THE CLOSEST PACKING OF CONVEX TWO-DIMENSIONAL DOMAINS.

BY

C. A. ROGERS

of LONDON.

1. The following theorem is among the results proved by L. Fejes Tóth[1] in a recent paper.

Theorem. *Let $K_1, \ldots, K_n$ be n convex domains, which lie without mutual overlapping in a hexagon*[2] *H of area $a(H)$, and each of which arises from a given convex domain K by an area-preserving affine transformation. Then*

$$n h(K) \leq a(H), \tag{1}$$

where $h(K)$ denotes the area of the smallest hexagon circumscribed about K.

Some time ago I obtained a similar result on the restrictive hypothesis that the domains $K_1, \ldots, K_n$ are all congruent and similarly situated.[3] Although my results are largely superseded by those of Fejes Tóth, they are slightly stronger than his when the above condition is satisfied (especially when the domains do not have a centre of symmetry), and are obtained by a very different method. So I hope that the following statements of the results together with indications of the methods of proof may have some interest.

2. Let $\boldsymbol{a}, \boldsymbol{b}, \ldots, \boldsymbol{z}$ denote the points in two-dimensional space with coordinates $(a_1, a_2), (b_1, b_2), \ldots, (z_1, z_2)$; $\boldsymbol{0}$ being the origin with coordinates $(0, 0)$. Let

---

[1] Acta Sci. Math. (Szeged), 12 (1950), 62—67, see Theorem 1 and the remarks on page 66.

[2] A convex polygon having at most six sides will be called a hexagon.

[3] My first result, Theorem 2, was obtained in 1947, and was described in seminars in London, Cambridge, Bristol and Princeton in the years 1948—49; its most important consequence was announced in a paper by J. H. H. CHALK and myself (J. L. M. S., 23 (1948), 178—187 (179)). Detailed proofs of the results were given in the version of the present paper originally submitted to Acta mathematica.

$\lambda \boldsymbol{a} + \mu \boldsymbol{b}$ denote the point with coordinates $(\lambda a_1 + \mu b_1, \lambda a_2 + \mu b_2)$ for all real λ, μ. If λ is any real number, $\boldsymbol{a}$ is any point and S is any set, $\lambda S + \boldsymbol{a}$ will denote the set of points of the form $\lambda \boldsymbol{x} + \boldsymbol{a}$ with $\boldsymbol{x}$ in S.

Let K and S be any open convex sets. If $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are points such that the sets $K + \boldsymbol{a}_1, \ldots, K + \boldsymbol{a}_n$ lie in S without mutual overlapping, then these sets will be said to form a non-overlapping packing of n sets congruent and similarly situated to K into S, or simply a packing of n sets K into S. If Λ is a lattice and the sets $K + \boldsymbol{x}$ with $\boldsymbol{x}$ in Λ form a packing of an infinite number of sets K into the whole plane, then these sets will be said to form a lattice packing of K; the determinant $d(\Lambda)$ of the lattice will be called the determinant of the lattice packing. The determinant of the closest lattice packing of K is defined to be the lower bound of the determinants of the lattice packings of K and will be denoted by $d(K)$.

Our main results is:

Theorem 1. *Let K and S be any open bounded convex sets with areas $a(K)$ and $a(S)$. If n sets K can be packed into S (with $n \geq 1$), then*

$$(n - 1)\, d(K) + a(K) \leq a(S). \tag{2}$$

When one restricts oneself to packings of congruent and similarly situated sets, this theorem is in some ways stronger than the result of Fejes Tóth, for in the first place it applies to packings into a general convex set S and in the second place we have[1]

$$d(K) \geq h(K), \tag{3}$$

with strict inequality in the general case. However, when K has a centre of symmetry, it follows by a result of K. Reinhardt[2] that $d(K) = h(K)$.

The proof of Theorem 1 depends on the following rather more complicated theorem (see Fig. 1).

---

[1] This inequality is not difficult to prove. By continuity considerations it suffices to prove the inequality in the case when K is strictly convex. If one considers a lattice Λ with determinant $d(K)$ giving a lattice packing of strictly convex sets K, it follows from the well known theory of MINKOWSKI (*Diophantische Approximationen* (Teubner, Berlin, 1947), § 4, or see K. MAHLER, Proc. London Math. Soc. (2) 49 (1946), 158) that each set $K + \boldsymbol{x}$ with $\boldsymbol{x}$ in Λ has a boundary point in common with the boundaries of just six of the other sets of this form. If tac-lines are drawn to K through these six points of contact, care being taken to ensure that opposite tac-lines are parallel, they bound an open hexagon H circumscribing K, and no two of the hexagons $H + \boldsymbol{x}$ with $\boldsymbol{x}$ in Λ have common points. Thus we see that $h(K) \leq a(H) \leq d(\Lambda) = d(K)$.

[2] Abh. Math. Sem. Hamb. Univ., 10 (1934), 216—230 or see K. MAHLER, Proc. K. Ned. Akad. v. Wet. (Amsterdam), 50 (1947), 692—703.

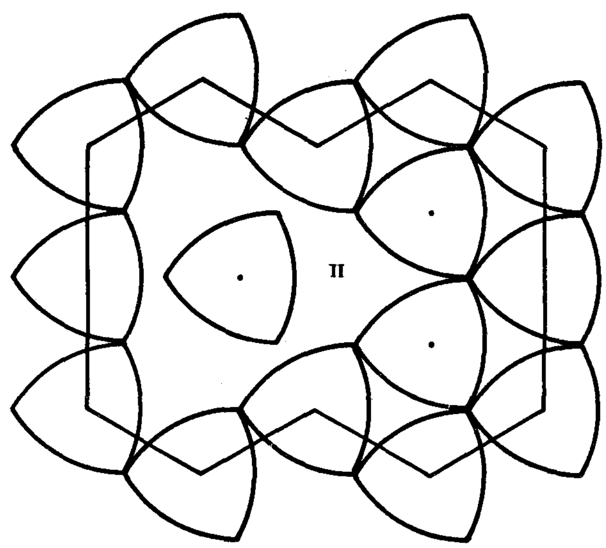

Fig. 1.

