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# Fundamental solutions of real homogeneous cubic operators of principal type in three dimensions

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## 1. Introduction

1.1. The operator $\partial_1^3+\partial_2^3+\partial_3^3$ was considered—to my knowledge—for the first time in 1913 in N. Zeilon's article [20], wherein he generalizes I. Fredholm's method of construction of fundamental solutions (see [5]) from homogeneous *elliptic* equations to arbitrary homogeneous equations in three variables with a *real-valued* symbol (cf. [20, II, pp. 14–22], [6, Chapter 11, pp. 146–148]). An explicit formula for a fundamental solution was given in [19]. The objective of this paper is to generalize the calculations in [19] to the operators $\partial_1^3+\partial_2^3+\partial_3^3+3a\partial_1\partial_2\partial_3$, $a\in\mathbf{R}\setminus\{-1\}$. As discussed below, this class of operators comprises all real homogeneous cubic operators of principal type in three dimensions.

According to Newton's classification of real elliptic curves, the non-singular real homogeneous polynomials $P(\xi)$ of third order in three variables are divided into two types according to whether the real projective curve $\{[\xi]\in\mathbf{P}(\mathbf{R}^3):P(\xi)=0\}$ consists of one or of two connected components, respectively. (For $\xi\in\mathbf{R}^n\setminus\{0\}$, $[\xi]\in\mathbf{P}(\mathbf{R}^n)$ denotes the corresponding projective point, i.e., the line $\{t\xi:t\in\mathbf{R}\}$.) In Hesse's normal form, all non-singular real cubic curves are—up to linear transformations—given by

$$P_a(\xi)=\xi_1^3+\xi_2^3+\xi_3^3+3a\xi_1\xi_2\xi_3,\quad a\in\mathbf{R}\setminus\{-1\}$$

(cf. [3, 7.3, Satz 4, p. 379; English transl., p. 293], [4, §7, (10), p. 39], [17, §1.4, p. 19]). Let $X_a:=\{[\xi]\in\mathbf{P}(\mathbf{R}^3):P_a(\xi)=0\}$ denote the real projective variety defined by $P_a$. For $a>-1$, $X_a$ is connected, whereas, for $a<-1$, $X_a$ consists of two components (cf. Figure 1). The corresponding operators $P_a(\partial)$ also differ from the physical viewpoint: For $a<-1$, every projective line through $[1,1,1]$ intersects $X_a$ in three different projective points, and thus $P_a$ is strongly hyperbolic in the direction $(1,1,1)$ ([1, 3.8, p. 129]); for $a>-1$, $P_a$ is not hyperbolic in any direction, nor is it an evolution operator (cf. [15, Example 1, p. 463] for the case of $a=0$).

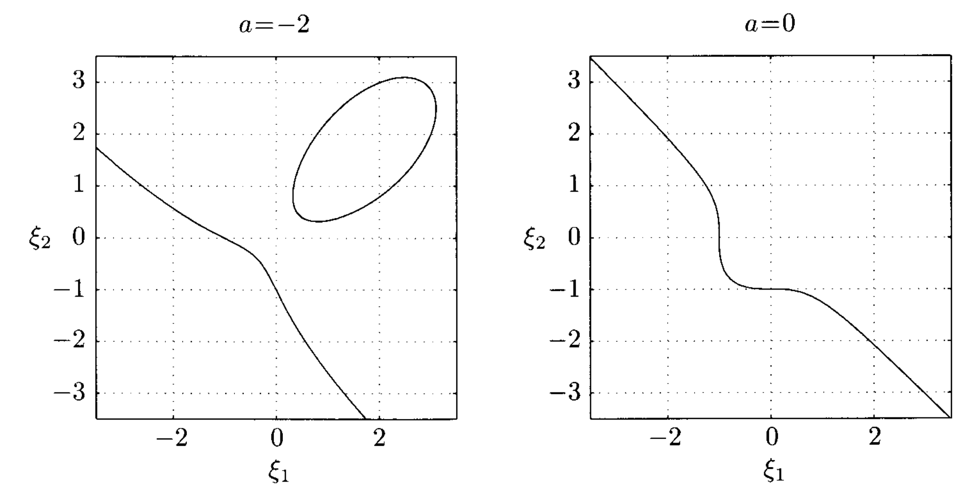

Fig. 1. $\{(\xi_1,\xi_2):[\xi_1,\xi_2,1]\in X_a\}$ for $a=-2$ and for $a=0$.

1.2. In §2 of this paper, we shall define the fundamental solution $E_a$ of $P_a(\partial)$ as Fourier transform of the homogeneous distribution which is of order $-3$ and has $\mathrm{vp}(1/P_a(\omega))\in\mathcal{D}'(\mathbf{S}^2)$ as its restriction to the sphere. From theorems on the wave front set of the Fourier transform of a homogeneous distribution ([11, Theorems 8.1.8, 8.4.18]), it immediately results that the (analytic) singular support of $E_a$ is the dual (see [1, p. 154]) of $X_a$, i.e.,

$$\operatorname{sing\,supp} E_a = \operatorname{sing\,supp}_{\mathrm{A}} E_a = \{t\nabla P_a(\xi):\xi\in\mathbf{R}^3,\ P_a(\xi)=0,\ t\in\mathbf{R}\}.$$

By the classical Plücker formulas (cf. [9, p. 280]), $[\operatorname{sing\,supp} E_a\setminus\{0\}]$ is an algebraic curve of degree 6. Its complexification has nine cusps, three of which are real in correspondence with the three flexes of $X_a$ (cf. Figure 2). Explicitly, we have $\operatorname{sing\,supp} E_a=\{x\in\mathbf{R}^3: A_a(x)=0\}$, where

$$\begin{aligned} A_a(x) := {} & 3a(a^3+4)x_1^2x_2^2x_3^2+4(a^3+1)(x_1^3x_2^3+x_1^3x_3^3+x_2^3x_3^3) \\ & +6a^2x_1x_2x_3(x_1^3+x_2^3+x_3^3)-(x_1^3+x_2^3+x_3^3)^2. \end{aligned} \tag{1}$$

If $a<-1$, then $P_a$ is hyperbolic with respect to $(1,1,1)$, and

$$W_a := \{x\in\mathbf{R}^3: A_a(x)=0,\ x_1+x_2+x_3\geqslant 0\} \quad (a<-1) \tag{$2_1$}$$

consists of *two* conical surfaces which are the respective duals of the two components of $X_a$. Let $F_a$ denote the unique fundamental solution of $P_a(\partial)$ with support in $\{x\in\mathbf{R}^3: x_1+x_2+x_3\geqslant 0\}$. Then $E_a=\frac{1}{2}(F_a-\check{F}_a)$, where the superscript $\check{}$ indicates reflection with

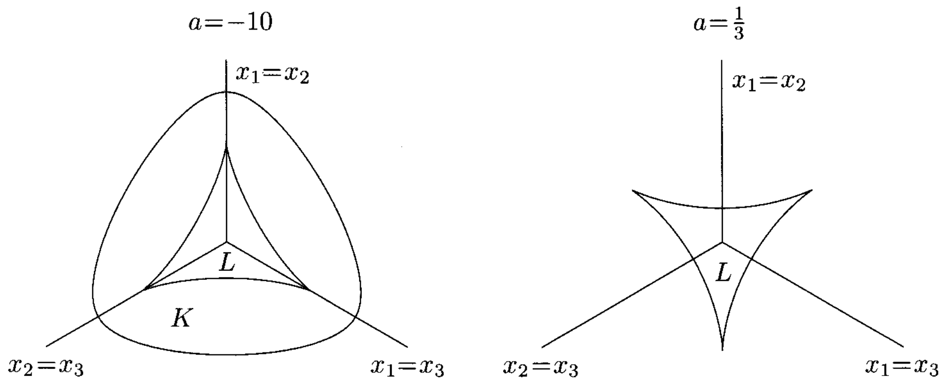


Fig. 2. $\{x \in W_a : x_1+x_2+x_3=1\}$ for $a=-10$ and for $a=\frac{1}{3}$.

