THE STRUCTURE OF FINITE DIMENSIONAL BANACH SPACES WITH THE 3.2. INTERSECTION PROPERTY

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1. Introduction

Let X be a Banach space over the real numbers. Let n and k be integers with $2 \leqslant k < n$. We say that X has the $n.k.$ intersection property ($n.k.$I.P.) if the following holds:

Any n balls in X intersect provided any k of them intersect.

In [2], O. Hanner characterized finite dimensional spaces with the 3.2.I.P. by the facial structure of their unit hall. He also proved that this property is preserved under l_1- and l_∞-summands, i.e. direct sums $X \oplus Y$ with the l_1-norm $\|x\| + \|y\|$ or the l_∞-norm $\max(\|x\|, \|y\|)$. We shall prove the converse of this result. Any finite dimensional Banach space X with the 3.2.I.P. is obtained from the real line by repeated l_1- and l_∞-summands. Hanner proved this for dimension at most 5.

In sections 2 to 4 we gradually introduce the concepts and theorems that we need. To become familiar with the techniques involved, we have included the proof of some of the results. In sections 5 and 6 we prove some technical lemmas and characterize the parallel-faces and split-faces among the faces of the unit balls of Banach spaces with the 3.2.I.P. These results are used in the proof of the main result in section 7.

Banach spaces are denoted X, Y, and Z. The closed ball in X with center x and radius r is denoted $B(x, r)$, but for the unit ball we write $X_1 = B(0, 1)$. The dual space of X is written X^*. The convex hull of a set S is written conv (S) and the set of extreme points

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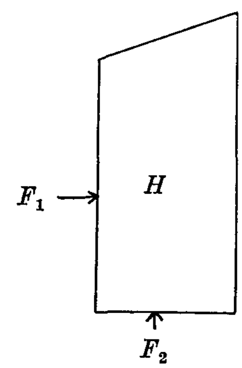

Fig. 2.1

of a convex set F is written $\partial_e F$. $(X \oplus Y)_{l_1}$ and $(X \oplus Y)_{l_\infty}$ denotes the direct sum of X and Y with the norms $\|(x, y)\| = \|x\| + \|y\|$ and $\|(x, y)\| = \max(\|x\|, \|y\|)$ respectively.

All spaces are assumed to be real.

2. Faces of the unit ball

If M is a subset of the unit ball X_1 of X, we denote by face (M) the smallest face of X_1 containing M. Recall the following fact:

LEMMA 2.1. *Let $M \subseteq X_1$ and let $y \in X_1$. Then the following two statements are equivalent:*

(1) $y \in \text{face}(M)$

(2) *There exist $x \in \text{conv}(M)$, $z \in X_1$ and $\alpha \in (0, 1]$ such that*

$$x = \alpha y + (1-\alpha)z.$$

The notion of parallel-faces will play a central role throughout this paper.

Definition 2.2. Let F and H be faces of X_1 with $F \subseteq H$. F is called a *parallel-face* of H if there exists another face G of H such that the following conditions are satisfied:

(1) $F \cap G = \emptyset$

(2) $H = \text{conv}(F \cup G)$

(3) Whenever $x_1, x_2 \in F$, $y_1, y_2 \in G$ and $\lambda_1, \lambda_2 \in [0, 1]$ are such that

$$\lambda_1 x_1 + (1-\lambda_1) y_1 = \lambda_2 x_2 + (1-\lambda_2) y_2$$

then $\lambda_1 = \lambda_2$.

Example 2.3. Assume H is the face in Fig. 2.1. Clearly F_1 is a parallel-face of H. The face F_2 satisfies (1) and (2) but not (3) in definition 2.2.

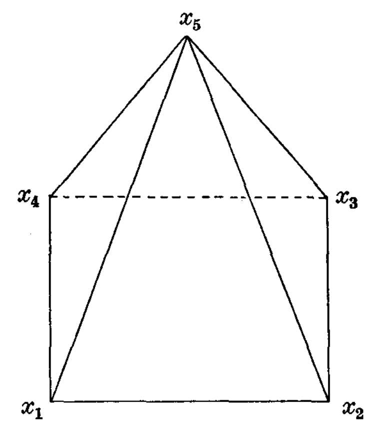

Fig. 2.2

It follows from Theorem 3.6 that if X has the 3.2.I.P., then (3) is a consequence of (1) and (2) in definition 2.2.

We denote by $P(H)$ the set of all proper parallel-faces of H when H is a face of X_1. $P_M(H)$ is the set of all maximal (with respect to inclusion) proper parallel-faces of H.

Definition 2.4. Let F and H be faces of X_1 such that $F \subseteq H$. F is called a *split-face* of H, if there exists another face G of H, such that the following conditions are satisfied:

(1) $F \cap G = \emptyset$

(2) $H = \operatorname{conv}(F \cup G)$

(3) Whenever $x_1, x_2 \in F$, $y_1, y_2 \in G$ and $\lambda_1, \lambda_2 \in [0, 1]$ are such that

$$\lambda_1 x_1 + (1-\lambda_1) y_1 = \lambda_2 x_2 + (1-\lambda_2) y_2$$

then $\lambda_1 = \lambda_2$ and if $\lambda_1 \neq 0, 1$ then also $x_1 = x_2$ and $y_1 = y_2$.

Obviously every split-face is also a parallel-face. The opposite is not true.

Example 2.5. Let $X = (l_\infty^3 \oplus \mathbf{R})_{l_1}$, and let H be the following maximal proper face of X_1: $H = \operatorname{conv}(x_1, x_2, x_3, x_4, x_5)$ where $x_1 = (1, 1, 1, 0)$, $x_2 = (1, -1, 1, 0)$, $x_3 = (1, -1, -1, 0)$, $x_4 = (1, 1, -1, 0)$ and $x_5 = (0, 0, 0, 1)$.

The vertex $\{x_5\}$ is a split-face of H, $\operatorname{conv}(x_1, x_2)$ is a parallel-face but not a split-face of H, and $\operatorname{conv}(x_1, x_5)$ is neither. See Fig. 2.2.

When F and H are faces of X_1 with $F \subseteq H$, we denote by F'_H the set

$$F'_H = \{x \in H: \operatorname{face}(x) \cap F = \emptyset\}.$$

Note that if F is norm-complete, then $H = \text{conv}\,(F \cup F'_H)$ [1]. If F is a parallel-face of H, then necessarily $H = \text{conv}\,(F \cup F'_H)$ and F'_H is convex. In fact, $F'_H = G$ in definition 2.2. Example 2.3 shows that F'_H can be convex even though F is not a parallel-face. Usually F'_H is non-convex. F'_H is convex if and only if it is a face.

THEOREM 2.6, [3]. *Let H be a face of X_1. Let F be a split-face of H and assume M is a face of F and N is a face of F'_H. Then* conv $(M \cup N)$ *is a face of H.*

Proof. Show this or look at [3].

Definition 2.7. Let F be a proper face of X_1. F is called an M-face if there exists a $G \in P_M(F)$ such that $G'_F \in P_M(F)$.

If H is a proper face of X_1, we denote by $m(H)$ the following number (if it exists)

$$m(H) = \sup\,\{\dim \operatorname{span} F\colon F \text{ is an } M\text{-face of } H\}.$$

$m(X)$ denotes the number (if it exists)

$$m(X) = \sup\,\{m(H)\colon H \text{ a proper face of } X_1\}.$$

Example 2.8. (a) Let H be as in example 2.5. The largest M-face of H is $F = \text{conv}\,(x_1, x_2, x_3, x_4)$, so $m(H) = \dim \operatorname{span} F = 3 < 4 = \dim X$.

(b) Let $X = (l_1^3 \oplus \mathbf{R})_{l_\infty}$. Let $F = \text{conv}\,(x_1, \ldots, x_6)$ where $x_1 = (1, 0, 0, 1)$, $x_2 = (0, 1, 0, 1)$, $x_3 = (0, 0, 1, 1)$, $x_4 = (1, 0, 0, -1)$, $x_5 = (0, 1, 0, -1)$, $x_6 = (0, 0, 1, -1)$. Then F is a maximal proper face of X_1. Both $G = \text{conv}\,(x_1, x_2, x_3) \in P_M(F)$ and $G'_F = \text{conv}\,(x_4, x_5, x_6) \in P_M(F)$. Hence F is an M-face. We have

$$m(F) = \dim \operatorname{span} F = 4 = \dim X.$$

(a) and (b) should be compared with the main result Theorem 7.3.

Definition 2.10. X is called a CL-*space* if $X_1 = \text{conv}\,(F \cup -F)$ whenever F is a maximal proper face of X_1.

