An analytic family of uniformly bounded representations of free groups

by

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1. Introduction

Harmonic analysis on a free group F has attracted considerable attention in the last ten years or so. There seem to be two reasons for that: one is the discovery of a deep analogy of certain aspects of harmonic analysis on a free group and harmonic analysis on $SL(2, \mathbf{R})$, cf. e.g. fundamental works of P. Cartier [4], A. Figà-Talamanca and M. A. Picardello [8], the other being the interest in the C^*-algebra generated by the regular representation of F, cf. A. Connes [5], J. Cuntz [6], U. Haagerup [10], M. Pimsner and D. Voiculescu [14].

In most of this work Hilbert space representations of F play an essential role. Clearly F has a lot of unitary representations since any collection of unitary operators corresponding to the free generators of F gives rise to a unitary representation of F. On the other hand, it is by no means as trivial to construct a representation π of F on a Hilbert space $\mathcal{H}$ such that $\sup_{x\in F}\|\pi(x)\|<+\infty$ and π cannot be made unitary by introducing another equivalent inner product in $\mathcal{H}$. Various series of such representations have been already constructed and used in harmonic analysis on F, cf. e.g. [12], [9].

This paper is devoted to the study of a new series of such bounded Hilbert space representations of F together with some applications of them.

For every complex number z, $|z|<1$, we are going to construct a representation π_z of F on $\ell^2(F)$ in such a way that:

(i) $$\sup_{x\in F}\|\pi_z(x)\|\leqslant 2\frac{|1-z^2|}{1-|z|}.$$

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(ii) $$\pi_z^*(x) = \pi_{\bar z}(x^{-1}).$$

(iii) If L is the left regular representation of F, then $\pi_z(x)-L_x$ is a finite dimensional operator for every x in F.

(iv) The map $z\to\pi_z(x)$ is holomorphic.

Moreover, $\pi_0=L$ and $\pi_1=\lim_{z\to 1}\pi_z=1\oplus\tilde{L}$, where 1 is the trivial representation of F and $\tilde{L}$ is a representation weakly equivalent to L.

One of the features of these representations is that the properties above do not depend on the number of the free generators of F and, in fact, π_z have even nicer properties if the number of the free generators is infinite. For instance, for every $z\neq 0$, π_z leaves no non-trivial closed subspace invariant and for $z\neq z'$ representations π_z and $\pi_{z'}$ are topologically inequivalent.

The formula $\pi_z^*=\pi_{\bar z}^{-1}$ implies that for real z π_z is unitary and so $[0,1]\ni t\to\pi_t$ is a continuous, even analytic, path of unitary representations each of which differs from the regular representation by operators of finite rank. Construction of such a path is an essential step in the proof of the theorem that the regular C^*-algebra of a free group on two generators has no non-trivial projections, cf. [5], [6], [14].

Other applications of the representations π_z we consider are towards the identification of the functions on F which are matrix coefficients of bounded Hilbert space representations. It is easy to check that if $|x|$ is the length of the word x in F, then

$$\langle \pi_z(x)\,\delta_e, \delta_e\rangle = z^{|x|}.$$

This is a generalization of a result of Haagerup [10] stating that $x\to r^{|x|}$, $r\in(0,1)$ is positive definite.

However, many more functions turn out to be the matrix coefficients of bounded Hilbert space representations of F, if instead of π_z we look at the representations

$$\pi_\gamma = \oplus\int_\gamma \pi_z|dz|,$$

where γ is a closed path in $\{z\colon |z|<1\}$. Cf. chapter 3 for the details.

Finally let us mention that for the free group with infinitely many free generators the representations defined in Theorem 4 seem to be of special importance at least as far as the matrix coefficients are concerned.

The authors would like to thank the referee for a simplification of the proof of Theorem 4 and remarks concerning the presentation.

2. The analytic family of representations

2.1. *Notation.* Let F be a free group with fixed set E of generators (not necessarily finite). Each element x of F may be uniquely expressed as a finite sequence of elements of $E\cup E^{-1}$ with no adjacent factors like aa^{-1} or $a^{-1}a$. It is called a reduced word. The number of letters in this word is called the *length* of x and is denoted $|x|$. Put $|e|=0$ for the identity element e of F. When $x\neq e$, denote by $\bar{x}$ the word obtained from x by deleting the last letter.

Define $\mathcal{K}(F)$ to be the space of all complex functions on F with finite support. This space consists of all linear combinations of δ_x (characteristic function of the one point set $\{x\}$), $x\in F$.

Introduce the linear operator P: $\mathcal{K}(F)\to\mathcal{K}(F)$ setting $P\delta_x=\delta_{\bar{x}}$ when $x\neq e$ and $P\delta_e=0$.

If $a\in F$ write L_a for the translation operator defined by $L_af(x)=f(a^{-1}x)$ where $x\in F$ and f is any complex function on F.

When $a\in F$ denote by F_a the finite set of the elements $\{a, \bar{a}, \bar{\bar{a}}, \ldots, e\}$ (a word x in F belongs to F_a if and only if $|x|=n$ with $n\leq|a|$ and x consists of the first n letters of a). Let $\mathcal{K}(F_a)$ be the space of all complex functions on F supported by F_a. The space $\mathcal{K}(F_a)$ may be identified with the finite dimensional space $C^{|a|+1}$ via the natural mapping:

$$C^{|a|+1}\ni(\alpha_0,\alpha_1,\ldots,\alpha_{|a|})\to\sum_{k=0}^{|a|}\alpha_kP^k\delta_a\in\mathcal{K}(F_a).$$

One may introduce the standard unilateral shift S and its conjugate S^* into $\mathcal{K}(F_a)$ with respect to above identification. Recall that when the operators S and S^* act on $C^{|a|+1}$ they are given by

$$S(\alpha_0,\alpha_1,\ldots,\alpha_{|a|})=(0,\alpha_0,\alpha_1,\ldots,\alpha_{|a|-1})$$

$$S^*(\alpha_0,\alpha_1,\ldots,\alpha_{|a|})=(\alpha_1,\alpha_2,\ldots,\alpha_{|a|},0).$$

2.2. *A series of representations on* ℓ^p. Start with a lemma.

LEMMA 1. *When $a\in F$ the space $\mathcal{K}(F_a)$ is invariant under both operators P and $L_aPL_a^{-1}$. The restrictions of these operators to $\mathcal{K}(F_a)$ coincide with S and S^* respectively. For any function $f\in\mathcal{K}(F)$ which is zero on F_a its images Pf and $L_aPL_a^{-1}f$ are equal.*

Proof. If $x\in F-F_a$ then $a\overline{a^{-1}x}=aa^{-1}\bar{x}=\bar{x}$. Thus $L_aPL_a^{-1}\delta_x=P\delta_x$, which proves the second part of the lemma. Suppose now $x\in F_a$. It is obvious that $P\delta_x=S\delta_x$. To see that

$L_a P L_a^{-1}\delta_x = S^*\delta_x$ assume that the word a has the reduced form $a=a_1a_2\dots a_n$. Then $x=a_1a_2\dots a_k$ for some $k\leq n$. If $k=n$ then $x=a$ so $L_aPL_a^{-1}\delta_x=S^*\delta_x=0$. If $k<n$ then $L_aPL_a^{-1}\delta_x=\delta_{a_1a_2\dots a_{k+1}}=S^*\delta_x$. This concludes the proof.

For any complex number z the operator $I-zP$ is invertible on $\mathcal{K}(F)$. For if $f\in\mathcal{K}(F)$, then $P^nf=0$ for n sufficiently large. Thus the series $\Sigma_{n=0}^{\infty} z^nP^nf$ has only finitely many non-zero terms.

