THE RELATIVELY EXPONENTIAL, LOGARITHMIC AND CIRCULAR FUNCTIONS IN RECURSIVE FUNCTION THEORY

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In this paper we develop a theory of relative integration in recursive function theory and apply it to establish the fundamental properties of the relatively exponential, logarithmic and circular functions which are the 'analogues' in the rational space of recursive function theory of the classical functions of the same names. The present account is self contained and may be read without reference to the literature on recursive function theory.[1]

Recursive function theory is a development of a free variable formalisation of arithmetic introduced by Th. Skolem.[2] The elementary formulae of this system are equations between 'terms', and the class of formulae is constructed from the elementary formulae by the operations of the propositional calculus. The terms are the free numeral variables, the sign 0, and the signs for functions. As function signs we have $S(x)$ for the successor function, and signs for functions introduced by recursion. The derivation rules comprise the propositional calculus, the substitution of terms for numeral variables, the schema

$$a = b \to (\mathfrak{a}(a) \to \mathfrak{a}(b)),$$

the induction schema

$$\frac{\mathfrak{a}(o),\ \mathfrak{a}(n) \to \mathfrak{a}(S(n))}{\mathfrak{a}(n)},$$

[1] A list of publications on recursive function theory is given in the Bibliography of the author's "Constructive Formalism" (Leicester, 1951).

[2] In his paper "Begründung der elementaren Arithmetik durch die rekurrierende Denkweise ...", *Videnskapsselskapets Skrifter* (Kristiania 1923), 2, Vol. I § 7 pp. 3–38.

explicit definition, and the recursive schema

$$f(0, a) = \alpha(a)$$
$$f(S(n), a) = \beta(n, a, f(\gamma(n), a))$$

in which $f(n, a)$ is the function introduced by the schema and $\alpha(a)$, $\beta(n, a, b)$ and $\gamma(n)$ are functions previously introduced. If $\gamma(n) = n$, the schema is called primitive recursion; if however $\gamma(n)$ is a predecessor of n in a series of natural numbers of transfinite ordinal Ω then the recursion is said to be transfinite of ordinal Ω.[1] Provided that the ordinal does not exceed $\omega^{\omega^{\omega}}$, it has been shown that the value, for any assigned argument, of a recursive function of ordinal Ω, may be determined by repeated substitution, the number of such substitutions being given by a function of ordinal less than Ω.

A codification of recursive arithmetic has been given in which all the foregoing axioms and axiom schema are *demonstrable*.[2]

To complete these introductory remarks it remains only to indicate the passage from the recursive arithmetic of natural numbers to the recursive arithmetic of rational numbers. Following Bernays, we define a rational number to be an ordered triplet $(p, q)/r$, with $r > 0$, such that $(p, q)/r \gtreqless (p', q')/r'$ according as $pr' + q'r \gtreqless p'r + qr'$; likewise a recursive function of one or more variables $(p, q)/r$ is a triplet of recursive functions of natural numbers

$$(P(p, q, r),\ Q(p, q, r))/R(p, q, r), \text{ with } R(p, q, r) > 0, \text{ and}$$

(writing P for $P(p, q, r)$, P' for $P(p', q', r')$, etc.)

$$(P', Q')/R' = (P, Q)/R \text{ when } (p', q')/r' = (p, q)/r.$$

We shall generally denote natural numbers by the letters k, m, n, p, q, r, with or without suffixes, rationals by x and y, and rational functions of a numeral variable n and a rational variable x by $f(n, x)$, $g(n, x)$.

The numbers $(p, 0)/1$, $(0, q)/1$ are the positive and negative integers respectively and will be denoted as usual by $+p$, $-q$, and we take for granted the further details which justify writing $(p, q)/r$ as $+(p-q)/r$ if $p > q$, $-(q-p)/r$ if $p < q$ and 0 if $p = q$.

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[1] Transfinite recursion was introduced by Ackermann in "Zur Widerspruchsfreiheit der Zahlentheorie", *Mathematische Annalen*, Vol. 177 (1940).

[2] R. L. GOODSTEIN, "Function theory in an axiom free equation calculus", *Proc. London Math. Soc.* Vol. 48 (1945).

The class of rational recursive functions is not increased by applying the original recursion schema to rational functions of a natural number argument. For instance, if we define the rational function $f(n, x)$ by the recursion

$$f(0, x) = 0, \; f(Sn, x) = \phi(n, x, f(n, x)) \tag{i}$$

where

$$\phi(n, (p, q)/r, (u, v)/w)$$
$$= \{a(n, p, q, r, u, v, w), \; b(n, p, \ldots, u, \ldots)\}/c(n, p, \ldots, u, \ldots),$$

then we can find recursive functions $u_n(p, q, r)$, $v_n(p, q, r)$, $w_n(p, q, r)$ so that

$$f(n, (p, q)/r) = (u_n, v_n)/w_n; \tag{ii}$$

for if u_n, v_n, w_n are defined by the simultaneous recursions $u_0 = v_0 = 0$, $w_0 = 1$

$$u_{n+1}(p, q, r) = a(n, u_n, v_n, w_n)$$
$$v_{n+1}(p, q, r) = b(n, u_n, v_n, w_n)$$
$$w_{n+1}(p, q, r) = c(n, u_n, v_n, w_n)$$

then u_n, v_n, w_n are recursive and therefore $f(n, x)$ defined by (ii) is recursive and satisfies the recursion (i).

We are now ready to describe the fundamental concepts of recursive function theory.

1. Notation

1.01. If $[x]$ denotes the whole part of x and if for some integer k, $[10^k x] = 0$, we write $x = 0(k)$, or $x - 0(k) = 0$ (so that $x = 0(k)$ is equivalent to the recursive relation $|x| < 10^{-k}$).

If there is a recursive function $N(m_1, m_2, \ldots, m_p)$ such that a relation $R(n, m_1, m_2, \ldots, m_p)$ holds for all $m_1, m_2, \ldots, m_p$ and all $n \geqq N(m_1, m_2, \ldots, m_p)$ then we say that R holds *for majorant n*.

More generally, if there are recursive functions

$$N_0(m_1, m_2, \ldots, m_p), \; N_1(n_1, m_1, m_2, \ldots, m_p)$$

such that $R(n_1, n_2, m_1, m_2, \ldots, m_p)$ holds for all $m_1, m_2, \ldots, m_p$ and

$$n_1 \geqq N_0(m_1, m_2, \ldots, m_p), \; n_2 \geqq N_1(n_1, m_1, m_2, \ldots, m_p)$$

then $R_1(n_1, n_2, m_1, \ldots, m_p)$ is said to hold for majorant n_1, n_2 (the order of n_1, n_2 being material).

1.1. Equivalence: The recursive functions $f(n, x)$, $g(n, x)$ are said to be equivalent functions of x, or equal relative to n, if $f(n, x) - g(n, x) = 0\,(k)$ for majorant n. Similarly $f(n_1, n_2, x)$, $g(n_1, n_2, x)$ are said to be equal relative to n_1, n_2 if $f(n_1, n_2, x) - - g(n_1, n_2, x) = 0\,(k)$ for majorant n_1, n_2.