PROPOSITION 2.11, [7]. *Let X be a finite dimensional space. Then the following statements are equivalent:*

(1) *X is a* CL-*space.*
(2) *For all $x \in \partial_e X_1$ and $f \in \partial_e X_1^*$, $f(x) = \pm 1$.*
(3) *X^* is a* CL-*space.*

Example 2.12. (a) If $X = l_\infty^n$ or $X = l_1^n$, then X is a CL-space.

(b) Assume X is a finite dimensional CL-space and let $\{f_1, \ldots, f_n\} \subseteq \partial_e X_1^*$ be a basis for X^*. Then the mapping $T: X \to l_\infty^n$ defined by

$$T(x) = (f_1(x), \ldots, f_n(x))$$

is a linear isomorphism that maps every extreme point of X_1 to a corner of the n-cube $(l_\infty^n)_1$. Hence the unit ball X_1 can be obtained as a convex hull of some subset of $\partial_e (l_\infty^n)_1$, where $n = \dim X$. This was observed in [7].

3. Intersection properties

Definition 3.1. Let n and k be integers with $2 \leqslant k < n$. X is said to have the $n.k.$ intersection property ($n.k.$I.P.) if the following condition is satisfied:

Any n balls in X intersect provided any k of them intersect.

Example 3.2. (a) Let $\{[x_i, y_i]\}_{i=1}^n$ be a set of n balls in $\mathbf{R}$ with $x_i \leqslant y_i$ for all i. If they intersect mutually, then $x_i \leqslant y_j$ for all i and j, such that there exists an $x \in \mathbf{R}$ with $x_i \leqslant x \leqslant y_j$ for all i and j. Thus $x \in \bigcap_{i=1}^n [x_i, y_i]$. Hence the real line has the $n.2.$I.P. for every $n \geqslant 2$. It follows that $\mathbf{R}$ has the $n.k.$I.P. for all $n > k \geqslant 2$.

(b) It follows from Helly's theorem that every Banach space X with $n = \dim X < \infty$ has the $(n+2).(n+1).$I.P.

We refer to [7] for an extensive study of the intersection properties. Let us mention here without proof the following results.

THEOREM 3.3, [7]. *X has the* 4.2.I.P. *if and only if X^* is isometric to the space $L_1(\mu)$ for some measure μ.*

COROLLARY 3.4, [7]. *Assume X is finite dimensional. X has the* 4.2.I.P. *if and only if $X = l_\infty^n$ where $n = \dim X$.*

THEOREM 3.5, [6]. *Assume X is finite dimensional. X has the* 4.3.I.P. *if and only if $X = (E_1 \oplus \ldots \oplus E_p)_{l_\infty}$ where $\dim E_i \in \{1, 2\}$.*

In the following we shall be concerned only with the 3.2.I.P. Hanner characterized the finite dimensional spaces with the 3.2.I.P. by their facial properties [2]. The following theorem which extends Hanner's results was proved by Lima.

THEOREM 3.6, [5]. *If X is a real Banach space, then the following statements are equivalent:*

(1) *X has the* 3.2.I.P.

(2) *If $x, y \in X$ with $\|x\| = \|y\| = 1$ and* face $(x) \cap$ face $(y) = \emptyset$, *then $\|x-y\| = 2$.*

(3) *If F_1 and F_2 are disjoint faces of X_1, then there exists a proper face F of X_1, such that $F_1 \subseteq F$, $F_2 \subseteq -F$ and $X_1 =$* conv $(F \cup -F)$.

(4) *If $x, y \in X$, then there exist $z, u, v \in X$ such that*

$$\begin{aligned} x &= z+u, & \|x\| &= \|z\| + \|u\| \\ y &= z+v, & \|y\| &= \|z\| + \|v\| \\ & & \|x-y\| &= \|u\| + \|v\| \end{aligned}$$

(5) *X^* has the* 3.2.I.P.

COROLLARY 3.7, [4]. *If X has the* 3.2.I.P., *then X is a* CL-*space.*

Example 3.8. (a) Since l_∞^n has the 3.2.I.P., we get from (5) of Theorem 3.6 that l_1^n has the 3.2.I.P.

(b) Assume Y and Z have the 3.2.I.P. Then $(Y \oplus Z)_{l_1}$ has the 3.2.I.P. by (4) of Theorem 3.6 and $(Y \oplus Z)_{l_\infty}$ has the 3.2.I.P. by (1) of Theorem 3.6.

PROPOSITION 3.9. *Assume X is a finite dimensional* CL-*space and that F and H are proper faces of X_1 such that $F \subseteq H$. If F is a maximal proper face of H, then F is a parallel-face of H.*

Proof. Since F is a proper face of H, there exists an $x \in \partial_e H \setminus F$. By Proposition 2.11 and [5; Proposition 3.2], there exists an $f \in \partial_e X_1^*$ such that $f(x) = -1$ and $f = 1$ on F. Let $G = \{y \in H: f(y) = 1\}$ and $M = \{y \in H: f(y) = -1\}$. By Proposition 2.11, we get $H =$ conv $(G \cup M)$. Hence G is a parallel-face of H. Since $F \subseteq G$ and F is a maximal proper face of H, we get $F = G$, such that F is a parallel-face of H.

PROPOSION 3.10. *Assume X is a finite dimensional space with the* 3.2.I.P. *and that F and H are proper faces of X_1 such that $F \subsetneqq H$. Then the following statements are equivalent:*

(1) *F is a parallel-face of H.*

(2) *F'_H is convex.*

(3) *There exists $f \in \partial_e X_1^*$ such that $F = \{x \in H: f(x) = 1\}$.*

Proof. Note that F'_H is convex if and only if it is a face. Since dim $X < \infty$, we always have $H =$ conv $(F \cup F'_H)$. It follows from Theorem 3.6 that if (1) and (2) in definition 2.2 is satisfied, then (3) is also satisfied. Now the equivalence of (1), (2), and (3) is obvious.

PROPOSITION 3.11. *Let X be a finite dimensional space with the* 3.2.I.P. *and let F be a proper face of X_1. Then $Y = \operatorname{span} F$ is a* CL*-space.*

Proof. Let $x \in Y$ with $\|x\| = 1$. Then we can write $x = y - z$ where $y, z \in \operatorname{cone}(F) = \bigcup_{\lambda \geqslant 0} \lambda F$. By (4) of Theorem 3.6 we may assume $\|x\| = \|y\| + \|z\|$. Hence $Y_1 = \operatorname{conv}(F \cup -F)$, and $\partial_e Y_1 = \partial_e F \cup -\partial_e F \subseteq \partial_e X_1$.

Let $x \in \partial_e Y_1$ and let $f \in \partial_e Y_1^*$. By the Hahn-Banach theorem, there exists a $g \in \partial_e X_1^*$ such that $g|_Y = f$. Hence, we get $f(x) = \pm 1$. Thus Y is a CL-space.

That most CL-spaces do not have the 3.2.I.P. was known by Hanner [2]. Here is an example which shows that Y in Proposition 3.11 need not have the 3.2.I.P.

Example 3.12. Let $X = (l_1^3 \oplus l_1^3)_{l_\infty}$. Let $f = (1, 1, 1, 0, 0, 0)$ and $g = (0, 0, 0, 1, 1, 1) \in \partial_e X_1^*$, and define a face G of X_1 by

$$G = \{x \in X_1 : f(x) = 1 = g(x)\}.$$

Then

$$G = \{(t_1, \ldots, t_6) \in X_1 : t_1 + t_2 + t_3 = t_4 + t_5 + t_6 = 1\}.$$

$Y = \operatorname{span} G$ is a CL-space by Proposition 3.11. Consider the following extreme points of G: $x_1 = (1, 0, 0, 1, 0, 0)$, $x_2 = (0, 1, 0, 0, 1, 0)$, $y_1 = (0, 0, 1, 0, 1, 0)$, $y_2 = (1, 0, 0, 0, 0, 1)$, $z_1 = (0, 0, 1, 1, 0, 0)$, and $z_2 = (0, 1, 0, 0, 0, 1)$. Then we have

$$x_1 + (y_1 - y_2) = x_2 + (z_1 - z_2)$$

and it is easy to see that (in Y)

$$\operatorname{face}\left(\frac{y_1 - y_2}{2}\right) \cap \operatorname{face}\left(\frac{z_1 - z_2}{2}\right) = \varnothing.$$

By (2) of Theorem 3.6, we get that Y does not have the 3.2.I.P.

4. L- and M-summands

Definition 4.1. Let P be a projection in X.

(1) P is called an *L-projection* if for all $x \in X$,

$$\|x\| = \|Px\| + \|x - Px\|.$$

(2) P is called an *M-projection* if for all $x \in X$,

$$\|x\| = \max(\|Px\|, \|x - Px\|).$$

(3) The range of an L-projection is called an *L-summand* of X.