Theorem 2. *Let K be an open bounded strictly convex*[1] *set. Let $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n = \boldsymbol{x}_0, \boldsymbol{x}_{n+1}, \ldots, \boldsymbol{x}_{n+m}$ be points, such that*

(1) *the polygon $\boldsymbol{x}_0 \boldsymbol{x}_1 \ldots \boldsymbol{x}_n$ is a Jordan polygon bounding a domain Π of area $a(\Pi)$;*

(2) *the sets $K + \boldsymbol{x}_{r-1}$ and $K + \boldsymbol{x}_r$ have a common boundary point, if $1 \leq r \leq n$;*

(3) *the points $\boldsymbol{x}_{n+1}, \ldots, \boldsymbol{x}_{n+m}$ lie in or on the boundary of Π; and*

(4) *the sets $K + \boldsymbol{x}_r$ and $K + \boldsymbol{x}_s$ have no points in common if $1 \leq r < s \leq n + m$.*

Then

$$(m + \tfrac{1}{2} n - 1)\, d(K) \leq a(\Pi). \tag{4}$$

While Theorem 2 only applies to strictly convex sets, it is in some respects more general and more precise than Theorem 1, since it does not assume that the polygon Π is convex and since the inequality (4) is satisfied with equality for many configurations of the sets.

3. Throughout this section K will denote an open strictly convex set. Further C will denote the boundary of K, and K' will denote the closure of K. We shall say that two sets touch if their closures have just one point in common,

[1] An open set K is said to be strictly convex if it is such that, for every pair of distinct points $\boldsymbol{a}$ and $\boldsymbol{b}$ on the boundary of K, every inner point $\boldsymbol{c}$ of the line segment joining $\boldsymbol{a}$ to $\boldsymbol{b}$ is in K.

the common point shall be called the point of contact. We will say that a system of sets $K + \boldsymbol{x}_1, \ldots, K + \boldsymbol{x}_n$ is connected if, for any integers r, w with $1 \leq r < w \leq n$, either $K + \boldsymbol{x}_r$ touches $K + \boldsymbol{x}_w$, or there is a sequence of positive integers $s, t, \ldots, v$ all less than or equal to n such that $K + \boldsymbol{x}_r$ touches $K + \boldsymbol{x}_s$, $K + \boldsymbol{x}_s$ touches $K + \boldsymbol{x}_t, \ldots, K + \boldsymbol{x}_v$ touches $K + \boldsymbol{x}_w$.

We first show that Theorem 2 is a consequence of a particular case of the following lemma.

Lemma 1. *Suppose that K has $\mathbf{0}$ as centre. Let n be an integer with $n \geq 3$ and let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be points such that:*

(1) *the polygon $\boldsymbol{x}_1 \boldsymbol{x}_2 \ldots \boldsymbol{x}_n \boldsymbol{x}_1$ is a Jordan polygon bounding a domain Π with area $a(\Pi)$;*

(2) *the sets $K + \boldsymbol{x}_r$ and $K + \boldsymbol{x}_{r+1}$ touch for $r = 1, \ldots, n-1$; and*

(3) *no two of the sets $K + \boldsymbol{x}_1, \ldots, K + \boldsymbol{x}_n$ have a common point.*

Suppose that for some integer $m \geq 0$ there exist points $\boldsymbol{s}$, $\boldsymbol{x}_{n+1} = \boldsymbol{y}_1, \ldots, \boldsymbol{x}_{n+m} = \boldsymbol{y}_m$ and a real number σ with $0 \leq \sigma < 1$ such that:

(4) *the points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ lie in the closure Π' of Π;*

(5) *no two of the sets $\sigma K + \boldsymbol{s}$, $K + \boldsymbol{x}_1, \ldots, K + \boldsymbol{x}_{n+m}$ have a common point;*

(6) *the system of sets $K + \boldsymbol{x}_1, \ldots, K + \boldsymbol{x}_{n+m}$ is connected; and*

(7) *the set $\sigma K + \boldsymbol{s}$ touches $K + \boldsymbol{x}_1$ and $K + \boldsymbol{x}_n$.*

Suppose further that m is such that:

(8) *it is not possible to find points $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{m+1}$ in Π', with no two of the sets $K + \boldsymbol{x}_1, \ldots, K + \boldsymbol{x}_n, K + \boldsymbol{z}_1, \ldots, K + \boldsymbol{z}_{m+1}$ having a common point.*

Then

$$a(\Pi) \geq (m + \tfrac{1}{2} n - 1)\, d(K). \tag{5}$$

Reduction of Theorem 2 to the case $\sigma = 0$ of Lemma 1. First suppose that K has the origin $\mathbf{0}$ as centre. We may also suppose without loss of generality that it is impossibe to find points $\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_{m+1}$ in the closure Π' of Π, such that no two of the sets $K + \boldsymbol{x}_1, \ldots, K + \boldsymbol{x}_n, K + \boldsymbol{z}_1, \ldots, K + \boldsymbol{z}_{m+1}$ have a common point. Also, if we regard the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ as fixed, we may suppose that the points $\boldsymbol{x}_{n+1}, \ldots, \boldsymbol{x}_{n+m}$ are chosen to make the sum of their second coordinates minimal subject to the conditions (3) and (4) of Theorem 2. Then it is easy to verify that the conditions of Lemma 1 are satisfied if $\boldsymbol{s} = \frac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_n)$ and $\sigma = 0$. So Lemma 1 implies the truth of Theorem 2 in this case.