respect to the origin. Further, we denote by $K_a$ the propagation cone of $P_a$ with respect to $(1,1,1)$, i.e.,

$$\begin{aligned} K_a &:= \text{dual cone of the component of } (1,1,1) \text{ in } \{x \in \mathbf{R}^3 : P_a(x) \neq 0\} \\ &= \text{convex hull of } W_a. \end{aligned} \tag{3}$$

From the Herglotz–Petrovsky–Leray formula (cf. [1, 7.16, p. 173]), we infer that $F_a$ has a Petrovsky lacuna (in the sense of [1, p. 185]) inside the cone

$$L_a := \{x \in K_a : A_a(x) > 0\} \quad (a < -1). \tag{4$_1$}$$

Hence $W_a$ consists of $\partial K_a$ and of $\partial L_a$, which bound a convex and a non-convex cone, respectively (cf. Figure 2).

If $a > -1$, then still $E_a$ has lacunas inside $L_a$ and $-L_a$, where now we define

$$L_a := \text{component of } (1,1,1) \text{ in } \{x \in \mathbf{R}^3 : A_a(x) > 0\} \quad (a > -1) \tag{4$_2$}$$

and

$$W_a := \partial L_a \quad (a > -1). \tag{2$_2$}$$

In both cases, the fundamental solutions $E_a$ are constant inside $L_a$ and $-L_a$, and we represent these constant values as *complete* elliptic integrals of the first kind. Finally, we show in §2 that $E_a$ is continuous outside the origin.

1.3. In §3, we shall derive an explicit representation for $E_a(x)$ by elliptic integrals of the first kind. Following N. Zeilon, we introduce first one of the complex zeros of the rational integrand in the Herglotz–Petrovsky–Leray formula as a new variable, and, using a substitution (also indicated by N. Zeilon already), we transform the resulting integral into Weierstrass' canonical form. Then we use the addition theorem for the $\wp$-function and the qualitative information from §2 in order to find a real-valued representation of $E_a$ symmetric in the variables $x_1, x_2, x_3$. The final result is contained in the following theorem. ($Y$ denotes Heaviside's function and $\mathcal{F}$ the Fourier transform, cf. 1.4.)

THEOREM. *Let* $a \in \mathbf{R} \setminus \{-1\}$. *The limit*

$$T_a := \lim_{\varepsilon \searrow 0} \frac{Y(|\xi_1^3+\xi_2^3+\xi_3^3+3a\xi_1\xi_2\xi_3|-\varepsilon)}{\xi_1^3+\xi_2^3+\xi_3^3+3a\xi_1\xi_2\xi_3}$$

*defines a distribution in* $\mathcal{S}'(\mathbf{R}^3)$. *If* $E_a := (i/2\pi)^3 \mathcal{F} T_a$, *and* $A_a, W_a, L_a$ *and, for* $a<-1$, $K_a$ *are as in* (1), (2$_1$), (2$_2$), (4$_1$), (4$_2$), (3), *respectively, then*:

(a) $E_a$ *is a fundamental solution of* $\partial_1^3+\partial_2^3+\partial_3^3+3a\partial_1\partial_2\partial_3$;

(b) $E_a$ *is homogeneous of degree* 0;

(c) $E_a$ *is odd and invariant under permutations of the co-ordinates*;

(d) $\operatorname{sing\,supp} E_a = \operatorname{sing\,supp}_{\mathrm{A}} E_a = W_a \cup -W_a$;

(e) $E_a$ *is continuous in* $\mathbf{R}^3 \setminus \{0\}$;

(f) *If* $a<-1$, *then* $E_a = \frac{1}{2}(F_a - \check{F}_a)$, $P_a(\partial)F_a = \delta$, $\operatorname{supp} F_a = K_a$;

(g) $E_a$ *is constant in* $L_a$ *and in* $-L_a$, *and the values* $E_a|_{L_a}$ *are given by the following complete elliptic integrals of the first kind*:

$$E_a|_{L_a} = \begin{cases} -\dfrac{1}{4\sqrt{3}\,\pi} \displaystyle\int_\varrho^\infty \frac{du}{\sqrt{p_a(u)}}, & a>-1, \\[2ex] -\dfrac{1}{4\sqrt{3}\,\pi} \displaystyle\int_{-\infty}^\varrho \frac{2\,du}{\sqrt{p_a(u)}}, & a<-1, \end{cases}$$

*where* $p_a(u) := 4(a^3+1)u^3+9a^2u^2+6au+1$ *and* $\varrho$ *is the smallest real root of* $p_a(u)$;

(h) *Let* $x \in U_a$, *where* $U_a := \mathbf{R}^3 \setminus (\bar{L}_a \cup -\bar{L}_a)$ *if* $a>-1$, *and* $U_a := K_a \setminus (L_a \cup W_a)$ *if* $a<-1$, *and denote by* $z(x)$ *the only simple real root or, if* $x$ *belongs to one of the co-ordinate axes, the triple root* 0, *respectively, of the cubic equation*

$$\begin{aligned} Q_a(x,z) := {} & A_a(x)z^3 + 9(ax_1^2+x_2x_3)(ax_2^2+x_1x_3)(ax_3^2+x_1x_2)z^2 \\ & + [9a^2x_1^2x_2^2x_3^2 + 6a(x_1^3x_2^3+x_1^3x_3^3+x_2^3x_3^3) + 3x_1x_2x_3(x_1^3+x_2^3+x_3^3)]z \\ & + 3ax_1^2x_2^2x_3^2 + x_1^3x_2^3 + x_1^3x_3^3 + x_2^3x_3^3 = 0. \end{aligned} \tag{5}$$

*Then* $z$ *is a real-analytic function in* $U_a$, *and*

$$E_a(x) = \tfrac{1}{2}Y(-1-a)E_a|_{L_a} + \frac{\operatorname{sign}(\widetilde{P}_a(x))}{4\sqrt{3}\,\pi} \int_\varrho^{z(x)} \frac{du}{\sqrt{p_a(u)}}$$

*where* $\widetilde{P}_a(x) := 3[(a^3-2)\varrho + a^2]x_1x_2x_3 - (3a\varrho+1)(x_1^3+x_2^3+x_3^3)$.

*Remark.* Before proceeding, let us comment on some of the properties of the polynomial $Q_a$, which, outside the lacunas, yields the level sets of $E_a$.

