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Area distortion of quasiconformal mappings

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1. Introduction

A homeomorphism $f:\Omega\to\Omega'$ between planar domains Ω and Ω' is called *K-quasiconformal* if it is contained in the Sobolev class $W^1_{2,\mathrm{loc}}(\Omega)$ and its directional derivatives satisfy

$$\max_\alpha |\partial_\alpha f(x)| \leqslant K \min_\alpha |\partial_\alpha f(x)| \quad \text{a.e. } x\in\Omega.$$

In recent years quasiconformal mappings have been an efficient tool in the study of dynamical systems of the complex plane. We show here that, in turn, methods or ideas from dynamical systems can be used to solve a number of open questions in the theory of planar quasiconformal mappings.

It has been known since the work of Ahlfors [A] and Mori [Mo] that K-quasiconformal mappings are locally Hölder continuous with exponent $1/K$. The function

$$f_0(z) = z|z|^{1/K-1} \tag{1}$$

shows that this exponent is the best possible. In addition to distance, quasiconformal mappings distort also the area by a power depending only on K, as shown first by Bojarski [Bj]. Since $|f_0B(r)|=\pi^{1-1/K}|B(r)|^{1/K}$, where $B(r)=\{z\in\mathbf{C}:|z|<r\}$, it is natural to expect that the optimal exponent in area distortion is similarly $1/K$.

In this paper we give a positive answer to this problem and prove the following result which was conjectured and precisely formulated as below by Gehring and Reich [GR]. We shall denote by Δ the open unit disk and by $|E|$ the area of the planar set E.

THEOREM 1.1. *Suppose $f:\Delta\to\Delta$ is a K-quasiconformal mapping with $f(0)=0$. Then we have*

$$|fE| \leqslant M|E|^{1/K} \tag{2}$$

for all Borel measurable sets $E \subset \Delta$. Moreover, the constant $M=M(K)$ depends only on K with $M(K)=1+O(K-1)$.

For the proof of (2) we consider families $\{B_i\}_1^n$ of disjoint disks $B_i=B_i(\lambda)$ which depend holomorphically on the parameter λ (in a sense to be defined later). After an approximation (2) now becomes equivalent to

$$\sum_{i=1}^{n} |B_i(\lambda)| \leqslant C\left(\sum_{i=1}^{n} |B_i(0)|\right)^{(1-|\lambda|)/(1+|\lambda|)}, \tag{3}$$

where C depends only on λ. Furthermore, iterating the configuration one is led to measures on Cantor sets and there we shall apply the Ruelle–Bowen thermodynamic formalism [Bw]; if we write (3) in terms of the topological pressure, then the proof comes out in a transparent way.

The function f_0 is extremal in the distortion of area as well as of distance, and therefore it is natural to ask [I, 9.2] if for quasiconformal mappings the Hölder continuity alone, rather than the dilatation, implies the inequality (2). However, this turns out to be false, as shown recently by P. Koskela [K].

As is well known the optimal control of area distortion answers several questions in this field. For example, in general domains Ω one can interpret (2) in terms of the local integrability of the Jacobian J_f of the quasiconformal mapping f. This leads to a solution of the well known problem [LV], [Ge] on the value of the constant

$$p(K) = \sup\{p : J_f \in L^p_{\text{loc}}(\Omega) \text{ for each } K\text{-quasiconformal } f \text{ on } \Omega\}.$$

COROLLARY 1.2. *In every planar domain Ω, $p(K)=K/(K-1)$.*

In other words, for each K-quasiconformal $f:\Omega\to\Omega'$,

$$f \in W^1_{p,\text{loc}}(\Omega), \quad p < \frac{2K}{K-1}.$$

The example (1) shows that this is false for $p \geqslant 2K/(K-1)$.

Theorem 1.1 governs also the distortion of the Hausdorff dimension $\dim(E)$ of a subset E.

COROLLARY 1.3. *Let $f:\Omega\to\Omega'$ be K-quasiconformal and suppose $E\subset\Omega$ is compact. Then*

$$\dim(fE) \leqslant \frac{2K\dim(E)}{2+(K-1)\dim(E)}. \tag{4}$$

This inequality, as well, is the best possible.

THEOREM 1.4. *For each $0<t<2$ and $K\geqslant 1$ there are a set $E\subset\mathbf{C}$ with $\dim(E)=t$ and a K-quasiconformal mapping f of $\overline{\mathbf{C}}$ such that*

$$\dim(fE)=\frac{2K\dim(E)}{2+(K-1)\dim(E)}.$$

The estimate (4) was suggested by Gehring and Väisälä [GV]. It can also be formulated [IM2] in the symmetric form

$$\frac{1}{K}\left(\frac{1}{\dim(E)}-\frac{1}{2}\right)\leqslant\frac{1}{\dim(fE)}-\frac{1}{2}\leqslant K\left(\frac{1}{\dim(E)}-\frac{1}{2}\right). \tag{5}$$

The results 1.3 and 1.4 are closely related to the removability properties of quasiregular mappings, since in plane domains they can be represented as compositions of analytic functions and quasiconformal mappings. The strongest removability conjecture, due to Iwaniec and Martin [IM1], suggests that sets of Hausdorff d-measure zero, $d=n/(K+1)$, are removable for bounded K-quasiregular mappings in $\mathbf{R}^n$. Here we obtain the following.

COROLLARY 1.5. *In planar domains sets E of Hausdorff dimension*

$$\dim(E)<\frac{2}{K+1}$$

are removable for bounded K-quasiregular mappings.

Conversely, for each $K\geqslant 1$ and $t>2/(K+1)$ there is a t-dimensional set $E\subset\mathbf{C}$ which is not removable for some bounded K-quasiregular mappings.

In addition to [IM1] removability questions have recently been studied for instance in [JV], [KM] and [Ri].

Finally, we mention the applications to the regularity results of quasiregular mappings. Recall that a mapping

$$f\in W^1_{q,\mathrm{loc}}(\Omega),\quad 1<q<2,$$

is said to be *weakly K-quasiregular*, if $J_f\geqslant 0$ almost everywhere and

$$\max_{|h|=1}|Df(x)h|\leqslant K\min_{|h|=1}|Df(x)h|\quad\text{a.e. } x\in\Omega.$$

Then f is K-quasiregular in the usual sense if $f\in W^1_{2,\mathrm{loc}}(\Omega)$, i.e. if J_f is locally integrable.

We can now consider the number $q(K)$, the infimum of the q's such that every weakly K-quasiregular mapping $f\in W^1_{q,\mathrm{loc}}(\Omega)$ is actually K-quasiregular.

COROLLARY 1.6. $q(K)=2K/(K+1)$.

Indeed, Lehto and Virtanen [LV] have proven that the precise estimate on the L^p-integrability, Corollary 1.2, implies that $q(K)\leqslant 2K/(K+1)$. The opposite inequality $q(K)\geqslant 2K/(K+1)$ was shown by Iwaniec and Martin in [IM1].