1.2. Recursive convergence. A recursive function $f(n, x)$ of a positive integral variable n and a rational variable x is said to be (recursive) convergent in n, for $a \leqq x \leqq b$, if there is a recursive $N(k, x)$[1] such that $N(k+1, x) \geq N(k, x) \geq k$ and for $n_1 \geqq n_2 \geqq N(k, x)$, and $a \leqq x \leqq b$,

$$f(n_1\, x) - f(n_2, x) = 0\,(k).$$

If $N(k, x)$ may be replaced by a recursive $N(k)$ independent of x then $f(n, x)$ is said to be uniformly recursive convergent; and if $N(k) = k$, $f(n, x)$ is in *standard form.*[2]

1.3. Relative convergence. $f(m, n)$ is said to be convergent in m, relative to n, if there is a recursive $M(k)$ such that, for $m_1, m_2 \geqq M(k)$,

$$f(m_1, n) - f(m_2, n) = 0\,(k), \text{ for majorant } n.$$

1.4. Relative continuity. $f(n, x)$ is said to be continuous for $a \leqq x \leqq b$, relative to n, if $f(n, x)$ is recursive convergent for $a \leqq x \leqq b$ and if there is a strictly increasing recursive function $c(k)$ such that, for all x_1, x_2 satisfying $a \leqq x_1 \leqq x_2 \leqq b$ and $x_1 - x_2 = 0\,(c(k))$,

$$f(n, x_1) - f(n, x_2) = 0\,(k), \text{ for majorant } n.$$

1.5. Relative differentiability. $f(n, x)$ is said to be differentiable for $a \leqq x \leqq b$, relative to n, with a relative derivative $f^1(n, x)$, if $f(n, x)$ and $f^1(n, x)$ are recursive convergent for $a \leqq x \leqq b$ and if there is a strictly increasing recursive $d(k)$ such that for all x, x^* satisfying $a \leqq x < x^* \leqq b$ and $x^* - x = 0\,(d(k))$ we have

$$\{f(n, x^*) - f(n, x)\}/(x^* - x) - f^1(n, x) = 0\,(k),$$

for majorant n.[3] We observe that, if $f(n, x)$ is relatively differentiable, then both $f(n, x)$ and $f^1(n, x)$ are relatively continuous in (a, b).

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[1] When it is rendered necessary by the context we denote the connection of this function N with the function f by attaching f to N as a suffix. In the same way the several functions $M(k)$, $c(k)$ and $d(k)$ of § 1.3, 1.4, 1.5 carry suffixes when needed to avoid ambiguity.

[2] Every uniformly convergent function has an equivalent in standard form, for if $f(n, x)$ is uniformly convergent then $f(N(k), x)$ is an equivalent in standard form.

[3] Relative differentiability is therefore uniform in x.

1.51. Relative continuity and relative differentiability are invariants of the equivalence relation and any recursive equivalent of a relative derivative of a function $f(n, x)$ is itself a relative derivative of $f(n, x)$.[1]

1.6. Recursive differentiability. A recursive $f(n, x)$ is said to be recursively differentiable (for each value of n) with derivative $g(n, x)$, for $a \leqq x \leqq b$, if there is a recursive $d(n, k)$ such that for all x, x^* satisfying $a \leqq x < x^* \leqq b$ and $x^* - x = 0(d(n, k))$ we have

$$\{f(n, x^*) - f(n, x)\}/(x^* - x) - g(n, x) = 0(k)$$

for each value of n.

1.7. Ruled functions. A recursive $f(n, x)$ is said to be *ruled* for $a \leqq x \leqq b$ if $f(n, x)$ is uniformly[2] recursive convergent for $a \leqq x \leqq b$ and if there are recursive functions a_r^n, v_r^n, $b(n)$ and $t(m, n, r)$ such that

$$a_0^n = a, \quad a_{b(n)}^n = b, \quad a_{r+1}^n > a_r^n, \quad a_r^n = a_{t(m,n,r)}^m \text{ for } m > n,$$

and

$$f(n, x) = v_r^n, \text{ for } a_r^n < x < a_{r+1}^n, \text{ and } 0 \leqq r \leqq b(n) \dot{-} 1.$$

A ruled function is absolutely bounded, for if M_0 is the greatest of $|f(0, a_r^0)|$, $0 \leqq r \leqq b(0)$, and $|v_r^0|$, $0 \leqq r < b(0)$, then $|f(0, x)| \leqq M_0$ for $a \leqq x \leqq b$. But (taking $f(n, x)$ in standard form) $|f(n, x) - f(0, x)| < 1$ and so $|f(n, x)| < M_0 + 1$ for all n, and x in (a, b).

2.

Theorem 1. *A relatively continuous function has a uniformly convergent equivalent.* For if $f(n, x)$ is relatively continuous then there are recursive functions $N(k, x)$ and $c(k)$ such that

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[1] If $f(n, x)$ is relatively differentiable with relative derivative $f^1(n, x)$ for $a \leq x \leq b$ and if $f^1(n, x) = 0$ relative to n, then $f(n, x) = f(n, a)$ relative to n, $a \leq x \leq b$. Let the points a_r^k, $0 \leq r \leq i(k)$ divide (a, b) into equal parts of length $\Delta_k = 0(d(k))$, so that $\{f(n, x) - f(n, a_r^k)\}/(x - a_r^k) = 2 \cdot 0(k)$ for $a_r^k < x \leq a_{r+1}^k$), $0 \leq r \leq i(k) \dot{-} 1$, and majorant n, whence, by addition, $f(n, x) - f(n, a) = 2(b-a) \cdot 0(k)$, for any x in (a, b) and majorant n.

[2] The uniformity is needed to ensure that if x_n converges then $f(n, x_n)$ converges. If $f(n, x)$ is a non-uniformly convergent sequence of step functions there may be sequences x_n for which $f(n, x_n)$ diverges. For instance, if i_n is a nest of intervals in $[0, 1)$ enclosing the point $1/\sqrt{2}$ (so that for every rational x in $(0, 1)$, x lies outside i_n from some n onwards) and if $f(n, x) = n$ in the closed interval i_n, and takes the value 0 outside, then for each n, $f(n, x)$ is a step function, and for rational x, $f(n, x) = 0$ from some n onwards, so that $f(n, x)$ converges; but if x_n is an end point of i_n, $f(n, x_n) = n$ so that $f(n, x_n) \to \infty$. Of course $f(n, x)$ is not uniformly convergent, since for any two p, q with $q > p$

$$f(q, x) - f(p, x) = q - p \geqq 1 \text{ for any } x \text{ in } i_q.$$

for $n_1 \geqq n_2 \geqq N(k, x)$, $f(n_1, x) - f(n_2, x) = 0(k)$, and

for $x^* - x = 0(c(k))$, $f(n, x^*) - f(n, x) = 0(k)$, for majorant n,

and therefore, if $\phi(k, x) = f(N(k, x), x)$ we have, for $q > p$, $\phi(p, x) - \phi(q, x) = 0(p)$ and

$$\begin{aligned}\phi(k, x^*) - \phi(k, x) &= f(N(k, x^*), x^*) - f(m, x^*) \\ &\quad + f(m, x^*) - f(m, x) + f(m, x) - f(N(k, x), x) \\ &= 3 \cdot 0(k), \ m \text{ majorant,}\end{aligned}$$

so that $\phi(k, x)$ is uniformly convergent and relatively continuous, and since

$$\begin{aligned}\phi(n, x) - f(n, x) &= \phi(n, x) - f(m, x) + f(m, x) - f(n, x) \\ &= 2.0(k), \ n \text{ majorant,}\end{aligned}$$

therefore $\phi(n, x)$ is equivalent to $f(n, x)$.

