ON THE CONSISTENCY OF BOREL'S CONJECTURE

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For X a subset of $[0, 1]$, there is a family of properties which X might have, each of which is stronger than X having Lebesgue measure zero, and each of which is trivially satisfied if X is countable. The main properties in this family (apart from the Lusin-type conditions, which are really meant to be stated in conjunction with $2^{\aleph_0} = \aleph_1$ or with Martin's axiom—see section 1) are

X has strong measure zero, i.e., if $\varepsilon_0, \varepsilon_1, ..., \varepsilon_n, ...$ $(n < \omega)$ is a sequence of positive reals then there exists a sequence $I_0, I_1, ..., I_n, ...$ $(n < \omega)$ of intervals with length $I_n \leqslant \varepsilon_n$ and $X \subseteq \bigcup_{n<\omega} I_n$.

X has universal measure zero, i.e., $f[X]$ has Lebesgue measure zero for each homeomorphism f: $[0, 1] \to [0, 1]$, equivalently, for each nonatomic, nonnegative real valued Baire measure μ on $[0, 1]$, $\mu(X) = 0$.

If X has strong measure zero then X has universal measure zero. These are strong restrictions to place on X—a nonempty perfect set, which can of course have measure zero, cannot have either of these properties, hence no uncountable analytic set can have either of these properties. The question thus arises as to whether there exist uncountable sets with these properties.

For universal measure zero sets, the answer is yes—Hausdorff ([8]) constructed in ZFC a universal measure zero set of power $\aleph_1$.

For strong measure zero sets the situation is different. The notion of strong measure zero is due to Borel ([2]), who conjectured ([2], page 123) that

all strong measure zero sets are countable.

In fact, though, uncountable strong measure zero sets can be constructed if the continuum hypothesis is assumed—the Lusin set ([11]), a set, constructed with the aid of the CH, having countable intersection with each first category set, has strong measure zero ([18]).

The question thus became (see, e.g., [9], page 527) whether the conjecture is consistent. An affirmative answer is the main result of this paper.

THEOREM. *If* ZFC *is consistent, then so* is ZFC$+$*Borel's conjecture.*

In the forcing proof, a countable transitive model $\mathcal{M}$ of ZFC $+2^{\aleph_0}=\aleph_1$ is started with, and a cofinality preserving Cohen extension $\mathcal{N}$ of $\mathcal{M}$ is found in which $2^{\aleph_0}=\aleph_2$ and there do not exist any uncountable strong measure zero sets.

A certain purely set theoretic condition for uncountable cardinals $\leqslant 2^{\aleph_0}$ holds in this Cohen extension—Sierpinski ([20]) proved that there exists no strong measure zero set of power $\varkappa$ if and only if for every family of sets $\{A_s: s\in(2)^{<\omega}\}$ such that $A_\phi=\varkappa$, $A_{s^\frown 0}\cup A_{s^\frown 1}=A_s$, $A_{s^\frown 0}\cap A_{s^\frown 1}=\phi$, with Card $\bigcap_{n<\omega}A_{f\restriction n}\leqslant 1$ all $f\in(2)^\omega$, there exists a sequence n_i, $i<\omega$, of nonnegative integers such that if $s_i\in(2)^{<\omega}$ and Card $s_i=n_i$, all $i<\omega$, then $\bigcup_{i<\omega}A_{s_i}\neq\varkappa$.

Regarding universal measure zero sets, Hausdorff's theorem is best possible in the sense that there is a model of ZFC$+2^{\aleph_0}>\aleph_1$ in which there are no universal measure zero sets of power $\aleph_2$ (add Sacks reals ([17]) or Solovay reals ([24]) to a model of $2^{\aleph_0}=\aleph_1$). This fact (not proved here) is an extension of an unpublished result of Baumgartner which states that adding Sacks reals or Solovay reals to a model of $2^{\aleph_0}=\aleph_1$ gives a model in which there are no strong measure zero sets of power $\aleph_2$.