First, if $q_i$ denote the coefficients of $Q_a$ with respect to $z$, i.e.,

$$Q_a(x,z)=\sum_{i=0}^{3} q_i(a,x)z^i,$$

then

$$q_3=A_a, \quad q_2=\frac{3}{4}\cdot\frac{\partial q_3}{\partial a}, \quad q_1=\frac{1}{3}\cdot\frac{\partial q_2}{\partial a}, \quad q_0=\frac{1}{6}\cdot\frac{\partial q_1}{\partial a}.$$

Second, let us investigate the relation between $Q_a$ and $p_a, \widetilde{P}_a$. We note that all $q_i$ belong to the four-dimensional subspace $V$ spanned by

$$B_1(x)=x_1^2x_2^2x_3^2, \qquad B_2(x)=x_1^3x_2^3+x_1^3x_3^3+x_2^3x_3^3,$$
$$B_3(x)=x_1x_2x_3(x_1^3+x_2^3+x_3^3), \quad B_4(x)=(x_1^3+x_2^3+x_3^3)^2$$

in the complex vector space of all symmetric polynomials in $x_1, x_2, x_3$ of degree six. The closure $C_a$ of $\{[Q_a(x,z)]:z\in\mathbf{C}\}$ in $\mathbf{P}(V)$ is a cubic curve: $Q_a(x,z)=\sum_{i=1}^4\beta_i(z)B_i(x)$ with

$$\beta_1(z)=3a(a^3+4)z^3+9(a^3+1)z^2+9a^2z+3a,$$
$$\beta_2(z)=4(a^3+1)z^3+9a^2z^2+6az+1,$$
$$\beta_3(z)=6a^2z^3+9az+3z,$$
$$\beta_4(z)=-z^3.$$

The square polynomials make up a quadric curve $S$ in $\mathbf{P}(V)$, namely

$$S=\text{closure of } \{[P_z(x)^2]:z\in\mathbf{C}\}=\left\{\left[\sum_{i=1}^{4}\alpha_iB_i(x)\right]:\alpha_2=0,\, \alpha_3^2-4\alpha_1\alpha_4=0\right\}.$$

The curves $C_a$ and $S$ meet at $[Q_a(x,z)]$ for those $z$ for which $p_a(z)=0$, since $\beta_2=p_a$ and

$$\beta_3^2-4\beta_1\beta_4=(12az^3+9z^2)\beta_2.$$

The polynomial $\widetilde{P}_a(x)$ fulfills $\widetilde{P}_a(x)^2=4(a^3+1)Q_a(x,\varrho)$, and hence $[\widetilde{P}_a(x)^2]$ is just one of the three intersection points of $C_a$ and $S$.

For a discussion of the zeros of $Q_a(x,z)$ with respect to $z$, we refer to 3.4.

1.4. Let us establish some notations. We consider $\mathbf{R}^n$ as a Euclidean space with the inner product $x\cdot y:=x_1y_1+\ldots+x_ny_n$ and write $|x|:=\sqrt{x\cdot x}$. $\mathbf{S}^{n-1}$ denotes the unit sphere $\{\omega\in\mathbf{R}^n : |\omega|=1\}$ in $\mathbf{R}^n$ and $d\sigma$ the Euclidean measure on $\mathbf{S}^{n-1}$. We write $\mathbf{P}(V)$ for the projective space corresponding to the vector space $V$ (over $\mathbf{R}$ or $\mathbf{C}$, respectively), and $[\zeta]\in\mathbf{P}(V)$ for the projective point corresponding to $\zeta\in V\setminus\{0\}$. By

$$\oint$$

we denote the Cauchy principal value.

When we make use of the theory of distributions, we adopt the notations from [11], [13], [18]. In particular, the Heaviside function is abbreviated by $Y$, i.e., $Y(t)=1$ for $t>0$ and 0 else, and $\langle\varphi, T\rangle$ stands for the value of the distribution $T$ on the test function $\varphi$. We use the Fourier transform $\mathcal{F}$ in the form

$$(\mathcal{F}\varphi)(x) := \int \exp(-ix\cdot\xi)\,\varphi(\xi)\,d\xi, \quad \varphi\in\mathcal{S}(\mathbf{R}^n).$$

## 2. Singular support and lacunas of $E_a$

2.1. Let us repeat first some elements from [19, §2]. If $P$ is a *real-valued, homogeneous* polynomial *of principal type* in $n$ variables and of degree $m$, then $\Phi:=\mathrm{vp}(1/P(\omega))\in\mathcal{D}'(\mathbf{S}^{n-1})$ defined by

$$\left\langle \varphi, \mathrm{vp}\,\frac{1}{P(\omega)}\right\rangle := \lim_{\varepsilon\searrow 0}\int_{|P(\omega)|>\varepsilon}\frac{\varphi(\omega)}{P(\omega)}\,d\sigma(\omega), \quad \varphi\in\mathcal{D}(\mathbf{S}^{n-1}),$$

solves the division problem $P(\omega)\cdot\Phi=1$ on the sphere $\mathbf{S}^{n-1}$, and

$$T := \operatorname*{Pf}_{\lambda=-m}\left[\Phi\left(\frac{\xi}{|\xi|}\right)|\xi|^{\lambda}\right]\in\mathcal{S}'(\mathbf{R}^n)$$

solves the division problem $P(\xi)\cdot T=1$ in $\mathbf{R}^n$. Hence $E:=(i^m/(2\pi)^n)\mathcal{F}T$ is a fundamental solution of $P(\partial)$. Theorems 8.1.8, 8.4.18 in [11] yield the representation

$$W := \operatorname{sing\,supp} E = \operatorname{sing\,supp}_{\mathrm{A}} E = \{t\nabla P(\xi) : \xi\in\mathbf{R}^n,\ P(\xi)=0,\ t\in\mathbf{R}\}$$

for the singular support of $E$.

2.2. Let us prove next, similarly as in [1], that, for $P$ as above and *odd* $n$, the Petrovsky condition on lacunas is valid (cf. [1, 10.3, p. 185]). First, radial integration in the Fourier integral for $E$ yields Borovikov's formulas (cf. [2, (5г), (5в), p. 204; English transl., (5c), (5d), p. 16], [7, Chapter I, 6.2, (5), (6), p. 129]):

$$\langle\varphi, E\rangle = \begin{cases} \dfrac{(-1)^{(n-1)/2}}{4(2\pi)^{n-1}(m-n)!}\left\langle \displaystyle\int \varphi(x)(\omega\cdot x)^{m-n}\operatorname{sign}(\omega\cdot x)\,dx, \Phi(\omega)\right\rangle, & m\geqslant n,\\[2ex] \dfrac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}}\langle\langle\varphi(x), \delta^{(n-m-1)}(\omega\cdot x)\rangle, \Phi(\omega)\rangle, & m<n, \end{cases} \tag{6}$$

where $\varphi\in\mathcal{S}(\mathbf{R}^n)$ and $\Phi:=\mathrm{vp}(1/P(\omega))$ as in 2.1.

Let $x\in\mathbf{R}^n\setminus W$ and set

$$v_x(\xi):=\nabla P(\xi)-\frac{x}{|x|^2}(x\cdot\nabla P(\xi)),\quad \xi\in\mathbf{R}^n.$$

If $\varepsilon$ is a small positive number, then $P(\omega\pm i\varepsilon v_x(\omega))\neq 0$ for all $\omega\in\mathbf{S}^{n-1}$ (since $x\notin W$), and

$$\Phi=\frac{1}{2}\lim_{\varepsilon\to 0}\left(\frac{1}{P(\omega)+i\varepsilon}+\frac{1}{P(\omega)-i\varepsilon}\right)=\frac{1}{2}\lim_{\varepsilon\to 0}\left(\frac{1}{P(\omega+i\varepsilon v_x(\omega))}+\frac{1}{P(\omega-i\varepsilon v_x(\omega))}\right).$$

If $P$ is hyperbolic in the direction $\theta$ and if $F$ denotes the unique fundamental solution of $P(\partial)$ with support in $\{x\in\mathbf{R}^n:\theta\cdot x\geqslant 0\}$, then

$$F=\frac{i^m}{(2\pi)^n}\mathcal{F}\left(\operatorname*{Pf}_{\lambda=-m}\left[\Psi\left(\frac{\xi}{|\xi|}\right)|\xi|^\lambda\right]\right)$$

with $\Psi=\lim_{\varepsilon\searrow 0}1/P(\omega+i\varepsilon\theta)\in\mathcal{D}'(\mathbf{S}^{n-1})$ (cf. [12, Theorem 12.5.1, p. 120], [16, Proposition 1, p. 530]), and hence $\Phi=\frac{1}{2}(\Psi+(-1)^m\check{\Psi})$ and $E=\frac{1}{2}(F+(-1)^m\check{F})$.