It follows that we may without loss of generality suppose any relatively continuous $f(n, x)$ to satisfy $f(p, x) - f(q, x) = 0(p)$, for $q \geqq p$,[1] i.e. to be in standard form.

Theorem 2. *A relatively continuous function has a ruled equivalent.*

If $f(n, x)$ is relatively continuous for $a \leqq x \leqq b$, and if γ is the least integer such that $b - a \leqq 10^{\gamma}$, it suffices to take $a_r^n = a + r \cdot (b-a)/10^{\gamma + c(n)}$, $b(n) = 10^{\gamma + c(n)}$ then $g(n, x)$, defined by the recursive conditions

$$g(n, a) = f(n, a), \ g(n, x) = f(n, a_{r+1}^n) \text{ for } a_r^n < x \leqq a_{r+1}^n,$$

is a ruled equivalent of $f(n, x)$.

Theorem 3 (a). *If, for each value of n, $f(n, x)$ is recursively differentiable for $a \leqq x \leqq b$, with a derivative $f^1(n, x)$ then there is a recursive c_k^n, such that $a < c_k^n < b$ and*

$$\{f(n, b) - f(n, a)\}/(b-a) \leqq f^1(n, c_k^n) + 0(k),$$

for each value of n.

For if $\mu_n(x, y)$ denotes $\{(f(n, x) - f(n, y)\}/(x-y)$, $x < y$, and if z is the midpoint of (x, y) then $\mu_n(x, y)$ lies between $\mu_n(x, z)$ and $\mu_n(z, y)$ and so $\mu_n(x, y)$ is exceeded by one of $\mu_n(x, z)$, $\mu_n(z, y)$; thus we may bisect (a, b) repeatedly choosing a succession of intervals (a, b), (a_1^n, b_1^n), (a_2^n, b_2^n), $\ldots$, (a_r^n, b_r^n), say, each of which is a half of its predecessor, and such that $\mu_n(a_r^n, b_r^n)$ is non-decreasing in r. But for a

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[1] Consequently, a relatively continuous function is absolutely bounded, for we may divide (a, b) into a finite number p of parts, each of length $0(c(1))$, such that for any two x_1, x_2 in the same part $f(n, x_1) - f(n, x_2) = 0(1)$, n majorant, and therefore $f(n, x_1) - f(n, x_2) = 3 \cdot 0(1)$ for $n \geqq 1$, so that for any x in (a, b), $|f(n, x) - f(n, a)| = p \cdot 0(1)$ for $n \geqq 1$.

suitable value of r, $\mu_n(a_r^n, b_r^n) = f^1(n, c_k^n) + 0(k)$, where we take c_k^n to be either a_r^n, or b_r^n, whichever is in the interior of (a, b).

Theorem 3 suffices for the purposes of the present paper though of course we could replace it by the more familiar mean value theorem.[1]

The same argument suffices to prove also Theorems 3 (b), 3 (c) and 3 (d) below.

Theorem 3 (b). *Under the conditions of Theorem 3 we can find a recursive c_k^n in (a, b) such that*

$$\{f(n, b) - f(n, a)\}/(b-a) \geqq f^1(n, c_k^n) + 0(k)$$

for each value of n.

Theorem 3 (c). *If $f(n, x)$ is relatively differentiable for $a \leqq x \leqq b$, with relative derivative $f^1(n, x)$, then there is a recursive c_k^n in (a, b) such that*

$$\{f(n, b) - f(n, a)\}/(b-a) \leqq f^1(n, c_k^n) + 0(k),$$

n majorant.

Theorem 3 (d). Theorem 3 (c) holds also with the inequality reversed.

Theorem 4. *If $s(n, x)$ is recursively differentiable for each n with derivative $\sigma(n, x)$, and if $\sigma(n, x)$ is uniformly recursive convergent, for $a \leqq x \leqq b$, then $s(n, x)$ is differentiable relative to n, with relative derivative $\sigma(n, x)$, for $a \leqq x \leqq b$.*

It follows from definition 1.6 that there is a recursive $d(n, k)$ and, from definition 1.2, a recursive $N(k)$ such that $\sigma(n, x^*) - \sigma(n, x) = 0(k)$ for $x^* - x = 0(d(n, k))$ and $\sigma(n, x) - \sigma(N(k), x) = 0(k)$ for $n \geqq N(k)$. By Theorem 3 (a) there is a c_k^n such that, for $a \leqq x < x^* < b$, $x < c_k^n < x^*$ and

$$\{s(n, x^*) - s(n, x)\}/(x^* - x) \leqq \sigma(n, c_k^n) + 0(k),$$

and therefore, for $n \geqq N(k)$ and $x^* - x = 0(d(N(k), k))$,

$$\begin{aligned}\{s(n, x^*) - s(n, x)\}/(x^* - x) - \sigma(n, x) \\ \leqq \{\sigma(n, c_k^n) - \sigma(N(k), c_k^n)\} + \{\sigma(N(k), c_k^n) - \sigma(N(k), x)\} \\ + \{\sigma(N(k), x) - \sigma(n, x)\} + 0(k) = 4 \cdot 0(k)\end{aligned}$$

and similarly by Theorem 3 (b), for the same n, x and x^*,

$$\{s(n, x^*) - s(n, x)\}/(x^* - x) - \sigma(n, x) \geqq 4 \cdot 0(k)$$

which completes the proof.

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[1] Vide R. L. GOODSTEIN "Mean value theorems in recursive function theory", *Proc. London Math. Soc.* Vol. 52 (1950), pp. 81–106.

12 – 543808 *Acta Mathematica*. 92. Imprimé le 30 décembre 1954.

3.

In preparation for some results on inverse functions we prove next

Theorem 5. If

(i) for $a \leqq x \leqq b$, $g(m, x)$ is continuous relative to m, and $\alpha = g(m, a) \leqq g(m, x) \leqq \beta$, for majorant m,

(ii) for $\alpha \leqq t \leqq \beta$, $f(n, t)$ is differentiable relative to n and $f^1(n, t) \geqq 10^{-\mu}$, for majorant n[1], and $f(n, \alpha) = a$,

(iii) for $a \leqq x \leqq b$, $f(n, g(m, x)) = x$, relative to (m, n), then

$$g(q, f(p, t)) = t, \text{ relative to } (p, q),$$

for any t such that, for an $x < b$, $\alpha \leqq t < g(m, x)$ for majorant m.

