ON SYMBOLIC CALCULUS OF TWO VARIABLES

BY

N. K. CHAKRAVARTY

in Calcutta

1. The notation

$$\phi(p, q) \leqslant h(x, y)$$

between the image and the object in the symbolic calculus of two variables is used to represent the convergent double integral

$$\phi(p, q) = p q \int_0^\infty \int_0^\infty e^{-px-qy} h(x, y)\, dx\, dy. \quad R(p) > 0, \; R(q) > 0.$$

The object of the present paper is to derive a theorem in the symbolic calculus of two variables, starting from a chain of relations in one variable, and to show some applications of the theorem.

2. **Theorem.** If $f(p) < x^{\nu-1} h(x)$ and $p^{\mu-\lambda} h\left(\frac{1}{p^\mu}\right) < \sigma(x)$, then

$$y^{\nu-\frac{\lambda}{\mu}} \sigma(x y^{\frac{1}{\mu}}) \geqslant p^{\mu\nu-\lambda} f(p^\mu q),$$

valid when $\mu > 0$, $R(\lambda) > -1$, $R(\nu) > -1$.

Proof. Let $J_\lambda^\mu(x)$ represent Wright's generalised Bessel function [1] defined by

$$J_\lambda^\mu(x) = \sum_{r=0}^\infty \frac{(-x)^r}{r!\,\Gamma(1+\lambda+\mu r)}, \quad \mu > 0, \; R(\lambda) > -1.$$

Then

$$x^\lambda y^\nu J_\lambda^\mu(x^\mu y) = \sum_{r=0}^\infty \frac{(-1)^r x^{\lambda+\mu r} y^{\nu+r}}{r!\,\Gamma(1+\lambda+\mu r)}$$

$$\geqslant \frac{1}{p^\lambda q^\nu} \sum_{r=0}^\infty \frac{\Gamma(\nu+r+1)}{r!} \left(-\frac{1}{p^\mu q}\right)^r$$

$$= \Gamma(\nu+1) \cdot \frac{p^{\mu(\nu+1)-\lambda} \cdot q}{(1+p^\mu q)^{\nu+1}}, \quad R(\nu) > -1. \tag{2.1}$$

1 — 543809. *Acta Mathematica*. 93. Imprimé le 7 mai 1955.

Let

$$f(p) \risingdotseq x^{\nu-1} h(x) \quad \text{imply} \quad h(p) \risingdotseq g(x). \tag{2.2}$$

Then, by Gupta [2], we have

$$p^{\mu-\lambda} h\left(\frac{1}{p^{\mu}}\right) \risingdotseq x^{\lambda} \int_0^{\infty} g(s) J_{\lambda}^{\mu}(s x^{\mu}) \, ds \tag{2.3}$$

where

$$g(s) = O(s^{-1+\varepsilon}), \quad \text{for small } s, \ \varepsilon > 0$$

and

$$g(s) = O[s^{\varkappa(\lambda+\frac{1}{2})-1-\varepsilon_1} \exp(-s^{\varkappa+\varepsilon_2 \cos \pi\varkappa} \cos \pi\varkappa)], \quad \varkappa = \frac{1}{\mu+1},$$

for large s; $\varepsilon_1, \varepsilon_2 > 0$, and $\int_0^{\infty} e^{-px} F(x) \, dx$ is absolutely convergent, $F(x)$ being the right-hand side expression of (2.3).

Thus

$$x^{-\lambda} \sigma(x) = \int_0^{\infty} g(s) J_{\lambda}^{\mu}(s x^{\mu}) \, ds. \tag{2.4}$$

Writing (2.4) in the form

$$y^{\nu-\frac{\lambda}{\mu}} \sigma(x y^{\frac{1}{\mu}}) = \int_0^{\infty} s^{-\nu} g(s) x^{\lambda} (ys)^{\nu} J_{\lambda}^{\mu}(x^{\mu} y s) \, ds,$$

and interpreting by (2.1), we obtain

$$y^{\nu-\frac{\lambda}{\mu}} \sigma(x y^{\frac{1}{\mu}}) \gg \Gamma(\nu+1) p^{\mu(\nu+1)-\lambda} q \int_0^{\infty} \frac{g(s) \, ds}{(p^{\mu} q + s)^{\nu+1}}, \tag{2.5}$$

provided the integral on the right of (2.5) is convergent. For this we should have $\lambda \geq \mu\nu + \mu + \nu + \frac{1}{2}$, which holds under the condition of validity stated in the theorem.

Now Shastri [3] has shown that

$$\text{if} \quad f(p) \risingdotseq h(x), \quad \frac{1}{p^m} h(p) \risingdotseq g(x), \quad \text{then}$$

$$f(p) = \Gamma(m+2) p \int_0^{\infty} \frac{g(t) \, dt}{(p+t)^{m+2}}, \quad R(m) > -2.$$

So that from (2.2), we have

$$f(p) = \Gamma(\nu+1) p \int_0^{\infty} \frac{g(t) \, dt}{(p+t)^{\nu+1}}, \quad R(\nu) > -1,$$

so that

$$f(p^{\mu} q) = \Gamma(\nu+1) p^{\mu} q \int_0^{\infty} \frac{g(t) \, dt}{(p^{\mu} q + t)^{\nu+1}}. \tag{2.6}$$

The theorem follows on comparison of (2.5) and (2.6).

The following corollaries are interesting.

(A) Put $\nu=1$, $\mu=1$ and $\lambda=k+1$ and we have:

If $f(p) \subset h(x)$, $\frac{1}{p^k} h\left(\frac{1}{p}\right) \subset \sigma(x)$, then $y^{-k}\sigma(xy) \supset p^{-k} f(pq)$—a result due to Delerue [4].

(B) Put $\nu=1$, $\mu=2$, $\lambda=2m+2$ and we obtain:

If $f(p) \subset h(x)$, $\frac{1}{p^{2m}} h\left(\frac{1}{p^2}\right) \subset \sigma(x)$, then $y^{-m}\sigma(x\sqrt{y}) \supset p^{-2m} f(p^2 q)$, $R(m) > -\frac{3}{2}$.