Now consider the general case when K does not necessarily have a centre. The difference set $\mathfrak{D} K$ of K is defined to be the set of all points of the form $\boldsymbol{x}-\boldsymbol{y}$ with $\boldsymbol{x}$ and $\boldsymbol{y}$ in K. As K is strictly convex it is well known (and easy to verify) that $\mathfrak{D} K$ is also strictly convex and has $\mathbf{0}$ as centre of symmetry. Further, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are any points, the sets $\frac{1}{2}\mathfrak{D} K+\boldsymbol{a}$ and $\frac{1}{2}\mathfrak{D} K+\boldsymbol{b}$ have a point in common if and only if the sets $K+\boldsymbol{a}$ and $K+\boldsymbol{b}$ have a point in common, and the sets $\frac{1}{2}\mathfrak{D} K+\boldsymbol{a}$ and $\frac{1}{2}\mathfrak{D} K+\boldsymbol{b}$ touch if and only if the sets $K+\boldsymbol{a}$ and $K+\boldsymbol{b}$ touch. Consequently we have $d(K)=d(\frac{1}{2}\mathfrak{D} K)$, and the conditions of Theorem 2 are satisfied by the set $\frac{1}{2}\mathfrak{D} K$ and the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n+m}$. Thus the general case of Theorem 2 is a consequence of the special case, when K has $\mathbf{0}$ as centre, considered above. This completes the reduction.

The proof of Lemma 1 is inductive; it is based on three lemmas.

Lemma 2. *If α and β are positive numbers and $\boldsymbol{a}$ and $\boldsymbol{b}$ are points, then the curves $\alpha C+\boldsymbol{a}$ and $\beta C+\boldsymbol{b}$ cannot have more than two points of intersection unless $\alpha=\beta$ and $\boldsymbol{a}=\boldsymbol{b}$.*

Proof. Suppose that the curves had three distinct points $\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3$ in common. Then, if

$$\boldsymbol{x}_r=\frac{1}{\alpha}\{\boldsymbol{z}_r-\boldsymbol{a}\}, \quad \boldsymbol{y}_r=\frac{1}{\beta}\{\boldsymbol{z}_r-\boldsymbol{b}\}, \qquad r=1,2,3,$$

the triangles $\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3$ and $\boldsymbol{y}_1\boldsymbol{y}_2\boldsymbol{y}_3$ are in direct similitude and are inscribed in C. Since C is strictly convex it is easy to see that this is impossible unless both triangles are proper and coincide with each other. In this case $\alpha=\beta$ and $\boldsymbol{a}=\boldsymbol{b}$.

Lemma 3. *Suppose that K has $\mathbf{0}$ as centre. Let σ be a number with $0<\sigma<1$, and let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{s}$ be points such that $\sigma K+\boldsymbol{s}$ touches both $K+\boldsymbol{a}$ and $K+\boldsymbol{b}$. Let Γ be a Jordan arc leading from $\boldsymbol{a}$ to $\boldsymbol{b}$, having no points other than $\boldsymbol{a}$ and $\boldsymbol{b}$ in or on the boundary of the triangle $\boldsymbol{asb}$, and having no point in $\sigma K'+\boldsymbol{s}$. Let Π be the domain bounded by Γ and the segment $\boldsymbol{ba}$. Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be two points, just one being in Π, such that neither $K'+\boldsymbol{x}_1$ nor $K'+\boldsymbol{x}_2$ has a point in common with Γ or $K+\boldsymbol{a}$ or $K+\boldsymbol{b}$, and such that $K+\boldsymbol{x}_1$ has no point in common with $\sigma K+\boldsymbol{s}$. Then $K'+\boldsymbol{x}_1$ and $K'+\boldsymbol{x}_2$ have no point in common.*

Proof. Suppose, for example, that $\boldsymbol{x}_1$ is in Π; the proof is similar when $\boldsymbol{x}_2$ is in Π. Then $\boldsymbol{x}_2$ is not in Π. Let T be the set of inner points of the triangle $\boldsymbol{asb}$ and let T' be the closure of T. Let Γ_0 be the contour consisting of the arc Γ the segment $\boldsymbol{bs}$ and segment $\boldsymbol{sa}$. Then Γ_0 is a Jordan contour bounding a domain Π_0. Since there is no point of $K' + \boldsymbol{x}_1$ in common with $K + \boldsymbol{a}$ or $K + \boldsymbol{b}$, the point $\boldsymbol{x}_1$ does not lie in the triangle $\boldsymbol{a s b}$. Hence $\boldsymbol{x}_1$ lies in Π_0. Similarly $\boldsymbol{x}_2$ is not in Π_0.

Let γ_1 and γ_2 be the two arcs of $\sigma C + \boldsymbol{s}$ leading from the point $\boldsymbol{t}$ of contact of $\sigma K + \boldsymbol{s}$ and $K + \boldsymbol{b}$ to the point $\boldsymbol{r}$ of contact of $\sigma K + \boldsymbol{s}$ and $K + \boldsymbol{a}$, the arc γ_1 lying in Π_0' and the arc γ_2 being outside Π_0. Then since $\sigma K + \boldsymbol{s}$ and $K + \boldsymbol{x}_1$ have no point in common the set $K + \boldsymbol{x}_1$ lies entirely in the domain Π_1 bounded by the arc Γ, the segment $\boldsymbol{bt}$, the arc γ_1, and the segment $\boldsymbol{ra}$. Further it follows by use of Lemma 2 that the set $K' + \boldsymbol{x}_2$ can have no point in Π_1'. Thus $K' + \boldsymbol{x}_1$ and $K' + \boldsymbol{x}_2$ have no point in common.

Lemma 4. *Suppose that K has $\mathbf{0}$ as centre. Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ be the vertices of a triangle T of area $a(T)$, let $\boldsymbol{s}$ be a point and let σ be a number satisfying $0 \leq \sigma < 1$. Suppose that no two of the sets $K + \boldsymbol{x}_r$, $r = 1, 2, 3$ have any common point, but that $\sigma K + \boldsymbol{s}$ touches $K + \boldsymbol{x}_r$ for $r = 1, 2, 3$. Then*

$$a(T) \geq \tfrac{1}{2} d(K). \tag{6}$$

Proof. If $(1 + \sigma)^{-1} \leq \lambda \leq 1$ the set $\{\lambda(1 + \sigma) - 1\} K + \lambda \boldsymbol{s}$ touches the sets $K + \lambda \boldsymbol{x}_r$, $r = 1, 2, 3$. When $\lambda = (1 + \sigma)^{-1}$ the point $\lambda \boldsymbol{s}$ is common to the boundaries of the three sets; when $\lambda = 1$ the sets have no common points. So we can choose such a λ so that no two of the sets have a common point, but at least two of them touch. The area of the triangle $\lambda \boldsymbol{x}_1, \lambda \boldsymbol{x}_2, \lambda \boldsymbol{x}_3$ is

$$\lambda^2 a(T) \leq a(T).$$

Thus it is clear that in proving the lemma we may suppose that two of the sets $K + \boldsymbol{x}_r$, $r = 1, 2, 3$, touch.