We start by proving that if $\alpha \leqq t < g(m, x)$ for majorant m and an $x < b$, then $a \leqq f(n, t) < b$ for majorant n.

For, by (ii), (and Theorem 3 (d)), $f(n, t)$ is strictly increasing (for majorant n), and so by (iii) if $\alpha \leqq t < g(m, x)$, for majorant m, and $x < b$ then

$$a = f(n, \alpha) \leqq f(n, t) < f(n, g(m, x)) = x + 0(k) < b$$

for a large enough k and for majorant m, n.

Using the uniform convergence of $f(n, x)$, $g(n, x)$ and the relative continuity of $f(n, x)$ it follows from (iii) that

$$f(n, g(m, x)) = x + 3 \cdot 0(k)$$

for $m \geqq c_f(k)$, $n \geqq k$ and $a \leqq x \leqq b$, and therefore, for the same m, n and for $p \geqq k$,

$$f(n, g(m, f(p, t))) = f(n, t) + 4 \cdot 0(k)$$

for $\alpha \leqq t < g(m, x)$ and $x < b$.

But by Theorem 3 (d) there is a recursive c_k^n such that

$$|f(n, g) - f(n, t)| \geqq |g - t| \cdot \{f^1(n, c_k^n) + 0(k)\}$$

for majorant n, and therefore, for $m \geqq c_f(k)$, $p \geqq k$,

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[1] In fact it is sufficient if $|f^1(n, t)| \geqq 10^{-\mu}$, for majorant n, for $f^1(n, t)$ is relatively continuous and so for any two points t_1, t_2 in (α, β) we can divide the interval (t_1, t_2) into a finite number of parts such that the values of $f^1(n, t)$ at any two points in the same part differ by less than $1/10^{\mu+1}$, relative to n, and therefore $f^1(n, t)$ has the same sign at t_1, and t_2, for majorant n.

$$g(m, f(p, t)) - t = 0(k - \mu - 1);$$

hence $g(q, f(p, t)) = t$, relative to p, q.

Of course the result is unchanged if $f(n, t)$, $g(m, x)$ are not in standard form.

4. The reduced function

Given a relatively continuous function $f(n, x)$ and a uniformly convergent $g(n, x)$, both in standard form, then the function

$$f(k+1,\ g(c_f(k+1), x))$$

is called the reduce of f on g and is denoted by $R_g^f(k, x)$.

The following two theorems are immediate consequences of this definition.

Theorem 6. For $n \geqq k+1$, $m \geqq c_f(k+1)$,

$$R_g^f(k, x) - f(n, g(m, x)) = 2.0(k+1)$$

and for $q > p$,

$$R_g^f(q, x) - R_g^f(p, x) = 2.0(p+1) = 0(p),$$

so that $R_g^f(n, x)$ is uniformly convergent in standard form.

Theorem 7. If $\phi(n, x)$, $\psi(n, x)$ are equivalents of a relatively continuous $f(n, x)$ and a uniformly convergent $g(n, x)$, all in standard form, then $R_\psi^\phi(k, x)$ is equivalent to $R_g^f(k, x)$.

4.01. If $f(n, x)$ is relatively continuous and $g(n, x)$ uniformly recursively convergent, neither necessarily in standard form, and if $\phi(n, x)$, $\psi(n, x)$ are standard form equivalents, we define $R_g^f(k, x)$ to be $R_\psi^\phi(k, x)$.

4.1. We remark that if $g(m, x)$ is a ruled function then so is $R_g^f(n, x)$, and if $g(m, x)$ is relatively continuous (as is $f(n, x)$) then $R_g^f(n, x)$ is relatively continuous.

Theorem 8. If $f(n, y)$, $g(m, x)$ are relatively differentiable with relative derivatives $f^1(n, y)$, $g^1(m, x)$ for $a \leqq x \leqq b$ and $h \leqq y \leqq H$, and if $h \leqq g(m, x) \leqq H$ for $a \leqq x \leqq b$, and majorant m, then $R_g^f(n, x)$ is relatively differentiable, with relative derivative

$$R_g^{f^1}(n, x) \cdot g^1(n, x), \text{ for } a \leqq x \leqq b.$$

Consider any two x, X such that $a \leqq x < X \leqq b$ and let $\xi(x, X)$ be the exponent of the least power of 10 which exceeds $1/(X-x)$; μ the exponent of the least power of 10 which exceeds $|g^1(m, x)| + 1$ throughout (a, b), and

$$d(k) = \max(d_g(k),\ d_f(k) + \mu).$$

Then, for $X-x=0\,(d\,(k))$,

$$\frac{f(n, g(m, X)) - f(n, g(m, x))}{X-x} - f^1(n, g(m, x))\cdot g^1(m, x)$$
$$= (f^1(n, g(m, x)) + g^1(m, x) + 0\,(k))\cdot 0\,(k)$$

and, for $p \geqq k + \xi(x, X)$,

$$\frac{R_g^f(p, X) - R_g^f(p, x)}{X-x} = \frac{f(n, g(m, X)) - f(n, g(m, x))}{X-x} + 2.0\,(k)$$

and

$$R_g^{f'}(p, x)\cdot g^1(p, x) - f^1(n, g(m, x))\cdot g^1(m, x)$$
$$= (f^1(n, g(m, x)) + g^1(m, x) + 0\,(k))\cdot 0\,(k),$$

all for sufficiently large m and n, whence Theorem 8 follows.

5. Integration

If $f(n, x)$ is a ruled function in (a, b) so that there are recursive functions a_r^n, v_r^n and $b(n)$ such that

$$f(n, x) = v_r^n, \text{ for } a_r^n < x < a_{r+1}^n,\ 0 \leqq r \leqq b(n) \dot{-} 1,$$

then the sum

$$\sum_{r=0}^{b(n)\dot{-}1} v_r^n\,(a_{r+1}^n - a_r^n)$$

is called a relative definite integral of $f(n, x)$ from a to b, $(a<b)$, and denoted by $I_f(n, a, b)$.

If $a=b$ we define

$$I_f(n, a, b) = 0$$

and if $a>b$,

$$I_f(n, a, b) = -I_f(n, b, a).$$

Taking $f(n, x)$ in standard form (and using the notation of definition 1.7) we see that, for $N>n$ and

$$t(N, n, r) \leqq s < t(N, n, r+1),\ v_s^N - v_r^n = 0\,(n)$$

and therefore

$$I_f(N, a, b) - I_f(n, a, b) = \sum_{r=0}^{b(n)\dot{-}1}\ \sum_{s=t(N,n,r)}^{t(N,n,r+1)\dot{-}1} (v_s^N - v_r^n)\,(a_{s+1}^N - a_s^N)$$
$$= (b-a)\cdot 0\,(n)$$

which proves that $I_f(n, a, b)$ converges.

5.1. Theorem 9. If $f(n, x)$ and $g(n, x)$ are equivalent ruled functions then $I_f(n, a, b)$ and $I_g(n, a, b)$ are equivalent.