(C) Put $\nu=1$, $\mu=\frac{1}{2}$, $\lambda=m+\frac{1}{2}$ and we obtain:

If $f(p) \subset h(x)$, $\frac{1}{p^m} h\left(\frac{1}{\sqrt{p}}\right) \subset \sigma(x)$, then $y^{-2m}\sigma(xy^2) \supset p^{-m} f(q\sqrt{p})$, $R(m) > -\frac{3}{2}$.

(1) *Application of the Theorem*

Let

$$h(x) = e^{-\frac{1}{2}x^2}, \quad \text{so} \quad x^{\nu-1} h(x) = x^{\nu-1} e^{-\frac{1}{2}x^2} \supset \Gamma(\nu)\, p\, e^{\frac{1}{4}p^2} D_{-\nu}(p) \equiv f(p)$$

and

$$\frac{1}{p^{\lambda-\mu}} h\left(\frac{1}{p^\mu}\right) = \frac{1}{p^{\lambda-\mu}} e^{-\frac{1}{2p^{2\mu}}} \subset x^{\lambda-\mu} J_{\lambda-\mu}^{2\mu}\left(\tfrac{1}{2} x^{2\mu}\right) \equiv \sigma(x).$$

Hence from the theorem, we obtain the operational representation

$$x^{\lambda-\mu} y^{\nu-1} J_{\lambda-\mu}^{2\mu}\left(\tfrac{1}{2} y^2 x^{2\mu}\right) \supset \Gamma(\nu)\, p^{\mu(1+\nu)-\lambda} \cdot q\, e^{\frac{1}{4}p^{2\mu}q^2} D_{-\nu}(p^\mu q) \tag{2.7}$$

which can be written in the form

$$x^m y^{n+k-2} J_m^{\frac{2m}{n-1}}\left(\tfrac{1}{2} y^2 x^{\frac{2m}{n-1}}\right) \supset \Gamma(n+k-1)\, q\, p^{\frac{mk}{n-1}} e^{\frac{1}{4}q^2 p^{\frac{2m}{n-1}}} D_{-(n+k-1)}\left(q\, p^{\frac{m}{n-1}}\right). \tag{2.8}$$

In particular,

$$x^{\frac{1}{2}m} y^{m+k-1} J_m(y\sqrt{2x}) \supset \frac{\Gamma(2m+k)}{2^{\frac{1}{2}m}} q\, p^{\frac{1}{2}k} e^{\frac{1}{4}pq^2} D_{-(2m+k)}(q\sqrt{p}) \tag{2.9}$$

$$x^{\frac{1}{2}m} y^{m-1} J_m(y\sqrt{2x}) \supset \frac{\Gamma(2m)}{2^{\frac{1}{2}m}} q\, e^{\frac{1}{4}pq^2} D_{-2m}(q\sqrt{p}) \tag{2.91}$$

$$x^{\frac{1}{2}} J_1(y\sqrt{2x}) \supset 2^{-\frac{1}{2}} q\, e^{\frac{1}{4}pq^2} D_{-2}(q\sqrt{p}) \tag{2.92}$$

$$x^{\frac{1}{2}m} y^m J_m(y\sqrt{2x}) \supset \frac{\Gamma(2m+1)}{2^{\frac{1}{2}m}} q\sqrt{p}\, e^{\frac{1}{4}pq^2} D_{-(2m+1)}(q\sqrt{p}) \tag{2.93}$$

$$J_0(y\sqrt{2x}) \supset q\sqrt{p}\, e^{\frac{1}{4}pq^2} D_{-1}(q\sqrt{p}) \tag{2.94}$$

$$\sin(y\sqrt{x}) \supset \sqrt{\pi p}\, q\, e^{\frac{1}{8}pq^2} D_{-2}(q\sqrt{2p})^{[4]} \tag{2.95}$$

and

$$y^{k-1} \sin (y\sqrt{2x}) \geqslant \sqrt{\frac{\pi}{2}}\,\Gamma(k+1)\, q\, p^{\frac{1}{2}k} e^{\frac{1}{4}pq^2} D_{-(k+1)}(q\sqrt{p}). \qquad (2.96)$$

Also

$$x^{\frac{1}{3}(m+1)} y^{\frac{1}{3}(m+3k-2)} J_{\frac{1}{3}(m-1),\frac{1}{3}m}[3\sqrt[3]{\tfrac{1}{8}x^2y^2}] \geqslant \sqrt{\frac{2}{\pi}}\,\Gamma(m+k)\, q\, p^k e^{\frac{1}{4}p^2q^2} D_{-(m+k)}(pq) \qquad (2.10)$$

with the following special cases

$$(xy)^{\frac{1}{3}(m+1)} J_{\frac{1}{3}(m-1),\frac{1}{3}m}[3\sqrt[3]{\tfrac{1}{8}x^2y^2}] \geqslant \sqrt{\frac{2}{\pi}}\,\Gamma(m+1)\, pq\, e^{\frac{1}{4}p^2q^2} D_{-(m+1)}(pq) \qquad (2.101)$$

$$x^{\frac{1}{3}} J_{-\frac{1}{3},\frac{1}{3}}[3\sqrt[3]{\tfrac{1}{8}x^2y^2}] \geqslant \sqrt{\frac{2}{\pi}}\, q\sqrt{p}\, e^{\frac{1}{4}p^2q^2} D_{-1}(pq) \qquad (2.102)$$

$$(xy)^{\frac{1}{3}} J_{-\frac{1}{3},\frac{1}{3}}[3\sqrt[3]{\tfrac{1}{8}x^2y^2}] \geqslant \frac{1}{\sqrt{2}}\, pq\, e^{\frac{1}{4}p^2q^2} D_{-\frac{3}{2}}(pq). \qquad (2.103)$$

(2) *Application of Cor. A*

(a) From

$$\frac{1}{1+x^2} > p\,\{\sin p\, Ci\,(p) - \cos p\, Si\,(p)\},$$

$$e^{-\alpha x}\frac{1}{1+\beta^2x^2} > \frac{p}{\beta}\left\{\sin\left(\frac{p+\alpha}{\beta}\right) Ci\left(\frac{p+\alpha}{\beta}\right) - \cos\left(\frac{p+\alpha}{\beta}\right) Si\left(\frac{p+\alpha}{\beta}\right)\right\}$$

and

$$\frac{1}{p^{n-2}}\,\frac{e^{-\frac{\alpha}{p}}}{p^2+\beta^2} < \frac{1}{\beta^n} U_n\,(2\beta x,\, 2\sqrt{\alpha x})^{[5]},$$

where U_n is the Lommel function for unrestricted values of n (Watson [5], p. 537).