We suppose that two of the sets $K + \boldsymbol{x}_r$, $r = 1, 2, 3$, touch. It is convenient to rename the points $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$, calling them $\boldsymbol{x}_{00}, \boldsymbol{x}_{10}, \boldsymbol{x}_{01}$ in a way which ensures that $K + \boldsymbol{x}_{00}$ and $K + \boldsymbol{x}_{10}$ touch. It is also convenient to suppose that our axes of coordinates are chosen so that the points $\boldsymbol{x}_{00}$, $\boldsymbol{x}_{10}$ and $\boldsymbol{x}_{01}$ have coordinates $(0, 0)$, $(1, 0)$ and $(0, 1)$ respectively, this being possible since it is clear that the three points form a proper triangle.

For any integers r, s we use $\boldsymbol{x}_{rs}$ to denote the point, given by

$$\boldsymbol{x}_{rs} = \boldsymbol{x}_{00} + r(\boldsymbol{x}_{10} - \boldsymbol{x}_{00}) + s(\boldsymbol{x}_{01} - \boldsymbol{x}_{00}),$$

and having coordinates (r, s). We show that the system of sets $K + \boldsymbol{x}_{rs}$ forms a lattice packing of K. We first prove inductively that $K + \boldsymbol{x}_{10}$ and $K + \boldsymbol{x}_{r1}$ have no common point for $r \leq 0$. We know that $K + \boldsymbol{x}_{10}$ and $K + \boldsymbol{x}_{01}$ have no common point, so we suppose that, for some $n \leq -1$, the sets $K + \boldsymbol{x}_{10}$ and $K + \boldsymbol{x}_{r1}$ have no common point for $r = n + 1, \ldots, -1, 0$, and prove that $K + \boldsymbol{x}_{10}$ and $K + \boldsymbol{x}_{n1}$ have no common point. As $K + \boldsymbol{x}_{10}$ and $K + \boldsymbol{x}_{(n+1)1}$ have no common point it is clear that $K + \boldsymbol{x}_{00}$ and $K + \boldsymbol{x}_{n1}$ have no common point. Also since $K + \boldsymbol{x}_{00}$ and $K + \boldsymbol{x}_{10}$ have no common point it is clear that $K + \boldsymbol{x}_{01}$ and $K + \boldsymbol{x}_{n1}$ have no common point.

Let the tac-lines to K parallel to the x_1-axis touch K at points $\pm \boldsymbol{z}$, where $z_2 > 0$. Then it is easy to see that we may join the points $\boldsymbol{x}_{00} - \boldsymbol{z}$ and $\boldsymbol{x}_{01} + \boldsymbol{z}$ by a Jordan arc γ having no points, other than its end points, in common with the sets $\sigma K' + \boldsymbol{s}$, $K' + \boldsymbol{x}_{00}$, $K' + \boldsymbol{x}_{10}$, $K' + \boldsymbol{x}_{01}, \ldots, K' + \boldsymbol{x}_{n1}$, and the triangle $\boldsymbol{x}_{00}\boldsymbol{s}\boldsymbol{x}_{01}$, and such that just one of the points $\boldsymbol{x}_{10}$, $\boldsymbol{x}_{n1}$ lies in the domain Π bounded by the Jordan contour consisting of the segment $\boldsymbol{x}_{00}(\boldsymbol{x}_{00} - \boldsymbol{z})$, the arc γ, the segment $(\boldsymbol{x}_{01} + \boldsymbol{z})\boldsymbol{x}_{01}$, and the segment $\boldsymbol{x}_{01}\boldsymbol{x}_{00}$. It now follows from Lemma 3 that $K + \boldsymbol{x}_{10}$ and $K + \boldsymbol{x}_{n1}$ have no point in common. Thus $K + \boldsymbol{x}_{10}$ and $K + \boldsymbol{x}_{r1}$ have no point in common if $r \leq 0$.

A similar argument, in which the roles played by $\boldsymbol{x}_{00}$ and $\boldsymbol{x}_{10}$ are interchanged, shows that $K + \boldsymbol{x}_{00}$ and $K + \boldsymbol{x}_{r1}$ have no common points for $r \geq 0$. So we see that $K + \boldsymbol{x}_{r0}$ and $K + \boldsymbol{x}_{t1}$ have no point in common for any integers r, t. Since the sets $K + \boldsymbol{x}_{rs}$, $r = 0, \pm 1, \ldots$ form a line of touching sets, it is clear that no two of the sets $K + \boldsymbol{x}_{rs}$, $r, s = 0, \pm 1, \ldots$, have a point in common. So in our new coordinate system the lattice of points with integral coordinates having determinant 1 gives a lattice packing of K. Hence $d(K) \leq 1$. But in the new coordinate system $a(T) = \frac{1}{2}$. Consequently we have shown that $a(T) \geq \frac{1}{2} d(K)$.

Proof of Lemma 1. We prove the lemma by induction on $(m + \frac{1}{2}n - 1)$. We suppose that for some $N \geq \frac{1}{2}$ the lemma is true if $m + \frac{1}{2}n - 1 \leq N - \frac{1}{2}$; we shall show that this supposition, which is vacuous when $N = \frac{1}{2}$, implies that the lemma is satisfied when $m + \frac{1}{2}n - 1 = N$. Suppose then that the conditions of the lemma are satisfied by some integers m and n with $m + \frac{1}{2}n - 1 = N$ and by some points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n+m}$.

We may without loss of generality suppose that the axes are chosen so that $\mathbf{0}$ is the mid-point of the segment $\boldsymbol{x}_1\boldsymbol{x}_n$, the x_1-axis being along this segment and the points of the negative x_2-axis near to $\mathbf{0}$ being in Π.