Quasiconformal mappings are also the homeomorphic solutions of the elliptic differential equations

$$\bar{\partial} f(z)=\mu(z)\partial f(z);$$

here μ is the complex dilatation or the Beltrami coefficient of f with

$$\|\mu\|_\infty=\frac{K-1}{K+1}<1.$$

Hence there are close connections to the singular integral operators and especially to the Beurling operator

$$S\omega(z)=-\frac{1}{\pi}\int_{\mathbf{C}}\frac{\omega(\zeta)\,dm(\zeta)}{(\zeta-z)^2},$$

see [I], [IM1], [IK] for example. In fact, this operator was the main tool in the work of Bojarski [Bj] and Gehring–Reich [GR], cf. also [IM2]. Below we shall use mostly different approaches and the role of the S operator remains implicit. Still the area distortion inequality has a number of implications on the properties of S. In particular, we have

COROLLARY 1.7. *There is a constant $\alpha\geqslant 1$ such that for any measurable set $E\subset\Delta$,*

$$\int_\Delta |S\chi_E|\,dm\leqslant |E|\log\frac{\alpha}{|E|}. \tag{6}$$

It is for this consequence that we must show the asymptotic estimate $M(K)=1+O(K-1)$ and then, actually, Theorem 1.1 is equivalent to 1.7, cf. [GR].(1)

If we consider general functions $\omega\in L^\infty(\Delta)$ then the inequality (6) implies the correct exponential decay for $|\{z\in\Delta:\operatorname{Re}S\omega>t\}|$ when $t\to\infty$. As a consequence, for each $\delta>1$ there is a constant $M_\delta<\infty$ such that

$$\int_\Delta |Sv|\,dm\leqslant\delta\int_\Delta |v|\log\left(1+M_\delta\frac{|v|}{|v|_\Delta}\right)dm,\quad v\in L\log L(\Delta). \tag{7}$$

Here $|v|_\Delta=(1/\pi)\int_\Delta |v|\,dm$ is the integral mean of $|v|$. It is a natural question whether (7) holds at $\delta=1$ as in (6) with characteristic functions. This would also imply the Iwaniec–Martin removability conjecture in the planar case. However, in the last section we show, again by considering the inequalities arising from the thermodynamic formalism, that in fact (7) fails when $\delta=1$.

(1) David Hamilton and Tadeusz Iwaniec have pointed out that now (6) holds with $\alpha=e\pi$.

Further results equivalent to the Gehring–Reich conjecture have been given by Iwaniec and Kosecki [IK]. These include applications to the L^1-theory of analytic functions, quadratic differentials and critical values of harmonic functions. Moreover, by results of Lavrentiev, Bers and others the solutions of the elliptic differential equations $\nabla \cdot A\nabla u=0$ can be interpreted in terms of quasiregular mappings f. Therefore Corollary 1.2 yields sharp exponents of integrability on the gradient ∇u; note that the dilatation of f and so necessarily the optimal integrability exponent depends in a complicated manner on all the entries of the matrix A rather than just on its ellipticity coefficient.

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2. Holomorphic deformations of Cantor sets

Let us first consider a family $\{B_i\}_{i=1}^n$ of nonintersecting subdisks of Δ. We shall study the quasiconformal deformation of such families and, in particular, estimate sums

$$\sum_{i=1}^{n} r(B_i)^t, \quad t \in \mathbf{R}, \tag{8}$$

where $r(B_i)$ denotes the radius of B_i. Looking for the extremal phenomena we can iterate the configuration $\{B_i\}_{i=1}^n$ and are thus led to Cantor sets. There one needs measures μ which reflect in a natural manner the properties of the sums (8). It turns out that such measures can, indeed, be found by using the thermodynamic formalism introduced by Ruelle and Bowen, cf. [Bw], [W].

To describe this in more detail suppose hence that we are given similarities γ_i, $1 \leqslant i \leqslant n$, for which $B_i = \gamma_i \Delta$. Since the γ_i are contractions, there is a unique compact subset J of the unit disk for which

$$\bigcup_{i=1}^{n} \gamma_i(J) = J.$$

Thus J is self-similar in the terminology of Hutchinson [H]. We can also reverse this picture and define the mapping $g: \bigcup_i B_i \to \Delta$ by $g|_{B_i} = \gamma_i^{-1}$. Now

$$J = \bigcap_{k=0}^{\infty} g^{-k} \Delta,$$

and g is a n-to-1 expanding mapping with J completely invariant, $J=gJ=g^{-1}J$. Furthermore, g represents the shift on J; in a natural manner we can identify the point $x\in J$ with the sequence $(j_k)_{k=0}^{\infty}\in\{1,\dots,n\}^{\mathbf{N}}$ by defining $j_k=i$ if $g^k(x)\in B_i$. Then, in this identification, $g\colon(j_k)_{k=0}^{\infty}\mapsto(j_{k+1})_{k=0}^{\infty}$.

In the sequel we use the notation $J=J(g)$ for our Cantor set and say also that it is generated by the similarities γ_i.

Next, let $s=\dim(J(g))$, the Hausdorff dimension of $J(g)$. Then the Hausdorff s-measure is nonzero and finite on $J(g)$ and after a normalization it defines a probability measure μ_s which is invariant under the shift g, i.e. $\mu_s(g^{-1}E)=\mu_s(E)$ for all Borelian $E\subset J(g)$. A general and systematic way to produce further invariant measures is provided by the Ruelle–Bowen formalism: Given a Hölder continuous and real valued function ψ on $J(g)$ there is a unique shift-invariant probability measure $\mu=\mu_\psi$, called the Gibbs measure of ψ, for which the supremum

$$P(\psi)=\sup\left\{h_\mu(g)+\int_{J(g)}\psi\,d\mu:\mu\text{ is }g\text{-invariant}\right\}\tag{9}$$

is attained, see [Bw] or [W]. Here $h_\mu(g)$ denotes the entropy of μ and the quantity $P(\psi)$ is called the *topological pressure* of ψ.

Let us then look for the Gibbs measures that are related to the sums (8). Recall that $s=\dim(J(g))$ is the unique solution of $P(-s\log|g'|)=0$, and this suggest the choices $\psi_t=-t\log|g'|$. It then readily follows from [Bw, Lemma I.1.20], that

$$P(-t\log|g'|)=\log\left(\sum_{i=1}^{n}|\gamma_i'|^t\right)=\log\left(\sum_{i=1}^{n}r(B_i)^t\right).\tag{10}$$

In fact, the functions $\psi=\psi_t$ are in our situation locally constant and therefore it can be shown that the system $g\colon(J(g),\mu_\psi)\to(J(g),\mu_\psi)$ is Bernoulli. In other words, the numbers $p_i=\mu_\psi(J\cap B_i)$ satisfy $\sum_{i=1}^{n}p_i=1$ and on $J(g)\cong\{1,\dots,n\}^{\mathbf{N}}$ μ_ψ is the product measure determined by the probability distribution $\{p_i\}_{i=1}^{n}$ on $\{1,\dots,n\}$. This enables one to make the dynamical approach more elementary, as pointed to us by Alexander Eremenko. We are grateful to him for letting us include this simplification here.

For the readers convenience let us recall the proof of the variational principle, the counterpart of (9), in the elementary setting of product measures. Then also the entropy of $\mu=\mu_\psi$ attains the simple form

$$h_\mu(g)=-\sum_{1}^{n}p_i\log p_i.$$

LEMMA 2.1. *Let ν be a product measure on $J(g)$ determined by the probability distribution $\{q_i\}_{i=1}^n$. Then for each $t\in\mathbf{R}$,*

$$h_\nu(g)-t\int_{J(g)}\log|g'|\,d\nu\leqslant\log\Bigl(\sum_{i=1}^n r(B_i)^t\Bigr)$$

with equality if and only if

$$q_i=\frac{r(B_i)^t}{\sum_{i=1}^n r(B_i)^t},\quad 1\leqslant i\leqslant n.$$

Proof. Since the logarithm is concave on $\mathbf{R}_+$,

$$h_\nu(g)-t\int_{J(g)}\log|g'|\,d\nu=\sum_{i=1}^n q_i\log\frac{|\gamma_i'|^t}{q_i}=\sum_{i=1}^n q_i\log\frac{r(B_i)^t}{q_i}\leqslant\log\Bigl(\sum_{i=1}^n r(B_i)^t\Bigr)$$

where the equality holds if and only if $q_i r(B_i)^{-t}$ has the same value for each $1\leqslant i\leqslant n$.