By combining the subdivisions on which f and g are constant for a given n, we determine a subdivision c_r^n, say, such that both $f(n, x)$ and $g(n, x)$ are constant in each subinterval (c_r^n, c_{r+1}^n); if μ_r^n denotes the mid-point of this subinterval, then

$$I_f(n, a, b) - I_g(n, a, b) = \Sigma (f(n, \mu_r^n) - g(n, \mu_r^n))(c_{r+1}^n - c_r^n);$$

but, for $N > n$, $f(n, \mu_r^n) - f(N, \mu_r^n) = 0(n)$,

$$g(n, \mu_r^n) - g(N, \mu_r^n) = 0(n)$$

and

$$f(N, \mu_r^n) - g(N, \mu_r^n) = 0(n), \text{ for majorant } N,$$

so that $I_f(n, a, b) - I_g(n, a, b) = 3(b-a) \cdot 0(n)$.

5.2. If a recursive function $f(n, x)$ has a ruled equivalent $f^*(n, x)$ in (a, b) then $f(n, x)$ is said to be relatively integrable in (a, b) with a relative integral $I_{f^*}(n, a, b)$; the integral of $f(n, x)$ is also denoted by $I_f(n, a, b)$.

In particular a relatively continuous function is relatively integrable.

In virtue of Theorem 9 any two integrals of $f(n, x)$ over (a, b) are equivalent.

A relatively integrable function is bounded, for if $f(n, x)$ is relatively integrable then it has a ruled equivalent $f^*(n, x)$ which is absolutely bounded, and if $|f^*(n, x)| < M$ for all x and n, then $|f(n, x)| < M + 1$ for majorant n.

Theorem 10. *Darboux's Theorem.* If $f(n, x)$ is relatively integrable in (a, b), with a bound M, and $f^*(n, x)$ is a ruled equivalent of $f(n, x)$, in standard form and constant (for each value of n) in the open intervals (a_r^n, a_{r+1}^n), $0 \leqq r \leqq b(n)$, and if (x_r), $0 \leqq r \leqq \nu$, is any subdivision of (a, b) into subintervals of smaller length than any of the intervals of the subdivision a_r^k, $0 \leqq r \leqq b(k)$, and also smaller than

$$(b-a)/\{2(M+1)10^k \cdot b(k)\},$$

and if ξ_r is any point in

$$(x_r, x_{r+1}), \; 0 \leqq r \leqq \nu - 1,$$

then

$$I_f(k, a, b) - \sum_{r=0}^{\nu-1} f(n, \xi_r)(x_{r+1} - x_r) = 6(b-a) \cdot 0(k), \text{ for majorant } n.$$

Since $f^*(n, x)$ is a ruled equivalent of $f(n, x)$, in standard form, therefore $M + 1$ is an absolute bound of $|f^*(n, x)|$ in (a, b), and

$$I_{f^*}(k, a, b) - I_f(k, a, b) = 3(b-a) \cdot 0(k),$$

$$\sum_{r=0}^{\nu-1} f^*(k, \xi_r)(x_{r+1}-x_r) - \sum_{r=0}^{\nu-1} f(n, \xi_r)(x_{r+1}-x_r) = 2(b-a)\cdot 0(k), \ n \text{ majorant},$$

and so it remains to prove that

$$I_{f^*}(k, a, b) - \sum_{r=0}^{\nu-1} f^*(k, \xi_r)(x_{r+1}-x_r) = (b-a)\cdot 0(k).$$

The proof is completed along familiar lines by combining the subdivisions (a_r^k) and (x_r), and we omit the details.

Theorem 11. If $f(n, x)$ is relatively integrable in (a, b) then $|f(n, x)|$ is relatively integrable in (a, b), and $|I_f(n, a, b)| \leqq I_{|f|}(n, a, b)$, relative to n.

Theorem 12. If $f(n, x)$ is relatively integrable in (a, b) and if c lies between a and b then $f(n, x)$ is relatively integrable in (a, c) and (c, b) and

$$I_f(n, a, c) + I_f(n, c, b) = I_f(n, a, b), \text{ relative to } n.$$

(In fact $I_f(n, a, c) + I_f(n, c, b) - I_f(n, a, b) = 9(b-a)\cdot 0(n)$.)

Theorem 13. If $f(n, x)$ and $g(n, x)$ are relatively integrable in (a, b) then $f(n, x) + (g, x)$ is relatively integrable and

$$I_f(n, a, b) + I_g(n, a\ b) = I_{f+g}(n, a, b), \text{ relative to } n.$$

(The difference between the integrals is also $9(b-a)\cdot 0(n)$.)

Theorem 14. If $f(n, x) = 0$ for $a \leqq x \leqq b$ and $n \geqq N$, independent of x, then $I_f(n, a, b) = 0$, for $n \geqq N$.

Theorem 15. If $f(n, x)$ is relatively integrable in (a, b) and $f(n, x) \geqq 0$ in (a, b) then $I_f(n, a, b) > -10^{-k}$ for n satisfying $10^n \geqq 3(b-a)10^k$, i.e $I_f(n, a, b) \geqq 0$, relative to n.

If $f(n, x) > 0$ in (a, b) then $I_f(n, a, b) > 0$, relative to n.

Similarly if $f(n, x)$ is relatively integrable and $f(n, x) = 0(k)$ in (a, b), for majorant n, then $I_f(n, a, b) = (b-a)\cdot 0(k)$, for majorant n.

Theorem 16. If $f(n, x)$ is relatively integrable in (a, b) then $I_f(n, a, x)$ is relatively continuous in (a, b).

The proofs of Theorems 11 to 16 are omitted.

Theorem 17. If $f(n, x)$ is relatively continuous in (a, b) then $I_f(n, a, x)$ is relatively differentiable in (a, b) with a relative derivative $f(n, x)$.

Let $a \leqq t < T \leqq b$ and $\phi(n, x) = f(n, x) - f(n, t)$ so that $\phi(n, x) = 0(k)$ for $t \leqq x \leqq T$, $T - t = 0(c(k))$ and majorant n.

Then, for $T-t=0\,(c\,(k))$,

$$\{I_f(n, a, T)-I_f(n, a, t)\}/(T-t)-f(n, t)$$
$$=\{I\phi\,(n, t, T)+18\,(b-a)\cdot 0\,(n)\}/(T-t), \text{ by Theorems 12, 13,}$$
$$=2.0\,(k), \text{ for majorant } n, \text{ which completes the proof.}$$

Theorem 18. If $f(n, x)$ has a relative derivative $f^1(n, x)$ in (a, b) then

$$I_{f^1}(n, a, b)=f(n, b)-f(n, a), \text{ relative to } n.$$

For $I^1_{f^1}(n, a, x)-f^1(n, x)=0$, relative to n.

Theorem 19. In the interval (t_0, t_1) the function $g(n, t)$ is relatively differentiable with a relative derivative $g^1(n, t)$, and $\alpha \leqq g(n, t) \leqq \beta$ for majorant n.