We consider the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n+m}$ as fixed but consider the point $\boldsymbol{s}$ and the number σ to vary subject to the conditions (5) and (7) and the inequality $0 \leq \sigma \leq 1$. Suppose $\boldsymbol{s}$ and σ are chosen subject to these conditions so that s_2 has its least possible value. Then, if σ were equal to 1, the points $\boldsymbol{s}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ would lie in Π' while no two of the sets $K + \boldsymbol{x}_1, \ldots, K + \boldsymbol{x}_n, K + \boldsymbol{s}, K + \boldsymbol{y}_1, \ldots, K + \boldsymbol{y}_m$ would have a common point, contrary to the condition (8). So we have $0 \leq \sigma < 1$. Now it follows from the choice of $\boldsymbol{s}$ and σ that, for some t with $t \neq n$ and $2 \leq t \leq n + m$, the set $\sigma K + \boldsymbol{s}$ touches $K + \boldsymbol{x}_t$ at a point, which either lies in Π, or perhaps lies in the triangle $\boldsymbol{x}_1 \boldsymbol{s} \boldsymbol{x}_n$. It is not difficult to prove that the open triangle T with vertices $\boldsymbol{x}_1 \boldsymbol{x}_t \boldsymbol{x}_n$ lies in Π (see Fig. 2).

By condition (6) it is clear that we can choose points $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r$ with $1 \leq r \leq m + 1$ such that

(a) $\boldsymbol{z}_1 = \boldsymbol{x}_t$;
(b) $K + \boldsymbol{z}_j$ touches $K + \boldsymbol{z}_{j+1}$ for $1 \leq j < r$;
(c) for each j with $1 \leq j < r$ there is an integer k with $\boldsymbol{z}_j = \boldsymbol{y}_k$ and there is an integer l with $1 \leq l \leq n$ for which $\boldsymbol{z}_r = \boldsymbol{x}_l$; and
(d) the points $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r$ are distinct.

It is not difficult to see that Π splits up into the triangle T with vertices $\boldsymbol{x}_1 \boldsymbol{z}_1 \boldsymbol{x}_n$ and the domains Π^* and Π^{**} (which will degenerate in some cases) bounded by the polygons $\boldsymbol{x}_1 \boldsymbol{z}_1 \ldots \boldsymbol{z}_r (= \boldsymbol{x}_l) \boldsymbol{x}_{l-1} \ldots \boldsymbol{x}_2 \boldsymbol{x}_1$ and $\boldsymbol{x}_n \boldsymbol{z}_1 \ldots \boldsymbol{z}_r (= \boldsymbol{x}_l) \boldsymbol{x}_{l+1} \ldots \boldsymbol{x}_n$.

Let the points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$, which are in the interior of Π^* and Π^{**}, be $\boldsymbol{y}_1^*, \ldots, \boldsymbol{y}_{m^*}^*$ and $\boldsymbol{y}_1^{**}, \ldots, \boldsymbol{y}_{m^{**}}^{**}$ respectively. Then $m^* + m^{**} + (r - 1) = m$, as none of the points other than $\boldsymbol{z}_1$ are in the closure T' of T. Denote the vertices of Π^* and Π^{**} by $\boldsymbol{x}_1^* = \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n^*}^* = \boldsymbol{z}_1$ and $\boldsymbol{x}_1^{**} = \boldsymbol{x}_n, \ldots, \boldsymbol{x}_{n^{**}}^{**} = \boldsymbol{z}_1$. Then we have

$$(m^* + \tfrac{1}{2} n^* - 1) + (m^{**} + \tfrac{1}{2} n^{**} - 1) + \tfrac{1}{2} = (m + \tfrac{1}{2} n - 1). \qquad (7)$$

Thus, as $n^* \geq 2$ and $n^{**} \geq 2$,

$$m^* + \tfrac{1}{2} n^* - 1 \leq N - \tfrac{1}{2}, \quad m^{**} + \tfrac{1}{2} n^{**} - 1 \leq N - \tfrac{1}{2}.$$

It is easy to verify that the points $\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_{n^*}^*, \boldsymbol{s}, \boldsymbol{y}_1^*, \ldots, \boldsymbol{y}_{m^*}^*$ and the number σ satisfy the conditions of this lemma. It is clear that the conditions (1)—(5) and (7) are satisfied; that the conditions (6) and (8) are satisfied follows by straight-forward applications of Lemma 3. So by the hypothesis of our induction

$$a(\Pi^*) \geq (m^* + \tfrac{1}{2} n^* - 1)\, d(K).$$

Similarly

$$a(\Pi^{**}) \geq (m^{**} + \tfrac{1}{2} n^{**} - 1)\, d(K).$$

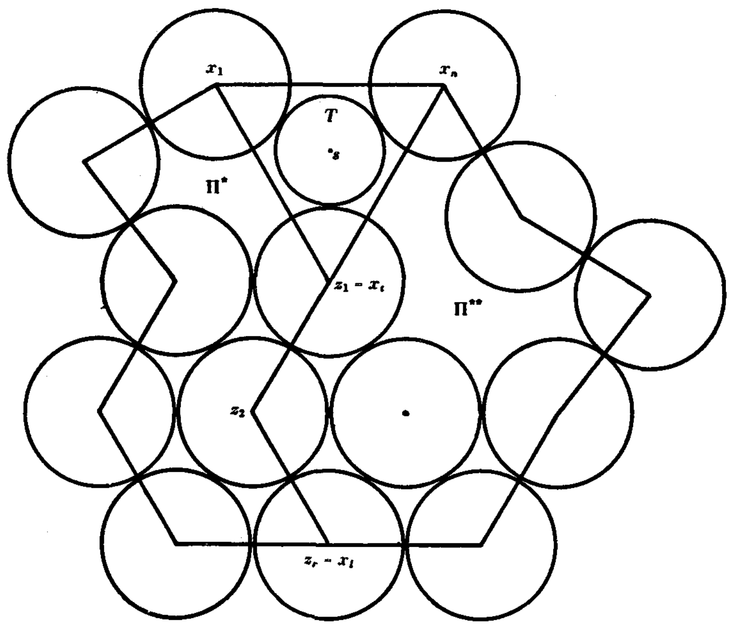

Fig. 2.