For $z\in C$ define the representation π_z° of F on the space $\mathcal{K}(F)$ by

$$\pi_z^\circ(a)=(I-zP)^{-1}L_a(I-zP),\quad a\in F.$$

It means π_z° is the conjugation of the left regular representation by the operator $I-zP$.

LEMMA 2. *Let $|z|<1$ and $1\leq p<\infty$. Then π_z° extends uniquely to a uniformly bounded representation of F on $\ell^p(F)$ with*

$$\|\pi_z^\circ(a)\|_{p,p}\leq\frac{1+|z|}{1-|z|},\quad a\in F. \tag{1}$$

The family of representations π_z° is analytic on $\{z\in C: |z|<1\}$. Moreover the operator $\pi_z^\circ(a)-L_a$ has finite rank for arbitrary a in F.

Proof. To prove (1) fix $a\in F$ and express the operator $\pi_z^\circ(a)L_a^{-1}$ in the form

$$\begin{aligned}\pi_z^\circ(a)L_a^{-1}&=(I-zP)^{-1}L_a(I-zP)L_a^{-1}\\&=I+\sum_{n=0}^{\infty} z^{n+1}P^n(P-L_aPL_a^{-1}).\end{aligned}$$

By Lemma 1 the operator $P-L_aPL_a^{-1}$ has finite rank and maps the space $\mathcal{K}(F)$ into $\mathcal{K}(F_a)$. Pointing out the relation between this operator and $S-S^*$ one has

$$\|(P-L_aPL_a^{-1})f\|_p\leq 2\|f\|_p$$

for any $p\geq 1$ and any $f\in\mathcal{K}(F)$. Since the function $(P-L_aPL_a^{-1})f$ lies in $\mathcal{K}(F_a)$ and the operator P is a contraction on $\mathcal{K}(F_a)$ in each ℓ^p-norm so

$$\|\pi_z^\circ(a)L_a^{-1}f\|_p\leq\|f\|_p+2\sum_{n=0}^{\infty}|z|^{n+1}\|f\|_p.$$

If $|z|<1$ then each $\pi_z^\circ(a)$, $a\in F$, extends uniquely to a bounded operator on $\ell^p(F)$ and

$$\|\pi_z^\circ(a)\|_{p,p} = \|\pi_z^\circ(a)\,L_a^{-1}\|_{p,p} \leqslant 1+2\sum_{n=0}^{\infty}|z|^{n+1} = \frac{1+|z|}{1-|z|}.$$

The last inequality guarantees also that the series

$$I+\sum_{n=0}^{\infty} z^{n+1}P^n(P-L_a P L_a^{-1})$$

is absolutely convergent in the operator norm and so it represents an analytic function.

Finally the operator $\pi_z^\circ(a)-L_a$ maps the entire space $\ell^p(F)$ into the finite dimensional space $\mathcal{H}(F_a)$ for each $p\geqslant 1$.

2.3. *The main result.* From now on we restrict our attention to the case $p=2$ only. We improve the representations π_z° to get a new class of representations with better properties.

Let T denote the orthogonal projection onto the one-dimensional subspace $C\delta_e$ in $\ell^2(F)$. For $|z|<1$ let T_z stand for the bounded invertible operator on $\ell^2(F)$ defined by

$$T_z = I-T+\sqrt{1-z^2}\,T,$$

where $\sqrt{1-z^2}$ denotes the principal branch of the square root.

For a complex number z with $|z|<1$ let us define the representation π_z by

$$\pi_z(a) = T_z^{-1}\pi_z^\circ(a)\,T_z, \quad a\in F. \tag{2}$$

THEOREM 1. *Let F be a free group on arbitrary many generators. The representations π_z, $z\in D=\{z\in C\colon |z|<1\}$, form an analytic family of uniformly bounded representations of F on the Hilbert space $\ell^2(F)$. Moreover:*

(i) $$\|\pi_z(a)\| \leqslant 2\frac{|1-z^2|}{1-|z|}.$$

(ii) $$\pi_z^*(a) = \pi_{\bar z}(a^{-1}).$$

(iii) *$\pi_z(a)-L_a$ is a finite rank operator.*

(iv) *If the group F has infinitely many generators then any representation π_z, $z\neq 0$, has no nontrivial closed invariant subspace. Any two different π_z's are topologically inequivalent.*

Proof. The first part of the theorem as well as point (iii) are obvious consequences of Lemma 2.

To get (i) and (ii) observe first that each $\mathcal{F}_a=\ell^2(F_a)$, $a\in F$, is a reducing subspace for each operator T_z, $|z|<1$. Also $\pi_z^\circ(a)$ maps $\mathcal{F}_{a^{-1}}$ onto $\mathcal{F}_a$ and coincides with L_a on $\mathcal{F}_{a^{-1}}$. Therefore $\mathcal{F}_a$ is a reducing subspace for both operators $\pi_z(a)L_a^{-1}$ and $L_a\pi_z(a^{-1})$. Let us examine these two operators more closely. We need to see only how they act on the space $\mathcal{F}_a$ because on the orthogonal complement $\mathcal{F}_a^\perp$ of $\mathcal{F}_a$ they coincide with the identity operator.

The operator $\pi_z(a)L_a^{-1}$ and $L_a\pi_z(a^{-1})$ are constructed by using operators $P, L_aPL_a^{-1}, T_z$ and $L_aT_zL_a^{-1}$ defined earlier. The subspace $\mathcal{F}_a$ is invariant for all of them and their restrictions to $\mathcal{F}_a$ can be expressed in terms of S and S^*. Namely

$$P|_{\mathcal{F}_a}=S$$

$$L_aPL_a^{-1}|_{\mathcal{F}_a}=S^*$$

$$T_z|_{\mathcal{F}_a}=S^*S+\sqrt{1-z^2}\,(I-S^*S)$$

$$L_aT_zL_a^{-1}|_{\mathcal{F}_a}=SS^*+\sqrt{1-z^2}\,(I-SS^*).$$

Therefore

$$\pi_z(a)L_a^{-1}|_{\mathcal{F}_a}=\left[S^*S+\frac{1}{\sqrt{1-z^2}}(I-S^*S)\right](I-zS)^{-1}(I-zS^*)\left[SS^*+\sqrt{1-z^2}(I-SS^*)\right].$$

Using the identities $S^*SS^*=S^*$ and $(I-zS^*)=(I-z^2SS^*)-z(I-zS)S^*$ we can write

$$\pi_z(a)L_a^{-1}|_{\mathcal{F}_a}=\left[\sqrt{1-z^2}\,S^*S+(I-S^*S)\right](I-zS)^{-1}\left[\sqrt{1-z^2}\,SS^*+(I-SS^*)\right]-zS^*. \quad (3)$$

Also

$$L_a\pi_z(a^{-1})|_{\mathcal{F}_a}=\left[\sqrt{1-z^2}\,SS^*+(I-SS^*)\right](I-zS^*)^{-1}\left[\sqrt{1-z^2}\,S^*S+(I-S^*S)\right]-zS. \quad (4)$$

It is easy to check now that

$$L_a\pi_z^*(a)|_{\mathcal{F}_a}=(\pi_z(a)L_a^{-1}|_{\mathcal{F}_a})^*=L_a\pi_z(a^{-1})|_{\mathcal{F}_a}$$

which proves (ii).