Hence, we have the operational representation

$$y^{-n}U_n(2\beta xy, 2\sqrt{\alpha xy}) \geqslant \left(\frac{\beta}{p}\right)^{n-1}\cdot q\left[\sin\frac{pq+\alpha}{\beta}\, Ci\left(\frac{pq+\alpha}{\beta}\right) - \cos\frac{pq+\alpha}{\beta}\, Si\left(\frac{pq+\alpha}{\beta}\right)\right]. \qquad (2.11)$$

(b) We have

$$\Gamma(1+n-\mu)\, p^{\mu} e^{\frac{1}{2}p} W_{-n,-\mu+\frac{1}{2}}(p) < \frac{x^{n-\mu}}{(1+x)^{n+\mu}} \equiv h(x),$$

$$R(1+n-\mu) \geq 0,$$

$$\frac{1}{p^{\mu+n-1}}\, h\left(\frac{1}{p}\right) = \frac{p^{\mu-n+1}}{(1+p)^{\mu+n}} < \frac{1}{\Gamma(2n)} e^{-\frac{1}{2}x} x^{n-1} M_{\mu,n-\frac{1}{2}}(x),$$

and we obtain the operational representation

$$x^{n-1} y^{-\mu} M_{\mu,n-\frac{1}{2}}(xy)\, e^{-\frac{1}{2}xy} > \Gamma(1+n-\mu)\,\Gamma(2n)\, p^{1-n} q^{\mu} e^{\frac{1}{2}pq} W_{-n,-\mu+\frac{1}{2}}(pq) \qquad (2.12)$$

and in particular,

$$e^{-xy} \geqslant pq\,e^{pq}\,E_i(pq), \tag{2.121}$$

E_i having the definition given in [6], p. 352, and

$$x^{\nu-1}\,e^{-xy} \geqslant \Gamma(\nu)\,q\,p^{2-\nu}\,S(\nu, pq),$$

known already, $S(\nu, x)$ being Schlömilch's function.

(c) From Adamoff's integral ([6], p. 353) we obtain the following two operational representations:

$$x^m \sin\sqrt{2x} > \frac{(-1)^m\sqrt{\pi}}{2^{m+\frac{1}{2}}}\,\frac{1}{p^m}\,e^{-\frac{1}{4p}}\,D_{2m+1}\left(\frac{1}{\sqrt{p}}\right) \tag{2.13}$$

and

$$x^{m-\frac{1}{2}} \cos\sqrt{2x} > \frac{(-1)^m\sqrt{\pi}}{2^m}\,\frac{1}{p^{m-\frac{1}{2}}}\,e^{-\frac{1}{4p}}\,D_{2m}\left(\frac{1}{\sqrt{p}}\right) \tag{2.14}$$

(m, positive integer in both cases).

Also,

$$\frac{1}{p^k}\sin\frac{2}{\sqrt{p}} < \sqrt{\pi}\,x^{\frac{4k+1}{6}}\,J_{\frac{2k+1}{2},\,\frac{1}{2}}(3\sqrt[3]{x}) \tag{2.15}$$

and

$$\frac{1}{p^k}\cos\frac{2}{\sqrt{p}} < \sqrt{\pi}\,x^{\frac{4k+1}{6}}\,J_{k,\,-\frac{1}{2}}(3\sqrt[3]{x}). \tag{2.16}$$

We then obtain the operational representations

$$x^{\frac{1}{6}(4k+1)}\,y^{\frac{1}{6}(6m-2k-2)}\,J_{k,\,-\frac{1}{2}}\left(3\sqrt[3]{\tfrac{1}{2}xy}\right) \geqslant \frac{(-1)^m}{2^{\frac{1}{6}(6m+2k-1)}}\,\frac{1}{p^k\,q^{m-\frac{1}{2}}}\,e^{-\frac{1}{4pq}}\,D_{2m}\left(\frac{1}{\sqrt{pq}}\right) \tag{2.17}$$

and

$$x^{\frac{1}{6}(4k+1)}\,y^{\frac{1}{6}(6m-2k+1)}\,J_{\frac{1}{2}(2k+1),\,\frac{1}{2}}\left(3\sqrt[3]{\tfrac{1}{2}xy}\right) \geqslant \frac{(-1)^m}{2^{\frac{1}{3}(3m+k+1)}}\,\frac{1}{p^k\,q^m}\,e^{-\frac{1}{4pq}}\,D_{2m+1}\left(\frac{1}{\sqrt{pq}}\right). \tag{2.18}$$

(d) From (2.13) and Mitra's [7] operational representation

$$\frac{(-1)^n\sqrt{2\pi}\,\Gamma(2n+2)}{\Gamma(n+1)}\,\frac{p^{n+1}}{(2p+1)^{n+3}} < e^{-\frac{1}{4}x}\,D_{2n+1}(\sqrt{x})$$

we obtain, from our corollary,

$$x^n \sin\sqrt{xy} \geqslant \frac{\pi\,\Gamma(2n+2)}{\Gamma(n+1)}\cdot\frac{p\,q^{n+1}}{(4pq+1)^{n+\frac{3}{2}}}. \tag{2.19}$$

Similarly, using (2.14) and

$$x^{-\frac{1}{2}} D_{2n}(2\sqrt{x}) > \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^n\,\Gamma(n+1)} \frac{p\,(1-p)^n}{(p+1)^{n+\frac{1}{2}}}, \quad \text{(Mitra, [7])},$$

we get

$$\frac{x^n \cos\sqrt{xy}}{\sqrt{xy}} > \frac{2\pi\cdot\Gamma(2n+1)}{\Gamma(n+1)} \frac{p q^{n+1}}{(4pq+1)^{n+\frac{1}{2}}} \tag{2.20}$$

whence,

$$\frac{\cos 2\sqrt{xy}}{\sqrt{xy}} > \frac{\pi p q}{(1+pq)^{\frac{1}{2}}},$$

a result, due to Delerue [4].