Remark. In this setup one can use (10) as the definition of the pressure $P(-t\log|g'|)$. Note also that if $s=\dim(J(g))$, then $\sum_{i=1}^n r(B_i)^s=1$, or $P(-s\log|g'|)=0$, and the extremal measure in Lemma 2.1 is again the normalized Hausdorff s-measure.

We shall next consider holomorphic families of Cantor sets or pairs $(g_\lambda, J(g_\lambda))$, $\lambda\in\Delta$. By this we mean that each set $J(g_\lambda)$ is generated as above by similarities $\gamma_{i,\lambda}(z)=a_i(\lambda)z+b_i(\lambda)$, $1\leqslant i\leqslant n$, where the coefficients $a_i(\lambda)\neq0$, $b_i(\lambda)$ now depend holomorphically on the parameter λ. On the other hand, we can also consider the $B_i(\lambda)=\gamma_{i,\lambda}\Delta$ and say that $\{B_i(\lambda)\}_1^n$ is a holomorphic family of disjoint disks in Δ.

Both of these configurations can be described as *holomorphic motions*; recall that a function $\Phi\colon\Delta\times A\to\overline{\mathbf{C}}$ is called a holomorphic motion of a set $A\subset\overline{\mathbf{C}}$ if

(i) for any fixed $a\in A$, the map $\lambda\mapsto\Phi(\lambda,a)$ is holomorphic in Δ,

(ii) for any fixed $\lambda\in\Delta$, the map $a\mapsto\Phi_\lambda(a)=\Phi(\lambda,a)$ is an injection, and

(iii) the mapping Φ_0 is the identity on A.

In fact, (global) quasiconformal mappings and holomorphic motions are just different expressions of the same geometric quantity. For instance, according to Slodkowski's generalized λ-lemma ([Sl], see also [AM, 3.3]) the correspondence $\gamma_{i,0}(z)\mapsto\gamma_{i,\lambda}(z)$ for $z\in\Delta$ and $1\leqslant i\leqslant n$, extends to a quasiconformal mapping $\Phi_\lambda\colon\overline{\mathbf{C}}\to\overline{\mathbf{C}}$ with

$$K(\Phi_\lambda)\leqslant\frac{1+|\lambda|}{1-|\lambda|}.$$

Therefore the estimate

$$\sum_{i=1}^n|B_i(\lambda)|\leqslant C\Bigl(\sum_{i=1}^n|B_i(0)|\Bigr)^{(1-|\lambda|)/(1+|\lambda|)}\tag{11}$$

is a special case of the Gehring–Reich conjecture. But after simplifying arguments, given later in Lemmas 3.1 and 3.3, we will see that the conjecture is in fact equivalent to (11).

Expressing this inequality now in terms of the topological pressure (10) we end up with the following formulation.

THEOREM 2.2. *Suppose that* $(g_\lambda, J(g_\lambda))$ *depends holomorphically on the parameter* $\lambda\in\Delta$. *Then*

$$\frac{1+|\lambda|}{1-|\lambda|}P(-2\log|g_0'|)\leqslant P(-2\log|g_\lambda'|)\leqslant\frac{1-|\lambda|}{1+|\lambda|}P(-2\log|g_0'|).$$

Proof. By the variational inequality 2.1 for each λ there is a unique (product) measure μ_λ such that

$$P(-2\log|g_\lambda'|)=h_{\mu_\lambda}(g_\lambda)-2\int_{J(g_\lambda)}\log|g_\lambda'|(z)\,d\mu_\lambda(z) \tag{12}$$

and clearly $\log|g_\lambda'|(z)$ is harmonic in λ. To use Harnack's inequality we "freeze" the measure μ_λ. In other words, given a probability distribution $\{p_i\}_1^n$ on $\{1,...,n\}$, define for each $\lambda\in\Delta$ a product measure $\bar\mu_\lambda$ on $J(g_\lambda)$ by the condition $\bar\mu_\lambda(J(g_\lambda)\cap B_i(\lambda))=p_i$; this is possible since the disks $B_i(\lambda)$ remain disjoint. By the construction, $h_{\bar\mu_\lambda}(g_\lambda)$ is then constant in λ.

Moreover, we have that $P(-2\log|g_\lambda'|)<0$, since $P(-s\log|g_\lambda'|)$ is strictly decreasing in s and it vanishes for $s=\dim(J(g))<2$. Alternatively, we may also use here the identity (10) to $P(-2\log|g_\lambda'|)=\log\big(\sum_{i=1}^n r(B_i(\lambda))^2\big)<0$.

If now the numbers $\{p_i\}$ are so chosen that $\bar\mu_0=\mu_0$ (the maximizing measure in (12) when the parameter $\lambda=0$), then Harnack's inequality with 2.1 implies that

$$\begin{aligned}\frac{1+|\lambda|}{1-|\lambda|}P(-2\log|g_0'|)&=\frac{1+|\lambda|}{1-|\lambda|}\left(h_{\bar\mu_0}(g_0)-2\int_{J(g_0)}\log|g_0'|\,d\bar\mu_0\right)\\&\leqslant h_{\bar\mu_\lambda}(g_\lambda)-2\int_{J(g_\lambda)}\log|g_\lambda'|\,d\bar\mu_\lambda\leqslant P(-2\log|g_\lambda'|)\end{aligned}$$

which proves the first of the required inequalities. The second follows similarly by symmetry in λ and 0.

When $t>2$ the same inequalities hold for $P(-t\log|g_\lambda'|)$ as well. However, smaller exponents must change with $|\lambda|$ and we shall later see how this reflects in the precise distortion of Hausdorff dimension under quasiconformal mappings.

COROLLARY 2.3. *If* $(g_\lambda, J(g_\lambda))$ *is as above and* $0<t\leqslant 2$, *set*

$$t(\lambda)=\frac{t(1+|\lambda|)}{1-|\lambda|+t|\lambda|}.$$

Then

$$\frac{1}{t(\lambda)}P(-t(\lambda)\log|g'_\lambda|)\leqslant\frac{1-|\lambda|}{1+|\lambda|}\frac{1}{t}P(-t\log|g'_0|).$$

Proof. If $\{\bar{\mu}_\lambda\}_{\lambda\in\Delta}$ is a family of product measures on $J(g_\lambda)$, all defined by a fixed probability distribution $\{p_i\}_1^n$ like in the previous theorem, then by 2.1

$$\begin{aligned}\frac{1}{t(\lambda)}h_{\bar{\mu}_\lambda}(g_\lambda)-\int_{J(g_\lambda)}\log|g'_\lambda|\,d\bar{\mu}_\lambda&=h_{\bar{\mu}_\lambda}(g_\lambda)\left(\frac{1}{t(\lambda)}-\frac{1}{2}\right)+\frac{1}{2}h_{\bar{\mu}_\lambda}(g_\lambda)-\int_{J(g_\lambda)}\log|g'_\lambda|\,d\bar{\mu}_\lambda\\&\leqslant\frac{1-|\lambda|}{1+|\lambda|}\left(h_{\bar{\mu}_0}(g_0)\left(\frac{1}{t}-\frac{1}{2}\right)+\frac{1}{2}h_{\bar{\mu}_0}(g_0)-\int_{J(g_0)}\log|g'_0|\,d\bar{\mu}_0\right)\\&\leqslant\frac{1-|\lambda|}{1+|\lambda|}\frac{1}{t}P(-t\log|g'_0|)\end{aligned}$$

and taking the supremum over the product measures on $J(g_\lambda)$ proves the claim.