On the other hand, for arbitrary $P$ as in 2.1 and for a multi-index $\nu\in\mathbf{N}_0^n$ which satisfies $n-m+|\nu|>0$, we obtain from (6), by differentiation,

$$\partial^\nu E(x)=\frac{(-1)^{(n-1)/2}}{4(2\pi)^{n-1}}\lim_{\varepsilon\to 0}\sum_{\pm}\left\langle\frac{\omega^\nu}{P(\omega\pm i\varepsilon v_x(\omega))},\delta^{(n-m+|\nu|-1)}(\omega\cdot x)\right\rangle.$$

Note that the two limits $\lim_{\varepsilon\searrow 0}P(\omega\pm i\varepsilon v_x(\omega))^{-k}$ exist in $\mathcal{D}'(\mathbf{S}_x^{n-2})$ if

$$\mathbf{S}_x^{n-2}:=\{\omega\in\mathbf{S}^{n-1}:\omega\cdot x=0\}$$

and $k\in\mathbf{N}$ (cf. [1, p. 121]). Therefore,

$$\begin{aligned}\partial^\nu E(x)&=\frac{(-1)^{(n-1)/2}}{4(2\pi)^{n-1}}\lim_{\varepsilon\to 0}\sum_{\pm}\int_{\mathbf{S}_x^{n-2}}\left(-\frac{x}{|x|^2}\cdot\nabla_\omega\right)^{n-m+|\nu|-1}\left(\frac{\omega^\nu}{P(\omega\pm i\varepsilon v_x(\omega))}\right)\frac{d\sigma_x(\omega)}{|x|}\\&=\frac{(-1)^{(n-1)/2}}{4(2\pi)^{n-1}}\lim_{\varepsilon\to 0}\sum_{\pm}\int_{\mathbf{S}_x^{n-2}}\left(-\frac{x}{|x|^2}\cdot\nabla_\zeta\right)^{n-m+|\nu|-1}\left(\frac{\zeta^\nu}{P(\zeta)}\right)\bigg|_{\zeta=\zeta_\pm(\omega)}\frac{d\sigma_x(\omega)}{|x|}\end{aligned}$$

where $\zeta_\pm(\omega)=\omega\pm i\varepsilon v_x(\omega)$ and $d\sigma_x$ is the surface measure on $\mathbf{S}_x^{n-2}$. Let $\eta_x(\zeta)$ be the Leray form on $\{\zeta\in\mathbf{C}^n:\zeta\cdot x=0\}$, i.e.,

$$d(\zeta\cdot x)\wedge\tilde{\eta}_x(\zeta)=\sum_{j=1}^n(-1)^{j-1}\zeta_j\,d\zeta_1\wedge\ldots\wedge d\zeta_{j-1}\wedge d\zeta_{j+1}\wedge\ldots\wedge d\zeta_n+O(\zeta\cdot x),$$

$\eta_x(\zeta)$ being the restriction of $\tilde{\eta}_x(\zeta)$ from $\mathbf{C}^n$ to $\{\zeta \in \mathbf{C}^n : \zeta \cdot x = 0\}$ and $O(\zeta \cdot x) \to 0$ for $\zeta \cdot x \to 0$, and put

$$\psi_{x,\nu}(\zeta) := \left(-\frac{x}{|x|^2} \cdot \nabla_\zeta\right)^{n-m+|\nu|-1} \left(\frac{\zeta^\nu}{P(\zeta)}\right) \eta_x(\zeta).$$

Then $\psi_{x,\nu}$ induces a holomorphic (and hence closed) $(n-2)$-form on $U_x := \{[\zeta] \in \mathbf{P}(\mathbf{C}^n) : \zeta \cdot x = 0,\ P(\zeta) \neq 0\}$, which we denote by $[\psi_{x,\nu}]$, and

$$\partial^\nu E(x) = \frac{(-1)^{(n-1)/2}}{4(2\pi)^{n-1}} \int_{c_x} [\psi_{x,\nu}], \tag{7}$$

where $c_x$ is the homology class of the $(n-2)$-chain $s_{x,\varepsilon} + \overline{s_{x,\varepsilon}}$, the cycle $s_{x,\varepsilon}$ being given by

$$s_{x,\varepsilon} : \mathbf{S}_x^{n-2} \to U_x, \quad \omega \mapsto [\omega + i\varepsilon v_x(\omega)],$$

and $\varepsilon$ is small. (We choose $\eta_x$ as orientation on $\mathbf{S}_x^{n-2}$.) Essentially, the representation in formula (7) is equivalent to [1, (7.17′), p. 173] (cf. also the proof on p. 176) or to [12, (12.6.10)‴, p. 131] for hyperbolic operators. Due to (7), $E$ coincides with a polynomial of the degree $m-n$ in those components of $\mathbf{R}^n \setminus W$ which contain a point $x$ with vanishing $c_x$ in the homology group $H_{n-2}(U_x)$. This is precisely the Petrovsky condition for lacunas.

2.3. We apply the foregoing discussion to $P_a(\xi) := \xi_1^3 + \xi_2^3 + \xi_3^3 + 3a\xi_1\xi_2\xi_3$, $a \in \mathbf{R} \setminus \{-1\}$. In this case, $\Phi$ is odd and thus $\lambda \mapsto \Phi(\xi/|\xi|)|\xi|^\lambda$ is analytic in $\lambda = -3$. Hence $T_a := \Phi(\xi/|\xi|)|\xi|^{-3}$ and $E_a := (i/2\pi)^3 \mathcal{F} T_a$ are also odd and homogeneous of the degrees $-3$ and 0, respectively. As in [19, 2.2], we obtain $T_a = \lim_{\varepsilon \searrow 0} Y(|P_a(\xi)| - \varepsilon)/P_a(\xi)$.

For $x = (1,1,1)$, all the three zeros of $P_a$ in $\{[\zeta] \in \mathbf{P}(\mathbf{C}^3) : \zeta \cdot x = 0\}$ are real. In fact, they are given by $[-1,0,1]$, $[0,1,-1]$, $[1,-1,0]$. Moreover, $s_{x,\varepsilon}$ and $\overline{s_{x,\varepsilon}}$ coincide since $v_x(-\omega) = v_x(\omega)$. Hence $c_x$ in 2.2 vanishes (cf. [1, (6.26), p. 167] and Figure 3), and $E_a$ is constant in the two components of $\mathbf{R}^3 \setminus W_a$ containing $(1,1,1)$ and $-(1,1,1)$, respectively, i.e., in $L_a$ and in $-L_a$. Of course, in the hyperbolic case $a < -1$, moreover $E_a$ vanishes in $\mathbf{R}^3 \setminus (K_a \cup -K_a)$, the so-called trivial lacuna (cf. [1, p. 115]).