Also by Lemma 4 we see that

$$a(T) \geq \tfrac{1}{2} d(K).$$

Adding these inequalities and using (7) we obtain (5).

It is easy to check that no difficulty arises in the cases (which we have tacitly ignored) when one or more of the polygons degenerate. We remark that in the initial case when $m + \frac{1}{2}n - 1 = \frac{1}{2}$, we have $m = 0$, $n = 3$, so that $r = 1$ and $\boldsymbol{z}_1 = \boldsymbol{x}_2 = \boldsymbol{x}_{n-1}$. Thus we obtain (5) from Lemma 4 without use of any inductive hypothesis. This completes the proof of Lemma 1.

4. Throughout this section K will denote an open convex set; we no longer suppose that K is necessarily strictly convex. Before we prove Theorem 1 we prove the following lemma.

Lemma 5. *Suppose that K is strictly convex. Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n+m}$ be points such that:*

(1) *the polygon $\boldsymbol{b}\boldsymbol{a}\boldsymbol{x}_1 \ldots \boldsymbol{x}_n\boldsymbol{b}$ is a Jordan polygon bounding a domain Π;*

(2) *the sets $K + \boldsymbol{x}_r$ and $K + \boldsymbol{x}_{r+1}$ touch for $r = 1, \ldots, n-1$ and the sets $K + \boldsymbol{x}_1$, $K + \boldsymbol{x}_n$ touch the line $\boldsymbol{a}\boldsymbol{b}$ at the points $\boldsymbol{a}$, $\boldsymbol{b}$ respectively;*

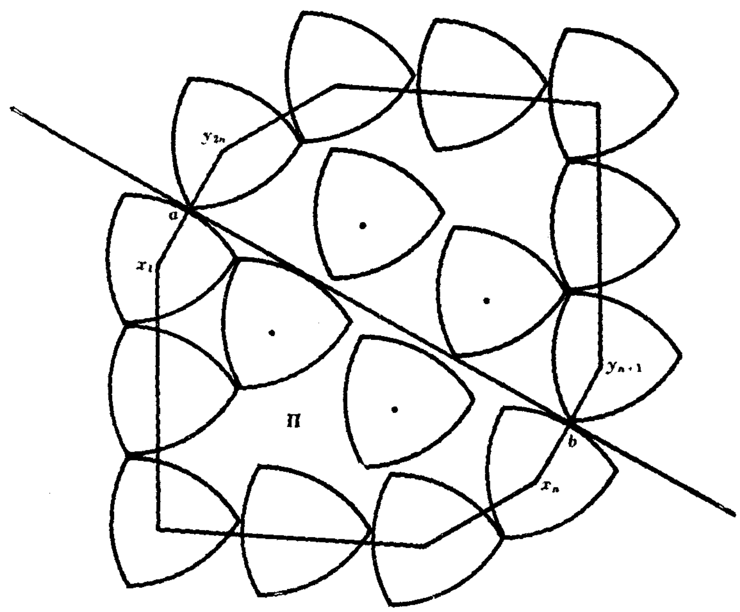

Fig. 3.

(3) *the sets* $K + \boldsymbol{x}_{n+1}, \ldots, K + \boldsymbol{x}_{n+m}$ *are contained in* Π; *and*

(4) *all the sets* $K + \boldsymbol{x}_1, \ldots, K + \boldsymbol{x}_{n+m}$ *lie on the same side of the line* $\boldsymbol{a}\boldsymbol{b}$ *and no two of these sets have a common point.*

Then

$$(m + \tfrac{1}{2} n - \tfrac{1}{2})\, d(K) \leq a(\Pi). \tag{8}$$

Proof.[1] We may without loss of generality take the point $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$ to be the origin. Let $\boldsymbol{c}$ and $\boldsymbol{d}$ be the points of contact of the tac-lines to K parallel to the line $\boldsymbol{a}\boldsymbol{b}$. We may suppose without loss of generality that $\frac{1}{2}(\boldsymbol{c} + \boldsymbol{d})$ coincides with $\boldsymbol{0}$. Write

$$\boldsymbol{y}_1 = \boldsymbol{x}_1, \ldots, \boldsymbol{y}_n = \boldsymbol{x}_n, \boldsymbol{y}_{n+1} = -\boldsymbol{x}_1, \ldots, \boldsymbol{y}_{2n} = -\boldsymbol{x}_n, \boldsymbol{y}_{2n+1} = \boldsymbol{x}_{n+1}, \ldots,$$
$$\boldsymbol{y}_{2n+m} = \boldsymbol{x}_{n+m}, \boldsymbol{y}_{2n+m+1} = -\boldsymbol{x}_{n+1}, \ldots, \boldsymbol{y}_{2n+2m} = -\boldsymbol{x}_{n+m}.$$

Then it is easy to verify that the sets $K + \boldsymbol{y}_1$ and $K + \boldsymbol{y}_{2n+2m}$ and the sets $K + \boldsymbol{y}_n$ and $K + \boldsymbol{y}_{n+1}$ touch at the points $\boldsymbol{a}$ and $\boldsymbol{b}$ (see Fig. 3). Further all the sets $K + \boldsymbol{y}_{n+1}, \ldots, K + \boldsymbol{y}_{2n}, K + \boldsymbol{y}_{2n+m+1}, \ldots, K + \boldsymbol{y}_{2n+2m}$ lie on the opposite side of $\boldsymbol{a}\boldsymbol{b}$ to the sets $K + \boldsymbol{y}_1, \ldots, K + \boldsymbol{y}_n, K + \boldsymbol{y}_{2n+1}, \ldots, K + \boldsymbol{y}_{2n+m}$. Also we see that if $1 \leq r \leq s \leq n + m$, the sets $K + \boldsymbol{x}_r, K + \boldsymbol{x}_s$ have a common point or a

[1] Compare with B. SEGRE and K. MAHLER, Amer. Math. Monthly, 51 (1944), 261—270, § 5.

common boundary point, if and only if, the sets $K - \boldsymbol{x}_r$, $K - \boldsymbol{x}_s$ have a common point or a common boundary point respectively. Thus the domain Σ bounded by the Jordan polygon $\boldsymbol{y}_1 \boldsymbol{y}_2 \ldots \boldsymbol{y}_{2n} \boldsymbol{y}_1$ has area $2\,a(\Pi)$, and the points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{2n+2m}$ satisfy the conditions of Theorem 2, with $2n$ and $2m$ in place of n and m. Hence by Theorem 2,

$$(2m + n - 1)\, d(K) \leq 2\, a(\Pi).$$

This proves the lemma.