(4) The range of an M-projection is called an *M-summand* of X.

Observe that if P is an L-projection in X, then $X=(Y\oplus Z)_{l_1}$ where $Y=P(X)$ and $Z=(I-P)(X)$. Similarly, if P is an M-projection in X, then $X=(Y\oplus Z)_{l_\infty}$ where $Y=P(X)$ and $Z=(I-P)(X)$. The following proposition was proved by Alfsen and Effros in [1]:

PROPOSITION 4.2, [1]. *Let P be a projection in X. Then P is an L-projection in X if and only if P^* is an M-projection in X^*.*

The same paper of Alfsen and Effros contains the following result.

PROPOSITION 4.3, [1]. *Assume X_1 contains a maximal proper face K such that $X_1=$ conv $(K\cup -K)$. Then the map $F\to$span F is a one-to-one correspondence between the proper split-faces of K and the proper L-summands of X.*

Since we will use one half of this result in section 7, we will indicate the proof of this part here.

So assume F is a proper split face of K. It follows from the definition of a split-face that $K=\operatorname{conv}(F\cup F'_K)$. (In fact, $F'_K=G$ in definition 2.4). Define $Y=\operatorname{span} F$ and $Z=\operatorname{span} F'_K$. Then $X=Y+Z$. Assume $x\in Y\cap Z$. Then $x=y_1-y_2=z_1-z_2$ where $y_1, y_2\in\operatorname{cone}(F)$ and $z_1, z_2\in\operatorname{cone}(F'_K)$. Hence $y_1+z_2=y_2+z_1$. Using that the norm is additive on cone (K) and (3) in definition 2.4 we get $y_1=y_2$. Hence $x=0$ and $Y\cap Z=(0)$. Thus $X=Y\oplus Z$.

Let now $y\in Y$ and $z\in Z$ and $x=y+z$. We can write $x=x_1-x_2$ where $x_1, x_2\in\operatorname{cone}(K)$ and $\|x\|=\|x_1\|+\|x_2\|$. Then use that $K=\operatorname{conv}(F\cup F'_K)$ and write $x_i=y_i+z_i$ where $y_i\in\operatorname{cone}(F)$ and $z_i\in\operatorname{cone}(F'_K)$; $i=1, 2$. Then $x=y+z=(y_1-y_2)+(z_1-z_2)$. Since $X=Y\oplus Z$, we get $y=y_1-y_2$ and $z=z_1-z_2$. Using that the norm is additive on cone (K) now gives

$$\begin{aligned}\|x\| = \|x_1\|+\|x_2\| &= \|y_1+z_1\|+\|y_2+z_2\| \\ &= \|y_1\|+\|z_1\|+\|y_2\|+\|z_2\| \\ &\geqslant \|y\|+\|z\| = \|x\|.\end{aligned}$$

Thus $X=(Y\oplus Z)_{l_1}$ and Y is the range of an L-projection in X.

PROPOSITION 4.4. *If X has the* 3.2.I.P. *and Y is an L- or M-summand of X, then Y also has the* 3.2.I.P.

Proof. Use that Y is the range of a norm-one projection in X.

5. The spaces $H^n(X)$

Definition 5.1. Let $n>2$ be an integer. We denote by $H^n(X)$ the space

$$H^n(X)=\left\{\mathbf{x}=(x_1, \ldots, x_n)\in X^n\colon \sum_{i=1}^n x_i=0\right\}$$

equipped with the norm

$$\|\mathbf{x}\| = \sum_{i=1}^{n} \|x_i\| \text{ for } \mathbf{x} = (x_1, \ldots, x_n) \in H^n(X)$$

Clearly $H^n(X)$ is a closed subspace of $(X \oplus \ldots \oplus X)_{l_1^n}$. In dealing with the intersection properties mentioned in section 3 the spaces $H^n(X)$ have shown to be very useful. This stems from Theorem 5.2 below which was proved in [4]. This theorem translates the intersection properties of balls in X into properties of the set of extreme points of the unit balls of the spaces $H^n(X^*)$. We shall refer to the following subsets of $H^n(X)$: For i, j integers with $1 \leqslant i < j \leqslant n$, let $S^n_{i,j}$ be defined by

$$S^n_{i,j} = \{\mathbf{x} = (x_1, \ldots, x_n) \in H^n(X) \colon \|\mathbf{x}\| = 1 \quad \text{and} \quad x_k = 0 \quad \text{when } k \neq i, j\}.$$

For a proof of the following theorem we refer to [4].

THEOREM 5.2, [4]. *Let $n > 2$ be an integer. The following statements are equivalent:*

(1) *X has the* n.2.I.P.

(2) $\partial_e H^n(X^*)_1 \subseteq \bigcup \{S^n_{i,j} \colon 1 \leqslant i < j \leqslant n\}$.

Example 5.3. Let $X = \mathbf{R}$. Then $H^3(X)$ is a subspace of l_1^3 of co-dimension 1. $H^3(X)_1$ is the convex hull of $\{\mathbf{x}_1, \ldots, \mathbf{x}_6\}$ where $\mathbf{x}_1 = (2^{-1}, 0, -2^{-1})$, $\mathbf{x}_2 = (2^{-1}, -2^{-1}, 0)$, $\mathbf{x}_3 = (0, -2^{-1}, 2^{-1})$, $\mathbf{x}_4 = -\mathbf{x}_1$, $\mathbf{x}_5 = -\mathbf{x}_2$ and $\mathbf{x}_6 = -\mathbf{x}_3$. Hence $H^3(X)_1$ is a regular hexagon. Since $\{\mathbf{x}_1, \ldots, \mathbf{x}_6\} \subseteq S^3_{1,2} \cup S^3_{1,3} \cup S^3_{2,3}$, X^* has the 3.2.I.P. by Theorem 5.2. This agrees with our earlier observations. In the same way, we can show that $\mathbf{R}$ has the n.2.I.P. for all $n \geqslant 3$.

Here we shall be concerned only with the 3.2.I.P. Let us include a proof of the following result.

LEMMA 5.4, [4]. *Assume X has the* 3.2.I.P. *and that* $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \partial_e H^4(X)_1$ *with* $\mathbf{x} \notin (\bigcup_{1 \leqslant i < j \leqslant 4} S^4_{i,j})$. *Then the following statements hold:*

(1) $\|x_i\| = 4^{-1}$ *for* $i = 1, 2, 3, 4$.

(2) $\|x_i + x_j\| = 2^{-1}$ *for* $1 \leqslant i \leqslant j \leqslant 4$.

(3) face $(4x_i) \cap$ face $(-4x_j) = \emptyset$ *in* X_1 *for* $1 \leqslant i < j \leqslant 4$.

Proof. By Theorem 3.6 there exist $z, u, v \in X$ such that

$$\begin{aligned} x_1 &= z + u, & \|x_1\| &= \|u\| + \|z\| \\ -x_2 &= z + v, & \|x_2\| &= \|v\| + \|z\| \\ & & \|x_1 + x_2\| &= \|u\| + \|v\|. \end{aligned}$$

Thus

$$\mathbf{x} = (x_1, x_2, x_3, x_4) = (u, -v, x_3, x_4) + (z, -z, 0, 0)$$

and

$$\begin{aligned}\|\mathbf{x}\| &= \sum_{i=1}^{4} \|x_i\| \\ &= \|u\| + \|v\| + \|x_3\| + \|x_4\| + 2\|z\| \\ &= \|(u, -v, x_3, x_4)\| + \|(z, -z, 0, 0)\|.\end{aligned}$$

Since $\mathbf{x} \notin S^4_{1,2}$, we get $z=0$. Hence $\|x_1+x_2\| = \|u\| + \|v\| = \|x_1\| + \|x_2\|$. By symmetry $\|x_i+x_j\| = \|x_i\| + \|x_j\|$ for all $i, j = 1, 2, 3, 4$. This now gives

$$\|x_1\| + \|x_2\| = \|x_1+x_2\| = \|x_3+x_4\| = \|x_3\| + \|x_4\|$$

and

$$\|x_1\| + \|x_3\| = \|x_1+x_3\| = \|x_2+x_4\| = \|x_2\| + \|x_4\|.$$

Adding these equations gives $\|x_1\| = \|x_4\|$. By symmetry and the fact that $\|\mathbf{x}\| = 1$, (1) and (2) follows. (3) follows from

$$\|x_i+x_j\| = \|x_i\| + \|x_j\|.$$

The next result will be used several times in sections 6 and 7.