In section 1 an account is given of the main results having to do with the existence of uncountable strong measure zero and universal measure zero sets.(1) In the places where the CH was originally used, we will note the more general results proved using Martin's axiom. In section 2 the main result is proved.

§ 1.

THEOREM 1.1 (Folklore). *No nonempty perfect set can have universal measure zero.*

Proof. Letting P be perfect, we may assume P is nowhere dense. Let O_n, $n<\omega$, be disjoint open intervals whose union is the complement of P. The theorem follows by inductively choosing intervals I_s, $s\in(2)^{<\omega}$, with $I_\phi=[0,1]$, $I_{s^\frown 0}$, $I_{s^\frown 1}$ disjoint intervals whose union is I_s, with Card $\bigcap_{n<\omega}I_{f\restriction n}\leqslant 1$ for all $f\in(2)^\omega$, in such a way that if μ is the measure defined by $\mu(I_s)=2^{-\mathrm{Card}\,s}$, then, e.g., $\mu(O_n)<2^{-(n+2)}$, whence $\mu(P)>0$.

THEOREM 1.2 (Hausdorff [8], see [23]). *There exists a universal measure zero set of power* $\aleph_1$.

(1) References are the ones in this paper plus the ones in [9], [20] and [22]. The strong measure zero property is classically referred to as "property C".

Proof. We work over the Cantor space $(2)^\omega$. For $f, g \in (2)^\omega$, let $f \leqslant g$ mean $\exists m\ \forall n > m$ $f(n) \leqslant g(n)$; if $f \leqslant g$ let $\delta(f, g)$ be the least such m. $f < g$ means $f \leqslant g$ and $g \not\leqslant f$. We will be done by two claims.

Claim 1. There is a set $\{f_\alpha: \alpha < \omega_1\} \cup \{g_\alpha: \alpha < \omega_1\}$ with $f_0 < f_1 < \ldots < f_\alpha < \ldots < g_\alpha < \ldots < g_0$ such that there is no $j \in (2)^\omega$ with $\forall \alpha\ f_\alpha < j < g_\alpha$.

Proof. Suppose $\alpha < \omega_1$ and we have constructed $F_{<\alpha}, G_{<\alpha} (= \{f_\beta: \beta < \alpha\}, \{g_\beta: \beta < \alpha\})$ such that the following additional condition holds: for all $\beta < \alpha$, $P(F_{<\beta}, g_\beta)$, that is, for each $m < \omega$ there are only finitely many $\gamma < \beta$ with $\delta(f_\gamma, g_\beta) = m$. We want to find f_α, g_α and make sure that $P(F_{<\alpha}, g_\alpha)$. Pick an h such that $\forall \beta < \alpha\ f_\beta < h < g_\beta$. For all $\beta < \alpha$, $P(F_{<\beta}, h)$; using this, construct a g_α such that $\forall \beta < \alpha\ f_\beta < g_\alpha < h$ and $P(F_{<\alpha}, g_\alpha)$, then choose f_α such that $\forall \beta < \alpha\ f_\beta < f_\alpha < g_\alpha$. This completes the construction. If there were a j with $\forall \alpha < \omega_1\ f_\alpha < j < g_\alpha$, then for some α, not $P(F_{<\alpha}, j)$, which would contradict $j < g_\alpha$ and $P(F_{<\alpha}, g_\alpha)$.

Claim 2. Any set satisfying the conditions of Claim 1 has universal measure zero.