The desired estimates for the norm of $\pi_z(a)$ follows from formula (3). In fact:

$$\|\pi_z(a)\|=\|\pi_z(a)L_a^{-1}\|=\max\{1,\|\pi_z(a)L_a^{-1}|_{\mathcal{F}_a}\|\},$$

but

$$\pi_z(a) L_a^{-1}|_{\mathscr{F}_a} = A+B+C$$

with

$$A = (1-z^2) S^*S(I-zS)^{-1}SS^*,$$
$$B = \sqrt{1-z^2}(I-S^*S)(I-zS)^{-1}SS^*+\sqrt{1-z^2}S^*S(I-zS)^{-1}(I-SS^*),$$
$$C = (I-S^*S)(I-zS)^{-1}(I-SS^*)-zS^*.$$

Since S^*S and SS^* both are orthogonal projections thus

$$\|A\| \leq |1-z^2|\,\|(I-zS)^{-1}\| \leq \frac{|1-z^2|}{1-|z|}.$$

Note that for $\xi \in C^{|a|+1}$

$$B(\xi) = \sqrt{1-z^2}\,[\langle \xi, u_1\rangle v_1 + \langle \xi, v_2\rangle u_2]$$

and

$$C(\xi) = z^{|a|}\langle \xi, v_2\rangle v_1 - zS^*(\xi),$$

where

$$u_1=(0, z^{|a|-1}, \ldots, z, 1), \quad u_2=(1, z, \ldots, z^{|a|-1}, 0), \quad v_1=(0,0,\ldots,0,1), \quad v_2=(1,0,\ldots,0,0).$$

This yields

$$\|B\| = |\sqrt{1-z^2}|\max\{\|u_1\|\,\|v_1\|, \|u_2\|\,\|v_2\|\} \leq \sqrt{\frac{|1-z^2|}{1-|z|^2}},$$

$$\|C\| = \max\{|z|^{|a|}\|v_1\|\,\|v_2\|, |z|\} = |z|$$

and consequently

$$\|\pi_z(a) L_a^{-1}|_{\mathscr{F}_a}\| \leq \frac{|1-z^2|}{1-|z|} + \sqrt{\frac{|1-z^2|}{1-|z|^2}} + |z| \leq 2\frac{|1-z^2|}{1-|z|}.$$

To prove the first part of (iv) we show first that any π_z is a cyclic representation of F with a cyclic vector δ_e. Next, under the assumption that the group F has infinitely many generators we show that the projection T belongs to the von Neumann algebra generated by $\pi_z(F)$. This will imply that every closed invariant and nonzero subspace for π_z contains δ_e, so it must be the whole of $\ell^2(F)$.

Let $x \in F$ and $x \neq e$. Then

$$\pi_z(x)\,\delta_e = z^{|x|}\delta_e + \sum_{k=0}^{|x|-1} z^k \sqrt{1-z^2}\, P^k \delta_x \tag{5}$$

and

$$z\pi_z(\bar{x})\,\delta_e = z^{|x|}\delta_e + \sum_{k=1}^{|x|-1} z^k \sqrt{1-z^2}\, P^k \delta_x,$$

thus

$$\pi_z(x)\,\delta_e - z\pi_z(\bar{x})\,\delta_e = \sqrt{1-z^2}\,\delta_x.$$

This implies that δ_e is a cyclic vector for π_z.

Assume now that the set E of free generators of the group F is infinite. Fix a sequence $x_1, x_2, \ldots$ in E and for a natural number n define the operator $S_{z,n}$ on $\ell^2(F)$ by

$$S_{z,n} = \frac{1}{n} \sum_{k=1}^{n} \pi_z(x_k). \tag{6}$$

Then the sequence $S_{z,1}, S_{z,2}, \ldots$ is strongly convergent to zT. Indeed, the sequence $S_{z,n}$ is bounded in the operator norm, thus we have to show only that $S_{z,n}\delta_e \to z\delta_e$ and $S_{z,n}\delta_x \to 0$ for $x \neq e$. We have

$$S_{z,n}\,\delta_e = z\delta_e + \frac{\sqrt{1-z^2}}{n} \sum_{k=1}^{n} \delta_{x_k}$$

which tends to $z\delta_e$ when $n \to +\infty$. Now for $x \neq e$, according to the case whether the first letter of x is one of x_k^{-1}, say $x_{k_0}^{-1}$, or not, $S_{z,n}\,\delta_x$ has one of the forms

$$S_{z,n}\,\delta_x = \frac{1}{n} \sum_{\substack{k=1 \\ k \neq k_0}}^{n} \delta_{x_k x} + \frac{1}{n}\pi_z(x_{k_0})\,\delta_x$$

or

$$S_{z,n}\,\delta_x = \frac{1}{n} \sum_{k=1}^{n} \delta_{x_k x}.$$

However in both cases it tends to zero.

Let $\mathscr{H}_0$ be a nonzero closed subspace in $\ell^2(F)$, invariant under π_z. If $f(e) \neq 0$ for a function f in $\mathscr{H}_0$ then $S_{n,z}f$ belongs to $\mathscr{H}_0$ for all n. But $S_{n,z}f$ tends to $zf(e)\,\delta_e$ and so $\delta_e \in \mathscr{H}_0$. Observe that we can always find a function $f \in \mathscr{H}_0$ for which $f(e) \neq 0$. In fact.

Take any nonzero function f in $\mathcal{H}_0$ and let a denote a shortest word in the support of f. Write f in the form $f=f(a)\delta_a+g$. Then $g\in\ell^2(F-F_a)$ and so

$$\pi_z(a^{-1})f=f(a)\pi_z(a^{-1})\delta_a+L_a^{-1}g.$$

In particular

$$(\pi_z(a^{-1})f)(e)=\sqrt{1-z^2}f(a)\neq 0.$$

Consider two representations π_z and $\pi_{z'}$ with $z, z'\neq 0$. If a bounded operator A intertwines them then $A^{-1}S_{z,n}A=S_{z',n}$ for each n, hence also $A^{-1}TA=(z'/z)T$. But since both T and $A^{-1}TA$ are projections, $z=z'$. Thus π_z and $\pi_{z'}$ are not similar for $z\neq z'$.

This finishes the proof of the theorem.

2.4. *Remarks*. (1) All the representations are cyclic with a cyclic vector δ_e. The representation π_0 is just the left regular representation of F. If the group F has only finitely many generators (say k) then P is a bounded operator on $\ell^2(F)$ and

$$\|P^n\|=\sqrt{2k(2k-1)^{n-1}},\quad n=1,2,\ldots.$$

Thus $(I-zP)^{-1}$ is also bounded for $|z|<(2k-1)^{-1/2}$. It means that representations π_z for all such z are similar to the left regular representation.

(2) By (ii) for real t the representations π_t are unitary. Thus the function

$$F\ni x\rightarrow\langle\pi_t(x)\delta_e,\delta_e\rangle=t^{|x|}\tag{7}$$

is positive definite. It gives an alternative proof of a result of Haagerup [10].

In connexion with Remark 1, if the number of generators in F is k then, comparing formula (7) with [10], Theorem 2.1, no representation π_t, $|t|>(2k-1)^{-1/2}$ is weakly, and so strongly, contained in the regular representation.

(3) Observe that it is possible to pass with z to the limit $+1$ or -1 in formula (3) and define two unitary representations π_1 and π_{-1}. It turns out that $\pi_1=\mathrm{tr}^+\oplus\lambda_0^-$ and $\pi_{-1}=\mathrm{tr}^-\oplus\lambda_0^+$, where tr^+ and tr^- are one-dimensional representations $F\ni x\rightarrow(\pm1)^{|x|}$ and λ_0^+, λ_0^- two representations acting on $\ell^2(F-\{e\})$ by

$$\lambda_0^\pm(a)\delta_x=\begin{cases}\delta_{ax} & \text{for } x\neq a^{-1}\\ \pm\delta_a & \text{for } x=a^{-1}\end{cases}$$

when $x\in F-\{e\}$ and a is one of the free generators.

The representation λ_0^+ was considered by Cuntz [6] and earlier in less explicit form by Pimsner and Voiculescu [14].

A construction of a continuous path of unitary representations connecting the regular representation to π_1, such that each representation in the path is congruent to the regular representation modulo compact operators was an essential step in the proof of the theorem (cf. [14], [5] and [6]) that the regular C^*-algebra of a free group has no nontrivial projections.