THEOREM 5.5. *Assume X is finite dimensional with the* 3.2.I.P. *Assume* $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \partial_e H^4(X)_4$ *with all* $x_i \neq 0$. *Then there exists a* $\mathbf{y} = (y_1, y_2, y_3, y_4) \in \partial_e H^4(X)_4$ *such that all* $y_i \in \partial_e X_1$ *and* $y_i \in \operatorname{face}(x_i)$ *for* $i = 1, 2, 3$.

Proof. $\|x_i\| = 1$ for all i by Lemma 5.4. Choose $y_1 \in \partial_e \operatorname{face}(x_1) \subseteq \partial_e X_1$. Then by Lemma 2.1 there exist $\alpha_1 \in (0, 1]$ and $z_1 \in X_1$ such that

$$x_1 = \alpha_1 y_1 + (1-\alpha_1) z_1.$$

Define

$$\mathbf{z} = (\alpha_1 y_1, x_2, x_3, (1-\alpha_1) z_1 + x_4).$$

By (2) of Lemma 5.4 we get $\|(1-\alpha_1)z_1 + x_4\| = (1-\alpha_1)+1$. Hence $\|\mathbf{z}\| = 4$, such that $\mathbf{z} \in H^4(X)_4$. If $4^{-1}\mathbf{z} \in \operatorname{conv}(\{S^4_{i,j}: 1 \leqslant i < j \leqslant 4\})$, then we can write $\mathbf{z}$ as follows

$$\begin{aligned}\mathbf{z} &= (\alpha_1 y_1, x_2, x_3, (1-\alpha_1) z_1 + x_4) \\ &= (b_1, -b_1, 0, 0) + (b_2, 0, -b_2, 0) + (b_3, 0, 0, -b_3) \\ &\quad + (0, b_4, -b_4, 0) + (0, b_5, 0, -b_5) + (0, 0, b_6, -b_6)\end{aligned}$$

where $\alpha_1 = \|\alpha_1 y_1\| = \|b_1\| + \|b_2\| + \|b_3\|$, $\|x_2\| = \|b_1\| + \|b_4\| + \|b_5\|$ and so on. By Lemma 5.4 face $(x_i) \cap$ face $(-x_j) = \emptyset$. Hence we must have $b_1 = b_2 = b_4 = 0$. But then

$$\begin{aligned} 2-\alpha_1 &= \|(1-\alpha_1)z_1 + x_4\| \\ &= \|b_3\| + \|b_5\| + \|b_6\| \\ &= \|\alpha_1 y_1\| + \|x_2\| + \|x_3\| \\ &= 2+\alpha_1 \end{aligned}$$

which is a contradiction since $\alpha_1 > 0$. Hence there exists a $\mathbf{y}^1 = (y_1^1, y_2^1, y_3^1, y_4^1) \in \text{face } (\mathbf{z}) \cap \partial_e H^4(X)_4$ with $\|y_i^1\| = 1$ for all i. Clearly $y_1^1 = y_1$ and $y_1^1 \in \text{face } (x_i)$ for $i = 2, 3$. We repeat the procedure on the second coordinate of $\mathbf{y}^1$, and then one more time on the third coordinate and find $\mathbf{y} = (y_1, y_2, y_3, y_4) \in \partial_e H^4(X)_4$ with $\|y_i\| = 1$ for all i and $y_i \in \partial_e$ face (x_i) for $i = 1, 2, 3$. Then clearly $y_4 = -y_1 - y_2 - y_3 \in \partial_e X_1$ since X is a CL-space.

It is a consequence of the main result Theorem 7.3, that if $\mathbf{x}$ is as in Theorem 5.5, then already $x_i \in \partial_e X_1$ for all i.

6. Characterizations of parallel- and split-faces

In this section X is a finite dimensional Banach space with the 3.2.I.P. and K is a proper face of X_1. With the tools of the previous chapters in hand, we are now able to characterize the parallel-faces and the split-faces among the faces of K

THEOREM 6.1. *Let F be a proper face of K. Then the following statements are equivalent:*

(1) *F is not a parallel-face of K.*

(2) *There exist $x_1 \in \partial_e F$ and $x_2, y_1, y_2 \in \partial_e K \cap F'_K$ such that*

$$x_1 + x_2 = y_1 + y_2.$$

Proof. (2) $\Rightarrow$ (1) follows by using Proposition 3.10.

Assume (1) holds. By Proposition 3.10 again there exist $a_1, a_2 \in F'_K$ such that $2^{-1}(a_1+a_2) \notin F'_K$, i.e.

$$F \cap \text{face } (2^{-1}(a_1+a_2)) \neq \emptyset.$$

Hence there exist $x_1 \in \partial_e F$, $\alpha \in (0, 1]$ and $a_4 \in K$ such that

$$2^{-1}(a_1+a_2) = \alpha x_1 + (1-\alpha) a_4.$$

Then

$$\mathbf{a} = (a_1, a_2, -2\alpha x_1, -2(1-\alpha)a_4) \in H^4(X)_4.$$

If $4^{-1}\mathbf{a} \in \operatorname{conv}(\{S^4_{i,j}: 1 \leq i < j \leq 4\})$, then we can write

$$\begin{aligned}\mathbf{a} &= (a_1, a_2, -2\alpha x_1, -2(1-\alpha)a_4)\\ &= (b_1, -b_1, 0, 0) + (b_2, 0, -b_2, 0) + (b_3, 0, 0, -b_3)\\ &\quad + (0, b_4, -b_4, 0) + (0, b_5, 0, -b_5) + (0, 0, b_6, -b_6)\end{aligned}$$

where $\|a_1\| = \|b_1\| + \|b_2\| + \|b_3\|$, $\|a_2\| = \|b_1\| + \|b_4\| + \|b_5\|$ and so on. Since $a_1, a_2 \in K$, we get $b_1 = 0$, and since $a_1, a_2 \in F'_K$ and $x_1 \in F$, we get $b_2 = b_4 = 0$. Hence

$$\begin{aligned}2(1-\alpha) &= \|2(1-\alpha)a_4\|\\ &= \|b_3\| + \|b_5\| + \|b_6\|\\ &= \|a_1\| + \|a_2\| + \|2\alpha x_1\|\\ &= 2(1+\alpha)\end{aligned}$$

which is a contradiction since $\alpha > 0$. Thus there exists $\mathbf{z} = (z_1, z_2, z_3, z_4) \in \partial_e \operatorname{face}(\mathbf{a}) \cap \partial_e H^4(X)_4$ with $\|z_i\| = 1$ for all i by Lemma 5.4. Clearly $z_3 = -x_1$ and $z_i \in \operatorname{face}(a_i)$ for $i = 1, 2$. Using Theorem 5.5 we find $y_i \in \partial_e \operatorname{face}(a_i)$ for $i = 1, 2$ and $x_2 \in \partial_e X_1$ such that

$$y_1 + y_2 = x_1 + x_2.$$

Clearly $x_2 \in K$ and since F is a face, we get $x_2 \in F'_K$. $y_i \in \operatorname{face}(a_i) \subseteq F'_K$ for $i = 1, 2$.

THEOREM 6.2. *Assume F is a proper parallel-face of K. Then the following statements are equivalent:*

(1) *F is not a split-face of K.*

(2) *There exist $y_1, y_3 \in \partial_e F$ and $y_2, y_4 \in \partial_e F'_K$ such that $y_1 \neq y_3$ and*

$$y_1 + y_2 = y_3 + y_4.$$

Proof. Note that we assume that F'_K is a face. (2) $\Rightarrow$ (1) is trivial, so assume (1). Since F is a parallel-face but not a split face of K, there exist $x_1, x_3 \in F$ and $x_2, x_4 \in F'_K$ such that $x_1 \neq x_3$ and

$$x_1 + x_2 = x_3 + x_4.$$

From Theorem 3.6. (4), it follows that there exist $z, u_1, u_3 \in X$ such that

$$\begin{aligned}x_1 &= z + u_1, \quad 1 = \|x_1\| = \|z\| + \|u_1\|\\ x_3 &= z + u_3, \quad 1 = \|x_3\| = \|z\| + \|u_3\|\\ &\|x_1 - x_3\| = \|u_1\| + \|u_3\| > 0.\end{aligned}$$

Similarly, there exist $v, u_2, u_4 \in X$ such that

$$x_2 = v + u_2, \quad 1 = \|x_2\| = \|v\| + \|u_2\|$$
$$x_4 = v + u_4, \quad 1 = \|x_4\| = \|v\| + \|u_4\|$$
$$\|x_2 - x_4\| = \|u_2\| + \|u_4\|.$$

Thus we get $\alpha = \|u_1\| = \|u_3\| = \|u_2\| = \|u_4\|$ and $u_1 + u_2 = u_3 + u_4$. Clearly

$$\mathbf{u} = \alpha^{-1}(u_1, u_2, -u_3, -u_4) \in H^4(X)_4.$$

Now choose any $\mathbf{v} \in \partial_e \operatorname{face}(\mathbf{u}) \subseteq H^4(X)_4$. By the above reduction, we get that $\mathbf{v} \notin (\bigcup_{1 \le i < j \le 4} S^4_{i,j})$. In fact, $\mathbf{v} \in S^4_{1,2}$ implies $\|x_1 + x_2\| < 2$, $\mathbf{v} \in S^4_{1,3}$ implies $\|u_1 - u_3\| < \|u_1\| + \|u_3\|$ and so on.