If $f(n, x)$ is relatively continuous in (α, β) and

$$g(n, t_0)=a, \qquad g(n, t_1)=b, \qquad \text{relative to } n,$$

then

$$I_f(n, a, b)=I_{R^f_g\cdot g^1}(n, t_0, t_1), \qquad \text{relative to } n.$$

We observe first that from the two conditions $g(n, t_0)=a$, relative to n, and $g(n, t_0)\geqq\alpha$ for majorant n, it follows that $a\geqq\alpha$, and similarly we can show that $b\leqq\beta$, so that $f(n, x)$ is continuous in (a, b), relative to n, and $I_f(n, a, x)$ exists for $a\leqq x\leqq b$. Since $R^f_g(n, t)$ and $g^1(n, t)$ are both relatively continuous in (t_0, t_1), therefore the integral $I_{R^f_g\cdot g^1}(n, t_0, t)$ exists.

If we denote $I_f(n, a, x)$ by $F(n, x)$, $R^F_g(n, t)$ by $G(n, t)$, and $I_{R^f_g\cdot g^1}(n, t_0, t)$ by $H(n, t)$ it readily follows that

$$G^1(n, t)-H^1(n, t)=0, \text{ relative to } n,$$

and so $G(n, t_1)-G(n, t_0)=H(n, t_1)$, relative to n, whence, since $G(n, t_0)=F(n, a)$ and $G(n, t_1)=F(n, b)$, relative to n, Theorem 19 follows.

Theorem 20. If

(i) for $0\leqq x\leqq b$, $g(m, x)$ is differentiable relative to m with a relative derivative $1/R^\phi_g(m, x)$, and

$$g(m, 0)=\alpha\leqq g(m, x)\leqq\beta, \text{ for majorant } m,$$

(ii) for $\alpha\leqq t\leqq\beta$, $\phi(n, t)$ is continuous relative to n and $\phi(n, t)\geqq 10^{-\mu}$, for majorant n,

then $g(n, x)$ and $I_\phi(n, \alpha, t)$ are inverse functions, relative to n.

Let $f(n, t)$ denote $I_{\phi}(n, \alpha, t)$ for $\alpha \leqq t \leqq \beta$; then, for $0 \leqq x \leqq b$, the relative derivative of $R_g^f(k, x)$ is

$$R_g^{\phi}(k, x) \cdot g^1(k, x) = 1, \text{ relative to } k,$$

and therefore

5.3. $$R_g^f(k, x) = x, \text{ relative to } k, \text{ for } 0 \leqq x \leqq b,$$

whence

$$f(n, g(m, x)) = x, \text{ relative to } m, n;$$

and hence, by Theorem 5, for $\alpha \leqq t < g(m, x)$, m majorant and $x < b$,

$$g(q, f(p, t)) = x, \text{ relative to } p, q$$

and so

5.4. $$R_f^g(k, t) = t, \text{ relative to } k.$$

Formulae 5.3 and 5.4 prove that $f(n, x)$ and $g(n, x)$ are relatively inverse functions.

6. The elementary functions

6.1. The relatively exponential function.

We define the *relatively exponential* function $E(n, x)$ by the recursive equations

$$E(0, x) = 1, \quad E(n+1, x) = E(n, x) + x^{n+1}/(n+1)!$$

$E(n, x)$ is uniformly convergent in any interval, for if N is any positive integer then, for $|x| \leqq N$ and $n \geqq 2N$,

$$\begin{aligned} |E(n+r+1, x) - E(n+r, x)| &\leqq \{x^n/n!\}\, 2^{-(r+1)} \\ &\leqq \{x^{2N}/(2N)!\} 2^{-(n-2N)} \cdot 2^{-(r+1)} \\ &\leqq \{(2N)^{2N}/(2N)!\}\, 2^{-n} \cdot 2^{-(r+1)} \end{aligned}$$

and so for $m \geqq n \geqq 2N$

6.101. $$|E(m, x) - E(n, x)| < \{(2N)^{2N}/(2N)!\} \cdot 2^{-n},$$

which proves that $E(n, x)$ is uniformly convergent for $|x| \leqq N$.

We observe next that

6.102. if $0 \leqq x < X$ then $(n+1)x^n < (X^{n+1} - x^{n+1})/(X - x) < (n+1)X^n$

and so

$$\begin{aligned} 0 &< \{(X^{n+1} - x^{n+1})/(X - x) - (n+1)x^n\}/(X - x) \\ &< (n+1)(X^n - x^n)/(X - x) < n(n+1)X^{n-1}. \end{aligned}$$

Whence if $0 \leqq x < X \leqq N$ then

$$\frac{1}{(n+1)!}\left\{\frac{X^{n+1}-x^{n+1}}{X-x}-(n+1)\,x^n\right\}/(X-x) < \frac{N^N}{N!}$$

The inequality is symmetrical in X, x and so holds also for $0 \leqq X < x \leqq N$. Hence, provided X, x have the same sign, positive or negative, and $|x| \leqq N$, $|X| \leqq N$ we have

$$\frac{1}{(n+1)!}\left|\frac{X^{n+1}-x^{n+1}}{X-x}-(n+1)\,x^n\right|/|X-x|$$
$$=\frac{1}{(n+1)!}\left|\frac{|X|^{n+1}-|x|^{n+1}}{|X|-|x|}-(n+1)\,|x|^n\right|/\big||X|-|x|\big| < \frac{N^N}{N!}.$$

It follows that if ν is the exponent of the least power of 10 which is not exceeded by $N^N/N!$ then

for $|x| \leqq N$, $|X| \leqq N$, $X/x > 0$ and $X-x=0\,(k+\nu)$ we have

$$\frac{1}{(n+1)!}\left\{\frac{X^{n+1}-x^{n+1}}{X-x}-(n+1)\,x^n\right\}=0\,(k).$$

On the other hand, whether x, X have the same sign or not, if $|x| \leqq \alpha < 1$, $|X| \leqq \alpha$ we have

$$\frac{1}{(n+1)!}\left|\frac{X^{n+1}-x^{n+1}}{X-x}-(n+1)\,x^n\right| \leqq \frac{2\,\alpha^n}{n!} \leqq 2\,\alpha, \text{ for any } n.$$

Hence for all X, x satisfying $|X| \leqq N$, $|x| \leqq N$, $X-x=0\,(k+\nu)$,

$$\frac{1}{(n+1)!}\left\{\frac{X^{n+1}-x^{n+1}}{X-x}-(n+1)\,x^n\right\}=0\,(k).$$

and therefore

$$\frac{E\,(n+1,\,X)-E\,(n+1,\,x)}{X-x}-E\,(n,\,x)=\frac{E\,(n,\,X)-E\,(n,\,x)}{X-x}-E\,(n-1,\,x)+0\,(k).$$

Since

$$\frac{E\,(1,\,X)-E\,(1,\,x)}{X-x}-E\,(0,\,x)=0,$$

it follows by induction that for any x, X in $(-N,\,+N)$ and $X-x=0\,(k+\nu)$

$$\frac{E\,(n,\,X)-E\,(n,\,x)}{X-x}-E\,(n-1,\,x)=(n-1)\cdot 0\,(k)$$

and thus, for $X-x=0\,(k+n+\nu)$,

$$\frac{E(n, X) - E(n, x)}{X - x} - E(n-1, x) = 0\,(k)$$

i.e. for each value of n, $E(n, x)$ is recursively differentiable with derivative $E(n-1, x)$. Hence, by Theorem 4, since $E(n, x) = E(n-1, x)$, relative to n, $E(n, x)$ is differentiable *relative to n* with derivative $E(n, x)$ in the interval $(-N, +N)$ for any N.