(e) Let

$$h(x) \equiv \cos\frac{3\nu\pi}{4}\cos x - \sin\frac{3\nu\pi}{4}\sin x > \frac{p}{1+p^2}\left[p\cos\frac{3\nu\pi}{4} - \sin\frac{3\nu\pi}{4}\right] \equiv f(p)$$

and

$$\frac{1}{p^\nu} h\left(\frac{1}{p}\right) \equiv \frac{1}{p^\nu}\cos\left(\frac{1}{p}+\frac{3\nu\pi}{4}\right) < x^{\frac{1}{2}\nu}\,\mathrm{ber}_\nu\,(2\sqrt{x}) \quad \text{(Mc Lachlan, [8])}.$$

So, from the theorem,

$$\left(\frac{x}{y}\right)^{\frac{1}{2}\nu} \mathrm{ber}_\nu\,(2\sqrt{xy}) > \frac{q}{p^{\nu-1}(1+p^2q^2)}\left[pq\cos\frac{3\nu\pi}{4} - \sin\frac{3\nu\pi}{4}\right].$$

Similarly, from

$$\frac{1}{p^\nu}\sin\left(\frac{1}{p}+\frac{3\nu\pi}{4}\right) < x^{\frac{1}{2}\nu}\,\mathrm{bei}_\nu\,(2\sqrt{x}),$$

$$\left(\frac{x}{y}\right)^{\frac{1}{2}\nu} \mathrm{bei}_\nu\,(2\sqrt{xy}) > \frac{q}{p^{\nu-1}(1+p^2q^2)}\left[\cos\frac{3\nu\pi}{4} + pq\sin\frac{3\nu\pi}{4}\right]$$

whence

$$\frac{1}{p^\nu}\frac{p^2q^2}{1+p^2q^2} < \left(\frac{x}{y}\right)^{\frac{1}{2}\nu}\left[\mathrm{ber}_\nu\,(2\sqrt{xy})\cos\frac{3\nu\pi}{4} + \mathrm{bei}_\nu\,(2\sqrt{xy})\sin\frac{3\nu\pi}{4}\right] \tag{2.21}$$

and

$$\frac{1}{p^\nu}\frac{pq}{1+p^2q^2} < \left(\frac{x}{y}\right)^{\frac{1}{2}\nu}\left[\mathrm{bei}_\nu\,(2\sqrt{xy})\cos\frac{3\nu\pi}{4} - \mathrm{ber}_\nu\,(2\sqrt{xy})\sin\frac{3\nu\pi}{4}\right]. \tag{2.22}$$

With $\nu=0$, familiar operational representations for ber $(2\sqrt{xy})$ and bei $(2\sqrt{xy})$ are obtained.

(3) *Applications of Cor. B*

(a) From the operational representations of $e^{\frac{1}{p}}$ and $I_0\left(\frac{1}{p}\right)$, it easily follows from the theorem that

$$\mathrm{ber}^2\left(2\sqrt{x\sqrt{y}}\right) + \mathrm{bei}^2\left(2\sqrt{x\sqrt{y}}\right) > e^{\frac{1}{p^2 q}}. \tag{2.23}$$

(b) Let

$$f(p)=\frac{(-1)^m\sqrt{\pi}}{2^{m+\frac{1}{2}}}\,\frac{1}{p^m}\,e^{-\frac{1}{4p}}\,D_{2m+1}\left(\frac{1}{\sqrt{p}}\right)$$

so

$$h(x)=x^m\sin\sqrt{2x}.$$

Hence $\frac{1}{p^{2\nu-2m}}\,h\left(\frac{1}{p^2}\right)=\frac{1}{p^{2\nu}}\sin\left(\frac{\sqrt{2}}{p}\right)$

$$>2^{-\frac{\nu}{2}}x^\nu\left[\mathrm{bei}_{2\nu}\left(2\sqrt{x\sqrt{2}}\right)\cos\frac{3\nu\pi}{2}-\mathrm{ber}_{2\nu}\left(2\sqrt{x\sqrt{2}}\right)\sin\frac{3\nu\pi}{2}\right].$$

Thus

$$x^\nu y^{m-\frac{\nu}{2}}\left[\mathrm{bei}_{2\nu}\left(2\sqrt{x\sqrt{2y}}\right)\cos\frac{3\nu\pi}{2}-\mathrm{ber}_{2\nu}\left(2\sqrt{x\sqrt{2y}}\right)\sin\frac{3\nu\pi}{2}\right]$$

$$\gg\frac{(-1)^m\sqrt{\pi}}{2^{m-\frac{\nu}{2}+\frac{1}{2}}}\cdot\frac{1}{p^{2\nu}q^m}\cdot e^{-\frac{1}{4p^2q}}\,D_{2m+1}\left(\frac{1}{p\sqrt{q}}\right). \qquad (2.24)$$

In particular

$$x^{2m}\,\mathrm{bei}_{4m}\left(2\sqrt{x\sqrt{2y}}\right)\gg\sqrt{\frac{\pi}{2}}\,\frac{(-1)^m}{p^{4m}q^m}\,e^{-\frac{1}{4p^2q}}\,D_{2m+1}\left(\frac{1}{p\sqrt{q}}\right) \qquad (2.241)$$

and

$$\mathrm{bei}\left(2\sqrt{x\sqrt{2y}}\right)\gg\sqrt{\frac{\pi}{2}}\,e^{-\frac{1}{4p^2q}}. \qquad (2.242)$$

Similarly, if we start with (2.14), we can obtain

$$x^\nu y^{m-\frac{1}{2}\nu-\frac{1}{2}}\left[\mathrm{ber}_{2\nu}\left(2\sqrt{x\sqrt{2y}}\right)\cos\frac{3\nu\pi}{2}+\mathrm{bei}_{2\nu}\left(2\sqrt{x\sqrt{2y}}\right)\sin\frac{3\nu\pi}{2}\right]$$

$$\gg\frac{(-1)^m\sqrt{\pi}}{2^{m-\frac{\nu}{2}}}\cdot\frac{1}{p^{2\nu}q^{m-\frac{1}{2}}}\,e^{-\frac{1}{4p^2q}}\,D_{2m}\left(\frac{1}{p\sqrt{q}}\right). \qquad (2.25)$$