From Lemma 5.4. and Theorem 5.5. it follows that there exists $\mathbf{y} = (y_1, y_2, -y_3, -y_4) \in \partial_e H^4(X)_4$ such that all $y_i \in \partial_e X_1$ and for $i = 1, 2, 3$, $y_i \in \operatorname{face}(\alpha^{-1}u_i) \subseteq \operatorname{face}(x_i)$. Hence $y_1, y_3 \in \partial_e F$ with $y_1 \neq y_3$ and $y_2 \in \partial_e F'_K$. Clearly also $y_4 \in \partial_e F'_K$.

Example 6.3. Let $K = H$ in example 2.5. Let $F = \operatorname{conv}(x_4, x_5)$. Since $x_1, x_2, x_3 \in F'_K$ and $x_1 + x_4 = x_2 + x_3$, it follows that F is not a parallel-face.

Let $G = \operatorname{conv}(x_3, x_4)$. Since $x_1, x_2 \in G'_K$ and $x_1 + x_4 = x_2 + x_3$, it follows that G is not a split-face.

COROLLARY 6.4. *Let F be a proper face of K. Then the following statements are equivalent:*

(1) *F is a split-face of K.*
(2) *For all $x \in \partial_e F$ and all $y \in \partial_e F'_K$,* $\operatorname{conv}(x, y)$ *is a face of K.*

COROLLARY 6.5. *Let $x_1 \in \partial_e K$. Then the following statements are equivalent:*

(1) *$\{x_1\}$ is a split-face of K.*
(2) *For all $y \in \partial_e K$,* $\operatorname{conv}(x_1, y)$ *is a face of K.*

COROLLARY 6.6. *Let $x_1 \in \partial_e K$. Then the following statements are equivalent:*

(1) *$\{x_1\}$ is not a split-face of K.*
(2) *There exist $x_2, x_3, x_4 \in \partial_e K \setminus \{x_1\}$ such that*

$$x_1 + x_2 = x_3 + x_4.$$

7. Finite dimensional Banach spaces with the 3.2.I.P

Throughout this section, let X be a finite dimensional Banach space with the 3.2.I.P. The first proposition is essential for the proof of our main theorem, which appears after the proposition. The rest of the section then consists of a series of lemmas which constitute the proof of the main theorem.

PROPOSITION 7.1. *Assume H is a proper face of X_1 and that F is a proper parallel-face of H. Then the following statements are equivalent:*

(1) *F is a maximal proper face of H.*

(2) *For all $x \in \partial_e F'_H$, $H = \text{face}\,(x, F)$.*

(3) *There exists an $x \in \partial_e F'_H$ such that $H = \text{face}\,(x, F)$.*

(4) *For all $x, y \in \partial_e F'_H$, there exist $u, v \in \partial_e F$ such that $x + u = y + v$.*

(5) $\dim \operatorname{span} H = 1 + \dim \operatorname{span} F$.

Note that face (x, F) means face $(\{x\} \cup F)$ and is described in Lemma 2.1.

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$ is trivial. To prove $(3) \Rightarrow (5)$ we can use the same arguments as those of $(2) \Rightarrow (4)$ and $(4) \Rightarrow (5)$ below.

$(2) \Rightarrow (4)$. Let $x, y \in \partial_e F'_H$. If $x = y$, then we can pick any $u = v \in \partial_e F$ and we are done. So assume $x \neq y$. By (2), $H = \text{face}\,(x, F)$ such that $y \in \text{face}\,(x, F)$. Since X is finite-dimensional we have $F = \text{face}\,(z)$ for some $z \in F$. Thus $y \in \text{face}\,(2^{-1}(x+z))$, such that for some $w \in X_1$ and some $\alpha > 0$

$$2^{-1}(x+z) = \alpha y + (1-\alpha) w.$$

Just as in the proof of Theorem 6.1 we can find $u \in \partial_e \text{face}\,(z) = \partial_e F$ and $v \in \partial_e X_1$ such that

$$x + u = y + v.$$

Then clearly $v \in H$ and since F is a parallel-face of H, we get $v \in \partial_e F$.

$(4) \Rightarrow (5)$. Let $x \in \partial_e F'_H$. For all $y \in \partial_e F'_H \setminus \{x\}$, there exist by (4), $u, v \in \partial_e F$ such that

$$x + u = y + v.$$

Hence $y \in \operatorname{span}(x, F)$. Since $H = \operatorname{conv}(F \cup F'_H)$ this gives that $H \subseteq \operatorname{span}(x, F)$ such that

$$\dim \operatorname{span} H = 1 + \dim \operatorname{span} F.$$

$(5) \Rightarrow (1)$. Suppose that F is not a maximal proper face of H. Then there exists a face G such that $F \subsetneqq G \subsetneqq H$. Thus

$$1 + \dim \operatorname{span} F \leqslant \dim \operatorname{span} G < \dim \operatorname{span} H.$$

(Note that if K is a maximal proper face of X_1 with $H \subseteq K$, then $(\operatorname{span} H) \cap K = H$ and similarly for F and G.)

The proof is complete.

COROLLARY 7.2. *Assume G and H are proper faces of X_1 such that $G \subseteq H$ and that there exists $x \in \partial_e H \setminus G$. If F is a maximal face of H such that $G \subseteq F$ and $x \notin F$, then F is a parallel-face of H and* $\dim \operatorname{span} F + 1 = \dim \operatorname{span} H$.

Proof. By the proof of Proposition 3.9, we get that F is a parallel-face of H. For each $y \in \partial_e H \setminus F$, we get $x \in \operatorname{face}(y, F)$. Now the proof of (4) $\Rightarrow$ (5) in Proposition 7.1 gives that $\dim \operatorname{span} F + 1 = \dim \operatorname{span} H$.

If $x, y \in \partial_e X_1$ are such that $\operatorname{conv}(x, y)$ is an edge of X_1, then clearly $\operatorname{conv}(x, y)$ is an M-face. More generally, it easily follows from Proposition 7.1 that if $x, y \in \partial_e X$ with $x + y \neq 0$, then face $(2^{-1}(x+y))$ is an M-face. Thus $m(X)$ is welldefined if $\dim X \geqslant 2$ and

$$2 \leqslant m(X) \leqslant \dim X.$$

Recall that $m(X)$ is the dimension of the largest subspace of X which is spanned by a proper M-face of X_1. Our main theorem follows:

THEOREM 7.3. *Assume X is a finte dimensional Banach space with* $\dim X > 2$. *If X has the* 3.2.I.P. *then the following statements are equivalent:*

(1) *X contains a proper L-summand.*
(2) *X^* contains a proper M-summand.*
(3) $m(X^*) = \dim X^*$.
(4) $m(X) < \dim X$.
(5) *There exists a maximal proper face of X_1 which contains a proper split face.*

Since either $m(X) < \dim X$, in which case X contains a proper L-summand by Theorem 7.3, or $m(X) = \dim X$, in which case X contains a proper M-summand by Theorem 7.3, the following corollary easily follows using Proposition 4.4 and induction.

COROLLARY 7.4. *Every finite dimensional Banach space with the* 3.2.I.P. *can be obtained by forming l_1- and l_∞-sums of the real line.*

Proof of Theorem 7.3. (1) $\Leftrightarrow$ (2) is Proposition 4.2. (5) $\Leftrightarrow$ (1) is Proposition 4.3. (2) $\Rightarrow$ (3) is Lemma 7.5 below. (3) $\Rightarrow$ (4) is Lemma 7.6 below. (4) $\Rightarrow$ (5) follows from the Lemmas 7.8, 7.9, and 7.10 below.

Lemma 7.5 is a generalization of example 2.8. Recall that we assume that X is a finite dimensional Banach space with the 3.2.I.P.