Proof. For $\alpha < \omega_1$, let $T_\alpha = \{h: f_\alpha \leqslant h \leqslant g_\alpha\}$. T_α is an F_σ set. Let μ be any nonatomic nonnegative real valued Baire measure on $(2)^\omega$, we will show $\exists \alpha\ \mu(T_\alpha) = 0$ (and that will give the claim). If there were no such α, then there would be an α_0 such that $\forall \alpha \geqslant \alpha_0$ $\mu(T_\alpha) = \mu(T_{\alpha_0}) = \varepsilon > 0$. Let $j \in (2)^\omega$ satisfy $\mu\{f \in T\alpha_0: f(n) = j(n)\} \geqslant \varepsilon/2$, all $n < \omega$. Pick an $\alpha > \alpha_0$ such that $j \notin T_\alpha$. Let $\{i_n: n < \omega\}$ enumerate the numbers i such that $f_\alpha(i) = g_\alpha(i)$. $T_\alpha = \bigcup_{m<\omega} T_{\alpha m}$, where $T_{\alpha m} = \{f: f(i_n) = f_\alpha(i_n)$ all $n > m\}$. Pick an m with $\mu(T_{\alpha m}) > \varepsilon/2$. Since $j \notin T_\alpha$ there is an i_n, $n > m$ such that $j(i_n) \neq g_\alpha(i_n)$. We have that $\mu\{f \in T_{\alpha_0}: f(i_n) \neq g_\alpha(i_n)\} = \mu\{f \in T_\alpha: f(i_n) \neq g_\alpha(i_n)\} \geqslant \varepsilon/2$, but this set is disjoint with $T_{\alpha m}$ and $\mu(T_{\alpha m}) > \varepsilon/2$, a contradiction.

THEOREM 1.3 (Folklore, see [9]). *If X has strong measure zero then X has universal measure zero.*

Proof. The theorem follows from the fact that if μ is a nonatomic, nonnegative real valued Baire measure on $[0, 1]$, then $\forall \varepsilon > 0\ \exists \delta > 0\ \forall$ intervals I (length $(I) < \delta \Rightarrow \mu(I) < \varepsilon$).

We will prove now the theorem of Martin and Solovay that if Martin's axiom holds, then all sets of power $< 2^{\aleph_0}$ have strong measure zero ([12]). We note a slight strengthening, the proof being essentially the same. The Rothberger property on $X \subseteq [0, 1]$ ([15], it is called property C'' in [9] and [15]) states that if I_{xn} is an interval around x, each $x \in X$, each $n < \omega$, then there exist members $x_1, x_2, \ldots, x_n, \ldots$ of X such that $X \subseteq \bigcup_{n<\omega} I_{x_n n}$. This property implies that X has strong measure zero (take length $I_{xn} = \varepsilon_n$, all x, n).

THEOREM 1.4. *If Martin's axiom, then every $X \subseteq [0, 1]$ of power $< 2^{\aleph_0}$ has the Rothberger property, and hence has strong measure zero.*

Proof. The consequence of MA needed is the strong Baire category property [12]: the union of $<2^{\aleph_0}$ first category sets is of first category. Given (rational) intervals I_{xn} as in the definition, choose for each $n<\omega$ an enumeration $\{I_{xn}: x \in X\} = \{I_n^0, I_n^1, I_n^2, \ldots\}$. For each $x \in X$ define $A_x = \{f \in (\omega)^\omega: x \notin \bigcup_{n<\omega} I_n^{f(n)}\}$. Since each A_x is nowhere dense in $(\omega)^\omega$, there is an $f \in (\omega)^\omega - \bigcup_{x \in X} A_x$. Then $X \subseteq \bigcup_{n<\omega} I_n^{f(n)}$. For each $n<\omega$ pick $x_n \in X$ with $I_n^{f(n)} = I_{x_n,n}$. Then $X \subseteq \bigcup_{n<\omega} I_{x_n n}$.

We now give (the Martin's axiom version of) the original construction of a strong measure zero set[1] of power $2^{\aleph_0}$. Say $X \subseteq [0, 1]$ is concentrated around a set $C \subseteq [0, 1]$ if whenever I_c is an interval around c all $c \in C$, then $X - \bigcup_{c \in C} I_c$ has power $<2^{\aleph_0}$. A generalized Lusin set(1) is an X which is concentrated around every dense subset of [0, 1] (equivalently, X is concentrated around the rationals, Card $(X \cap A) < 2^{\aleph_0}$ all A of first category).