The above estimates for the topological pressure hold actually in a much greater generality. We can consider, for instance, polynomial-like mappings of Douady and Hubbard [DH]. More precisely, suppose we have a family of holomorphic functions f_λ defined on the open sets U_λ, $\lambda\in\Delta$, such that $\overline{U_\lambda}\subset f_\lambda U_\lambda$. We need to assume that

$$J(f_\lambda)=\bigcap_{n=0}^{\infty}{f_\lambda}^{-n}U_\lambda$$

is a mixing repeller for f_λ. That is, $f'_\lambda\neq 0$ for $z\in J(f_\lambda)$ and $J(f_\lambda)$ is compact in $\mathbf{C}$ with no proper f_λ-invariant relatively open subsets. Then the f_λ are expanding on $J(f_\lambda)$ and the thermodynamic formalism extends to $f_\lambda\colon J(f_\lambda)\to J(f_\lambda)$, see [Bw] or [Ru].

To consider the dependence on the parameter, let U_λ depend continuously on λ and let $(\lambda,a)\mapsto f_\lambda(a)$ be holomorphic whenever defined. Because the functions are expanding, we have a holomorphic motion of the periodic points ([MSS], p. 198). Since these are dense in the repeller $J(f_\lambda)$, by the λ-lemma of Mañé, Sad and Sullivan we obtain a holomorphic motion Φ of $J(f_0)$ such that $J(f_\lambda)=\Phi_\lambda J(f_0)$ and $f_\lambda\circ\Phi_\lambda=\Phi_\lambda\circ f_0$.

Combining these facts we conclude that

$$\frac{1}{t(\lambda)}P(-t(\lambda)\log|f'_\lambda|)\leqslant\frac{1-|\lambda|}{1+|\lambda|}\frac{1}{t}P(-t\log|f'_0|).\tag{13}$$

Namely, since the variational principle generalizes to this setting, the proof of (13) is as in Lemma 2.3. In this case to show that $P(-t(\lambda)\log|f'_\lambda|)<0$ we may use Manning's formula [M]

$$\dim(\mu)=\frac{h_\mu(f_\lambda)}{\int_{J(f_\lambda)}\log|f'_\lambda|\,d\mu}$$

and the fact [Su] that $\dim(\mu)\equiv\inf\{\dim(E):\mu(E)=1\}\leqslant\dim(J(f_\lambda))<2$. These hold for any ergodic f_λ-invariant measure on $J(f_\lambda)$. Especially, starting from a measure μ on $J(f_0)$ we can take the images $\mu_\lambda=\Phi_\lambda^*\mu$ under the holomorphic motion, and since the entropy is an isomorphism invariant, (13) follows.

On the other hand, if one looks for the minimal approach to the quasiconformal area distortion, then the above leads also to a proof for (11) that avoids the thermodynamic formalism. In fact, this was shown to us by A. Eremenko and J. Fernández, who independently pointed out the following result on the (nonharmonic!) function $\log\|f(z)\|$.

COROLLARY 2.4. *Let $B^n=\{z\in\mathbf{C}^n:\|z\|<1\}$. If $f:\Delta\to B^n$ is a holomorphic mapping such that all of its coordinate functions f_i are everywhere nonzero, then*

$$\frac{1+|z|}{1-|z|}\log\|f(0)\|\leqslant\log\|f(z)\|\leqslant\frac{1-|z|}{1+|z|}\log\|f(0)\|.$$

Proof. If $f=(f_1,\ldots,f_n)$, consider numbers $p_i>0$ with $\sum_1^n p_i=1$ and set

$$u(z)=\sum_{i=1}^n p_i\log\frac{p_i}{|f_i(z)|^2}.$$

Then $u(z)$ is harmonic and by Jensen's inequality, $e^{-u(z)}\leqslant\sum_1^n p_i|f_i(z)|^2/p_i<1$, u is also positive. Hence using the concavity of the logarithm and the Harnack's inequality we may deduce

$$\log\|f(z)\|^2\geqslant\sum_{i=1}^n p_i\log\frac{|f_i(z)|^2}{p_i}\geqslant\frac{1+|z|}{1-|z|}\sum_{i=1}^n p_i\log\frac{|f_i(0)|^2}{p_i}.$$

Choosing finally $p_i=|f_i(0)|^2/\|f(0)\|^2$ proves the first inequality. The second follows by symmetry.

3. Distortion of area

We shall reduce the proof of the area distortion estimate $|fE|\leqslant M|E|^{1/K}$ into two distinct special cases. In the first, where we use the inequalities of the previous section, let us assume that E is a finite union of nonintersecting disks $B_i=B(z_i,r_i)\subset\Delta$, $1\leqslant i\leqslant n$.

LEMMA 3.1. *Suppose that $f:\Delta\to\Delta$ is K-quasiconformal with $f(0)=0$. If f is conformal in $E=\bigcup_1^n B_i$, then*

$$\sum_{i=1}^n|fB_i|\leqslant C(K)\Big(\sum_{i=1}^n|B_i|\Big)^{1/K},$$

where the constant $C(K)$ depends only on K. Moreover, $C(K)=1+O(K-1)$.

Proof. Extend f first to $\overline{\mathbf{C}}$ by a reflection across $\mathbf{S}^1$ and assume without loss of generality that $f(1)=1$. Then we can embed f to a holomorphic family of quasiconformal mappings of $\overline{\mathbf{C}}$. However, in order to control the distortion as $K\to\infty$ we need to modify f near ∞. Thus, if μ is the Beltrami coefficient of (the extended) f , define for each $\lambda\in\Delta$ new dilatations by

$$\mu_\lambda(z)=\begin{cases}\lambda\dfrac{K+1}{K-1}\mu(z), & |z|\leqslant 2,\\ 0, & |z|>2.\end{cases}$$

By the measurable Riemann mapping theorem there are unique μ_λ-quasiconformal mappings $f_\lambda\colon\overline{\mathbf{C}}\to\overline{\mathbf{C}}$ normalized by the condition

$$f_\lambda(z)-z=O(1/|z|)\quad\text{as } |z|\to\infty.$$

Then f_λ is conformal in E, $f_\lambda(z)$ and its derivatives (when $z\in E$) depend holomorphically on λ [AB, Theorem 3], $f_0(z)\equiv z$ and if $\lambda_0=(K-1)/(K+1)$, then

$$f_{\lambda_0}=\Phi\circ f, \tag{14}$$

where Φ is conformal in $fB(0,2)$.

To apply Theorem 2.2 note that by Koebe's $\frac14$-theorem

$$D_i(\lambda)\equiv B\big(f_\lambda(z_i),\tfrac14 r_i|f'_\lambda(z_i)|\big)\subset f_\lambda B(z_i,r_i)$$

and similarly $f_\lambda B(0,2)\subset B(0,8)$. Also $D_i(\lambda)=\psi_{\lambda,i}D_i(0)$, where

$$\psi_{i,\lambda}(z)=f'_\lambda(z_i)(z-z_i)+f_\lambda(z_i),$$

and thus $\{D_i(\lambda)\}_1^n$ is a holomorphic family of disjoint disks contained in $B(0,8)$. Therefore we need only choose extra similarities $\phi_i\colon B(0,8)\to D_i(0)$, $1\leqslant i\leqslant n$, set $\gamma_{i,\lambda}=\psi_{i,\lambda}\circ\phi_i$ and note that these generate a holomorphic family of Cantor sets $J(g_\lambda)\subset B(0,8)$. By Theorem 2.2

$$P(-2\log|g'_\lambda|)\leqslant\frac{1-|\lambda|}{1+|\lambda|}P(-2\log|g'_0|)$$

or, in other words, by (10)

$$\sum_{i=1}^n|f'_\lambda(z_i)|^2r_i^2\leqslant 32^{4|\lambda|/(1+|\lambda|)}\left(\sum_{i=1}^n r_i^2\right)^{(1-|\lambda|)/(1+|\lambda|)}.$$

The lemma will then be completed by simple approximation arguments. Since the images of circles under global quasiconformal mappings have bounded distortion,

$$\begin{aligned}|f_\lambda B_i| &\leqslant \pi \max_{z\in\partial B_i} |f_\lambda(z)-f_\lambda(z_i)|^2 \leqslant \pi C_0(|\lambda|) \min_{z\in\partial B_i} |f_\lambda(z)-f_\lambda(z_i)|^2 \\ &\leqslant \pi C_0(|\lambda|)|f'_\lambda(z_i)|^2 r_i^2,\end{aligned}$$

where the last estimate follows from the Schwarz lemma. Moreover, the correct expression for the constant $C_0(|\lambda|)$, see [L, p. 16], shows that $C_0(|\lambda|)=1+O(|\lambda|)$. If we choose $\lambda_0=(K-1)/(K+1)$, it then follows that $|f_{\lambda_0}E|\leqslant C_1(K)|E|^{1/K}$ with $C_1(K)=1+O(K-1)$.