(4) Let us state also that

$$\sup_{x \in F} \|\pi_z(x)\,\delta_e\| = |1-z^2|^{1/2}(1-|z|^2)^{-1/2} \tag{8}$$

which follows directly from (5). This formula will be used later.

3. Direct integrals of representations and multipliers

3.1. *Preliminaries*. Starting with the family of representations $\{\pi_z\colon |z|<1\}$ by integration on closed paths we obtain many other uniformly bounded representations. This yields a wide class of coefficients. Identification of functions which are the coefficients of hilbertian representations is especially useful when we study multiplier algebras. In this context, for locally compact groups, mainly three algebras were investigated: the Fourier-Stieltjes algebra $B(G)$ of all coefficients of unitary representations, the algebra $M(A(G))$ of multipliers of the Fourier algebra $A(G)$ and the algebra $B_2(G)$ of Herz multipliers.

A function φ in $L^\infty(G)$ is called a *Herz multiplier* if for any bounded operator A on $L^2(G)$ with kernel $A(x,y)$, $x, y \in G$, the function $\varphi(y^{-1}x)A(x,y)$ is again a kernel of a bounded operator on $L^2(G)$. The set $B_2(G)$ of all Herz multipliers, equipped with the multiplier norm is a Banach algebra under pointwise addition and multiplication.

PROPOSITION 1 (Schur). *Let π be a uniformly bounded representation of G on a Hilbert space $\mathcal{H}_\pi$. Then for any ξ, $\eta \in \mathcal{H}_\pi$ the coefficient*

$$\varphi(x) = \langle \pi(x)\,\xi, \eta\rangle, \quad x \in G,$$

of the representation π belongs to $B_2(G)$. Moreover

$$\|\varphi\|_{B_2} \leq \sup_{x\in G} \|\pi(x)\,\xi\| \sup_{x \in G} \|\pi^*(x)\,\eta\|.$$

We always have $B(G)\subset B_2(G)\subset M(A(G))$ with continuous inclusions. For amenable groups these algebras coincide. On the other hand both inclusions are proper for free groups (cf. [10], [11], [1], [7], [13]).

Remark. It has been shown in [2] that for any locally compact group G the algebra $B_2(G)$ coincides with the algebra $M_0(A(G))$ of all completely bounded multipliers of the algebra $A(G)$. This algebra was introduced and studied in [3].

Two uniformly bounded representations π_1 and π_2 of a locally compact group G on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ are called *similar* or *topologically equivalent* if there exists a bounded invertible operator $A\colon \mathscr{H}_1\to\mathscr{H}_2$ such that $A\pi_1(x)=\pi_2(x)A$ for any $x\in G$. Representations π_1 and π_2 are called *weakly similar* if they have the same closure in $B_2(G)$ of sets of their coefficients.

Note that for an amenable group G every uniformly bounded representation of G on a Hilbert space is similar to a unitary representation and if two unitary representations are weakly similar they are weakly equivalent.

3.2. *Integration on paths.* Let γ be a piecewise smooth curve contained in the unit disc $|z|<1$. Consider a representation of F

$$\pi_\gamma=\oplus\int_\gamma \pi_z|dz|$$

acting on the Hilbert space $\mathscr{H}_\gamma=\oplus\int_\gamma \ell^2(F)\,|dz|$. Clearly π_γ is a uniformly bounded representation with

$$\sup_{x\in F}\|\pi_\gamma(x)\|\leq 2\max_{z\in\gamma}\frac{|1-z^2|}{1-|z|}.$$

PROPOSITION 2. *Let f be a holomorphic function in a neigbourhood of γ. Then the complex function φ defined on F by*

$$\varphi(x)=\int_\gamma z^{|x|}f(z)\,dz$$

is a coefficient of the representation π_γ and

$$\|\varphi\|_{B_2}\leq\int_\gamma |f(z)|\frac{|1-z^2|}{1-|z|^2}\,|dz|.$$

Proof. Take two functions g and h on γ so that $|g(z)|=|h(z)|$ and $g(z)\overline{h(z)}=f(z)\chi(z)$, where $\chi(z)$ denotes the Radon-Nikodym derivative $dz/|dz|$. Define two vectors G and H in $\mathscr{H}_\gamma$ by

$$G = \oplus\int_\gamma g(z)\,\delta_e|dz|, \quad H = \oplus\int_\gamma h(z)\,\delta_e|dz|. \tag{9}$$

Then for $x\in F$,

$$\begin{aligned}\langle\pi_\gamma(x)\,G, H\rangle &= \int_\gamma \langle\pi_z(x)\,\delta_e, \delta_e\rangle\, g(z)\,\overline{h(z)}\,|dz| \\ &= \int_\gamma z^{|x|}f(z)\,\chi(z)\,|dz| = \int_\gamma z^{|x|}f(z)\,dz = \varphi(x).\end{aligned}$$

By Proposition 1, using the formula (8) and the fact that $|\chi(z)|=1$, $z\in\gamma$, we get an estimate for the norm $\|\varphi\|_{B_2}$:

$$\begin{aligned}\|\varphi\|_{B_2} &\leqslant \sup_{x\in F}\|\pi_\gamma(x)\,G\|\,\sup_{x\in F}\|\pi_\gamma^*(x)\,H\| \\ &\leqslant \left(\int_\gamma |g(z)|^2\frac{|1-z^2|}{1-|z|^2}\,|dz|\right)^{1/2}\left(\int_\gamma |h(z)|^2\frac{|1-z^2|}{1-|z|^2}\,|dz|\right)^{1/2} \\ &= \int_\gamma |f(z)|\frac{|1-z^2|}{1-|z|^2}\,|dz|.\end{aligned}$$

COROLLARY 1. *For $m=0,1,2,\ldots$ let χ_m denote the characteristic function of the set $\{x\in F\colon |x|=m\}$. Let $\alpha_0,\alpha_1,\alpha_2,\ldots$ be a sequence of complex numbers such that*

$$\sum_{m=0}^{\infty}|\alpha_m-\alpha_{m+2}|\,(m+2)<+\infty.$$

Then the function

$$\varphi=\sum_{m=0}^{\infty}\alpha_m\chi_m \tag{10}$$

belongs to $B_2(F)$.

Proof. For $m=0,1,2,\ldots$ define a function φ_m on F as $\chi_0+\chi_2+\ldots+\chi_m$ if m is even and $\chi_1+\chi_3+\ldots+\chi_m$ if m is odd.

If γ is a circle $\{z\in C\colon |z|=r\}$, $0<r<1$, then each φ_m is a coefficient of the representation π_γ because

$$\varphi_m(x)=\frac{1}{2\pi i}\int_\gamma z^{|x|}f_m(z)\,dz, \quad x\in F,$$

where

$$f_m(z)=\frac{1}{z^{m+1}(1-z^2)}.$$

By Proposition 2

$$\|\varphi_m\|_{B_2}\leqslant\frac{1}{2\pi}\int_\gamma\frac{|dz|}{|z|^{m+1}(1-|z|^2)}=\frac{1}{r^m(1-r^2)}.$$

Taking $r=(m/(m+2))^{1/2}$ we get $\|\varphi_m\|_{B_2}\leqslant\frac{1}{2}e(m+2)$.