In particular

$$x^{2m}y^{-\frac{1}{2}}\,\mathrm{ber}_{4m}\left(2\sqrt{x\sqrt{2y}}\right)\gg\sqrt{\pi}\cdot\frac{(-1)^m}{p^{4m}q^{m-\frac{1}{2}}}\,e^{-\frac{1}{4p^2q}}\,D_{2m}\left(\frac{1}{p\sqrt{q}}\right) \qquad (2.251)$$

and

$$\frac{1}{\sqrt{y}}\,\mathrm{ber}\left(2\sqrt{x\sqrt{2y}}\right)\gg\sqrt{\pi q}\,e^{-\frac{1}{4p^2q}}. \qquad (2.252)$$

(c) From

$$S(\nu,z)=\int_0^\infty e^{-zt}(1+t)^{-\nu}\,dt,$$

we have

$$\frac{1}{(1+x)^\nu}>p\cdot S(\nu,p).$$

Since

$$\frac{2^\nu \Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}} \cdot \frac{p}{(1+p^2)^{\nu+\frac{1}{2}}} < x^\nu J_\nu(x), \quad R(\nu) > -1,$$

we have,

$$x^\nu y^{-\frac{\nu}{2}} J_\nu(x\sqrt{y}) \geqslant \frac{2^\nu \Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}} \cdot \frac{q}{p^{2\nu-2}} \cdot S(\nu+\tfrac{1}{2}, p^2 q). \tag{2.26}$$

Since $e^{-p} S(1, p) = -E_i(-p)$, we obtain, in particular,

$$-pq\, e^{p^2 q} E_i(-p^2 q) \leqslant \frac{\sin x\sqrt{y}}{\sqrt{y}}. \tag{2.261}$$

Similarly, if we utilise

$$\frac{2^{\nu+1} \Gamma(\nu+\frac{3}{2})}{\sqrt{\pi}} \cdot \frac{p^2}{(1+p^2)^{\nu+\frac{3}{2}}} < x^{\nu+1} J_\nu(x), \qquad R(\nu) > -1,$$

we obtain,

$$x^{\nu+1} y^{-\frac{\nu}{2}} J_\nu(x\sqrt{y}) \geqslant \frac{2^{\nu+1} \Gamma(\nu+\frac{3}{2})}{\sqrt{\pi}} \frac{q}{p^{2\nu-1}} S(\nu+\tfrac{3}{2}, p^2 q). \tag{2.27}$$

In particular,

$$\cos(x\sqrt{y}) \geqslant -p^2 q\, e^{p^2 q} E_i(-p^2 q). \tag{2.271}$$

(4) *Applications of Cor. C*

(a) From

$$J_m(\tfrac{1}{2}x^2) > \pi^{-\frac{1}{2}} \Gamma(m+\tfrac{1}{2})\, p\, D_{-m-\frac{1}{2}}(p\, e^{\frac{1}{4}\pi i})\, D_{-m-\frac{1}{2}}(p\, e^{-\frac{1}{4}\pi i}), \quad R(m) > -\tfrac{1}{2}$$

and

$$J_m(\sqrt{x})\, I_m(\sqrt{x}) > J_m\left(\frac{1}{2p}\right),$$

we obtain from the theorem,

$$J_m(y\sqrt{x})\, I_m(y\sqrt{x}) \geqslant \pi^{-\frac{1}{2}} \Gamma(m+\tfrac{1}{2})\, q\sqrt{p}\, D_{-m-\frac{1}{2}}(q\sqrt{p}\, e^{\frac{1}{4}\pi i})\, D_{-m-\frac{1}{2}}(q\sqrt{p}\, e^{-\frac{1}{4}\pi i}). \tag{2.28}$$

(b) Again, if we proceed with

$$(-1)^n \sqrt{\pi}\, I_{n+\frac{1}{2}}(2x^2) > \tfrac{1}{2}\Gamma(n+1)\, p\, D_{-n-1}(\tfrac{1}{2}p)\, D_{-n-1}(-\tfrac{1}{2}p)$$

and

$$I_{n+\frac{1}{2}}\left(\frac{2}{p}\right) < \mathrm{ber}^2_{n+\frac{1}{2}}(2\sqrt{x}) + \mathrm{bei}^2_{n+\frac{1}{2}}(2\sqrt{x}),$$

we derive

$$\frac{(-1)^n 2\sqrt{\pi}}{\Gamma(n+1)} [\mathrm{ber}^2_{n+\frac{1}{2}}(2y\sqrt{x}) + \mathrm{bei}^2_{n+\frac{1}{2}}(2y\sqrt{x})] \geqslant q\sqrt{p}\, D_{-n-1}(\tfrac{1}{2}q\sqrt{p})\, D_{-n-1}(-\tfrac{1}{2}q\sqrt{p}). \tag{2.29}$$

(c) We have the operational relations

$$2\,p\,K_{2m}\,(\sqrt{2\,p}\,e^{\frac{1}{4}\pi i})\,K_{2m}\,(\sqrt{2\,p}\,e^{-\frac{1}{4}\pi i}) \subset x^{-1}\,K_{2m}\left(\frac{2}{x}\right),$$

deduced by a modification of McDonald's integral [5]; and

$$x^{-k}\,\exp\left(-\frac{1}{2\,x}\right) W_{k,m}\left(\frac{1}{x}\right) \supset 2\,p^{k+\frac{1}{2}}\,K_{2m}\,(2\sqrt{p}),$$

deduced from Goldstein's relation [9]

$$q^{\mu}\,K_{\nu}\,(q\,r) \subset \frac{1}{r\,t^{\frac{1}{2}(\mu-1)}}\,e^{-\frac{r^2}{4t}}\,W_{\frac{1}{2}\mu-\frac{1}{2},\,\frac{1}{2}\nu}\left(\frac{r^2}{4t}\right),$$

where $q^2 = p$ and r is complex with $|\arg r| < \frac{1}{4}\pi$.