It follows that the relative derivative of

$$E(n, -x) \cdot E(n, x+a)$$

is zero, relative to n, and so,

$$E(n, -x) \cdot E(n, x+a) = E(n, 0) \cdot E(n, a) = E(n, a),$$

relative to n; in particular

$$E(n, -x) \cdot E(n, x) = 1, \text{ relative to } n,$$

and so

$$E(n, x) \cdot E(n, a) = E(n, x+a), \text{ relative to } n,$$

which is the fundamental relation for the relative exponential function.

6.11. We note that, for any N and for $|x| \leqq N$, $E(n, x)$ is continuous in x, uniformly in x and n.

For if $0 \leqq x < X \leqq N$, then by 6.102

$$0 < E(n, X) - E(n, x) \leqq (X - x) E(n, X) \leqq (X - x) E(n, N).$$

By 6.101, if $n \geqq 2N$,

$$E(n, N) - E(2N, N) \leqq N^{2N}/(2N)!\,;$$

moreover, since $N^N/N!$ is the greatest of $N^r/r!$ for $o \leqq r \leqq 2N$, therefore

$$E(2N, N) \leqq (2N+1)\,N^N/N!\,,$$

and so

$$0 < E(n, X) - E(n, x) < (X - x)\{N^{2N}/(2N)! + (2N+1)\,N^N/N!\}$$

whence 6.11 readily follows.

We have incidentally shown that

6.12. for $|x| \leqq N$, and any m, $0 < E(m, x) < N^{2N}/(2N)! + (2N+1)\,N^N/N!$ and we denote this upper bound by E_N.

6.2. The relatively logarithmic function.

6.21. The relatively logarithmic function $\log(n, x)$ is defined by the equation

$$\log(n, x) = I_{\text{rec}}(n, 1, x), \; x > 0,$$

where $\text{rec}(n, x) = 1/x$, $x \neq 0$.

Accordingly $\log(n, 1) = 0$, $\log^1(n, x) = 1/x$.

By means of the substitution $x = bt$ we readily prove that

$$\log(n, ab) = \log(n, a) + \log(n, b), \text{ relative to } n.$$

6.22. We observe next that, since $E^1(n, x) = E(n, x)$, therefore

$$E^1(n, x) = 1/R_E^{\text{rec}}(n, x), \text{ relative to } n;$$

thus given any $N > 0$, for $0 \leqq x \leqq N$, $E(m, x)$ is differentiable relative to m, with a relative derivative $1/R_E^{\text{rec}}(m, x)$ and (in the notation of § 6.12)

$$E(m, 0) = 1 \leqq E(m, x) < E_N.$$

Furthermore, for $1 \leqq t \leqq E_N$, $1/t$ is uniformly continuous and $1/t \geqq 1/E_N$.

Hence, by Theorem 20, $E(m, x)$ and $\log(n, x)$ are inverse functions, and for $o \leqq x \leqq N$,

(i) $\log(n, E(m, x)) = x$, relative to m, n; and for $1 \leqq t < E(m, x)$, m majorant and $x < N$,

(ii) $E(m, \log(n, t)) = t$, relative to n, m.

Since N is arbitrary, and $E(m, x)$ is arbitrarily great with arbitrarily great x, $(m > 0)$, therefore (i) holds for all $x \geqq 0$ and (ii) for all $t \geqq 1$. But

$$E(m, -x) = 1/E(m, x), \text{ relative to } m,$$

and

$$\log(n, 1/t) = -\log(n, t), \text{ relative to } n,$$

so that $\log(n, E(m, -x)) = -x$, relative to m, n and all $x \geqq 0$, and (i) holds for all (rational) values of x.

Similarly we can show that (ii) holds for all $t > 0$.

6.3. The relatively circular functions.

We define the relatively circular functions $\sin(n, x)$ and $\cos(n, x)$ by the recursive equations

$$\sin(0, x) = 0, \ \sin(n+1, x) = \sin(n, x) + (-1)^n x^{2n+1}/(2n+1)!$$
$$\cos(0, x) = 0, \ \cos(n+1, x) = \cos(n, x) + (-1)^n x^{2n}/(2n)!$$

A trivial induction shows that, for all n, $\sin(n, 0) = 0$, $\cos(n, 0) = 1$. Exactly as for $E(n, x)$ we can prove that, for any x, $\sin(n, x)$, $\cos(n, x)$ are differentiable relative to n, with relative derivatives $\cos(n, x)$, $-\sin(n, x)$ respectively. It follows that a relative derivative of $\sin^2(n, x) + \cos^2(n, x)$ is zero, and so

$$\sin^2(n, x) + \cos^2(n, x) = 1, \text{ relative to } n.$$

for any x.

So, too, a relative derivative of the function

$$\cos(n, c-x)\cos(n, x) - \sin(n, c-x)\sin(n, x)$$

is zero, and therefore

$$\cos(n, c) = \cos(n, c-x)\cos(n, x) - \sin(n, c-x)\sin(n, x), \text{ relative to } n,$$

or, writing $x+y$ for c,

$$\cos(n, x+y) = \cos(n, x)\cos(n, y) - \sin(n, x)\sin(n, y), \text{ relative to } n,$$

whence differentiating (as functions of x)

$$\sin(n, x+y) = \sin(n, x)\cos(n, y) + \cos(n, x)\sin(n, y).$$

6.31. For $|x| \leqq 4$ and $p \geqq 1$

$$(\sin(p+2, x) - \sin(p, x))/(\sin(p+2, x) - \sin(p+1, x))$$
$$= 1 - (2p+2)(2p+3)/x^2 < 0$$

so that $\sin(p+2, x)$ lies between $\sin(p, x)$ and $\sin(p+1, x)$ and therefore $\sin(n, x)$ lies between $\sin(p, x)$ and $\sin(p+1, x)$ for $n \geqq p+2 \geqq 3$.

Similarly, for $n \geqq p+2 \geqq 4$, and $|x| \leqq 4$,

$$\cos(n, x) \text{ lies between } \cos(p, x) \text{ and } \cos(p+1, x).$$

It follows that (for $|x| \leqq 4$), $\sin(n, x)$ lies between x and $x - \frac{x^3}{6}$ and, in particular, for $0 \leqq x \leqq 1\cdot 6$, between $\frac{1}{2}x$ and x, for $n \geqq 3$.