In order to obtain an equation for the wave front surface $W_a$, we take into account that $W_a$ is the set of $x$ where the two equations $\xi \cdot x = 0$, $P_a(\xi) = 0$ have multiple solutions $[\xi] \in P(\mathbf{R}^3)$. Hence $W_a$ is the zero set of the discriminant of the polynomial $P_a(u, -(ux_1 + x_3)/x_2, 1)$ with respect to $u$. This discriminant is $27A_a(x)/x_2^6$ with $A_a$ as in (1).

2.4. Let us calculate next the constant values $E_a|_{L_a}$. Upon application of some obvious estimates and of Lebesgue's dominated convergence theorem (cf. [19, 2.2]), formula (6) implies that $E_a$ is a locally integrable function given by

$$E_a(x) = -\frac{1}{8\pi^2} \lim_{\varepsilon \searrow 0} \int_{|P_a(u,v,1)| > \varepsilon} \frac{\operatorname{sign}(ux_1 + vx_2 + x_3)}{P_a(u,v,1)}\, du\, dv. \tag{8}$$

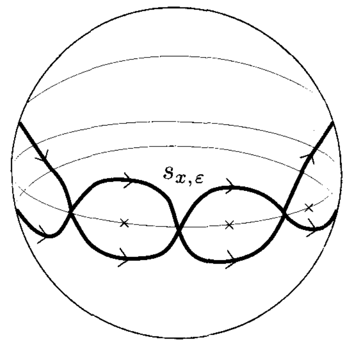


Fig. 3. The path $s_{x,\varepsilon}$ in $\{[\zeta]\in\mathbf{P}(\mathbf{C}^3):\zeta\cdot x=0\}$ for $x=(1,1,1)$. (Each $\times$ denotes one of the three real zeros of $P_a$ in the complex projective line considered.)

Employing the substitution $w=u+v$, we infer

$$E_a(1,1,1)=-\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\operatorname{sign}(w+1)\,dw\oint_{-\infty}^{\infty}\frac{du}{3u^2(w-a)-3uw(w-a)+w^3+1}.$$

The quadratic polynomial of $u$ in the last integral has no real zeros if and only if $(w-a)(w^3+3aw^2+4)$ is positive, and the inner integral yields

$$\frac{2\pi}{\sqrt{3}}\cdot\frac{\operatorname{sign}(w-a)}{\sqrt{(w-a)(w^3+3aw^2+4)}}$$

in this case and 0 else, i.e.,

$$E_a(1,1,1)=-\frac{1}{4\sqrt{3}\,\pi}\int_{-\infty}^{\infty}\frac{\operatorname{sign}((w+1)(w-a))\,dw}{[(w-a)(w^3+3aw^2+4)]_+^{1/2}},$$

where $x_+:=Y(x)x$ for $x\in\mathbf{R}$. With $p_a(u):=4(a^3+1)u^3+9a^2u^2+6au+1$, the substitution $u=1/(w-a)$ furnishes

$$E_a(1,1,1)=-\frac{1}{4\sqrt{3}\,\pi}\int_{-\infty}^{\infty}\operatorname{sign}(1+u(a+1))\,p_a(u)_+^{-1/2}\,du.$$

The discriminant of $p_a$ is $-2^4\cdot 3^3\cdot(a^3+1)$, and hence $p_a$ has one or three real roots according to the sign of $a+1$.

If $a>-1$, then the only real root $\varrho$ of $p_a$ satisfies $-1/(a+1)<\varrho<0$, since $p_a(0)=1$ and $p_a(-1/(a+1))=-3/(a+1)^2$. Thus $1+u(a+1)>0$ if $p_a(u)>0$ and

$$E_a|_{L_a}=-\frac{1}{4\sqrt{3}\,\pi}\int_{\varrho}^{\infty}\frac{du}{\sqrt{p_a(u)}}\qquad(a>-1).$$

If $a<-1$, then $p_a$ has three real roots, say $\varrho<\sigma<\tau$. From $p_a(0)>0$, $p_a'(0)<0$, $p_a''(0)>0$, and $p_a(-1/(a+1))<0$, $p_a'(-1/(a+1))=-6(a-2)/(a+1)<0$, $p_a''(-1/(a+1))=-6(a-2)^2<0$, we conclude that $0<\varrho<\sigma<\tau<-1/(a+1)$, and thus again $1+u(a+1)>0$ if $p_a(u)>0$. By [10, 222.2b], this implies

$$E_a|_{L_a}=-\frac{1}{4\sqrt{3}\,\pi}\left[\int_{-\infty}^{\varrho}+\int_{\sigma}^{\tau}\right]\frac{du}{\sqrt{p_a(u)}}=-\frac{1}{4\sqrt{3}\,\pi}\int_{-\infty}^{\varrho}\frac{2\,du}{\sqrt{p_a(u)}}\qquad(a<-1).$$

2.5. Let us finally show in this section that $E_a$ is continuous outside the origin. From formula (8) we infer (substituting $w=u+v$ for $v$ as in 2.4)

$$E_a(x)=-\frac{1}{8\pi^2}\int_{-\infty}^{\infty}dw\oint_{-\infty}^{\infty}\frac{\operatorname{sign}(u(x_1-x_2)+wx_2+x_3)\,du}{3u^2(w-a)-3uw(w-a)+w^3+1}.$$

For real values $\alpha,\beta,\gamma,\delta$, [10, 131.3] yields

$$\oint_{\delta}^{\infty}\frac{du}{\alpha u^2+2\beta u+\gamma}=\begin{cases}\dfrac{1}{\sqrt{|\beta^2-\alpha\gamma|}}\cdot\dfrac{1}{2}\ln\left|\dfrac{\beta+\alpha\delta+\sqrt{\beta^2-\alpha\gamma}}{\beta+\alpha\delta-\sqrt{\beta^2-\alpha\gamma}}\right|, & \alpha\gamma<\beta^2,\\[2ex] \dfrac{\operatorname{sign}\alpha}{\sqrt{|\beta^2-\alpha\gamma|}}\cdot\operatorname{arccot}\left(\dfrac{\beta\operatorname{sign}(\alpha)+\alpha\delta}{\sqrt{\alpha\gamma-\beta^2}}\right), & \alpha\gamma>\beta^2.\end{cases}$$

In our case, $\beta^2-\alpha\gamma=-\frac{3}{4}(w-a)(w^3+3aw^2+4)$ and $\delta=(wx_2+x_3)/(x_2-x_1)$. If $x_1\neq x_2$, then Lebesgue's dominated convergence theorem can be applied in order to show that $E_a$ is continuous in $x$. Since $(1,1,1)\notin W_a$, we conclude, by the symmetry of $E_a$ with respect to the co-ordinates $x_1,x_2,x_3$, that $E_a$ is continuous in $\mathbf{R}^3\setminus\{0\}$.

Let us note, by the way, that, for $a=-1$, $P_a$ decomposes:

$$P_{-1}(\xi)=(\xi_1+\xi_2+\xi_3)(\xi_1^2+\xi_2^2+\xi_3^2-\xi_1\xi_2-\xi_1\xi_3-\xi_2\xi_3).$$

From this factorization, one can see that $\lim_{\varepsilon\searrow 0}Y(|P_{-1}(\omega)|-\varepsilon)/P_{-1}(\omega)$ diverges in $\mathcal{D}'(\mathbf{S}^2)$, and hence $E_{-1}$ is not defined. But it is easy to check that

$$\frac{\operatorname{sign}(x_1+x_2+x_3)}{12\sqrt{3}\,\pi}\ln(x_1^2+x_2^2+x_3^2-x_1x_2-x_1x_3-x_2x_3)$$

is a fundamental solution of $P_{-1}(\partial)$.