So, on application of the theorem, we get

$$x^{-k}\,e^{-\frac{1}{2xy^2}}\,W_{k,m}\left(\frac{1}{xy^2}\right) \gtrdot 4\,p^{k+\frac{1}{2}}\cdot q\cdot K_{2m}\,(\sqrt{2\,q\sqrt{p}}\,e^{\frac{1}{4}\pi i})\,K_{2m}\,(\sqrt{2\,q\sqrt{p}}\,e^{-\frac{1}{4}\pi i}) \qquad (2.30)$$

with the following special cases

(i) $$\frac{e^{-\frac{1}{2xy^2}}\,K_m\left(\frac{1}{2\,xy^2}\right)}{y\sqrt{x}} \gtrdot 4\sqrt{\pi\,p}\cdot q\,K_{2m}\,(\sqrt{2\,q\sqrt{p}}\,e^{\frac{1}{4}\pi i})\,K_{2m}\,(\sqrt{2\,q\sqrt{p}}\,e^{-\frac{1}{4}\pi i}) \qquad (2.301)$$

(ii) $$\frac{1}{x^{2m+1}\,y^{2m+1}}\,e^{-\frac{1}{xy^2}} \gtrdot 4\,p^{m+1}\,q\,K_{2m}\,(\sqrt{2\,q\sqrt{p}}\,e^{\frac{1}{4}\pi i})\,K_{2m}\,(\sqrt{2\,q\sqrt{p}}\,e^{-\frac{1}{4}\pi i}) \qquad (2.302)$$

(iii) $$\frac{1}{x\,y}\,e^{-\frac{1}{xy^2}} \gtrdot 4\,p\,q\,K_0\,(\sqrt{2\,q\sqrt{p}}\,e^{\frac{1}{4}\pi i})\,K_0\,(\sqrt{2\,q\sqrt{p}}\,e^{-\frac{1}{4}\pi i}) \qquad (2.303)$$

(iv) $$e^{-\frac{1}{xy^2}} \gtrdot 4\,q\sqrt{p}\,K_1\,(\sqrt{2\,q\sqrt{p}}\,e^{\frac{1}{4}\pi i})\,K_1\,(\sqrt{2\,q\sqrt{p}}\,e^{-\frac{1}{4}\pi i}) \qquad (2.304)$$

(v) $$\frac{1}{\sqrt{x\,y}}\,e^{-\frac{1}{xy^2}} \gtrdot \pi\sqrt{2\,p\,q}\,e^{-2\sqrt{q\sqrt{p}}} \qquad (2.305)$$

and

$$x^{\mu-\frac{1}{2}}\,y^{-\frac{1}{2}}\,e^{-\frac{1}{2xy^2}}\,D_{-2\mu-\frac{1}{2}}\left(\sqrt{\frac{2}{xy^2}}\right) \gtrdot \pi\cdot 2^{\frac{1}{2}-\mu}\,p^{\frac{1}{2}-\mu}\,q^{\frac{1}{2}}\,e^{-2\sqrt{q\sqrt{p}}} \qquad (2.306)$$

where D_{ν} is the parabolic Cylinder function.

Since

$$D_{-1}(\sqrt{2x}) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{\frac{1}{2}x} (1 - \operatorname{erf} \sqrt{x}), \qquad \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du,$$

we have

$$y^{-\frac{1}{2}} \left[1 - \operatorname{erf}\left(\frac{1}{y\sqrt{\pi}}\right)\right] \geqslant \sqrt{2\pi q}\, e^{-2\sqrt{q}\sqrt{p}}. \tag{2.307}$$

(d) By taking $f(p) = \sqrt{p}\, E_i(-\sqrt{p})$, so that $h(x) = \frac{1}{2\sqrt{\pi x}} E_i\left(-\frac{1}{4x}\right)$, and $\sigma(x) = \frac{1}{4\pi\sqrt{x}} E_i\left(-\frac{1}{64x}\right)$ in the theorem, we easily obtain the operational representation

$$\frac{1}{\sqrt{xy}} E_i\left(-\frac{1}{64xy^2}\right) \geqslant 4\pi\sqrt{pq}\, E_i(-\sqrt{q\sqrt{p}}). \tag{2.31}$$

(e) We have the operational representations

$$\operatorname{ber}^{(2)}(3\sqrt[3]{x}) > \operatorname{ber}\left(\frac{2}{\sqrt{p}}\right) \text{ and } \operatorname{bei}^{(2)}(3\sqrt[3]{x}) > \operatorname{bei}\left(\frac{2}{\sqrt{p}}\right),$$

where $\operatorname{ber}^{(n)}$ and $\operatorname{bei}^{(n)}$ are defined by

$$I^{(n)}_{0,\ldots,0}(x\, i^{\frac{1}{n+1}}) = \operatorname{ber}^{(n)}(x) + i \operatorname{bei}^{(n)}(x), \qquad I^{(n)}_{0,\ldots,0}(x)$$

being the Hyperbesselian function (Delerue [4]) and

$$\operatorname{ber}(x) > \frac{p}{\sqrt{2}} \left[\frac{p^2}{1+p^4} + \frac{1}{\sqrt{1+p^4}}\right]^{\frac{1}{2}}$$

$$\operatorname{bei}(x) > \frac{p}{\sqrt{2}} \left[\frac{1}{\sqrt{1+p^4}} - \frac{p^2}{1+p^4}\right]^{\frac{1}{2}}.$$

The theorem then gives the operational relations

$$\operatorname{ber}^{(2)}\left[3\sqrt[3]{\tfrac{1}{4}x\sqrt{y}}\right] \geqslant \frac{q\sqrt{p}}{\sqrt{2}(1+p^2q^4)^{\frac{1}{2}}} [\sqrt{1+p^2q^4} + pq^2]^{\frac{1}{2}} \tag{2.32}$$

and

$$\operatorname{bei}^{(2)}\left[3\sqrt[3]{\tfrac{1}{4}x\sqrt{y}}\right] \geqslant \frac{q\sqrt{p}}{\sqrt{2}(1+p^2q^4)^{\frac{1}{2}}} [\sqrt{1+p^2q^4} - pq^2]^{\frac{1}{2}}. \tag{2.33}$$