LEMMA 7.5. *If X contains a proper M-summand, then $m(X)=\dim X$.*

Proof. By assumption X contains an M-projection P with $P\neq 0, I$. By Proposition 4.3 P^* is an L-projection in X^*. Then clearly $P^*f=f$ or 0 for all $f\in\partial_e X_1^*$. Choose $f_0\in\partial_e X_1^*$. By replacing P by $I-P$ if necessary, we may assume $P^*f_0=f_0$. Let $K=\{x\in X_1: f_0(x)=1\}$. Since X is a CL-space we get $X_1=\operatorname{conv}(K\cup -K)$. Define $U=2P-I$. Then U is an isometry and $U^2=I$. If $x\in K$, then

$$f_0(Ux)=U^*f_0(x)=f_0(x)=1.$$

Hence $U(K)=K$.

Since $P\neq I$, there exists an $f\in\partial_e X_1^*$ with $P^*f=0$, i.e. $U^*f=-f$. Define $F=\{x\in K: f(x)=1\}$. Then F is a parallel-face of K and $F'_K=\{x\in K: f(x)=-1\}$. Since $U^*f=-f$, we get

$$U(F)=F'_K \quad\text{and}\quad U(F'_K)=F.$$

Let G be a maximal proper face of K containing F. By Proposition 3.9, G is a parallel-face of K. Hence there exists a $g\in\partial_e X_1^*$ such that $G=\{x\in K: g(x)=1\}$. If $P^*g=g$, then $U(G)=G$ such that $F'_K=U(F)\subseteq U(G)=G$ and thus $G=K$. This contradicts that G is a proper face of K. Hence $P^*g=0$. But then $U(G)=G'_K$ and $F'_K=U(F)\subseteq U(G)=G'_K$ such that $G\subseteq F$. Hence $F=G$ and F is a maximal proper face of K. Similarly we show that F'_K is a maximal proper face of K.

We have shown that every maximal proper face of X_1 is an M-face and the proof is complete.

LEMMA 7.6. *If $\dim X>2$ and $m(X)=\dim X$, then $m(X^*)<\dim X^*$.*

Proof. Since $m(X)=\dim X$, there exists a maximal proper face K of X_1 which is an M-face. Assume for contradiction that $m(X^*)=\dim X^*$. Then there exists a maximal proper face K^* of X_1^* such that K^* is an M-face. By replacing K^* by $-K^*$ if necessary, we may assume that there exist $x_0\in\partial_e K$ such that $K^*=\{f\in X_1^*: f(x_0)=1\}$.

Let F be a maximal proper face of K such that F'_K is a maximal proper face of K. By interchanging F and F'_K if necessary we can find $f_0, f\in\partial_e K^*$ such that

$$K=\{x\in X_1: f_0(x)=1\} \quad\text{and}\quad F=\{x\in K: f(x)=1\}.$$

Since $\dim X>2$, there exists a $g\in\partial_e K^*\setminus\{f_0, f\}$. Let $G=\{x\in K: g(x)=1\}$. G is a parallel-face of K and $G\neq F, F'_K, K, \emptyset$. This implies that $G\cap F\neq\emptyset$, $G\cap F'_K\neq\emptyset$, $G'_K\cap F\neq\emptyset$, and

$G'_K \cap F'_K \neq \emptyset$. (Indeed, if $G \cap F = \emptyset$, then $F \subseteq G'_K$ such that $F = G'_K$ and $f = -g$ since F is a maximal proper face of K.) Thus for all choices of signs

$$\|f_0 \pm f \pm g\| = 3.$$

Note that this holds for all $g \in \partial_e K^* \setminus \{f_0, f\}$. We shall show that this implies that K^* cannot be an M-face.

We assume for contradiction that K^* is an M-face, and let H be a maximal proper face of K^* such that also $H' = H'_{K^*}$ is a maximal proper face of K^*. We look at three cases.

(i) Assume $f_0, f \in H$ (or both are in H'). Then by Proposition 7.1 there exist $g, h \in \partial_e H'$ (or $\partial_e H$) with $f_0 + g = f + h$. But then

$$1 = \|h\| = \|f_0 - f + g\| = 3$$

which is a contradiction.

(ii) There exists a $g \in \partial_e \operatorname{face}(2^{-1}(f_0+f)) \setminus \{f_0, f\}$. Then there exist $\alpha > 0$ and an $h \in X_1^*$ such that

$$\alpha g + (1-\alpha) h = 2^{-1}(f_0 + f).$$

By chosing α as large as possible in $(0, 1]$, we can assume $g \notin \operatorname{face}(h)$. By Theorem 3.6 there exists an $x \in \partial_e X_1$ such that $g(x) = 1$ and $h(x) = -1$. Since X is a CL-space, we get

$$2\alpha - 1 = \alpha g(x) + (1-\alpha) h(x) = 2^{-1}(f_0(x) + f(x)) \in \{1, 0, -1\}.$$

Hence $\alpha = 2^{-1}$, and $f_0 + f = g + h$. But then

$$1 = \|h\| = \|f_0 + f - g\| = 3$$

which is a contradiction.

Thus it only remains to consider case (iii).

(iii) $f_0 \in H$, $f \in H'$ and $\operatorname{face}(2^{-1}(f_0+f)) = \operatorname{conv}(f_0, f)$. Let N be a maximal proper face of H such that $f_0 \notin N$. (Here we use $\dim X > 2$ to ensure that $N \neq \emptyset$.) But then $N \cap \operatorname{face}(2^{-1}(f_0+f)) = \emptyset$. By Theorem 3.6 there exists a parallel-face M of K^* such that $N \subseteq M$ and $f_0, f \in M' = M'_{K^*}$. If $M \cap H' = \emptyset$, then $H' \subseteq M'$, so $H' = M'$ and $H = M$. This is a contradiction. Hence $M \cap H' \neq \emptyset$. Thus we get by Proposition 7.1

$$\begin{aligned} \dim X^* &> \dim \operatorname{span} M \\ &\geqslant \dim \operatorname{span} N + 1 \\ &= \dim \operatorname{span} H \\ &= \dim X^* - 1. \end{aligned}$$

Hence dim span $M = \dim X^* - 1$, such that M is a maximal proper face of K by Proposition 7.1. Again by Proposition 7.1 there exist $g, h \in \partial_e M$ such that

$$f_0 + g = f + h.$$

Hence

$$1 = \|h\| = \|f_0 - f + g\| = 3$$

which is a contradiction. The proof is complete.

We shall now give a short proof of (1) $\Rightarrow$ (4) in Theorem 7.3.

LEMMA 7.7. *Assume* $\dim X > 2$ *and that* X *contains a proper* L*-summand. Then* $m(X) < \dim X$.

Proof. Assume for contradiction that $m(X) = \dim X$. Then some maximal proper face K of X_1 is an M-face. Let F and $F' = F'_K$ be maximal proper faces of K.

Since X contains a proper L-summand, K contains a proper split-face G by Proposition 4.3. Also $G' = G'_K$ is a split face and

$$m(X) = \dim X = \dim \operatorname{span} G + \dim \operatorname{span} G'.$$

We can assume $G' \cap F' \neq \emptyset$. By Theorem 2.6, $H = \operatorname{conv}(G \cup (F \cap G'))$ is a proper face of K. Since $F \subseteq H$, we get $F = H$. Thus $F' \subseteq G'$ and hence $F' = G'$ such that $F = G$. By Proposition 7.1 we get

$$\begin{aligned} \dim X &= \dim \operatorname{span} G + \dim \operatorname{span} G' \\ &= \dim \operatorname{span} F + \dim \operatorname{span} F' \\ &= 2(\dim X - 1) \end{aligned}$$

such that $\dim X = 2$. A contradiction. The proof is complete.

It remains to prove (4) $\Rightarrow$ (5) in Theorem 7.3. This follows from the following three lemmas. Note that once we have shown that one maximal proper face of X_1 contains a proper split-face, then it follows that all maximal proper faces of X_1 have this property.

LEMMA 7.8. *Let* M *be a proper face of* X_1 *and let* F *be a proper parallel-face of* M. *Assume* F *is a maximal* M*-face in* M *and that* G *and* H *are disjoint maximal proper faces of* F. *Then* G *and* H *are parallel-faces of* M.

Proof. Let $x \in \partial_e H$. Choose a maximal proper face S of M such that $G \subseteq S$ and $x \in S' = S'_M$. Then S is a parallel-face of M by Proposition 3.9. Since G is a maximal proper face of F, we get $S \cap F = G$. Let $F' = F'_M$ and assume there exists a $z \in \partial_e F' \cap S'$. Then $x \in \text{face}(z, S)$ such that by Proposition 7.1 there exist $a, b \in \partial_e S$ such that

$$x + a = z + b.$$

Define $F_z = \text{face}(z, F)$. Since F is a parallel-face of M, we get by Proposition 7.1 that F is a maximal proper face of F_z. F is a parallel-face and $x \in F$ and $z \in F'$. Hence $a \in F'$ and $b \in F \cap S = G$. But then $G \cup \{a\} \subseteq S \cap F_z$ and $H \cup \{z\} \subseteq S' \cap F_z$. Hence $S \cap F_z$ and $S' \cap F_z$ are maximal proper faces of F_z. Thus F_z is an M-face of M containing F. This is a contradiction such that we have $F' \subseteq S$. But then $S' = H$ is a parallel-face of M. Similarly we show that G is a parallel-face of M.