Express the function $\varphi=\Sigma_{m=0}^\infty \alpha_m\chi_m$ in the form $\varphi=\Sigma_{m=0}^\infty(\alpha_m-\alpha_{m+2})\varphi_m$. Then

$$\|\varphi\|_{B_2}\leqslant\sum_{m=0}^\infty|\alpha_m-\alpha_{m+2}|\,\|\varphi_m\|_{B_2}\leqslant\frac{e}{2}\sum_{m=0}^\infty(m+2)\,|\alpha_m-\alpha_{m+2}|.$$

As a special case of Corollary 1 we get

COROLLARY 2. *Let $\alpha_0,\alpha_1,\ldots$ be a decreasing sequence of positive numbers. If the series $\Sigma_{m=0}^\infty\alpha_m$ is convergent then the function $\Sigma_{m=0}^\infty\alpha_m\chi_m$ belongs to $B_2(F)$.*

3.3. *Remarks.* (1) If the group F has infinitely many generators then none of the functions $\varphi=\Sigma_{m=0}^N\alpha_m\chi_m$ belongs to the Fourier-Stieltjes algebra $B(F)$, except $\varphi=\alpha_0\delta_e$. Indeed, if the function φ belongs to $B(F)$ then $\varphi|_{F_k}$ belongs to $B(F_k)$ for $k=1,2,\ldots$, where F_k is a subgroup in F generated by k among the free generators. Moreover

$$\|\varphi\|_{B(F)}\geqslant\|\varphi|_{F_k}\|_{B(F_k)}=\|\varphi|_{F_k}\|_{A(F_k)}.$$

(The last equality holds since $\varphi|_{F_k}$ has finite support.) On the other hand, it follows by [9], VIII.1.1 that for $n=1,2,\ldots,N$

$$\|\varphi|_{F_k}\|_{A(F_k)}\geqslant\frac{|\alpha_n|}{n+1}\|\chi_n|_{F_k}\|_2=\frac{|\alpha_n|}{n+1}\sqrt{2k(2k-1)^{n-1}}.$$

Comparing these two inequalities and letting k tend to infinity we get $\alpha_n=0$ for $n=1,2,\ldots,N$.

Even if the group F has only finite number of generators, the same argument shows that there exists a function in $B_2(F)-B(F)$.

(2) If the group F has no more than countably many generators then there exists a sequence of functions with finite supports on F which is an approximate unit for $A(F)$ and is uniformly bounded in the $B_2(F)$ norm. This is a result of de Canniere and

Haagerup ([3], 3.9). It can be seen easily, applying Corollary 1, that the approximate unit in $A(F)$ constructed by Haagerup in the earlier paper [10] has the desired property.

3.4. *Some estimates from below.* A complex function φ on F is called *radial* if the value $\varphi(x)$, $x \in F$, depends only on $|x|$, the length of x. Any radial function has unique expression of the form (10).

Proposition 2 and the method presented in the proof of Corollary 1 give a tool to estimate from above the $B_2(F)$ norm of radial functions. The next theorem gives estimates from below.

THEOREM 2. *Let F be a free group on infinitely many generators. For any radial function $\varphi = \Sigma_{m=0}^{\infty} \alpha_m \chi_m$ with only a finite number of α_m's different from zero we have*

$$\|\varphi\|_{B_2(F)} \geq \|\varphi\|_{M(A(F))} \geq \sup_{t \in (0,\pi)} \frac{2}{\pi} \int_0^{\pi} \left| \sum_{m=0}^{\pi} \alpha_m \frac{\sin(m+1)s \sin s \sin(m+1)t}{\sin t} \right| ds. \tag{11}$$

Proof. Fix a sequence $x_1, x_2, \ldots$ of free generators in F and let F_k denote the subgroup in F generated by $x_1, x_2, \ldots, x_k$. Let $VN(F_k)$ denote the von Neumann algebra of operators on $\ell^2(F_k)$ generated by the left regular representation. Denote also $VN_r(F_k)$ the subalgebra in $VN(F_k)$ of these operators T for which $T\delta_e$ is a radial function on F_k.

For $k, n = 1, 2, \ldots$ define $\chi_{n,k}$ to be the characteristic function of the set $\{x \in F_k : |x| = n\}$ and $\hat{\chi}_{n,k}$ the function on $(0,\pi)$ defined as

$$\hat{\chi}_{n,k}(s) = (2k-1)^{n/2} \left(\frac{\sin(n+1)s}{\sin s} - \frac{1}{2k-1} \frac{\sin(n-1)s}{\sin s} \right)$$

$$\hat{\chi}_{0,k}(s) = 1.$$

It follows from [15], Theorem 5.1 that the correspondence $\chi_{n,k} \to \hat{\chi}_{n,k}$, $n = 0, 1, 2, \ldots$, may be uniquely extended to an isometric isomorphism of $VN_r(F_k)$ onto $L^{\infty}(0,\pi)$. For a function $f = \Sigma_{m=0}^{\infty} \beta_m \chi_{m,k}$ in $VN_r(F_k)$, the function $\hat{f}$ has the form

$$\hat{f}(s) = \sum_{m=0}^{\infty} (2k-1)^{m/2} (\beta_m - \beta_{m+2}) \frac{\sin(m+1)s}{\sin s}, \quad 0 < s < \pi. \tag{12}$$

Let $\varphi = \Sigma_{m=0}^{\infty} \alpha_m \chi_m$ be a radial function on F with only a finite number of α_m's different from zero. For any natural number k multiplication by φ defines a bounded operator on $VN_r(F_k)$ with norm not exceeding $\|\varphi\|_{M(A(F))}$. Thus

$$\|\varphi\|_{M(A(F))} \geq \sup\{\|(\varphi f)^\wedge\|_\infty : f \in VN_r(F_k), \|\hat{f}\|_\infty \leq 1\}.$$

Since by (12) we have

$$\frac{2}{\pi}\int_0^\pi \hat{f}(s)\sin s \sin(m+1)s\,ds = (2k-1)^{m/2}(\beta_m-\beta_{m+2}), \quad m=0,1,2,\ldots,$$

thus

$$\beta_m = (2k-1)^{-m/2}\frac{2}{\pi}\int_0^\pi \hat{f}(s)\sin s\sum_{r=0}^\infty \frac{\sin(m+2r+1)s}{(2k-1)^r}\,ds$$

and so

$$\begin{aligned}(\varphi f)^\wedge(t) &= \sum_{m=0}^\infty (\alpha_m\beta_m-\alpha_{m+2}\beta_{m+2})(2k-1)^{m/2}\frac{\sin(m+1)t}{\sin t}\\ &= \frac{2}{\pi}\int_0^\pi \hat{f}(s)\sin s\sum_{m=0}^\infty\Bigg[\Bigg(\alpha_m\sin(m+1)s\\ &\quad+\sum_{r=1}^\infty\frac{\alpha_m-\alpha_{m+2}}{(2k-1)^r}\sin(m+2r+1)s\Bigg)\frac{\sin(m+1)t}{\sin t}\Bigg]\,ds.\end{aligned}$$

If we put an arbitrary function g in $L^\infty(0,\pi)$ with $\|g\|_\infty \leq 1$ instead of $\hat{f}$ and pass to the limit with k tending to infinity we get

$$\|\varphi\|_{M(A(F))} \geq \sup_{t\in(0,\pi)}\frac{2}{\pi}\left|\int_0^\pi g(s)\sum_{m=0}^\infty \alpha_m\frac{\sin(m+1)s\sin s\sin(m+1)t}{\sin t}\,ds\right|.$$

This implies (11).

Consider the system U_m, $m=1,2,\ldots$ of the second type Tschebyshev polynomials

$$U_m(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin((m+1)\arccos x)}{\sqrt{1-x^2}}, \quad x\in(-1,1).$$

This system is an orthonormal basis in $L^2((-1,1),\mu)$, where $d\mu(x)=\sqrt{1-x^2}\,dx$.