(f) Let

$$f(p) = \sqrt{\pi}\, \Gamma(\tfrac{1}{2} - \mu)(\tfrac{1}{2}p)^{1+\mu} \{H_{-\mu}(p) - Y_{-\mu}(p)\}$$

so that

$$h(x) = \frac{1}{(1+x^2)^{\mu+\frac{1}{2}}}.$$

Then

$$\frac{1}{p^{2\lambda}} h\left(\frac{1}{\sqrt{p}}\right) = \frac{p^{\mu-2\lambda+\frac{1}{2}}}{(1+p)^{\mu+\frac{1}{2}}} < \frac{e^{-\frac{1}{2}x} x^{\lambda-\frac{1}{2}} M_{\mu-\lambda,\lambda}(x)}{\Gamma(1+2\lambda)}, \quad \lambda > -\tfrac{1}{2}, \quad \equiv \sigma(x) \text{ say.}$$

We then obtain

$$e^{-\frac{1}{2}xy^2} x^{\lambda-\frac{1}{2}} y^{-2\lambda-1} M_{\mu-\lambda,\lambda}(xy^2) \gg \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\mu)\,\Gamma(2\lambda+1)}{2^{\mu+1}} \cdot q^{1+\mu} \cdot p^{\frac{1}{2}(1+\mu-4\lambda)} \cdot$$

$$\cdot [H_{-\mu}(q\sqrt{p}) - Y_{-\mu}(q\sqrt{p})]. \qquad (2.34)$$

In particular,

$$x^{2m} e^{-xy^2} \gg \frac{\sqrt{\pi}\,\Gamma(-2m)\,\Gamma(1+2m)}{2^{2m+\frac{3}{2}}} p^{\frac{1}{2}(\frac{3}{2}-2m)} \cdot q^{\frac{3}{2}+2m} \cdot$$

$$\cdot [H_{-2m-\frac{1}{2}}(q\sqrt{p}) - Y_{-2m-\frac{1}{2}}(q\sqrt{p})], \quad -\tfrac{1}{2} < m < 0. \quad (2.341)$$

(g) Again, if we take

$$f(p) = \Gamma(m+\tfrac{1}{2})\, p^{\frac{1}{2}(n-m+1)} e^{\frac{1}{2}p}\, W_{-\frac{1}{2}(n+m),\,-\frac{1}{2}(n-m)}(p), \quad R(m) > -\tfrac{1}{2}$$

so that

$$h(x) = \frac{x^{m-\frac{1}{2}}}{(1+x)^{n+\frac{1}{2}}},$$

and

$$\sigma(x) = \sqrt{\frac{2}{\pi}}\,(2x)^{\frac{1}{2}(n-\frac{1}{2})} e^{\frac{x}{2}} D_{-n-\frac{1}{2}}(\sqrt{2x}), \quad R(n) > -\tfrac{3}{2},$$

in the theorem, we obtain

$$x^{\frac{1}{2}(n-\frac{1}{2})} y^{m-\frac{1}{2}} e^{\frac{1}{2}xy^2} D_{-n-\frac{1}{2}}(y\sqrt{2x})$$

$$\gg \frac{\sqrt{\pi}\,\Gamma(m+\frac{1}{2})}{2^{\frac{1}{2}(n+\frac{1}{2})}} \cdot p^{\frac{1}{2}(m-n+1)} q^{\frac{1}{2}(n-m+1)} e^{\frac{1}{2}q\sqrt{p}} \cdot W_{-\frac{1}{2}(n+m),\,-\frac{1}{2}(n-m)}(q\sqrt{p}) \qquad (2.35)$$

where $R(n-m) > -3$.

In particular,

$$y^{-\frac{1}{2}} e^{xy^2}[1 - \operatorname{erf}(y\sqrt{x})] \gg \pi p^{\frac{1}{2}} q\, e^{q\sqrt{p}}\left[1 - \operatorname{erf}\sqrt{q\sqrt{p}}\right]. \qquad (2.351)$$

3. In the following, we shall obtain the sum of certain series and evaluate certain definite integrals by the help of the symbolic calculus of two variables.

(1) We have

$$L_n^{\alpha}(x) = \sum_{r=0}^{n} \binom{n+\alpha}{n-r} \frac{(-x)^r}{r!} \tag{3.1}$$

and

$$L_n^{-\frac{1}{2}}(x) = \frac{(-1)^n}{2^n\, n!} e^{\frac{x}{2}} D_{2n}(\sqrt{2x}).$$

So, writing (3.1) in the form

$$\frac{1}{n!\, 2^{\frac{1}{6}(4k+1)}} \cdot \frac{(-1)^n}{2^{\frac{1}{6}(6n+2k-1)}} \cdot \frac{1}{p^k q^{n-\frac{1}{2}}} e^{-\frac{1}{4pq}} D_{2n}\left(\frac{1}{\sqrt{pq}}\right)$$

$$= \sum_{r=0}^{n} \frac{(-1)^n}{r!} \binom{n-\frac{1}{2}}{n-r} \cdot \frac{1}{(2p)^{k+r} q^{n+r-\frac{1}{2}}} e^{-\frac{1}{2pq}}$$

and interpreting both sides by (2.17) and Delerue's [4] operational representation

$$x^{\frac{2m-n}{3}} y^{\frac{2n-m}{3}} J_{m,n}(3\sqrt[3]{xy}) \gg \frac{1}{p^m q^n} e^{-\frac{1}{pq}},$$

we obtain, on reduction,

$$\frac{1}{n!} (\tfrac{1}{2} xy)^{\frac{n}{3}} J_{k,-\frac{1}{2}}(3\sqrt[3]{\tfrac{1}{2} xy}) = \sum_{r=0}^{n} \frac{(-1)^r}{r!} \binom{n-\frac{1}{2}}{n-r} (\tfrac{1}{2} xy)^{\frac{1}{3} r} J_{k+r,\, n+r-\frac{1}{2}}(3\sqrt[3]{\tfrac{1}{2} xy}).$$