LEMMA 7.9. *Assume $m(X) < \dim X$ and let F be a proper M-face of X_1 with $m(X) =$ $\dim \operatorname{span} F$. Let K be a maximal proper face of X_1 with $F \subseteq K$. Then F is a parallel-face of K.*

Proof. Assume for contradiction that F is not a parallel-face of K. Then, by Proposition 3.10, F'_K is non-convex. There exists a face M such that $F \subseteq M \subseteq K$ and M is minimal with the following properties: F'_M is non-empty and non-convex. ($F'_M \neq \emptyset$ simply means that F is a proper subface of M.) Then by Theorem 6.1 there exist $x \in \partial_e F$ and $y, y_1, y_2 \in$ $\partial_e M \cap F'_M$ such that

$$x + y = y_1 + y_2.$$

Clearly $M = \text{face}(y, F)$ since M is minimal with $F'_M \neq \emptyset$ and non-convex.

Since F is an M-face, there exist a pair of disjoint maximal proper faces G and H of F. We can assume $x \in H$ since $F = \text{conv}(G \cup H)$.

We want to show that $T = \text{face}(y, H)$ is an M-face with $\dim \operatorname{span} T > \dim \operatorname{span} F$. This will be our final contradiction.

Let N_1 and N_2 be maximal proper faces of M such that $F \subseteq N_1 \cap N_2$ and $y_2 \notin N_1$ and $y_1 \notin N_2$. Then, by Corollary 7.2, N_1 and N_2 are parallel-faces of M. We have $x, y_1 \in N_1$ and $y, y_2 \in (N_1)'_M$, and $x, y_2 \in N_2$ and $y, y_1 \in (N_2)'_M$.

Since N_1 is a proper face of M containing F, we have that F is a parallel-face in N_1. Then, by Lemma 7.8, H is a parallel-face of N_1 such that $S = H'_{N_1} = \text{conv}(F'_{N_1} \cup G)$ is a parallel-face of N_1. We can thus choose a maximal proper face F_1 of M such that $S \subseteq F_1$ and $x \notin S$. Then clearly $F_1 \cap N_1 = S$ and $(F_1)'_M \cap N_1 = H$.

Since $y_1 \in T \cap N_1$ and $y \in T$, we get $H \subsetneq T \cap N_1 \subsetneq T$. Hence

$$\begin{aligned} \dim \operatorname{span} F &= \dim \operatorname{span} H + 1 \\ &\leqslant \dim \operatorname{span} (T \cap N_1) \\ &< \dim \operatorname{span} T. \end{aligned}$$

Since $(F_1)'_M \cap T$ is a parallel-face of T containing H, we get $T = \text{face}\,(y, T \cap (F_1)'_M)$. Hence, by Proposition 7.1, $T \cap (F_1)'_M$ is a maximal proper face of T.

Thus it remains to show that $F_1 \cap T$ is a maximal proper face of T.

Let us draw a picture. We look upon M from above G.

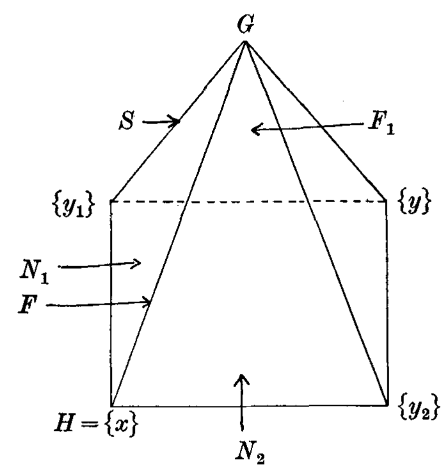

Fig. 3.1

Assume for contradiction that there exists

$$z \in \partial_e M \cap \text{face}\,(F, F_1 \cap N_2 \cap F'_M) \cap ((F_1)'_M \cap N_2 \cap F'_M).$$

Then there exist $z_1 \in F$, $z_2 \in F_1 \cap N_2 \cap F'_M$, $u \in M$ and $\alpha \in (0, 1]$ such that

$$2^{-1}(z_1 + z_2) = \alpha z + (1-\alpha) u.$$

The argument used to prove Theorem 6.1 shows that we may assume z_1, z_2 and u are extreme points and $\alpha = 2^{-1}$. Hence

$$z_1 + z_2 = z + u.$$

$F \subseteq N_2$ such that $z_1, z_2, u \in N_2$. We have $z_1 \in F$ and $z \in F'_M$. Hence $u \in F$ since F is a parallel-face of N_2. Furthermore $z_2 \in F_1$ and $z \in (F_1)'_M$. Hence $u \in F_1$, and then $u \in F \cap F_1 = G$. Also $z_1 \in (F_1)'_M \cap F = H$. Using that G and H are parallel-faces of N_2 which follows from Lemma 7.8, we get a contradiction. Hence $(F_1)'_M \cap N_2 \cap F'_M$ and face $(F, F_1 \cap N_2 \cap F'_M)$ are disjoint. By Theorem 3.6 there exists a parallel-face N_3 of M such that

$$x \in F \cup (F_1 \cap N_2 \cap F'_M) \subseteq N_3$$

and

$$y_2 \in (F_1)'_M \cap N_2 \cap F'_M \subseteq (N_3)'_M.$$

(If $F_1 \cap N_2 \cap F'_M = \varnothing$, we can take $N_3 = N_1$.)

Define $S_1=N_1\cap(N_2)'_M\cap N_3$ and $S_2=(N_1)'_M\cap N_2\cap(N_3)'_M$. Clearly $y_1\in S_1$ and $y_2\in S_2$. We want to show that $T=\text{face}\,(S_1,S_2)$.

Since

$$\begin{aligned}S_2\cap F_1&=F_1\cap(N_1)'_M\cap N_2\cap(N_3)'_M\\&\subseteq(F_1\cap F'_M\cap N_2)\cap(N_3)'_M\\&=N_3\cap(N_3)'_M=\varnothing\end{aligned}$$

we get $S_2\subseteq(F_1)'_M$ and $F_1\cap(N_1)'_M\cap(N_3)'_M\subseteq(N_2)'_M$. Let $u\in\partial_eH$. Then $u\in\text{face}\,(y_2,F_1)$, such that by Proposition 7.1 there exist $s,t\in\partial_eF_1$ such that

$$u+s=y_2+t.$$

We have $u\in N_1\cap N_2\cap N_3$ and $y_2\in(N_1)'_M\cap N_2\cap(N_3)'_M=S_2$. Hence $s\in F_1\cap(N_1)'_M\cap(N_3)'_M\subseteq(N_2)'_M$. But then $t\in N_1\cap(N_2)'_M\cap N_3=S_1$. Hence $T\subseteq\text{face}\,(S_1,S_2)$. Let next $t\in\partial_eS_1$. Then $t\in\text{face}\,(y,N_2)$, such that, by Proposition 7.1, there exist $a,b\in\partial_eN_2$ such that

$$t+a=y+b.$$

$y\in F_1\cap(N_1)'_M\cap(N_2)'_M\cap(N_3)'_M$ and $t\in N_1\cap(N_2)'_M\cap N_3$ implies that $a\in(N_1)'_M\cap N_2\cap(N_3)'_M\subseteq(F_1)'_M$ such that $b\in N_1\cap(F_1)'_M=H$. Hence $S_1\subseteq T$, and it follows from the computation that $S_1\subseteq F_1\cap T$. Thus in order to show that $T=\text{face}\,(S_1,S_2)$ and that $F_1\cap T$ is a maximal proper face of T, it suffices to show that $S_2\subseteq\text{face}\,(x,F_1\cap T)$.

Thus let $u\in\partial_eS_2\subseteq(F_1)'_M$. Then $u\in\text{face}\,(y,N_1)$. By Proposition 7.1 there exist $a,b\in\partial_eN_1$ such that

$$u+a=y+b.$$

Now $u\in S_2\subseteq(F_1)'_M$ and $y\in F_1$. Hence $b\in(F_1)'_M\cap N_1=H$. Thus $S_2\subseteq\text{face}\,(y,H)=T$. Also $u\in\text{face}\,(x,F_1)$, so, by Proposition 7.1, there exist $a,b\in\partial_eF_1$ such that

$$u+a=x+b.$$

Here $u\in(N_1)'_M\cap N_2\cap(N_3)'_M\subseteq(F_1)'_M$ and $x\in N_1\cap N_2\cap N_3$. Hence $b\in F_1\cap(N_1)'_M\cap(N_3)'_M\subseteq(N_2)'_M$. Thus $a\in F_1\cap N_1\cap(N_2)'_M\cap N_3=F_1\cap S_1\subseteq F_1\cap T$. Hence $b\in T=\text{face}\,(S_1,S_2)$ such that $b\in T\cap F_1$, and we have proved that $S_2\subseteq\text{face}\,(x,F_1\cap T)$.