COROLLARY 3. *Let F be a free group on infinitely many generators. For a radial function $\varphi=\sum_{m=0}^\infty \alpha_m\chi_m$ in $B_2(F)$ define an operator on $L^1((-1,1),\mu)$ by*

$$(T_\varphi f)(x) = \int_{-1}^1 K_\varphi(x,y)f(y)\,d\mu(y),$$

where

$$K_\varphi(x,y)=\sum_{m=0}^{\infty}\alpha_m U_m(x)\,U_m(y).$$

Then T_φ is a bounded operator and

$$\|T_\varphi\|\leq\|\varphi\|_{M(A(F))}\leq\|\varphi\|_{B_2(F)}. \tag{13}$$

The correspondence $\varphi\to T_\varphi$ is an algebra homomorphism.

Proof. Changing variables $x=\cos s$, $y=\cos t$ we get that the right hand side of (11) is equal to

$$\sup_{x\in(-1,1)}\int_{-1}^{1}|K_\varphi(x,y)|\,d\mu(y).$$

It means that T_φ is a bounded operator on $L^1((-1,1),\mu)$ and (13) holds. To see that $T_{\varphi\psi}=T_\varphi T_\psi$ observe that $T_{\chi_m}U_m=U_m$ and $T_{\chi_m}U_n=0$ for $m\neq n$.

COROLLARY 4. *Assume that the free group F has infinitely many generators. For a complex number z, $|z|<1$, define $\varphi_z(x)=z^{|x|}$ for $x\in F$. Then*

$$\|\varphi_z\|_{B_2(F)}=\|\varphi_z\|_{M(A(F))}=\|T_{\varphi_z}\|=\frac{|1-z^2|}{1-|z|^2}.$$

Proof. The estimate $\|\varphi_z\|_{B_2(F)}\leq|1-z^2|/(1-|z|^2)$ follows from (8) and Proposition 1. On the other side

$$\|T_{\varphi_z}\|\geq\frac{2}{\pi}\int_0^\pi\left|\sum_{m=0}^{\infty}(m+1)\,z^m\sin(m+1)s\sin s\right|ds$$

by Theorem 2. But

$$\sum_{m=0}^{\infty}(m+1)\,z^m\sin(m+1)s=\frac{1}{2i}\left(e^{is}\sum_{m=0}^{\infty}(m+1)\,z^m e^{ims}-e^{-is}\sum_{m=0}^{\infty}(m+1)\,z^m e^{-ims}\right)$$
$$=\frac{1}{2i}\left(\frac{e^{is}}{(1-z\,e^{is})^2}-\frac{e^{-is}}{(1-z\,e^{-is})^2}\right)=\frac{(1-z^2)\sin s}{(1-z\,e^{is})^2(1-z\,e^{-is})^2}.$$

Thus

$$\left|\sum_{m=0}^{\infty}(m+1)\,z^m\sin(m+1)s\sin s\right|=|1-z^2|\frac{\sin^2 s}{|(1-z\,e^{is})(1-z\,e^{-is})|^2}.$$

Denote

$$f_z(s) = \frac{\sin s}{(1-z\,e^{is})(1-z\,e^{-is})} = \frac{1}{2iz}\left(\frac{1}{1-z\,e^{is}} - \frac{1}{1-z\,e^{-is}}\right) = \sum_{m=0}^{\infty} z^m \sin(m+1)\,s.$$

Then

$$\|T_{\varphi_z}\| \geqslant \frac{2}{\pi}|1-z^2| \int_0^{\pi} f_z(s)\,\overline{f_z(s)}\,ds = |1-z^2| \sum_{m=0}^{\infty} |z|^{2m} = \frac{|1-z^2|}{1-|z|^2}.$$

3.5. *Remarks.* (1) The functions φ_z, $|z|<1$, play a fundamental part in present theory. They are analogues of the spherical functions on $SL(2,\mathbf{R})$. The function φ_z is the unique, up to a constant multiple, radial coefficient of the representation π_z (cf. Theorem 3). The explicit formula for the kernel K_{φ_z} of the operator T_{φ_z} is

$$K_{\varphi_z}(\cos s, \cos t) = \frac{1-z^2}{(1-2z\cos(s+t)+z^2)(1-2z\cos(s-t)+z^2)}.$$

(2) Applying Proposition 2 and Theorem 2 to the function $\varphi=\chi_m$, $m=1,2,\ldots$, we get the following estimate

$$\frac{8}{\pi^2}(m+1) \leqslant \|\chi_m\|_{B_2(F)} \leqslant \frac{4e}{\pi}(m+1).$$

3.6. *Characterization of radial coefficients.* As we have seen in Proposition 2, if we take two vectors G, H of the special form (9) in the representation space $\mathscr{H}_\gamma$ for a path γ then the corresponding coefficient of π_γ is a radial function. Conversely, if φ is a radial coefficient of the representation π_γ then we can always find two vectors G, H in $\mathscr{H}_\gamma$ of the form (9) such that $\varphi(x)=\langle \pi_\gamma(x)\,G, H\rangle$. We prove it only for circles, although the proof works generally. In this case we obtain especially simple characterization of radial coefficients.

THEOREM 3. *Let $C(r)$ denote the circle $\{z\in C\colon |z|=r\}$, $0<r<1$. Let F be a free group on infinitely many generators. A radial function φ is a coefficient of the representation $\pi_{C(r)}$ if and only if there exists a function f in $L^1(T^1)$, where T^1 is the unit circle, such that*

$$\varphi(x) = r^n \hat{f}(n), \quad x\in F,\ n=|x|. \tag{14}$$

Proof. Let $G, H \in \mathscr{H}_{C(r)}$, i.e.

$$G = \oplus\int_{C(r)} G_z|dz|, \quad H = \oplus\int_{C(r)} H_z|dz|,$$

where $G_z, H_z \in \ell^2(F)$. Suppose that the function

$$\varphi(x) = \langle \pi_{C(r)}(x) G, H\rangle = \int_{C(r)} \langle \pi_z(x) G_z, H_z\rangle \, |dz|$$

is radial. Write $\varphi = \Sigma_{m=0}^{\infty} \alpha_m \chi_m$. Choose a sequence $x_1, x_2, \ldots$ of free generators in F and let $S_{z,k}$ be the operator defined in (6). Let T be the orthogonal projection onto the one-dimensional subspace $C\delta_e$. Then

$$\alpha_n = \int_{C(r)} \langle S^n_{z,k} G_z, H_z\rangle \, |dz|$$

for any $k, n = 1, 2, \ldots$. Since the sequence of operators $\{S_{z,k}\}_{k=1}^{\infty}$ strongly converges on $\ell^2(F)$ to the operator zT then by the Dominated Convergence Principle we get

$$\alpha_n = \int_{C(r)} z^n \langle TG_z, H_z\rangle \, |dz|, \quad n = 1, 2, \ldots .$$

Define a function f on T^1 by

$$f(z) = r\langle TG_{r\bar z}, H_{r\bar z}\rangle = r\langle TG_{r\bar z}, TH_{r\bar z}\rangle, \quad |z| = 1.$$

Then $f \in L^1(T^1)$ with $\|f\|_1 \leq \|G\|_{\mathscr{H}_{C(r)}} \|H\|_{\mathscr{H}_{C(r)}}$ and f fulfills (14) for $n = 1, 2, \ldots$. To get (14) also for $n = 0$ take the function $f - \hat f(0) + \alpha_0$ instead of f.

The converse implication is actually shown in the proof of Proposition 2.