This leads to the expansion

$$\frac{1}{n!} x^n J_{k,-\frac{1}{2}}(3x) = \sum_{r=0}^{n} \frac{(-1)^r}{r!} \binom{n-\frac{1}{2}}{n-r} x^r J_{k+r,\, n+r-\frac{1}{2}}(3x). \tag{3.2}$$

Similarly, if we proceed with (3.1) and

$$L_n^{\frac{1}{2}}(x) = \frac{(-1)^n}{2^{n+\frac{1}{2}}\, n!} x^{-\frac{1}{2}} e^{\frac{1}{2}x} D_{2n+1}(\sqrt{2x}),$$

and interpret by the help of (2.18) and Delerue's result quoted above, we obtain

$$\frac{2}{n!} (\tfrac{1}{2} xy)^{\frac{1}{3}(n+1)} J_{\frac{1}{3}(2k+1),\, \frac{1}{3}}(3\sqrt[3]{\tfrac{1}{2} xy}) = \sum_{r=0}^{n} \frac{(-1)^r}{r!} \binom{n+\frac{1}{2}}{n-r} (\tfrac{1}{2} xy)^{\frac{1}{3} r} J_{k+r-\frac{1}{2},\, n+r-\frac{1}{2}}(3\sqrt[3]{\tfrac{1}{2} xy}).$$

This gives the expansion

$$\frac{2}{n!} x^{n+1} J_{\frac{1}{3}(2k+1),\, \frac{1}{3}}(3x) = \sum_{r=0}^{n} \frac{(-1)^r}{r!} \binom{n+\frac{1}{2}}{n-r} x^r J_{k+r-\frac{1}{2},\, n+r-\frac{1}{2}}(3x). \tag{3.3}$$

(2) We have

$$\frac{1}{p^m q^n} e^{-(\alpha+\beta)\frac{1}{pq}} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \beta^r \frac{1}{p^{m+r} q^{n+r}} e^{-\frac{\alpha}{pq}}.$$

Hence, on interpretation, by Delerue's operational representation and subsequent simplification, we obtain

$$J_{m,n}[3\sqrt[3]{(\alpha+\beta)xy}] = \left(1+\frac{\beta}{\alpha}\right)^{\frac{m+n}{3}} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{\beta}{\sqrt[3]{\alpha^2}}\right)^r (xy)^{\frac{1}{3}r} J_{m+r,\, n+r}(3\sqrt[3]{\alpha x y}).$$

This can be written in the form

$$J_{m,n}[3x\sqrt[3]{\alpha+\beta}] = \left(1+\frac{\beta}{\alpha}\right)^{\frac{m+n}{3}} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{\beta}{\sqrt[3]{\alpha^2}}\right)^r x^r J_{m+r,\, n+r}(3x\sqrt[3]{\alpha}). \tag{3.4}$$

Where, for convergence of the series on the right, we take $\beta^3 < \alpha^2$.

(3) We have

$$\frac{1}{pq} \cdot \frac{p^{\mu(\nu+1)-\lambda} \cdot q}{(1+p^\mu q)^{\nu+1}} \cdot \frac{p^{\mu(\beta+1)-\gamma} \cdot q}{(1+p^\mu q)^{\beta+1}} = \frac{p^{\mu(\nu+\beta+2)-(\lambda+\gamma+1)} \cdot q}{(1+p^\mu q)^{\nu+\beta+2}}.$$

So, on interpretation by the product theorem and the operational representation (2.1) we obtain

$$\int_0^x \int_0^y (x-\xi)^\lambda (y-\eta)^\nu \xi^\gamma \eta^\beta J_\lambda^\mu[(x-\xi)^\mu (y-\eta)] J_\gamma^\mu(\xi^\mu \eta)\, d\xi\, d\eta$$
$$= \frac{\Gamma(\beta+1)\Gamma(\nu+1)}{\Gamma(\nu+\beta+2)} x^{\lambda+\gamma+1} y^{\nu+\beta+1} J_{\lambda+\gamma+1}^\mu(x^\mu y). \tag{3.5}$$

In particular,

$$\int_0^x \int_0^y (x-\xi)^{\frac{\lambda+1}{3}} \xi^{\frac{\gamma+1}{3}} (y-\eta)^{\frac{1}{6}(6\nu-2\lambda+1)} \cdot \eta^{\frac{1}{6}(6\beta-2\gamma+1)} \cdot J_{\frac{\lambda}{2}-\frac{1}{2},\frac{\lambda}{2}}[3\sqrt[3]{\tfrac{1}{4}(x-\xi)^2(y-\eta)}] \cdot$$
$$\cdot J_{\frac{\gamma}{2}-\frac{1}{2},\frac{\gamma}{2}}[3\sqrt[3]{\tfrac{1}{4}\xi^2\eta}]\, d\xi\, d\eta$$
$$= \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(\beta+1)\Gamma(\nu+1)}{\Gamma(\nu+\beta+2)} x^{\frac{\lambda+\gamma+2}{3}} y^{\frac{1}{6}(6\nu+6\beta-2\lambda-2\gamma+5)} \cdot J_{\frac{\lambda+\gamma-1}{2},\, \frac{\lambda+\gamma+1}{2}}\left\{3\sqrt[3]{\tfrac{1}{4}x^2 y}\right\}. \tag{3.51}$$

Department of Mathematics, Central Calcutta College, Calcutta.

References

[1]. E. M. Wright, *Proc. Lond. Math. Soc.* 46 (1940).
[2]. H. C. Gupta, *Proc. Nat. Inst. Sci. India* 14 (1948), 131–156.
[3]. N. A. Shastri, *Proc. Ind. Acad. Sci.*, 3 (1944), 211.
[4]. P. Delerue, *Sur le calcul symbolique à n variables et sur les fonctions hyperbesseliennes.* Thèse. Montpellier, 1951.
[5]. G. N. Watson, *Bessel Functions*, 1948.
[6]. E. T. Whittaker and G. N. Watson, *Course of Modern Analysis*, 1940.
[7]. S. C. Mitra, *Proc. Edin. Math. Soc.* 4 (1933), 33.
[8]. N. W. Mc Lachlan, *Modern Operational Calculus*, 1948.
[9]. S. Goldstein, *Proc. Lond. Math. Soc.* 34 (1932).