The proof is complete.

LEMMA 7.10. *Assume $m(X)<\dim X$. Let F be a proper M-face of X_1 with $m(X)=\dim\operatorname{span}F$. Then there exists a maximal proper face K of X_1 such that F is a split-face of K.*

Proof. Choose a maximal proper face K of X_1 such that $F \subseteq K$. Assume for contradiction that F is not a split-face of K. By Lemma 7.9 F is a parallel-face of K, such that, by Theorem 6.2, there exist $x_1, x_2 \in \partial_e F$ and $y_1, y_2 \in \partial_e F'_K$ such that

$$x_1 + y_1 = x_2 + y_2.$$

Choose F_1 a maximal proper face of F such that $x_2 \in F_1$ and $x_1 \notin F_1$. Then choose a maximal proper face F_2 of F such that $(F_1)'_F \subseteq F_2$ and $x_2 \notin F_2$. Then $(F_1)'_F \cap (F_2)'_F = \emptyset$. If $F_1 \cap F_2 = \emptyset$ then, by Lemma 7.8, F_1 is a parallel-face of K. This is impossible since $x_1, y_1 \notin F_1$ and $x_2 \in F_1$. Hence $F_1 \cap F_2 \neq \emptyset$.

Choose N_1 and N_2 maximal proper faces of K such that $x_2 \in F_1 \subseteq N_1$ and $x_1 \notin N_1$ and $x_1 \in F_2 \subseteq N_2$ and $x_2 \notin N_2$. Then clearly $N_1 \cap F = F_1$ and $N_2 \cap F = F_2$. Assume there exists a $y \in \partial_e((N_1)'_K \cap (N_2)'_K) \cap \operatorname{face}(F, N_1 \cap N_2 \cap F'_K)$. Then, as in the proof of Theorem 6.1, there exist $a \in \partial_e K$, $b \in \partial_e F$ and $c \in \partial_e(N_1 \cap N_2 \cap F'_K)$ such that

$$y + a = b + c.$$

We get $b \in F \cap (N_1)'_K \cap (N_2)'_K = F \cap (F_1)'_F \cap (F_2)'_F = \emptyset$. This shows that $((N_1)'_K \cap (N_2)'_K)$ and face $(F, N_1 \cap N_2 \cap F'_K)$ are disjoint faces. By Theorem 3.6, there exists $\varphi \in \partial_e X_1^*$ such that $\varphi = 1$ on face $(F, N_1 \cap N_2 \cap F'_K)$ and $\varphi = -1$ on $(N_1)'_K \cap (N_2)'_K$. Let $S = K \cap \varphi^{-1}(1)$ and let $K_1 = \operatorname{conv}(S \cup -S'_K)$. Then K_1 is a maximal proper face of X_1 and $F \subseteq K_1$. F is also a parallel-face of K_1 by Lemma 7.9. If $y_1, y_2 \in S'_K$, we replace them by $-y_2$ and $-y_1$.

Let $f_i \in \partial_e X_1^*$ such that $N_i = K \cap f_i^{-1}(1)$ for $i = 1, 2$. Then

$$\begin{aligned} &f_1^{-1}(-1) \cap f_2^{-1}(-1) \cap K_1 \\ &\quad = \operatorname{conv}[(f_1^{-1}(-1) \cap f_2^{-1}(-1) \cap \partial_e S) \cup (f_1^{-1}(-1) \cap f_2^{-1}(-1) \cap (-\partial_e S'_K))] \\ &\quad = -f_1^{-1}(1) \cap f_2^{-1}(1) \cap S'_K = \emptyset. \end{aligned}$$

Let M_i be maximal proper faces of K_1 such that $K_1 \cap f_i^{-1}(1) \subseteq M_i$ for $i = 1, 2$ and $x_1 \notin M_1$ and $x_2 \notin M_2$. Then $(M_1)'_{K_1} \cap (M_2)'_{K_1} = \emptyset$. Denoting K_1 by K and M_i by N_i, we have shown that we can assume $(N_1)'_K \cap (N_2)'_K = \emptyset$.

Let G and H be a pair of maximal proper faces of F. By Lemma 7.8, G and H are parallel-faces of K. Hence we have $x_1, x_2 \in G$ or $x_1, x_2 \in H$. Thus we can assume $x_1, x_2 \in H$. Let $f_1, f_2, f_3, f_4 \in \partial_e X_1^*$ such that $N_1 = K \cap f_1^{-1}(1)$, $N_2 = K \cap f_2^{-1}(1)$, $G = K \cap f_3^{-1}(1)$ and $H = K \cap f_4^{-1}(1)$. Let $f = 2^{-1}(f_1 + f_3)$ and $g = 2^{-1}(f_2 + f_4)$. $g(x_1) = 1$ gives $\|g\| = 1$. If $G \cap N_1 = \emptyset$, then $G \subseteq (F_1)'_F \subseteq F_2$, such that $G = F_2$. Hence $x_1 \in G$. This is a contradiction. Thus $G \cap N_1 \neq \emptyset$ and $\|f\| = 1$.

Assume now that face $(-f) \cap$ face $(g) = \emptyset$. Then there exists by Theorem 3.6 an $x_0 \in \partial_e X_1$ with $f(x_0) = g(x_0) = 1$. If $x_0 \in K$, then $x_0 \in (N_1 \cap G)) \cap (N_2 \cap H) \subseteq G \cap H = \emptyset$. If $x_0 \in -K$, then $-x_0 \in ((N_1)'_K \cap G'_K) \cap ((N_2)'_K \cap H'_K) \subseteq (N_1)'_K \cap (N_2)'_K = \emptyset$. Hence face $(-f) \cap$ face $(g) \neq \emptyset$.

Choose $h \in \partial_e$ face $(-f) \cap$ face (g). Just as in the proof of (ii) in the proof of Lemma 7.6, we find $h_1, h_2 \in \partial_e X_1^*$ such that

$$-f_1 - f_3 = h + h_1 \quad \text{and} \quad f_2 + f_4 = h + h_2.$$

Let now $T = K \cap h^{-1}(1)$. Then $(N_1 \cap G) \cup ((N_2)'_K \cap H'_K) \subseteq T'_K$ and $((N_1)'_K \cap G'_K) \cup (N_2 \cap H) \subseteq T$. We have $x_1 \in N_2 \cap H \subseteq T$. Furthermore $x_1 \in (N_1)'_K$ and $x_2 \in N_1$ gives $y_2 \in (N_1)'_K \cap G'_K \subseteq T$. Similarly $x_1 \in N_2$ and $x_2 \in (N_2)'_K$ gives $y_1 \in (N_2)'_K \cap H'_K \subseteq T'_K$. Hence $x_2 \in T'_K$.

We have shown that $F \cap N_1 \cap N_2 \neq \emptyset$. Assume now that $H \cap N_1 \cap N_2 = \emptyset$. Then there exists a $w \in \partial_e G \cap N_1 \cap N_2$. Clearly $G \cap (N_1)'_K = \emptyset$ implies $F_1 \subseteq G$. This is impossible because $x_2 \in F_1 \cap H$. Hence we may choose a $v \in \partial_e G \cap (N_1)'_K \subseteq N_2$. H is a maximal proper face of F, so by Proposition 7.1, there exist $a, b \in \partial_e H$ such that

$$a + v = b + w.$$

$v \notin N_1$ and $w \in N_1$ gives $a \in N_1$ and $b \notin N_1$. $(N_1)'_K \cap (N_2)'_K = \emptyset$ gives $b \in N_2$. Hence $a \in H \cap N_1 \cap N_2$, which is a contradiction.

Choose $y \in \partial_e H \cap N_1 \cap N_2 \subseteq T$. Then $y \in$ face $(x_2, G) = F$. Hence, by Proposition 7.1, there exist $c, d \in \partial_e G$ such that

$$c + y = x_2 + d.$$

Here $y \in N_1 \cap N_2$ and $x_2 \in N_1 \cap (N_2)'_K$ such that $c \in G \cap (N_2)'_K \subseteq N_1$. Thus $d \in N_1 \cap G \subseteq T'_K$. But then $y \in T \cap T'_K = \emptyset$. This is a contradiction. The lemma is proved.

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