It remains to show that the function Φ in (14) satisfies

$$|\Phi'(z)| \geqslant C_2(K) = (1+O(K-1))^{-1} \quad \text{for all } z\in\Delta.$$

First, since the diameter of $f_{\lambda_0}B(0,2)$ is at least four [P, 11.1], the basic bounds on the circular distortion, see [L, I.2.5], imply that

$$B(f_{\lambda_0}(0), \varrho(K)) \subset f_{\lambda_0}\Delta = \Phi\Delta$$

for a $\varrho(K)>0$ depending only on K. As above, $|\Phi'(0)|\geqslant\varrho(K)$ by the Schwarz lemma. Yet another application of the Schwarz lemma, this time to the function $\lambda\mapsto f_\lambda(z)-z$, gives

$$|f_\lambda(z)-z| \leqslant 10|\lambda|, \quad z\in B(0,2).$$

This shows that we may choose $\varrho(K)=(1+O(K-1))^{-1}$.

Furthermore, as $f\mathbf{S}^1=\mathbf{S}^1$ and f^{-1} is uniformly Hölder continuous with constants depending only on K, $fB(0,2)\supset B(0,R)$ for an $R=R(K)>1$. Then Koebe's distortion theorem combined with Lehto's majorant principle [L, II.3.5] proves that

$$|\Phi'(z)| \geqslant |\Phi'(0)| \left(\frac{(1-|z/R|)^3}{1+|z/R|}\right)^{(K-1)/(K+1)}, \quad z\in\Delta,$$

and the required estimates follow.

Remark 3.2. The above proof gives us the following "variational principle" for planar quasiconformal mappings:

Suppose we are given numbers $p_i>0$ with $\sum_{i=1}^n p_i=1$ and disjoint disks $B_i\subset\Delta$. Then for each K-quasiconformal mapping $f\colon\Delta\to\Delta$ for which $f(0)=0$ and

$$f|_{\bigcup_1^n B_i} \textit{ is conformal}, \tag{15}$$

we have the inequality

$$\sum_{i=1}^{n} p_i \log \frac{|fB_i|}{p_i} \leqslant \frac{1}{K} \sum_{i=1}^{n} p_i \log \frac{|B_i|}{p_i} + C(K) \tag{16}$$

where $C(K)=O(K-1)$ *depends only on* K.

In fact, choosing $p_i=|fB_i|/(\sum_{i=1}^{n}|fB_i|)$ shows that (16) generalizes Lemma 3.1.

Somewhat curiously, the variational inequality (16) is not true for general quasiconformal mappings, for mappings which do not satisfy (15); we shall return to this in §5 where it will have implications to the estimates of the $L\log L$-norm of the Beurling operator.

To prove the complementary case in the area distortion inequality we use therefore a different method. We shall apply here the approach due to Gehring and Reich [GR] based on a parametric representation.

LEMMA 3.3. *Let* $f\colon \Delta\to\Delta$ *be* K*-quasiconformal with* $f(0)=0$. *If* $E\subset\Delta$ *is closed and* f *is conformal outside* E, *then*

$$|fE| \leqslant b(K)|E|$$

where $b(K)=1+O(K-1)$ *depends only on* K.

Proof. As in [GR] define the Beltrami coefficients

$$\nu_t(z) = \operatorname{sgn}(\mu(z)) \tanh\left(\frac{t}{T} \operatorname{arctanh} |\mu(z)|\right), \quad t\in\mathbf{R}_+,$$

where μ is the complex dilatation of f, $T=\log K$ and $\operatorname{sgn}(w)=w/|w|$ if $w\neq 0$ with $\operatorname{sgn}(0)=0$. By the measurable Riemann mapping theorem we can find ν_t-quasiconformal $h_t\colon \Delta\to\Delta$ with $h_t(0)=0$.

If $A(t)=|h_tE|$, then Gehring and Reich show that

$$\frac{d}{dt}A(t) = \int_\Delta \phi S(\chi_{h_tE})\,dx\,dy + c(t)|h_tE| \tag{17}$$

where S is the Beurling operator and $|c(t)|$ is uniformly bounded. The function ϕ depends only on the family $\{h_t\}$, not on E, and from [GR, (2.6) and (3.6)] we conclude that $\|\phi\|_\infty\leqslant 1$ and that $\phi(w)=0$ whenever $\mu(h_t^{-1}(w))=0$.

Suppose now that f is conformal outside the compact subset $E\subset\Delta$. Then $\mu\equiv 0$ in $\Delta\setminus E$ and, in particular, we obtain

$$|\phi(z)| \leqslant \chi_{h_tE}(z).$$

But $S: L^2 \to L^2$ is an isometry and therefore for any set $F \subset \mathbf{C}$,

$$\int_F |S\chi_F|\,dx\,dy \leqslant |F|^{1/2}\left(\int_{\mathbf{C}} |S\chi_F|^2\,dx\,dy\right)^{1/2} = |F|. \tag{18}$$

Thus

$$\frac{d}{dt}A(t) \leqslant C_0 A(t), \quad 0 < t < \infty,$$

and an integration gives $|h_t E| = A(t) \leqslant e^{C_0 t} A(0) = e^{C_0 t}|E|$. Taking $t = \log K$ shows that $|fE| \leqslant e^{C_0 \log K}|E| = b(K)|E|$, where $b(K) = 1 + O(K-1)$.

Remark. On the other hand, as kindly pointed out by the referee, if one considers in $\overline{\mathbf{C}}$ the solution $f(z) = z + O(|z|^{-1})$ of the Beltrami equation $f_{\bar{z}} = \mu f_z$ where μ is supported on E, then in that situation the following argument gives a direct proof for a very precise estimate

$$|f(E)| \leqslant K|E|.$$

Namely, for $\omega = f_{\bar{z}}$ we have $f_z = 1 + S\omega$ with $\omega = \mu(1 + S\mu + S(\mu S\mu) + \dots)$. In view of $\|S\|_2 = 1$ we obtain

$$|f(E)| = \int_E |1+S\omega|^2 - |\omega|^2 \leqslant |E| + 2\,\mathrm{Re}\int_E S\omega,$$

where for the kth iterate $\int_E |S\mu S\mu \dots S\mu| \leqslant \|\mu\|_\infty^k |E|$ as in (18). Thus

$$|f(E)| \leqslant |E| + 2\|\mu\|_\infty |E| + 2\|\mu\|_\infty^2 |E| + \dots = |E|\left(-1 + \frac{2}{1-\|\mu\|_\infty}\right) = K|E|.$$

The area distortion inequality is now an immediate corollary of the two previous Lemmas 3.1 and 3.3.

Proof of Theorem 1.1. Suppose that $f: \Delta \to \Delta$ is K-quasiconformal and $f(0) = 0$. In proving the estimate

$$|fE| \leqslant M|E|^{1/K}$$

it suffices to study sets of the type $E = \bigcup_1^n B_i$, where the B_i are subdisks of Δ with pairwise disjoint closures. The general case follows then from Vitali's covering theorem.