## 3. Representation of $E_a$ by elliptic integrals

3.1. Let us consider now formula (7) in the case of $P=P_a$ and $x\in K_a\setminus L_a$. Then $P_a$ has two complex conjugate zeros in $\{[\zeta]\in\mathbf{P}(\mathbf{C}^3):\zeta\cdot x=0\}$, say $p,\bar{p}$. The residue theorem implies

$$\begin{aligned}\partial_j E_a(x) &= -\frac{1}{16\pi^2}\int_{c_x}[\psi_{x,j}]\\ &= \pm\frac{i}{4\pi}\big(\operatorname*{Res}_p[\psi_{x,j}] - \operatorname*{Res}_{\bar p}[\psi_{x,j}]\big) = \pm\frac{1}{2\pi}\operatorname{Im}\big(\operatorname*{Res}_p[\psi_{x,j}]\big),\end{aligned}$$

where $\psi_{x,j} := \zeta_j \eta_x(\zeta)/P_a(\zeta)$. Using $\zeta_1$ as variable on $\{\zeta\in\mathbf{C}^3 : \zeta\cdot x = 0, \zeta_3 = 1\}$ yields successively

$$\begin{aligned}\tilde\eta_x(\zeta) &= \frac{1}{|x|^2}\det(\zeta, x, d\zeta),\\ \eta_x(\zeta)|_{\zeta_3=1} &= \frac{1}{|x|^2}\det\left(\zeta, x, \begin{pmatrix} d\zeta_1\\ -x_1\,d\zeta_1/x_2\\ 0\end{pmatrix}\right) = -\frac{d\zeta_1}{x_2},\\ \operatorname*{Res}_p[\psi_{x,3}] &= \operatorname*{Res}_{y_1}(\psi_{x,3}|_{\zeta_3=1}) = -\frac{1}{x_2 R'(y_1)},\end{aligned}$$

where $R(u) := P_a(u, -(ux_1+\lambda)/x_2, 1)$ and $p = [y_1, y_2, 1]$. Next we substitute $\lambda$ by $y_1$ (cf. [20, p. 16]) in the integral $E_a(x) = \int^{x_3}(\partial_3 E_a)(x_1, x_2, \lambda)\,d\lambda$. From

$$\begin{aligned}P_a\Big(y_1, \frac{-(y_1x_1+\lambda)}{x_2}, 1\Big) = 0 \quad &\Rightarrow \quad \Big(\partial_1 P_a - \frac{x_1}{x_2}\partial_2 P_a\Big)\,dy_1 = \frac{\partial_2 P_a}{x_2}\,d\lambda\\ &\Rightarrow \quad \frac{d\lambda}{R'(y_1)} = \frac{x_2\,dy_1}{(\partial_2 P_a)(y_1, y_2, 1)} = \frac{x_2\,dy_1}{3(y_2^2 + a y_1)}\end{aligned}$$

we infer

$$E_a(x) = \text{constant} \pm \frac{1}{6\pi}\operatorname{Im}\int_{\gamma(x)}\frac{dy_1}{y_2^2 + a y_1},$$

where $\gamma(x)$ is a path in the Riemannian surface $\{(y_1, y_2)\in\mathbf{C}^2 : P_a(y_1, y_2, 1) = 0\}$ ending at the point $(y_1(x), y_2(x))$, which fulfills $y_1(x)x_1 + y_2(x)x_2 + x_3 = 0$, and $\operatorname{Im} y_1(x) > 0$, say.

Let us observe that

$$\Omega := \frac{dy_1}{y_2^2 + a y_1} = \frac{3\,dy_1}{\partial P_a(y_1, y_2, 1)/\partial y_2} = -3\,\text{P.R.}\left(\frac{dy_1\wedge dy_2}{P_a(y_1, y_2, 1)}\right)$$

spans the space $\Omega^1(X_a^c)$ of holomorphic 1-forms on the elliptic curve $X_a^c := \{[\zeta]\in\mathbf{P}(\mathbf{C}^3) : P_a(\zeta) = 0\}$ (with the co-ordinates $y_1 = \zeta_1/\zeta_3$, $y_2 = \zeta_2/\zeta_3$), and that $E_a$ can be expressed more symmetrically as $E_a(x) = \text{constant} \pm (1/6\pi)\int_{[\gamma(x)]}\Omega$. (Here P.R. denotes the Poincaré residue map as in [9, pp. 147, 221].)

The elliptic integral above is transformed into standard form with the help of the substitution $w=(1+y_2)/y_1$ (cf. [20, p. 60]). In fact,

$$\begin{aligned}
y_1^3+y_2^3+1+3ay_1y_2=0 \quad &\Rightarrow \quad (y_1^2+ay_2)\,dy_1+(y_2^2+ay_1)\,dy_2=0\\
&\Rightarrow \quad dw=\frac{dy_2}{y_1}-\frac{(1+y_2)\,dy_1}{y_1^2}\\
&\qquad\quad =-\frac{dy_1}{(y_2^2+ay_1)y_1^2}\,[y_1^3+y_2^3+2ay_1y_2+y_2^2+ay_1]\\
&\qquad\quad =\frac{(1+y_2-ay_1)(1-y_2)}{(y_2^2+ay_1)y_1^2}\,dy_1\\
&\Rightarrow \quad \frac{dy_1}{y_2^2+ay_1}=\frac{y_1^2\,dw}{(1+y_2-ay_1)(1-y_2)}
\end{aligned}$$

and, on the other hand,

$$\begin{aligned}
(w-a)(w^3+3aw^2+4)&=\frac{1+y_2-ay_1}{y_1^4}\,[(1+y_2)^3+3ay_1(1+y_2)^2+4y_1^3]\\
&=\frac{1+y_2-ay_1}{y_1^4}\,[(1+y_2)^3+3ay_1(1+y_2)^2-4(1+y_2^3)-12ay_1y_2]\\
&=-3\,\frac{(1+y_2-ay_1)^2(1-y_2)^2}{y_1^4}.
\end{aligned}$$

Hence we obtain

$$E_a(x)=\text{constant}\pm\frac{1}{2\sqrt{3}\,\pi}\ \mathrm{Re}\int_{\widetilde{\gamma(x)}}\frac{dw}{\sqrt{(w-a)(w^3+3aw^2+4)}}.$$

As in 2.4, we eventually transform this elliptic integral into Weierstrass' form by setting $u=1/(w-a)$. This furnishes

$$\begin{aligned}
E_a(x)&=\text{constant}\pm\frac{1}{2\sqrt{3}\,\pi}\ \mathrm{Re}\int_{u(x)}^{-1/(a+1)}\frac{du}{\sqrt{p_a(u)}}\\
&=\text{constant}\pm\frac{1}{2\sqrt{3}\,\pi}\ \mathrm{Re}\int_{u(x)}^{\infty}\frac{du}{\sqrt{p_a(u)}},
\end{aligned}\tag{10}$$

where $p_a$ is as in (g) of the Theorem, and $u(x)$ is determined by the equations $u(x)=1/(w(x)-a)$, $w(x)=(1+y_2(x))/y_1(x)$, $y_1(x)x_1+y_2(x)x_2+x_3=0$, $P_a(y_1(x),y_2(x),1)=0$, and by the condition $\mathrm{Im}\,y_1(x)>0$.