6.32. We consider the behaviour of $\cos(n, x)$ in the interval $(1\cdot 5, 1\cdot 6)$. A simple calculation shows that $\cos(4, 1\cdot 5) > 0$, and so *a fortiori* $\cos(3, 1\cdot 5) > 0$, so that $\cos(n, 1\cdot 5) > 0$ for all $n \geqq 3$. Similarly $\cos(n, 1\cdot 6) < 0$ for all $n \geqq 2$. Let λ_n be the least integer between 15.10^{n-1} and 16.10^{n-1} such that $\cos(n, (\lambda_n + 1)10^{-n}) \leqq 0$ and $\cos(n, \lambda_n 10^{-n}) > 0$. λ_n is a primitive recursive function for which we may take $\lambda_0 = 0$. If we write σ_n for $\lambda_n 10^{-n}$ then (using Theorem 3 b)

$$0 < \cos(n, \sigma_n) - \cos(n, \sigma_n + 10^{-n}) < 16/10^{n+1}$$

whence

$$0 < \cos(n, \sigma_n) < 16/10^{n+1}$$

and so

$$-\frac{1}{10^{n+1}} < \cos(n+1, \sigma_n) < \frac{17}{10^{n+1}}$$

(for $(1\cdot 6)^{2n}/(2n)!$ is very much less than $1/10^{n+1}$, $n \geqq 2$) and therefore $|\cos(N, \sigma_n)| < < 17/10^{n+1}$ for all $N \geqq n \geqq 1$. Since $\sin(n, x) \geqq 3/4$ for $1\cdot 5 \leqq x \leqq 1\cdot 6$ and $n \geqq 3$ it follows that

$$|\cos(N, \sigma_N) - \cos(N, \sigma_n)| \geqq (3/4)\,|\sigma_N - \sigma_n|$$

and so $|\sigma_N - \sigma_n| < 5/10^n$ which proves that σ_n converges. Since $\cos(n, \sigma_n) = 0$, relative to n, therefore $\sin(n, \sigma_n) = 1$ relative to n, and from the addition formulae we derive, in turn, $\cos(n, 2\sigma_n) = -1$, $\sin(n, 2\sigma_n) = 0$, relative to n, and $\cos(n, 4\sigma_n) = 1$, $\sin(n, 4\sigma_n) = 0$, relative to n and so (by the addition formulae)

$$\cos(n, x + 4\sigma_n) = \cos(n, x), \quad \sin(n, x + 4\sigma_n) = \sin(n, x)$$

relative to n, which proves that $\cos(n, x)$, $\sin(n, x)$ are periodic relative to n, with relative period $4\sigma_n$. Furthermore, since $\cos(n, \sigma_n - x) = \sin(n, x)$, relative to n, therefore $\cos(n, \sigma_n - x) > \frac{1}{3}x$, for majorant n and $0 < x \leqq 1\cdot 6$.

6.4. It follows that if we define

$$\tan(n, x) = \sin(n, x)/\cos(n, x)$$

then $\tan(n, x)$ is defined for $|x| \leqq b < \sigma_n - \delta$ for majorant n, and any $\delta > 0$, and is bounded above by $3/\delta$ in this range, and differentiable relative to n with relative derivative $1 + \tan^2(n, x)$. Writing $\phi(n, t) = 1/(1 + t^2)$ for any n and t, so that $\phi(n, t)$ is relatively integrable with a relative integral from 0 to t which we denote by $\arctan(n, t)$, then, by Theorem 20, $\tan(n, x)$ and $\arctan(n, t)$ are inverse functions and, for $|x| \leqq b$,

(i) $$\arctan(n, \tan(m, x)) = x, \text{ relative to } m, n,$$

and

(ii) $$\tan(m, \arctan(n, t)) = t, \text{ relative to } n, m,$$

for $0 \leqq t < \tan(m, x)$, m majorant and $x \leqq b < \sigma_n - \delta$ for majorant n. Since σ_n converges, given any $k \geqq 1$ we can find n_k so that

$$\sigma_{n_k} - 1/k < \sigma_n < \sigma_{n_k} + 1/k \text{ for } n \geqq n_k;$$

taking

$$x = \sigma_{n_k} - 3/k, \quad b = \sigma_{n_k} - 2/k, \quad \delta = 1/k;$$

then

$$x < b < \sigma_n - \delta, \text{ for } n \geqq n_k,$$

and

$$\cos(n, x) < 4/k, \quad \sin(n, x) > 1/2, \text{ whence}$$

$$\tan(n, x) > k/8, \text{ relative to } n, \text{ which proves}$$

that (ii) holds for all values of t.

6.5. The function which is relatively inverse to $\sin(n, x)$ can of course be defined in terms of $\arctan(n, t)$ but it is simpler to proceed as follows:

We define $\arcsin(0, x) = 0$

$$\arcsin(n+1, x) = \arcsin(n, x) + \frac{(2n)!}{2^{2n}(n!)^2} \frac{x^{2n+1}}{2n+1}.$$

For $|x| \leqq 1$ it may readily be shown that $\arcsin(n, x)$ is recursively convergent, and further that, for $|x| < 1$, the derivative of $\arcsin(n, x)$ for any n, is uniformly convergent; we denote this derivative by $\varrho(n, x)$ and observe that, for $|x| < 1$, $\varrho(n, x)$ is also differentiable with a uniformly convergent derivative $\varrho^1(n, x)$. A simple induction shows that, for $|x| < 1$,

$$x \cdot \varrho(n, x) = (1 - x^2) \cdot \varrho^1(n, x), \text{ relative to } n.$$

Now for $0 \leqq x \leqq \sigma_n - \delta$, $1 > \delta > 0$ and n majorant, $|\sin(m, x)| < 1 - \frac{1}{3}\delta^2$ and therefore,

$$\sin(m, x) \cdot \varrho(n, \sin(m, x)) = \cos^2(m, x) \cdot \varrho^1(n, \sin(m, x)), \text{ relative to } m, n,$$

which proves that $\varrho(n, \sin(m, x)) . \cos(m, x) = 1$, relative to m, n, and from this in turn follows

$$\arcsin(n, \sin(m, x)) = x, \text{ relative to } m, n,$$

for $|x| < \sigma_n - \delta$, $\delta > 0$ and n majorant.

Since $\varrho(n, x) \geqq 1$, for $x \geqq 0$, it follows now from Theorem 5 that, for $|x| < 1$.

$$\sin(m, \arcsin(n, x)) = x, \text{ relative to } n, m,$$

and the proof that $\sin(m, x)$ and $\arcsin(n, t)$ are relatively inverse functions is complete.

By a method which we have described elsewhere[1] for proving the recursive irrationality of π, we can replace σ_n by an equivalent recursive τ_n for which there is a recursive n_p such that

$$\tau_{n+1} \geqq \tau_n, \ \tau_{p+n_p} \geqq \tau_p + 10^{-p-n_p}.$$

With τ_n in place of σ_n the rather awkward condition '$|x| \leqq b \leqq \sigma_n - \delta$, for majorant n', becomes simply '$|x| \leqq \tau_p$ for some integer p'.

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[1] In a forthcoming paper "The recursive irrationality of π", *Journal of Symbolic Logic.*