To factor f we find by the measurable Riemann mapping theorem a K-quasiconformal mapping $g: \Delta \to \Delta$, $g(0) = 0$, with complex dilatation $\mu_g = \chi_{\Delta \setminus E}\mu_f$. Then g is conformal in E and $f = h \circ g$, where $h: \Delta \to \Delta$ is also K-quasiconformal, $h(0) = 0$, but now h is conformal outside gE. Since quasiconformal mappings preserve sets of zero area, $|h(\partial gE)| = |\partial gE| = 0$, and then Lemmas 3.1 and 3.3 imply

$$|fE| = |h(gE)| \leqslant b(K)|gE| \leqslant b(K)C(K)|E|^{1/K},$$

where $M(K)\equiv b(K)C(K)=1+O(K-1)$ as required.(2)

One of the equivalent formulations of Theorem 1.1 is the statement that for a K-quasiconformal f the Jacobian J_f belongs to the class weak-L^p, $p=K/(K-1)$.

COROLLARY 3.4. *If $f:\Delta\to\Delta$ is K-quasiconformal, $f(0)=0$, then for all $s>0$,*

$$|\{z\in\Delta:J_f(z)\geqslant s\}|\leqslant\left(\frac{M}{s}\right)^{K/(K-1)},$$

where M depends only on K. Moreover the exponent $p=K/(K-1)$ is the best possible.

Proof. If $E_s=\{z\in\Delta:J_f(z)\geqslant s\}$, then by Theorem 1.1

$$s|E_s|\leqslant\int_{E_s}J_f\,dm=|fE_s|\leqslant M(K)|E_s|^{1/K}.$$

No p larger than $K/(K-1)$ will do, since $|E_s|=\pi(Ks)^{-K/(K-1)}$ for $f(z)=z|z|^{1/K-1}$.

Proof of Corollary 1.2. If D is a compact disk in the domain Ω and $f:\Omega\to\Omega'$ is K-quasiconformal, choose conformal ψ,ϕ which map neigbourhoods of D and fD, respectively, onto the unit disk. As $\psi|_D$ and $\phi|_{fD}$ are bilipschitz, applying Corollary 3.4 to $\phi\circ f\circ\psi^{-1}$ proves that $J_f\in L^p_{\mathrm{loc}}(\Omega)$ for all $p<K/(K-1)$.

4. Distortion of dimension

In the previous section we determined the quasiconformal area distortion from the properties of the pressure $P(-2\log|g'_\lambda|)$. Similarly Corollary 2.3, or the variational inequality (16) with a suitable choice of the probabilities p_i, admits the following geometric interpretation:

If $f:\Delta\to\Delta$ is K-quasiconformal with $f(0)=0$ and if, in addition, f is conformal in the union of the disks $B_i\subset\Delta$, $1\leqslant i\leqslant n$, then

$$\sum_i|fB_i|^{tK/(1+t(K-1))}\leqslant C(K)\Big(\sum_i|B_i|^t\Big)^{1/(1+t(K-1))},\quad 0<t\leqslant 1,$$

where the constant $C(K)$ depends only on K.

Since the complementary Lemma 3.3 fails for exponents $t<2$, in the general case we content with slightly weaker inequalities.

(2) David Hamilton has informed us that the same methods can also be used to obtain good bounds for the constant M in $|fE|\leqslant M|E|^{1/K}$ if one considers instead of the case $f:\Delta\to\Delta$ those mappings f which are conformal outside Δ with $f(z)-z=O(1/|z|)$.

LEMMA 4.1. *If* $0<t<1$, $f:\Delta\to\Delta$ *is* K*-quasiconformal,* $f(0)=0$ *and* $\{B_i\}_1^n$ *are pairwise disjoint sets in* Δ, *then*

$$\sum_i |fB_i|^p \leqslant C_K(t,p)\Big(\sum_i |B_i|^t\Big)^{1/(1+t(K-1))}$$

whenever $(1+t(K-1))^{-1}tK<p\leqslant 1$.

Proof. We use the integrability of the Jacobian J_f as in [GV]. Since $p(1+t(K-1))>tK$ we can choose an exponent $1<p_0<K/(K-1)$ such that

$$\frac{1}{K}q_0<p\frac{1+t(K-1)}{tK}, \tag{19}$$

where $q_0=p_0/(p_0-1)$ is the conjugate exponent. Then using Hölder's inequality twice one obtains

$$\begin{aligned}\sum_i |fB_i|^p &= \sum_i\Big(\int_{B_i} J_f\,dm\Big)^p \leqslant \sum_i\Big(\int_{B_i} {J_f}^{p_0}\,dm\Big)^{p/p_0}|B_i|^{p/q_0}\\ &\leqslant \Big(\sum_i\int_{B_i} {J_f}^{p_0}\,dm\Big)^{p/p_0}\Big(\sum_i |B_i|^{(p/q_0)p_0/(p_0-p)}\Big)^{(p_0-p)/p_0}\\ &\leqslant \Big(\int_{\Delta} {J_f}^{p_0}\,dm\Big)^{p/p_0}\Big(\sum_i |B_i|^{(p/q_0)p_0/(p_0-p)}\Big)^{(p_0-p)/p_0}.\end{aligned}$$

On the other hand, as $p_0(K-1)/K<1<p(1+t(K-1))/tK$, it follows that $p_0/(p_0-p)>1+t(K-1)$. Combining this with Corollary 3.4 (or 1.2) yields

$$\sum_i |fB_i|^p \leqslant M\Big(\sum_i |B_i|^{(p/q_0)(1+t(K-1))}\Big)^{1/(1+t(K-1))},$$

where M depends only on p_0 and K. Since by (19) also $t<(p/q_0)(1+t(K-1))$, the claim follows.

Proof of Corollary 1.3. If $f:\Omega\to\Omega'$ is K-quasiconformal, let $E\subset\Omega$ be a compact subset with $\dim(E)<2$. Choose also a number $\frac{1}{2}\dim(E)<t\leqslant 1$ and cover E by squares B_i with pairwise disjoint interiors.

According to [LV, Theorems III.8.1 and III.9.1], $\operatorname{dia}(fB_i)^2\leqslant C_0|fB_i|$, where the constant C_0 depends only on K,E and Ω. Hence we conclude from Lemma 4.1 that

$$\sum_i \operatorname{dia}(fB_i)^\delta \leqslant C_1\Big(\sum_i \operatorname{dia}(B_i)^{2t}\Big)^{1/(1+t(K-1))}, \quad \delta>\frac{2tK}{1+t(K-1)}.$$

With a proper choice of the covering $\{B_i\}$ the sum on the right hand side can be made arbitrarily small and thus $\dim(fE)\leqslant\delta$. Consequently,

$$\dim(fE)\leqslant\frac{2K\dim(E)}{2+(K-1)\dim(E)} \tag{20}$$

which proves the corollary.

In the special case of K-quasicircles Γ, i.e. images of $\mathbf{S}^1$ under global K-quasiconformal mappings, Corollary 1.3 reads as

$$\dim(\Gamma)\leqslant 1+\frac{K-1}{K+1}=2-\frac{2}{K+1}.$$

This sharpens recent results due to Jones–Makarov [JM] and Becker–Pommerenke [BP]. On the other hand, Becker and Pommerenke showed that if the dilatation $K\sim1$, then

$$1+0.09\left(\frac{K-1}{K+1}\right)^2\leqslant\dim(\Gamma)\leqslant 1+37\left(\frac{K-1}{K+1}\right)^2.$$

These results suggest the following

Question 4.2. If Γ is a K-quasicircle, is it true that

$$\dim(\Gamma)\leqslant 1+\left(\frac{K-1}{K+1}\right)^2.$$

In the positive case, is the bound sharp?

