is closed.* Suppose not. We already know that any convergent sequence in $\mathcal{E}_m$ converges to a discrete nonelementary group, we wish to show such limits are again in $\mathcal{E}_m$. Consider such a sequence $G_k\to G$. Suppose that G contains an elliptic g of order $p>m$. The element g has finite word length in the generators of G and so the corresponding words in G_k, call them g_k, must converge to g. Since the order of g_k is either infinite or less than or equal to m, it must be the case for sufficiently large k that, g_k is not elliptic. However, we must have $g_k^p\to g^p=\mathrm{Id}$. Thus we have a sequence of elements of infinite order converging to the identity. The usual trick of taking two loxodromics in G with distinct fixed points will now produce the desired contradiction.

Unfortunately it may not be the case that $\mathcal{E}_m$ is also (relatively) open. We need to show:

(2) *Each component of $\mathcal{E}_m$ consists of isomorphic groups.* Let $G\in\mathcal{E}_m$ and let R be a relation in G. Lemma 6.2 implies that all groups sufficiently close to G also have the relation R. Let F be that subspace of $\mathcal{E}_m$ consisting of groups with the relation R. Lemma 6.2 implies that F is (relatively) open while clearly F is closed. Since F is both open and closed it is the union of components of $\mathcal{E}_m$. Next let H be a point of F which is in the same component as G. The above argument shows that every relation in H (in terms of the canonical generators) also occurs in G (in terms of the canonical

generators). Thus G and H have the same presentation and represent isomorphic groups.

We would now actually like to conclude that for some m, $\mathcal{E}_m=\mathcal{E}$ without using the hypothesis that $\mathcal{E}$ is compact. This would then imply that each component of $\mathcal{D}(n:r)$ consists entirely of isomorphic groups. We see that since every element of $\mathcal{E}$ is virtually torsion free

$$\mathcal{E}=\bigcup_{m=0}^{\infty}\mathcal{E}_m$$

and each $\mathcal{E}_m$ is closed and disjoint from $\mathcal{E}_n, m\neq n$. Now a continuum (connected compact set) cannot be written as the countable union of closed disjoint nonempty sets. This is Sierpinski's theorem, see [Ku, §47, III, Theorem 6]. This fact is not true if we are merely asuming that $\mathcal{E}$ is closed (i.e. a component). This last observation establishes the theorem, for then $\mathcal{E}_m=\mathcal{E}$ for sufficiently large m, and so $\mathcal{E}$ consists entirely of isomorphic groups.

It is not difficult to see from the above theorem that every path component of $\mathcal{D}(n:r)$ must consist entirely of isomorphic groups (Sierpinski's result is true for closed subsets or $\mathbf{R}$). By a *continuous deformation* of a discrete group $G\in\mathcal{D}(n:r)$ *through discrete groups* we mean a path in $\mathcal{D}(n:r)$ passing through the group G.

Evidently we have shown

COROLLARY 6.4. *Let G be a finitely generated discrete nonelementary group and let $\{G_t: t\in\mathbf{R}\}$ be a continuous deformation of G through discrete groups. Then the family $\{G_t\}$ consists entirely of isomorphic groups.*

Notice that a continuous family of representations of the group G is an analytic space, [Ra]. In particular this result implies that the hypothesis that the group be finitely presented and the family of homomorphisms be injective is unnecessary in Weil's rigidity theorem (in this case), see Theorem 6.19 in [Ra]. Of course Mostow's rigidity theorem completely covers this situation (that the quotient is cocompact) but we point it out as we are interested in finding a version of this rigidity theorem in the case that the group is not cocompact (or even cofinite volume). Such aims are the topic of the next section.

§7. Deformations of geometrically finite groups

This section consists of applying the results of Tukia [Tu] together with what we have found so far. We will show that for the most part and in all dimensions, a deformation of a geometrically finite Möbius group through discrete groups, consists of a deformation through isomorphic groups each canonically quasisymmetrically conjugate on the limit set. The term quasisymmetry is the necessary generalization of the notion of quasiconformality. To obtain such geometric results we must turn back to the picture of the action of the group on the ball. We need to recall some notation and definitions.

A discrete Möbius group G acts properly discontinuously on the ball $\mathbf{B}^n$ and extends naturally to the boundary sphere $\mathbf{S}^{n-1}$. The limit set $L(G)$ of G lies in this sphere. One can define in the usual manner, the *Dirichlet fundamental polyhedron* for the action of the group on $\mathbf{B}^n$

$$\mathcal{P}=\{y\in\mathbf{B}^n\colon \varrho(y,x)<\varrho(g(y),x) \text{ for all } g\in G\},$$

where ϱ is the hyperbolic metric of the ball. This fundamental polyhedron $\mathcal{P}$ will consist of the locally finite intersection of geodesic half spaces and the boundary will lie in the countable union of affine subspaces. If this countable union is actually finite, and G is finitely generated, we say that G is *geometrically finite* and that the fundamental domain $\mathcal{P}$ has a *finite number of faces*. There is an important special case for us. If G is geometrically finite and contains no parabolics we say that G is of *compact type*. In view of Corollary 2.5 of [Tu] this is equivalent to the condition that the *orbit space* $M_G=(\mathbf{B}^n-L(G))/G$ is compact.