3.7. *Invariant subspaces for* $\pi_{C(r)}$. Let $\mathscr{H}$ denote the Hilbert space

$$\mathscr{H} = \oplus \int_{T^1} \ell^2(F) \, |dz|.$$

This space may be realized also as one of the spaces $L^2(T^1 \times F)$ or $L^2(T^1) \bar\otimes \ell^2(F)$, where the symbol $\bar\otimes$ means the completion of $L^2(T^1) \otimes \ell^2(F)$ in the unique Hilbert space norm. For any r, $0 < r < 1$, the space $\mathscr{H}_{C(r)}$ is isometrically isomorphic to $\mathscr{H}$, the isomorphism being $\mathscr{H} \ni f \to f_r \in \mathscr{H}_{C(r)}$, where $f_r(z, x) = r^{-1/2} f(r^{-1} z, x)$, $|z| = r$, $x \in F$. In this manner we may treat each $\pi_{C(r)}$ acting on $\mathscr{H}$. We get the following formula for this action

$$(\pi_{C(r)}(a) f \otimes g)(z, x) = f(z)(\pi_{rz}(a) g)(x) \tag{15}$$

with $f \in L^2(T^1)$, $g \in \ell^2(F)$, $z \in T^1$ and $a, x \in F$.

Let $\mathcal{H}^\circ$ be the subspace $H^2\bar{\otimes}\ell^2(F)\subset\mathcal{H}$, where H^2 is the Hardy space of analytic functions in $L^2(T^1)$. Then since $\{\pi_z: |z|<1\}$ is an analytic family of representations, the space $\mathcal{H}^\circ$ is invariant under each representation $\pi_{C(r)}$, $0<r<1$. Denote the restriction of $\pi_{C(r)}$ to $\mathcal{H}^\circ$ by $\pi^\circ_{C(r)}$.

LEMMA 3. *Let F be a free group on infinitely many generators. Fix a number r, $0<r<1$. Then the representation $\pi^\circ_{C(r)}$ is indecomposable, weakly similar to $\pi_{C(r)}$ and $1\otimes\delta_e$ is a cyclic vector for $\pi^\circ_{C(r)}$.*

Proof. It is clear by (15) that the multiplication by z commutes with $\pi_{C(r)}$. It follows that each of the spaces $z^m\mathcal{H}^\circ=z^mH^2\bar{\otimes}\ell^2(F)$, $m=0,\pm1,\pm2,\dots$ is invariant under $\pi_{C(r)}$. The restriction of $\pi_{C(r)}$ to any of them is isometrically equivalent to $\pi^\circ_{C(r)}$. To prove that $\pi^\circ_{C(r)}$ is weakly similar to $\pi_{C(r)}$ it suffices to show that any coefficient of $\pi_{C(r)}$ is a limit in $B_2(F)$ of a sequence of coefficients of $\pi^\circ_{C(r)}$.

Let f and g be two functions in $\mathcal{H}$. There exist two sequences $f_1, f_2, \dots$ and $g_1, g_2, \dots$ in $\mathcal{H}$ such that $f_m, g_m\in z^{-m}\mathcal{H}^\circ$, $m=1,2,\dots$, and

$$\lim_{m\to\infty}\|f-f_m\|=\lim_{m\to\infty}\|g-g_m\|=0.$$

It follows that the coefficient

$$\varphi(x)=\langle\pi_{C(r)}(x)f,g\rangle,\quad x\in F,$$

is a limit in $B_2(F)$ of coefficients

$$\varphi_m(x)=\langle\pi_{C(r)}(x)f_m,g_m\rangle,\quad x\in F.$$

But φ_m is a coefficient of the representation $\pi_{C(r)}|_{z^{-m}\mathcal{H}^\circ}$ and so of $\pi^\circ_{C(r)}$ too.

To see that $\pi^\circ_{C(r)}$ is indecomposable let P be a projection in $\mathcal{H}^\circ$ which commutes with $\pi^\circ_{C(r)}$. As we have seen in the proof of Theorem 3 the operator $rzI\otimes T$ is a strong limit of a sequence $\oplus\int_{T^1}S_{rz,k}|dz|$ when $k\to\infty$. Therefore $zI\otimes T$ belongs to the von Neumann algebra generated by $\pi^\circ_{C(r)}$ and so it commutes with P. This means that $H^2\bar{\otimes}C\delta_e$ is an invariant subspace for P and $P(H^2\otimes C\delta_e)$ reduces the operator $zI\otimes T|_{H^2\bar{\otimes}C\delta_e}$. But the multiplication by z is an irreducible operator on H^2 (cf. [16], Theorem 5.3), thus the restriction of P to $H^2\bar{\otimes}C\delta_e$ must be 0 or I. In particular $P(1\otimes\delta_e)=1\otimes\delta_e$ or $P(1\otimes\delta_e)=0$. This implies that

$$P\pi^\circ_{C(r)}(x)(1\otimes\delta_e)=\pi^\circ_{C(r)}(x)(1\otimes\delta_e)$$

for any $x \in F$ or

$$P\pi^{\circ}_{C(r)}(x)(1 \otimes \delta_e) = 0$$

for any $x \in F$. If we prove that $1 \otimes \delta_e$ is a cyclic vector for $\pi^{\circ}_{C(r)}$ we get then $P=0$ or $P=I$.

Let M denote the closed cyclic subspace in $\mathscr{H}^{\circ}$ generated by $1 \otimes \delta_e$. To prove that $1 \otimes \delta_e$ is cyclic for $\pi^{\circ}_{C(r)}$ it suffices to show that $f \otimes \delta_x \in M$ for any $f \in H^2$ and any $x \in F$.

Applying the operator $zI \otimes T$ to $1 \otimes \delta_e$ several times we get $z^n \otimes \delta_e \in M$ for $n=0,1,2,\ldots$, and thus also that $f \otimes \delta_e \in M$ for any $f \in H^2$.

Let now $x \in F$. If $|x|=1$ then for any $f \in H^2$

$$\pi^{\circ}_{C(r)}(x)\left(\frac{f(z)}{1-r^2z^2} \otimes \delta_e\right) = \frac{rzf(z)}{\sqrt{1-r^2z^2}} \otimes \delta_e + f(z) \otimes \delta_x$$

by (5) and (15). But since

$$\frac{f(z)}{\sqrt{1-r^2z^2}} \otimes \delta_e \quad \text{and} \quad \frac{rzf(z)}{\sqrt{1-r^2z^2}} \otimes \delta_e$$

both are in M, we have $f \otimes \delta_x \in M$. For all other $x \in F$ the proof is similar and goes by induction on the length of x.

Remark. The only non-trivial closed subspaces in $\mathscr{H}^{\circ}$, invariant under $\pi^{\circ}_{C(r)}$ are

$$z^{m-1}H^2 \bar{\otimes} M + z^m H^2 \bar{\otimes} \ell^2(F),$$

where $m=1,2,\ldots$ and M is a closed subspace in $\ell^2(F)$ invariant under left translations.

This can be shown in four steps as follows. If for a non-zero function f in $\mathscr{H}^{\circ}$, $\mathscr{H}_f$ denotes the closed subspace in $\mathscr{H}^{\circ}$ generated by $\pi^{\circ}_{C(r)}(a)f$, $a \in F$, then

(i) $\mathscr{H}_f \cap H^2 \bar{\otimes} C\delta_e \neq \{0\}$,

(ii) $\mathscr{H}_f \cap H^2 \bar{\otimes} C\delta_e = z^m H^2 \bar{\otimes} C\delta_e$ for some m, $m=1,2,\ldots$,

(iii) $z^m H^2 \bar{\otimes} \ell^2(F) \subset \mathscr{H}_f \subset z^{m-1} H^2 \bar{\otimes} \ell^2(F)$,

(iv) restrictions of operators $\pi^{\circ}_{C(r)}(a)$ and $I \otimes L_a$, $a \in F$, to the space $z^{m-1}H^2 \bar{\otimes} \ell^2(F)$ are equal modulo $z^m H^2 \bar{\otimes} \ell^2(F)$, i.e. $(\pi^{\circ}_{C(r)}(a)f - I \otimes L_a f) \perp z^m H^2 \bar{\otimes} \ell^2(F)$ for any $f \in z^{m-1} H^2 \bar{\otimes} \ell^2(F)$.