Let us next show that the equality can occur in (20) for any value of K and $\dim(E)$. Note first that in terms of the holomorphic motions Corollary 1.3 obtains the following form.

COROLLARY 4.3. *Let $\Phi:\Delta\times E\to\overline{\mathbf{C}}$ be a holomorphic motion of a set $E\subset\overline{\mathbf{C}}$ and write $d(\lambda)=\dim(\Phi_\lambda(E))$. Then*

$$d(\lambda)\leqslant\frac{2d(0)}{(2-d(0))(1-|\lambda|)/(1+|\lambda|)+d(0)}. \tag{21}$$

Proof. By Slodkowski's extended λ-lemma Φ_λ is a restriction of a K-quasiconformal mapping of $\overline{\mathbf{C}}$, $K\leqslant(1+|\lambda|)/(1-|\lambda|)$, and hence the claim follows from 1.3.

For the converse, we start by constructing holomorphic motions of Cantor sets such that the equality holds in (21) up to a given $\varepsilon>0$. Thus for each, say, $n\geqslant10$ find disjoint disks $B(z_i,r)\subset\Delta$ all of the same radius $r=r_n$, such that $\frac{1}{2}\leqslant nr^2\leqslant1$. If $0<t<2$, let also

$$\beta(t)=\log(n^{1/t}r). \tag{22}$$

For n large enough, $\beta(t)>0$ and

$$\tfrac{1}{2}t \leqslant \frac{\log r}{\log r-\beta(t)} \leqslant \tfrac{1}{2}t+\varepsilon. \tag{23}$$

Set then

$$a_t(\lambda)=\exp\left(-\beta(t)\frac{1-\lambda}{1+\lambda}\right).$$

Clearly a_t is holomorphic in Δ with $a_t(\Delta)=\Delta\setminus\{0\}$. Therefore we can consider the holomorphic family of similarities

$$\gamma_{i,\lambda}(z)=ra_t(\lambda)z+z_i.$$

Since the disks $\gamma_{i,\lambda}\Delta\subset B(z_i,r)$ are disjoint, the similarities $\gamma_{i,\lambda}$ generate Cantor sets $(g_\lambda, J(g_\lambda))$ as in §2. Furthermore, the derivatives $|\gamma'_{i,\lambda}|$ do not depend on i and so the dimension $d(\lambda)=\dim(J(g_\lambda))$ is determined from the equation $n(r|a_t(\lambda)|)^{d(\lambda)}=1$. By (22) $n(r|a_t(0)|)^t=1$ and therefore

$$d(0)=t.$$

Similarly, if $0<\lambda<1$, it follows from (23) that

$$\begin{aligned}\frac{d(0)}{d(\lambda)}=\frac{\log(r|a_t(\lambda)|)}{\log(r|a_t(0)|)}&=\frac{\log r-\beta(t)(1-\lambda)/(1+\lambda)}{\log r-\beta(t)}\\ &\leqslant \tfrac{1}{2}d(0)+\left(1-\tfrac{1}{2}d(0)\right)\frac{1-\lambda}{1+\lambda}+\varepsilon.\end{aligned} \tag{24}$$

Proof of Theorem 1.4. Choose a countable collection $\{B_k\}_1^\infty$ of pairwise disjoint subdisks of Δ and define, using the argument above, in each disk B_k a holomorphic motion Φ of a Cantor set J_k with $\Phi_\lambda(J_k)\subset B_k$. If $d(\lambda)=\dim(\Phi_\lambda(J_k))$, we may assume that $d(0)=t$ and that for each k (24) holds with $\varepsilon=1/k$.

Clearly this construction determines a holomorphic motion Ψ of the union $J=\bigcup_k J_k$. Writing still $d(\lambda)=\dim(\Psi_\lambda(J))$ we have

$$d(\lambda)=\frac{2d(0)}{(2-d(0))(1-\lambda)/(1+\lambda)+d(0)},\quad 0\leqslant\lambda<1.$$

Now Slodkowski's generalized λ-lemma applies and Ψ extends to a K-quasiconformal mapping f of $\overline{\mathbf{C}}$, where $K=(1+\lambda)/(1-\lambda)$, $0\leqslant\lambda<1$. In other words, if $E=J$, then $\dim(E)=t$ and $\dim(fE)=2K\dim(E)/(2+(K-1)\dim(E))$.

Finally, Corollary 1.5 is an immediate consequence of 1.3 and 1.4 since K-quasiregular mappings f can be factored as $f=\phi\circ g$, where ϕ is holomorphic and g K-quasiconformal; for holomorphic ϕ sets E with $\dim(E)<1$ are removable by Painlevé's theorem while those with $\dim(E)>1$ are never removable [Ga, III.4.5].

Therefore in considering the removability questions for K-quasiregular mappings, the dimension $d_K=2/(K+1)$ is the border-line case and there we have the Iwaniec–Martin conjecture that all sets of zero Hausdorff d_K-measure are removable. More generally, it is natural to ask whether the precise bound on the dimension

$$\dim(fE)\leqslant\frac{2K\dim(E)}{2+(K-1)\dim(E)}$$

given by Corollary 1.3 is still correct on the level of *measures*.

Question 4.4. Let $0<\tau<2$ and $\delta=\delta_K(\tau)=2K\tau/(2+\tau(K-1))$. If f is a planar K-quasiconformal mapping, is it true that

$$H^\tau(E)=0 \implies H^\delta(fE)=0.$$

If not, what is the optimal Hausdorff measure H_h or measure function h such that $f^*H_h\ll H^\tau$?

5. Estimates for the Beurling operator

As we saw earlier quasiconformal mappings have important connections to the singular integrals and in particular to the Beurling operator, the complex Hilbert transform

$$S\omega(z)=-\frac{1}{\pi}\int_{\mathbf{C}}\frac{\omega(\zeta)\,dm(\zeta)}{(\zeta-z)^2}.$$

There are even higher dimensional counterparts, see [IM1] and the references there.

In fact, many properties of the S operator can be reduced to the distortion results of quasiconformal mappings. We shall here consider only the operation of S on the function space $L\log L$ and refer to the work of Iwaniec and Kosecki [IK] for further results.

In case of the characteristic functions $\omega=\chi_E$ we have then by Corollary 1.7 that

$$\int_B|S\chi_E|\,dm\leqslant|E|\log\frac{\alpha|B|}{|E|}\tag{25}$$

for all Borel subsets E of a disk $B\subset\overline{\mathbf{C}}$: the constant α does not depend on E or B. This translates also to the L^∞ setting:

COROLLARY 5.1. *Let $B\subset\mathbf{C}$ be a disk. If ω is a measurable function such that $|\omega(z)|\leqslant\chi_B(z)$ a.e., then*

$$|\{z\in B:|\operatorname{Re}S\omega(z)|>t\}|\leqslant2\alpha|B|e^{-t}.\tag{26}$$

Proof. Let $E_+=\{z\in B:\operatorname{Re} S\omega>t\}$. Since S has a symmetric kernel,

$$t|E_+|\leqslant \operatorname{Re}\int_{E_+} S\omega\, dm=\operatorname{Re}\int_B \omega S\chi_{E_+}\, dm\leqslant |E_+|\log\frac{\alpha|B|}{|E_+|}$$

by (25). Thus $|E_+|\leqslant\alpha|B|e^{-t}$ and since by the same argument $E_-=\{z\in B:\operatorname{Re} S\omega<-t\}$ satisfies $|E_-|\leqslant\alpha|B|e^{-t}$, the inequality (26) follows.

The estimate (26) is sharp since for $\omega=(z/\bar z)\chi_\Delta(z)$ we have

$$S\omega=(1+2\log|z|)\chi_\Delta(z).$$

For the modulus $|S\omega|$ Iwaniec and Kosecki [IK, Proposition 12] have shown that (25) implies

$$|\{z\in B:|S\omega(z)|>t\}|\leqslant\alpha(1+19t)|B|e^{-t}. \tag{27}$$

It remains open if the linear term $19t$ can be replaced by a constant.