Proof of Theorem 1. Since every convex set K can be approximated arbitrarily closely by inscribed strictly convex sets with packing determinants arbitrarily close to $d(K)$, it clearly suffices to prove the theorem on the assumption that K is strictly convex. So we suppose that K is strictly convex. It is clear (from Blaschke's selection theorem[1]) that we may suppose that it is not possible to pack n sets K into any open bounded convex set having area less than that of S. Then, if the sets $K + \boldsymbol{x}_1, \ldots, K + \boldsymbol{x}_n$ form a packing of n sets K into S, it is clear, from the minimal property of the area of S, that S is the least convex cover of the union of the sets $K + \boldsymbol{x}_1, \ldots, K + \boldsymbol{x}_n$.

Let $\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_s = \boldsymbol{z}_0$ be those of the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, for which $C + \boldsymbol{x}_t$ and the boundary B of S have a proper arc in common; the order being such that these arcs are in cyclic order on B (see Fig. 4). Let the end-points of the arc common to $C + \boldsymbol{z}_\varrho$ and B be $\boldsymbol{c}_\varrho$ and $\boldsymbol{d}_\varrho$ for $\varrho = 0, 1, \ldots, s$, the ends being named so that B consists of the line segment $\boldsymbol{d}_{\varrho-1}\boldsymbol{c}_\varrho$ and the arc $\boldsymbol{c}_\varrho \boldsymbol{d}_\varrho$ for $\varrho = 1, \ldots, s$. It follows without difficulty from the minimal property of the area of S that there is a connected sub-system of the sets $K + \boldsymbol{x}_1, \ldots, K + \boldsymbol{x}_n$ containing the sets $K + \boldsymbol{z}_1, \ldots, K + \boldsymbol{z}_s$.

Let Σ_0 be the domain bounded by the polygon $\boldsymbol{z}_0 \boldsymbol{d}_0 \boldsymbol{c}_1 \boldsymbol{z}_1 \boldsymbol{d}_1 \ldots \boldsymbol{c}_s \boldsymbol{d}_s$. By the above connectedness property it is possible to make an inductive choice, from the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, of sequences

$$\boldsymbol{v}_1^{(\sigma)} = \boldsymbol{z}_0,\ \boldsymbol{v}_2^{(\sigma)}, \ldots, \boldsymbol{v}_{r(\sigma)}^{(\sigma)} = \boldsymbol{z}_\sigma, \quad \sigma = 1, \ldots, s-1,$$

such that, for $\sigma = 1, \ldots, s - 1$, we have:

(a) the sets $K + \boldsymbol{v}_{\varrho-1}^{(\sigma)}$, $K + \boldsymbol{v}_\varrho^{(\sigma)}$ touch for $\varrho = 2, \ldots, r(\sigma)$; and

(b) the broken line $\boldsymbol{v}_{p(\sigma)}^{(\sigma)} \boldsymbol{v}_{p(\sigma)+1}^{(\sigma)} \ldots \boldsymbol{v}_{r(\sigma)}^{(\sigma)}\ (= \boldsymbol{z}_\sigma)$ is a Jordan arc dividing the set $\Sigma_{\sigma-1}$ bounded by the polygon $\boldsymbol{v}_1^{(\sigma-1)}\ (= \boldsymbol{z}_0)\ \boldsymbol{v}_2^{(\sigma-1)} \ldots \boldsymbol{v}_{r(\sigma-1)}^{(\sigma-1)}\ (= \boldsymbol{z}_{\sigma-1})\ \boldsymbol{d}_{\sigma-1} \boldsymbol{c}_\sigma \boldsymbol{z}_\sigma \ldots \boldsymbol{c}_s \boldsymbol{z}_s\ (= \boldsymbol{z}_0)$ [here $r(0) = 1$ and $\boldsymbol{v}_1^{(0)} = \boldsymbol{z}_0$] into the domain Π_σ bounded by

[1] W. Blaschke, *Kreis und Kugel* (Leipzig, 1916), § 18.

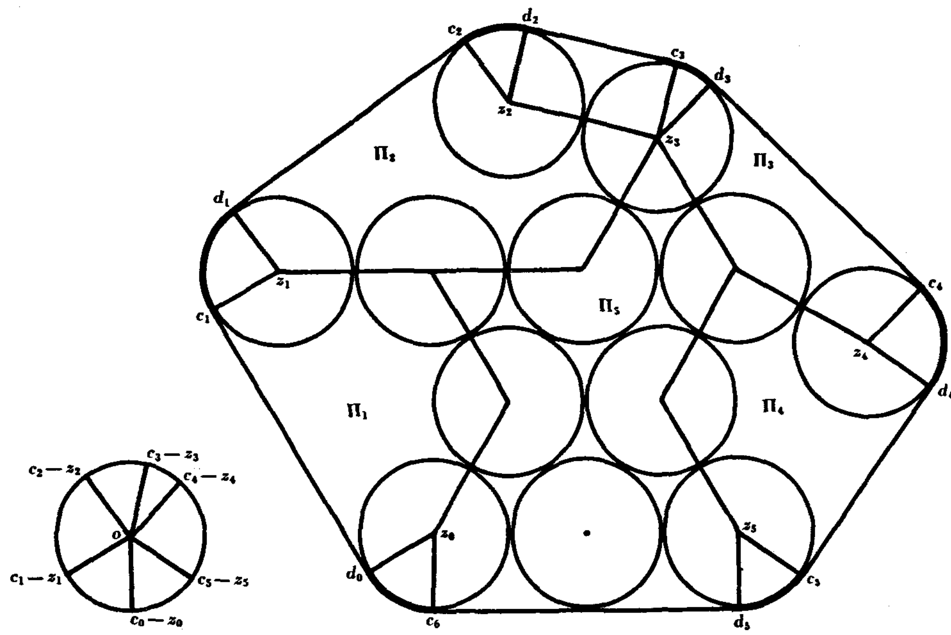

Fig. 4.