3.2. In order to obtain an integral representation of $E_a$ over a path on the real axis, let us employ the addition theorem of Weierstrass' $\wp$-function, i.e.,

$$\wp(s+t)=-\wp(s)-\wp(t)+\frac{1}{4}\left(\frac{\wp'(s)-\wp'(t)}{\wp(s)-\wp(t)}\right)^2,\tag{11}$$

cf. [8, 8.166.2]. Since $\wp'(s)=\sqrt{4\wp(s)^3-g_2\wp(s)-g_3}$, we have

$$\left[\int_\sigma^\infty+\int_\tau^\infty\right]\frac{du}{\sqrt{4u^3-g_2u-g_3}}=\int_z^\infty\frac{du}{\sqrt{4u^3-g_2u-g_3}} \tag{12}$$

if $\wp(s)=\sigma$, $\wp(t)=\tau$ and $\wp(s+t)=z$, which, by (11), amounts to

$$z=\frac{(4\sigma\tau-g_2)(\sigma+\tau)-2g_3-2\sqrt{4\sigma^3-g_2\sigma-g_3}\,\sqrt{4\tau^3-g_2\tau-g_3}}{4(\sigma-\tau)^2}.$$

Here we suppose that $\sigma\neq\tau$ are sufficiently large real numbers. By a shift of the integration variable, the following slightly more general addition theorem ensues from (12):

$$\left[\int_\sigma^\infty+\int_\tau^\infty\right]\frac{du}{\sqrt{q(u)}}=\int_z^\infty\frac{du}{\sqrt{q(u)}} \tag{13}$$

where $q(u)=\alpha u^3+\beta u^2+\gamma u+\delta$, $\alpha>0$, $\beta,\gamma,\delta\in\mathbf{R}$, $\sigma,\tau\in\mathbf{C}$, $\operatorname{Re}\sigma,\operatorname{Re}\tau$ are sufficiently large, $\sigma\neq\tau$, and

$$z=\frac{\alpha\sigma\tau(\sigma+\tau)+2\beta\sigma\tau+\gamma(\sigma+\tau)+2\delta-2\sqrt{q(\sigma)q(\tau)}}{\alpha(\sigma-\tau)^2}.$$

We apply formula (13) to (10) with $q=p_a$, $\sigma=u(x)$ and $\tau=\overline{u(x)}$. This yields

$$E_a(x)=\text{constant}\pm\frac{1}{4\sqrt{3}\,\pi}\int_{z(x)}^\infty\frac{du}{\sqrt{p_a(u)}} \tag{14}$$

with

$$\begin{aligned} z(x)&=\frac{4(a^3+1)u\bar{u}(u+\bar{u})+18a^2u\bar{u}+6a(u+\bar{u})+2-2\sqrt{p_a(u)p_a(\bar{u})}}{4(a^3+1)(u-\bar{u})^2}\\ &=\frac{2(a^3+1)(w+\bar{w}-2a)+(w-a)(\bar{w}-a)(w+2a)(\bar{w}+2a)-S}{2(a^3+1)(w-\bar{w})^2} \end{aligned} \tag{15}$$

wherein $u=u(x)$ and $w=w(x)$ are specified at the end of 3.1, and

$$S:=\sqrt{(w-a)(w^3+3aw^2+4)(\bar{w}-a)(\bar{w}^3+3a\bar{w}^2+4)}. \tag{16}$$

3.3. Let us next derive the cubic equation (5) for $z(x)$. Since

$$x_1y_1(x)+x_2y_2(x)+x_3=0 \quad\text{and}\quad P_a(y_1(x),y_2(x),1)=0,$$

$y_1(x)$ is a root of the following cubic polynomial in $u$:

$$x_2^3P_a(u,-(ux_1+x_3)/x_2,1)=x_2^3(u^3+1)-(ux_1+x_3)^3-3ax_2^2u(ux_1+x_3).$$

This implies that

$$w(x)=\frac{1+y_2(x)}{y_1(x)}=-\frac{x_1}{x_2}+\frac{x_2-x_3}{x_2y_1(x)}$$

is a solution of the cubic equation

$$B(w):=(x_2^2+x_2x_3+x_3^2)w^3+3(x_1x_2+x_1x_3-ax_2x_3)w^2 \\ +3x_1(x_1-ax_2-ax_3)w+(x_2-x_3)^2-3ax_1^2=0.$$

Furthermore, $B(w)=0$ implies

$$(w-a)(w^3+3aw^2+4)=-\frac{3(w-a)^2(2x_1+wx_2+wx_3)^2}{(x_2-x_3)^2},$$

and thus, for $w=w(x)$, the square root $S$ defined in (16) fulfills

$$S=3\,\frac{(w-a)(\bar{w}-a)(2x_1+wx_2+wx_3)(2x_1+\bar{w}x_2+\bar{w}x_3)}{(x_2-x_3)^2}. \tag{17}$$

We now consider $B$ as a polynomial over $K:=\mathbf{Q}(a,x_1,x_2,x_3)$, assuming $a,x_1,x_2,x_3$ transcendental over $\mathbf{Q}$. If $L$ is a splitting field of $B$ over $K$, then $B$ has three roots $w_1,w_2,w_3$ in $L$, and $z(x)$ is, according to (15) and (17), a rational function of $w_1,w_2$ say. Although the dimension of $L$ over $K$ is six, $z(x)$ satisfies a *cubic* equation over $\mathbf{Q}(a,x_1,x_2,x_3)$, since $z(x)$ is mapped to itself by that element of the Galois group of $L$ over $K$ which exchanges $w_1$ and $w_2$ (cf. [14]).

In order to determine the cubic equation for $z(x)$ over $K$, we first express $z(x)$ in (15) by $w_3$. Since $w_1,w_2,w_3$ are the roots of $B$, we have

$$w_1+w_2=-w_3-3\,\frac{x_1x_2+x_1x_3-ax_2x_3}{x_2^2+x_2x_3+x_3^2}$$

and

$$w_1w_2=-w_3(w_1+w_2)+3\,\frac{x_1(x_1-ax_2-ax_3)}{x_2^2+x_2x_3+x_3^2}.$$

Inserting these equations into (15) and (17), and making use of $B(w_3)=0$, a symbolic calculation program yields

$$z(x)=\frac{x_2x_3(x_2^2+x_2x_3+x_3^2)w_3-3ax_2^2x_3^2-x_1(x_2+x_3)(x_2-x_3)^2}{4(x_1x_2+ax_3^2)(x_1x_3+ax_2^2)-[(x_2^2+x_2x_3+x_3^2)w_3+x_1x_2+x_1x_3-ax_2x_3]^2}. \tag{18}$$

If $N$ and $D$ denote the numerator and the denominator, respectively, of the quotient in (18), then $z(x)$ is a root of the resultant of $B(w_3)$ and $Dz-N$ with respect to $w_3$, which resultant is $-(x_2^3-x_3^3)^2(x_2^2+x_2x_3+x_3^2)^2Q_a(x,z)$.