COROLLARY 5.2. *For each $\delta>1$ there is a constant $M(\delta)<\infty$ such that*

$$\int_B |Sv|\, dm\leqslant\delta\int_B |v(z)|\log\left(1+M(\delta)\frac{|v(z)|}{|v|_B}\right) dm(z)$$

whenever v is supported on B.

Proof. Let ω be a function, unimodular in B and vanishing in $\overline{\mathbf{C}}\setminus B$, such that

$$\int_B |Sv|\, dm=\int_B \omega Sv\, dm=|v|_B\int_B S\omega\frac{v}{|v|_B}\, dm.$$

We apply then the elementary inequality $ab\leqslant a\log(1+a)+e^b-1$. Since by (27)

$$\int_B e^{|S\omega|/\delta}-1\, dm\leqslant |B|\frac{\alpha}{\delta-1}\left(1+\frac{19\delta}{\delta-1}\right)=M_1(\delta)|B|,$$

it follows that

$$\int_B |Sv|\, dm\leqslant M_1(\delta)\int_B |v|\, dm+\delta\int_B |v|\log\left(1+\delta\frac{|v|}{|v|_B}\right) dm.$$

Define now $E_0=\{z\in B:|v(z)|<(1/e)|v|_B\}$. As $t\mapsto t\log(1/t)$ is increasing on $(0,1/e)$,

$$-e\int_{E_0} |v|\log\left(\frac{|v|}{|v|_B}\right) dm\leqslant |v|_B|E_0|\leqslant\int_B |v|\, dm,$$

where we use the convention $0\log 0=0$. Thus

$$\begin{aligned}\int_B |v|\log\left(1+\delta\frac{|v|}{|v|_B}\right)dm &\leqslant \int_{B\setminus E_0}|v|\log\left(\frac{|v|}{|v|_B}(e+\delta)\right)dm+\int_{E_0}|v|\log\left(1+\frac{\delta}{e}\right)dm\\ &\leqslant \int_B |v|\log\left(M_2\frac{|v|}{|v|_B}\right)dm,\end{aligned}\tag{28}$$

where $M_2=e^2+e\delta$. In conclusion, if $M=M_2\exp(M_1(\delta))$,

$$\int_B |Sv|\,dm\leqslant \delta\int_B |v|\log\left(M\frac{|v|}{|v|_B}\right)dm\leqslant \delta\int_B |v|\log\left(1+M\frac{|v|}{|v|_B}\right)dm,$$

which completes the estimation.

Since the variational inequality (16),

$$\sum_{i=1}^n p_i\log\frac{|fB_i|}{p_i}\leqslant\frac{1}{K}\sum_{i=1}^n p_i\log\frac{|B_i|}{p_i}+C(K)$$

with $C(K)=O(K-1)$ and $f|_{\bigcup B_i}$ conformal, was the key in the area distortion Theorem 1.1 it is of interest to know whether the inequality is valid without any conformality assumptions. Another natural question is whether Corollary 5.2 still holds at $\delta=1$; for characteristic functions this is true and (25) with [IK, Proposition 19] implies that for nonnegative functions v,

$$\int_{B\setminus E}|Sv|\,dm\leqslant\int_E |v(z)|\log\left(1+\alpha\frac{|v(z)|}{|v|_B}\right)dm(z),\quad\text{if }\operatorname{supp}(v)\subset E.$$

Indeed, it can be shown that these two questions are equivalent (if $v\geqslant 0$ in Corollary 5.2). However, it turns out that the answer to them is the negative. We omit here the proof of the equivalence; instead we give first a simple counterexample to the general variational inequality and then show how this reflects in the $L\log L$ estimates of the complex Hilbert transform.

Example 5.3. Choose $0<\varrho<1$. For $1\leqslant i\leqslant n$ consider the disjoint disks $B_i=B(\varrho^i,a\varrho^i)$ where $0<a<(1-\varrho)/(1+\varrho)$. Let also $p_i=1/n$ and $f_0(z)=z|z|^{1/K-1}$. Then

$$\sum_{i=1}^n p_i\log\frac{|B_i|}{p_i}=(n+1)\log\varrho+\log n+\log\pi a^2$$

while

$$\sum_{i=1}^n p_i\log\frac{|f_0B_i|}{p_i}\geqslant\frac{n+1}{K}\log\varrho+\log n+C_0=\frac{1}{K}\sum_{i=1}^n p_i\log\frac{|B_i|}{p_i}+\frac{K-1}{K}\log n+C_1,$$

where C_0,C_1 depend only on K and a. Letting $n\to\infty$ shows that the variational inequality fails for f_0.

PROPOSITION 5.4. *For each $M<\infty$ there is an $\varepsilon>0$ and a nonnegative function $v\in L\log L(\Delta)$ such that*

$$\int_\Delta |Sv|\,dm > (1+\varepsilon)\int_\Delta |v(z)|\log\left(1+M\frac{|v(z)|}{|v|_\Delta}\right) dm(z).$$

Proof. By inequality (28) it suffices to show that for no $M<\infty$ does

$$\int_\Delta |Sv|\,dm \leqslant \int_\Delta |v(z)|\log\left(M\frac{|v(z)|}{|v|_\Delta}\right) dm(z) \tag{29}$$

hold for all nonnegative functions $v\in L\log L(\Delta)$.

We argue by contradiction. Hence consider first the mapping $f(z)=z|z|^{K-1}$ and imbedd it to a one parameter family of quasiconformal mappings $h_t\colon\Delta\to\Delta$, as in the proof of Lemma 3.3. Thus for $t=\log K$, $h_t=f$.

Suppose next that we have disjoint open sets $\{D_i\}_1^n\subset\Delta$ and numbers $p_i>0$ with $\sum_1^n p_i=1$. Set then

$$v_t(z)=\sum_{i=1}^n \frac{p_i}{|h_tD_i|}\chi_{h_tD_i}(z).$$

Clearly $\int_\Delta v_t\,dm=1$ and if $\psi(t)=\int_\Delta v_t\log(Mv_t)\,dm$ then by Jensen's inequality $\psi(t)\geqslant 0$ for $M\geqslant\pi$. Furthermore, we can deduce from the Gehring–Reich identity (17) that

$$\psi'(t)=-\sum_{i=1}^n \frac{p_i}{|h_tD_i|}\frac{d}{dt}|h_tD_i| = -\int_\Delta \phi Sv\,dm - c(t)$$

where $|c(t)|$ is uniformly bounded. Thus if (29) holds, then $\psi'(t)\leqslant\psi(t)-c(t)$ and after integration $\psi(t)\leqslant e^t\psi(0)-e^t\int_0^t c(s)e^{-s}\,ds=e^t\psi(0)+c_1(t)$. Taking $t=\log K$ we obtain

$$\sum_{i=1}^n p_i\log\left(\frac{p_i}{|fD_i|}\right)\leqslant K\sum_{i=1}^n p_i\log\left(\frac{p_i}{|D_i|}\right)+C(K),$$

where $C(K)=(K-1)\log M+c_1(\log K)$.

Finally, if B_i, p_i are as in the previous example with $f^{-1}(z)=f_0(z)=z|z|^{1/K-1}$, we can choose $D_i=f_0B_i$. But this would mean that

$$\sum_{i=1}^n p_i\log\frac{|f_0B_i|}{p_i}\leqslant\frac{1}{K}\sum_{i=1}^n p_i\log\frac{|B_i|}{p_i}+\frac{C(K)}{K},$$

contradicting Example 5.3. Therefore (29) cannot hold and so the estimate of Corollary 5.2 is sharp.

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