It is important to notice that this is not the same as $\mathrm{int}(\mathbf{B}^n)/G$ being compact, that is G is cocompact (certainly though, cocompact implies compact type). Compact type means that the fundamental polyhedron $\mathcal{P}$ is compactly supported away from the limit set. Typical examples of such groups are Schottky groups and as another example every bordered Riemann surface without punctures arises as the quotient of such a group acting on the disk.

Given two geometrically finite groups G and H and an isomorphism $\psi\colon G\to H$ between them, we say that ψ is *type preserving* if ψ has the following property:

$$\psi(g) \text{ is parabolic in } H \text{ if and only if } g \text{ is parabolic in } G.$$

A mapping $f\colon L(G)\to L(H)$ is said to *induce* or to be *induced by* an isomorphism $\psi\colon G\to H$, if

$$f(g(x))=\psi(g)(f(x)) \quad \text{for all} \quad g\in G.$$

A *quasisymmetric* mapping between subsets E and F of $\mathbf{S}^n$ is a homeomorphism with the property that it distorts cross ratios by a bounded amount. For a more precise definition see [Tu], the point to make is that it is the natural generalization of the notion of quasiconformality. The main theorem of [Tu] is

THEOREM (Tukia). *A type preserving isomorphism $\psi: G \to H$ between two geometrically finite Möbius groups G and H is induced by a canonical quasisymmetric mapping*

$$f: L(G) \to L(H).$$

The map is canonical given the isomorphism as it identifies loxodromic fixed point pairs in the obvious manner.

It is important to note that if G and H are of cofinite volume, then $L(G) = L(H) = \mathbf{S}^{n-1}$, and so the two groups would be canonically quasiconformally conjugate on the boundary sphere. This is an important step in Mostow's proof of the rigidity theorem. The final step is to observe that such a quasiconformal conjugacy is linearizeable to a conformal conjugacy.

A *parabolic subgroup* of G is a subgroup which stabilizes exactly one point of $\mathbf{S}^{n-1}$. Such a group is elementary and consists entirely of parabolics and elliptics. The *rank* of a parabolic subgroup is the rank of maximal Abelian torsion free subgroup. In particular, the rank is one if and only if the parabolic subgroup is virtually cyclic. Notice that the rank of a parabolic subgroup is an isomorphism invariant as soon as it is larger than one. Also Tukia shows that if G is of compact type, then G contains no parabolics.

COROLLARY 7.1. *Let G and H be geometrically finite groups lying in the same connected compact subset of $\mathcal{D}(n:r)$. Suppose that G and H have no parabolic subgroups of rank one. Then G and H are isomorphic and canonically quasisymmetrically conjugate on their respective limit set.*

In particular, if G and H lie in the same path component of discrete finitely generated groups and neither G nor H have rank one parabolic subgroups (for instance if G and H are of compact type), then G and H are canonically conjugate on their limit sets.

Notice the implication that if one of G or H has $\mathbf{S}^{n-1}$ as its limit set, then both do and they are canonically quasiconformally conjugate. Also if G and H are cofinite volume, then each parabolic subgroup has maximal rank, greater than two, if the dimension $n \geq 3$ and so all the hypotheses are satisfied.

It is easy to construct examples (in all dimensions) where one can continuosly deform a compact type group through discrete groups into a geometrically finite group which contains a parabolic. For instance, one can do this in such a manner that the limit set changes from a Cantor set to a circle. Clearly then the hypothesis on the rank of the parabolic subgroups is necessary. In the compact type case we have what we consider to be a fair generalization of Weil's rigidity theorem (to the special case of $\mathcal{O}^+(1,n)$).

THEOREM 7.2. *Let G be a finitely generated discrete Möbius group of compact type. Let $\{G_t: t\in[-1,1]\}$ be a continuous deformation of $G=G_0$ through discrete groups each of compact type. Then each G_t is isomorphic to G_0 and the groups G_t and G_0 are canonically quasisymmetrically conjugate on their respective limit sets.*

Notice that there is no assumption that G_t is the homomorphic image of G_0. If G_0 is of compact type and G_t is the homomorphic image of G_0, then there is good reason to suspect that the arguments of Marden, see § 9 [Ma], will work in all dimensions and imply that at least for small time the group G_t is discrete and again of compact type, and consequently isomorphic and canonically quasisymmetrically conjugate. This would be a nice complement to the above result and we hope to return to this question along with the question of when there is a conjugacy defined on the whole ball $\mathbf{B}_n$ instead of just the limit sets at a later date.

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