Correction to

Upper semi-continuous set-valued functions

by

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§ 1. Introduction

We have found a mistake in the proof of Lemma 3 of [4]. In the penultimate paragraph of the proof, we seem to assume tacitly that the closed sets $Q_H(\xi_j^*)$ are contained in $\Sigma(H)$; but this is not necessarily so, and we can only conclude that the sets of constancy of the restriction of $F(x) \cap H$ to $\Xi(H)$ are relative $\mathscr{F}_\sigma$-sets in $\Xi(H)$.

In this note we give an additional argument that enables us to give a correct proof of this lemma. Since only the conclusion of the lemma is used in the rest of [4], no use of the incorrect details of the proof being made, even tacitly, all the theorems and lemmas of [4] hold good.

In a second paper [5] we have made extensive use of the details of the proofs of [4] to obtain selection results for upper semi-continuous set-valued functions taking their values in Banach spaces with their weak topology. The proofs of these results can also be corrected; this will be done elsewhere.

We take this occasion to mention an improved version of the selection theorem of [4]; proofs will be given elsewhere.

A function from a space X to a space Y is said to be of the *first Baire class* if it can be obtained as the point-wise limit of a sequence of continuous functions from X to Y. The higher Baire classes are defined by the condition that a function of the *α-th Baire* class, with α any countable ordinal, can be obtained as a point-wise limit of a sequence of functions from X to Y of the previous Baire classes. A space Y is said to have the *extension property* for a space X, if each continuous function to Y from a closed subset of X has a continuous extension mapping X into Y. Using these concepts and some results of Dugundji [1] and of Hansell [2, 3] we can obtain the following improved selection theorem.

THEOREM. *Let F be an upper semi-continuous set-valued map from a metric space X to the non-empty subsets of a metric space Y.*

(a) *The map F has a Borel measurable selector f of the second Borel class, the restriction of f to a set of the form $X \setminus P$, with P of the first category in X, being continuous.*

(b) *If F takes only values that are complete in the metric on Y, then F has a Borel measurable selector f of the first Borel class, the set of points of discontinuity of f being an $\mathcal{F}_\sigma$-set of the first category in X.*

(c) *If Y has the extension property for X, then f will be of the second Baire class in case* (a) *and of the first Baire class in case* (b).

§ 2. Corrected proof of Lemma 3

In this section we give the corrected version of the proof of Lemma 3 of [4]. We first recall some of the notation used in [4]. We use F to denote an upper semi-continuous set-valued map from a metric space X to a metric space Y. The boundary K of F is the set-valued map from X to Y defined by the formula

$$K(x) = \mathrm{proj}_Y([\mathrm{cl}\, E(x)] \cap [\{x\} \times F(x)]),$$

(note the unfortunate misprint in the corresponding formula in [4, p. 88]) with

$$E(x) = [(X \setminus \{x\}) \times (Y \setminus F(x))] \cap T,$$

and

$$T = \bigcup \{\{x\} \times F(x) \colon x \in X\}.$$

If H is any closed subset of Y, we use F_H to denote the set-valued function F_H: $x \mapsto F(x) \cap H$. This set-valued function F_H is automatically upper semi-continuous; we use K_H to denote its boundary. We also write

$$\Xi(H) = \{x \colon K(x) \cap H = \emptyset\},$$

and

$$\Delta(H) = \{x \colon K_H(x) = \emptyset\}.$$

Using this terminology, we need to prove: *If H is a closed subset of Y the sets of constancy of the restriction of $F(x)\cap H$ to $\Xi(H)$ form a disjoint family that is discretely σ-decomposable in the completion X^* of X, each set of constancy being an $\mathscr{F}_\sigma$-set in X.*

Now the "proof" of Lemma 3 of [4] only goes wrong in the last sentence of its penultimate paragraph where it is tacitly assumed that $Q_H(\xi_j^*)$ is contained in $\Xi(H)$, see p. 97. Earlier we established that $Q_H(\xi)$ is just the set of all x such that

$$F(x)\cap H=F(\xi)\cap H$$

and such that

$$F(\sigma)\cap H\subset F(\xi)$$

for all σ in $B(x;1/i(\xi))$. These conditions ensure that $x\in\Delta(H)$ so that $Q_H(\xi)\subset\Delta(H)$. In the special case when H coincides with Y, we conclude that

$$Q_Y(\xi)\subset\Delta(Y)=\Xi(Y).$$

It is now easy to verify that Lemma 3 does hold in the special case when $H=Y$.

It remains to verify that the general case of Lemma 3 is, in fact, a consequence of this special case. Take H to be a closed set in Y. Applying the special case to the set-function F_H we find that the sets of constancy of the restriction of F_H to $\Delta(H)$ form a disjoint family of $\mathscr{F}_\sigma$-sets that is discretely σ-decomposable in the completion X^* of X. In particular, $\Delta(H)$ is an $\mathscr{F}_\sigma$-set. To obtain the corresponding result for the restriction of F_H to $\Xi(H)$, it clearly suffices to show that $\Xi(H)$ is an $\mathscr{F}_\sigma$-set in X contained in $\Delta(H)$.

First note that, if $K(x)\cap H=\emptyset$, for some x in X, then $K_H(x)=\emptyset$, for this x. Thus $\Xi(H)\subset\Delta(H)$.

Since H is a closed set in the metric space Y, we can write

$$H=\bigcap_{r=1}^{\infty}H(r),$$

with $H(1)$, $H(2),\ldots$ a decreasing sequence of closed sets, each containing H in its interior. Then, since H is contained in the interior of $H(r)$, for each x in X, we have

$$K(x)\cap H\subset K_{H(r)}(x)\subset K(x)\cap H(r),\quad r\geq 1,$$

so that

$$K(x)\cap H=\bigcap_{r=1}^{\infty}K_{H(r)}(x).$$

Since the sequence $K_{H(r)}(x)$, $r=1,2,\ldots$, is a decreasing sequence of compact sets, for each x in X, it follows that

$$\operatorname{proj}_X(K \cap H) = \operatorname{proj}_X \bigcap_{r=1}^{\infty} K_{H(r)} = \bigcap_{r=1}^{\infty} \operatorname{proj}_X K_{H(r)}.$$

Thus

$$X \setminus \Xi(H) = \bigcap_{r=1}^{\infty} (X \setminus \Delta(H(r)))$$

and

$$\Xi(H) = \bigcup_{r=1}^{\infty} \Delta(H(r)).$$

Hence $\Xi(H)$ is an $\mathscr{F}_\sigma$-set, as required.

References

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