Point (i) holds because $zI \otimes T(\pi^{\circ}_{C(r)}(a)f) \neq 0$ for a suitable $a \in F$. Point (ii) because the only non-trivial closed subspace in H^2 invariant under multiplication by z are $z^m H^2$, $m=1,2,\ldots$ (cf. [16], Theorem 5.3). Point (iii) because $z^m \otimes \delta_e$ is a cyclic vector for the restriction of $\pi^{\circ}_{C(r)}$ to $z^m H^2 \bar{\otimes} \ell^2(F)$ and because $\mathcal{H}_f \subset (zI \otimes T)^{-1}(z^m H^2 \otimes C\delta_e)$. Finally point (iv) follows directly from (5) (use the Taylor expansion of $\sqrt{1-r^2z^2}$).

3.8. *An unexpected realization of the representation $\pi^{\circ}_{C(r)}$.*

THEOREM 4. *Let F be a free group on infinitely many generators and let $\mathcal{K}(F)$ denote the set of all complex functions on F with finite supports. Fix a number r, $0<r<1$, and for f, g in $\mathcal{K}(F)$ define*

$$\langle f, g\rangle_r = \sum_{k=0}^{\infty} \sum_{|x|=|y|=k} f(x)\overline{g(y)}\, r^{|y^{-1}x|}. \tag{16}$$

Then $\langle\ ,\ \rangle_r$ is a non-degenerate hermitian form on $\mathcal{K}(F)$.

Let $\mathcal{H}_r$ be the Hilbert space produced from $(\mathcal{K}(F), \langle\ ,\ \rangle_r)$ in the standard way. Then the left regular representation L of the group F on $\mathcal{K}(F)$ extends to a uniformly bounded representation of F on $\mathcal{H}_r$. This representation is indecomposable, similar to $\pi^{\circ}_{C(r)}$ and weakly similar to $\pi_{C(r)}$.

Proof. First we show that

$$\langle f, f\rangle_r \geq (1-r^2)\,\|f\|_2^2 \tag{17}$$

for any $f \in \mathcal{K}(F)$. This will prove the first part of the theorem.

Put $f_k = f\chi_k$ for $k=0,1,2,\ldots$ (recall that χ_k denotes the characteristic function of the set $\{x \in F: |x|=k)$. Then

$$\langle f, f\rangle_r = \sum_{k=0}^{\infty} \langle f_k, f_k\rangle_r.$$

We have $\langle f_0, f_0\rangle_r = |f(e)|^2$ and for $k=1,2,\ldots$ by an elementary computation we get

$$\langle f_k, f_k\rangle_r = (1-r^2)\sum_{n=0}^{k-1} r^{2n}\|P^n f_k\|_2^2 + r^{2k}\|P^k f_k\|_2^2 \geq (1-r^2)\,\|f_k\|_2^2.$$

Therefore

$$\langle f, f\rangle_r \geq (1-r^2)\sum_{k=0}^{\infty}\|f_k\|_2^2 = (1-r^2)\,\|f\|_2^2,$$

which shows (17).

Define a linear map T_r from $\mathcal{K}(F)$ into $\mathcal{H}^{\infty}$ putting

$$T_r(\delta_x) = \sqrt{\frac{1-r^2}{1-r^2z^2}}\,\pi^\circ_{C(r)}(x)\,(1\otimes\delta_e)$$

for $x\in F$. Since $1\otimes\delta_e$ is a cyclic vector for the representation $\pi^\circ_{C(r)}$ (Lemma 3) and multiplication by the function $(1-r^2)^{1/2}(1-r^2z^2)^{-1/2}$ is an invertible operator on $\mathcal{H}^\circ$ hence $T_r(\mathcal{H}(F))$ is dense in $\mathcal{H}^\circ$. Also

$$\pi^\circ_{C(r)}(x)\,T_r = T_r L_x \tag{18}$$

for any $x\in F$.

The set E of free generators is infinite by the assumption. Fix a sequence $x_1, x_2, \dots$ of distinct elements in E and put $x(j)=x_j\dots x_j$ (j factors). For $j=1,2,\dots$ define also a hermitian form h_j on $\mathcal{H}(F)$ by

$$\begin{aligned} h_j(f,g) &= \langle \pi^\circ_{C(r)}(x(j))\,T_r f, \pi^\circ_{C(r)}(x(j))\,T_r g\rangle \\ &= \langle T_r L_{x(j)} f, T_r L_{x(j)} g\rangle . \end{aligned}$$

Then

$$\langle f,g\rangle_r = \lim_{j\to\infty} h_j(f,g). \tag{19}$$

Of course it suffices to show (19) only for functions $f=\delta_a$, $g=\delta_b$ with arbitrary a and b in F.

By (5) we have

$$T_r(\delta_x) = (1-r^2)^{1/2}(1-r^2z^2)^{-1/2}r^{|x|}z^{|x|}\delta_e + (1-r^2)^{1/2}\sum_{k=0}^{|x|-1} r^k z^k P^k\delta_x$$

for any $x\in F$. Write $h_j(\delta_a,\delta_b)=h_j'(\delta_a,\delta_b)+h_j''(\delta_a,\delta_b)$ where

$$h_j'(\delta_a,\delta_b) = (1-r^2)\,r^{|xa|+|xb|}\int_{T^1} |1-r^2z^2|^{-1}\,z^{|xa|-|xb|}|dz|$$

and

$$h_j''(\delta_a,\delta_b) = (1-r^2)\sum_{k=0}^{n} r^{2k}\langle P^k\delta_{xa}, P^k\delta_{xb}\rangle$$

with $x=x(j)$ and $n=\min\{|x(j)\,a|, |x(j)\,b|\}-1$. Since $|x(j)\,a|=j+|a|$ and $|x(j)\,b|=j+|b|$ for large j, we get $\lim_{j\to\infty} h_j'(\delta_a,\delta_b)=0$. To compute $\lim_{j\to\infty} h_j''(\delta_a,\delta_b)$ first consider the case $|a|\neq|b|$. Then $|x(j)\,a|\neq|x(j)\,b|$ for large j and so $\langle P^k\delta_{x(j)\,a}, P^k\delta_{x(j)\,b}\rangle=0$ for any $k=0,1,2,\dots$. Thus $\lim_{j\to\infty} h_j''(\delta_a,\delta_b)=0$ in this case. Now let $|a|=|b|$. Observe that $\langle P^k\delta_{x(j)\,a}, P^k\delta_{x(j)\,b}\rangle$ takes only value 0 or 1 and it takes the value 1 exactly when

$2k \geqslant |(x(j)\,b)^{-1}(x(j)\,a)| = |b^1 a|$. Thus $h_j''(\delta_a, \delta_b) = r^{|b^{-1}a|} - r^{2j+2|a|}$, and so

$$\lim_{j\to\infty} h_j''(\delta_a, \delta_b) = r^{|b^{-1}a|} = \langle \delta_a, \delta_b \rangle_r.$$

Put

$$C = \sup_{x\in F} \|\pi^{\circ}_{C(r)}(x)\| \leqslant \sup_{x\in F} \max_{|z|=r} \|\pi_z(x)\| \leqslant 2\frac{1+r^2}{1-r^2}$$

and observe that $h_j(f,f) \leqslant C^2 \langle T_r f, T_r f\rangle$ and $\langle T_r f, T_r f\rangle \leqslant C^2 h_j(f,f)$ for any f in $\mathcal{H}(F)$ and any $j=1,2,\ldots$. Together with (19) it follows that T_r extends to an isomorphism of $\mathcal{H}_r$ onto $\mathcal{H}^{\circ}$. Formula (18) shows that L extends to a uniformly bounded representation of F on $\mathcal{H}_r$, similar the representation $\pi^{\circ}_{C(r)}$. The rest of Theorem 4 follows now from Lemma 3.

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