3.4. Let us finally verify the representation of $E_a$ announced in (h) of the Theorem. Define

$$U_a := \begin{cases} \mathbf{R}^3 \setminus (\bar{L}_a \cup -\bar{L}_a), & a > -1, \\ K_a \setminus (L_a \cup W_a), & a < -1, \end{cases}$$

and consider $x \in U_a$. The discriminant of $Q_a(x,z)$ with respect to $z$ is

$$27(x_1^3 - x_2^3)^2 (x_1^3 - x_3^3)^2 (x_2^3 - x_3^3)^2 A_a(x),$$

and this is negative in $U_a$ except for the planes $x_1 = x_2$, $x_1 = x_3$ and $x_2 = x_3$. Hence $Q_a(x,z)$ has exactly one *real* root $z(x)$ if $x \in U_a$ and $x$ does not belong to one of these planes. When $x_2 = x_3$, say, then there is a double root $z = -x_2/(x_1 + a x_2)$ (note that $A_a(-a,1,1) = 0$, so $x_1 + a x_2 \neq 0$), and since the discriminant of $\partial Q_a(x,z)/\partial z$ is

$$36(x_1^3 - x_2^3)^2 (x_1 + a x_2)^4 x_2^2,$$

and $(1,1,1)$ belongs to $L_a$, we have precisely one simple zero except on the co-ordinate axes. This simple zero is real-analytic in the whole set $U_a$, for if say $x_1 = 1$ and $x_2 = x_3$ is small, we have just given the double zero explicitly, and it follows that the simple zero is also analytic there. If $\varepsilon$ and $\delta$ are small enough, it follows that the simple zero $z$ can be continued uniquely analytically to $\{(x_2,x_3) \in \mathbf{C}^2 : 0 < |x_3| < \varepsilon, |x_2 - x_3| < \delta\}$, and since it is bounded, it extends analytically also to $x_3 = 0$. Therefore, if $z(x)$ is defined as in the Theorem, then it is a real-analytic function of $x \in U_a$.

If $\varrho$ denotes the smallest real root of $p_a$ (cf. 2.4), then a calculation shows that

$$Q_a(x,\varrho) = \frac{\widetilde{P}_a(x)^2}{4(a^3+1)}, \quad \widetilde{P}_a(x) := 3[(a^3-2)\varrho + a^2] x_1 x_2 x_3 - (3a\varrho + 1)(x_1^3 + x_2^3 + x_3^3).$$

Hence $z(x) = \varrho$ if $\widetilde{P}_a(x) = 0$. Let us investigate $\widetilde{P}_a$. From $(3a\varrho+1)^2 = -4(a^3+1)\varrho^3$, we conclude that $\operatorname{sign}\varrho = -\operatorname{sign}(a+1)$ and that $3a\varrho + 1 \neq 0$ for $a \in \mathbf{R} \setminus \{-1\}$. Hence

$$\widetilde{P}_a = -(3a\varrho+1) P_{\tilde{a}}, \quad \tilde{a} := -\frac{(a^3-2)\varrho + a^2}{3a\varrho + 1},$$

and we have to decide on the sign of $\tilde{a} + 1$. Using two values of $a$, say $a = 0$ and $a \nearrow -1$, and the continuity of $\varrho(a)$, we obtain that $3a\varrho + 1$ is always positive. $b = \tilde{a}$ is a root of the resultant

$$4(a^3+1)(b^3 + 3a^2 b^2 - 3ab + a^3 + 2)$$

of the two polynomials $p_a(u)$ and $(3au+1)b + (a^3-2)u + a^2$ with respect to $u$, and therefore $\tilde{a} \neq -1$ when $a \neq -1$. A test on two values of $a$ as above reveals that $\tilde{a} + 1$ is negative, and hence $X_{\tilde{a}}$ always consists of *two* components.

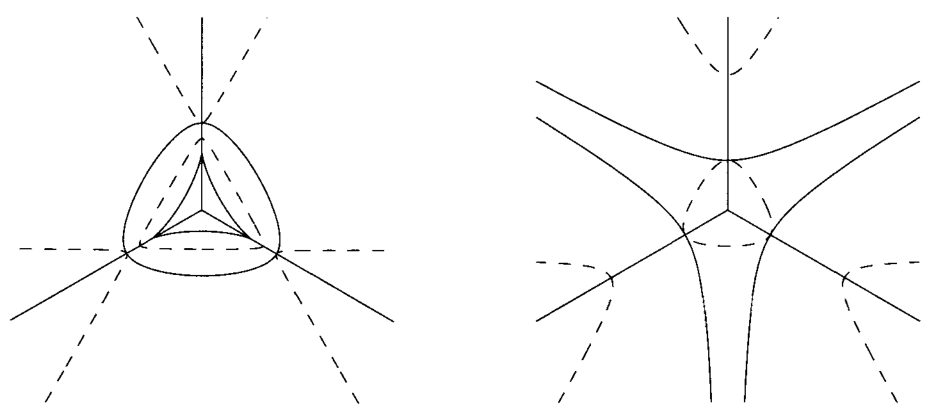


Fig. 4. $W_a$ (solid) and $X_{\tilde{a}}$ (dashed) on the plane $x_1+x_2+x_3=1$ for $a=-10$ and for $a=2$.

The curves $[W_a\backslash\{0\}]$ and $X_b$ intersect on the lines $[t,t,1]$, $t\in\mathbf{R}$, if and only if the resultant of $A_a(t,t,1)$ and $P_b(t,t,1)$ vanishes. This resultant is given by

$$108(b^3+3a^2b^2-3ab+a^3+2)(a^3+3ab-2)^3,$$

and thus $[W_a\backslash\{0\}]$ and $X_{\tilde{a}}$ touch at points on the three projective lines $x_1=x_2$, $x_1=x_3$ and $x_2=x_3$. Using two values of $a$ as above then shows: If $a>-1$, then the convex component of $X_{\tilde{a}}$ lies inside $[L_a\backslash\{0\}]$, and the non-convex component belongs to $[U_a]$; if $a<-1$, then the convex component of $X_{\tilde{a}}$ belongs to $[U_a]$, and the non-convex component lies in $\mathbf{P}(\mathbf{C}^3)\backslash[K_a]$ (cf. Figure 4).

Next let us discuss the behaviour of $z(x)$ for $x$ tending to $\partial U_a$ from inside $U_a$. Evidently, $z(x)\to\pm\infty$, and we can decide on the sign of the limit by noticing that it coincides with the sign of $Q_a(x,0)$ since $A_a(x)<0$ in $U_a$. For $x=(-a,1,1)\in\partial L_a$, $Q_a(x,0)=a^3+1$, and hence

$$z(x)\to\begin{cases}\infty, & a>-1,\\ -\infty, & a<-1,\end{cases}$$

if $x\to\partial L_a$ from inside $U_a$. On the other hand, if $a<-1$ and $x=(0,0,1)$, then $z(x)=0<\varrho$ and $\widetilde{P}_a(x)=-(3a\varrho+1)<0$ (whereas $\widetilde{P}_a(-a,1,1)>0$), and this implies that $z(x)\to-\infty$ if $x\to\partial K_a$ from inside $U_a$, $a<-1$. Hence, for all $x\in U_a$, $z(x)\geqslant\varrho$ if $a>-1$, and $z(x)\leqslant\varrho$ if $a<-1$. From this we conclude that $\int_\varrho^{z(x)}du/\sqrt{p_a(u)}$ is real-analytic in $[U_a]$ except possibly on $X_{\tilde{a}}\cap[U_a]$. A Taylor series argument as in [19, Remark] shows that

$$\operatorname{sign}(\widetilde{P}_a(x))\int_\varrho^{z(x)}\frac{du}{\sqrt{p_a(u)}}$$

is real-analytic on $X_{\tilde{a}} \cap [U_a]$ also.

Combining now the continuity of $E_a$ in $\mathbf{R}^3 \setminus \{0\}$, the values of $E_a|_{L_a}$ calculated in 2.4, the limit behaviour of $z(x)$ on the border of $U_a$ analyzed above, $\lim_{x \to \partial K_a} E_a(x) = 0$ for $a < -1$, and the representation of $E_a$ in (14) with the principle of analytic continuation, furnishes a proof of the assertion (h) of the Theorem.

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