the Jordan polygon $\boldsymbol{v}^{(\sigma-1)}_{r(\sigma-1)-1}(=\boldsymbol{z}_{\sigma-1})\,\boldsymbol{v}^{(\sigma-1)}_{r(\sigma-1)-1}\ldots\boldsymbol{v}^{(\sigma-1)}_{p(\sigma)}(=\boldsymbol{v}^{(\sigma)}_{p(\sigma)})\,\boldsymbol{v}^{(\sigma)}_{p(\sigma)+1}\ldots$ $\boldsymbol{v}^{(\sigma)}_{r(\sigma)}(=\boldsymbol{z}_\sigma)\,\boldsymbol{c}_\sigma\,\boldsymbol{d}_{\sigma-1}\,\boldsymbol{z}_{\sigma-1}$ [here $p(1)=1$ and $\boldsymbol{v}^{(0)}_1=\boldsymbol{z}_0$] and a set Σ_σ consisting of a finite number of domains and bounded by the polygon $\boldsymbol{v}^{(\sigma)}_1(=\boldsymbol{z}_0)\,\boldsymbol{v}^{(\sigma)}_2\ldots$ $\boldsymbol{v}^{(\sigma)}_{r(\sigma)}(=\boldsymbol{z}_\sigma)\,\boldsymbol{d}_\sigma\,\boldsymbol{c}_{\sigma+1}\,\boldsymbol{z}_{\sigma+1}\ldots\boldsymbol{c}_s\,\boldsymbol{z}_s(=\boldsymbol{z}_0)$.

We note that the polygon $\boldsymbol{v}^{(s-1)}_1(=\boldsymbol{z}_0)\,\boldsymbol{v}^{(s-1)}_2\ldots\boldsymbol{v}^{(s-1)}_{r(s-1)}(=\boldsymbol{z}_{s-1})\,\boldsymbol{d}_{s-1}\,\boldsymbol{c}_s\,\boldsymbol{z}_s(=\boldsymbol{z}_0)$ is a Jordan polygon. Hence Σ_{s-1} is a domain; we write $\Pi_s=\Sigma_{s-1}$. Thus we have found broken lines which split the domain Σ_0 up into the domains $\Pi_1,\ldots,\Pi_s$ bounded by the Jordan polygons: $\boldsymbol{v}^{(\sigma-1)}_{r(\sigma-1)}(=\boldsymbol{z}_{\sigma-1})\,\boldsymbol{v}^{(\sigma-1)}_{r(\sigma-1)-1}\ldots\boldsymbol{v}^{(\sigma-1)}_{p(\sigma)}$ $(=\boldsymbol{v}^{(\sigma)}_{p(\sigma)})\,\boldsymbol{v}^{(\sigma)}_{p(\sigma)+1}\ldots\boldsymbol{v}^{(\sigma)}_{r(\sigma)}(=\boldsymbol{z}_\sigma)\,\boldsymbol{c}_\sigma\,\boldsymbol{d}_{\sigma-1}\,\boldsymbol{z}_{\sigma-1}$, for $\sigma=1,\ldots,s$, where we take $p(s)=$ $=r(s)=1$ and $\boldsymbol{v}^{(s)}_1=\boldsymbol{z}_s=\boldsymbol{z}_0$.

Now the number $n(\sigma)$ of the points $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n$ on the boundary of Π_σ is given by

$$n(\sigma)=\{r(\sigma-1)-p(\sigma)+1\}+\{r(\sigma)-p(\sigma)\}.$$

Let $m(\sigma)$ be the number of the points $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n$ in Π_σ. It is clear that the points $\boldsymbol{c}_\sigma$, $\boldsymbol{d}_{\sigma-1}$ and those of the points $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n$ in and on the boundary of Π_σ satisfy the conditions of Lemma 5. Hence

$$a(\Pi_\sigma)\geq\{m(\sigma)+\tfrac{1}{2}n(\sigma)-\tfrac{1}{2}\}\,d(K)=\{m(\sigma)+\tfrac{1}{2}r(\sigma-1)+\tfrac{1}{2}r(\sigma)-p(\sigma)\}\,d(K).$$

Thus

$$a(\Sigma_0) \geq \sum_{\sigma=1}^{s} \{m(\sigma) + \tfrac{1}{2} r(\sigma - 1) + \tfrac{1}{2} r(\sigma) - p(\sigma)\} d(K).$$

But the total number of points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ is n, so that

$$n = r(1) + \sum_{\sigma=2}^{s} \{r(\sigma) - p(\sigma)\} + \sum_{\sigma=1}^{s} m(\sigma).$$

Since $r(0) = r(s) = p(0) = 1$, it follows that

$$a(\Sigma_0) \geq (n - 1) d(K).$$

Now the domain S is the union of the domain Σ_0 with certain of its boundary points and the sectorial domains $T_1, \ldots, T_s$; the domain T_σ being bounded by the segments $\boldsymbol{c}_\sigma \boldsymbol{z}_\sigma$, $\boldsymbol{z}_\sigma \boldsymbol{d}_\sigma$ and the arc $\boldsymbol{c}_\sigma \boldsymbol{d}_\sigma$ common to $C + \boldsymbol{z}_\sigma$ and B. But it is clear that the domains $T_\sigma - \boldsymbol{z}_\sigma$, $\sigma = 1, \ldots, s$, together with some of their boundary points fit together to make up K (see Fig. 4). Hence

$$a(S) = a(\Sigma_0) + \sum_{\sigma=1}^{s} a(T_\sigma) = a(\Sigma_0) + a(K)$$

and so

$$a(S) \geq (n - 1) d(K) + a(K).$$

This completes the proof of Theorem 1.