THEOREM 1.5. (Martin and Solovay, [12]) *Martin's axiom implies that there is a generalized Lusin set X of power $2^{\aleph_0}$.*

Proof. Let D_α, $\alpha < 2^{\aleph_0}$, enumerate the dense open subsets of [0, 1]. Let $X = \{x_\alpha: \alpha < 2^{\aleph_0}\}$, where x_α, chosen by the strong Baire category property, satisfies $x_\alpha \neq x_\beta (\beta < \alpha)$, $x_\alpha \in \bigcap_{\beta \in \alpha} D_\beta$.

THEOREM 1.6. *Martin's axiom implies that there is a strong measure zero set of power $2^{\aleph_0}$.*

Proof. It suffices to show that if MA and X is concentrated around $C = \{c_n: n < \omega\}$, then X has strong measure zero. Given $\varepsilon_n > 0$ $(n < \omega)$, pick for each n an interval J_n around c_n of length ε_{2n}, then choose intervals K_n of length ε_{2n+1} to cover the strong measure zero set $X - \bigcup_{n<\omega} J_n$.

Finally, a word about the nonreversibility of the implications

$$X \text{ is concentrated around a countable set} \underset{M}{\Rightarrow} X \text{ has}$$
$$\text{strong measure zero} \Rightarrow X \text{ has universal measure zero.}$$

Besicovitch ([1]) proved that if $2^{\aleph_0} = \aleph_1$ then there is a strong measure zero set which is not concentrated around any countable set. By going through his proof or the proof given by Darst in [5], it may be seen that such a set may also be constructed using Martin's axiom.

Sierpinski ([19], [21]) proved that if $2^{\aleph_0} = \aleph_1$ then there is a universal measure zero set which does not have strong measure zero, by the following method. Clearly if X has strong measure zero (respectively, is concentrated) and f: $[0, 1] \to [0, 1]$ is continuous then $f[X]$ has strong measure zero (respectively, is concentrated). Sierpinski constructed, with

(1) These are the MA versions of the classical definition, obtained by replacing $<\aleph_1$ by $<2^{\aleph_0}$ at the relevant place in the definition of concentrated.

CH, an X of universal measure zero and a continuous f such that $f[X]$ does not have universal measure zero. Darst ([7]) found such a construction where the f is also of bounded variation. These results also hold under the assumption of Martin's axiom.

§ 2.

The letter $\mathcal{M}$ varies over countable transitive models of ZFC. Unless otherwise stated, in a statement about forcing the ground model is denoted by $\mathcal{M}$. The method for getting an extension in which all strong measure zero sets are countable is as follows. A partial ordering $\mathcal{J}$ is chosen such that adding a $\mathcal{J}$-generic set $G_{\mathcal{J}}$ is the same as adding a certain real number canonically obtained from $G_{\mathcal{J}}$. This real kills all the ground model's uncountable strong measure zero sets (for each uncountable $X \subseteq [0, 1]$, $X \in \mathcal{M}$, there is a tail end of the real which codes up a sequence ε_n, $n < \omega$, such that there is no sequence I_n, $n < \omega$, in $\mathcal{M}(G_{\mathcal{J}}]$ with length $I_n \leqslant \varepsilon_n$, $X \subseteq \bigcup_{n<\omega} I_n$). Since there are new reals in $\mathcal{M}(G_{\mathcal{J}}]$, new uncountable strong measure zero sets may now appear; for example, $\mathcal{M}[G_{\mathcal{J}}]$, will be a model of $2^{\aleph_0} = \aleph_1$ if $\mathcal{M}$ is. So we will do an iterated forcing construction, successively adding $\mathcal{J}$-generic reals ω_2 times. In the original iterated forcing argument of Solovay and Tennenbaum ([25]), a condition gave information about finitely many of the generic sets occurring in the steps of the iteration. Since then, Jensen (see [7]) and Silver (see [14]) have used iterated forcing arguments with conditions not subject to the finiteness restriction. The conditions we will work with give information about countably many of the generic sets occurring in the steps of the iteration.

As Solovay and Tennenbaum say in [25] about the procedure for proving the consistency of Souslin's hypothesis, "Once a Souslin tree is killed, it stays dead." Strong measure zero sets are different; after an uncountable strong measure zero set has been killed, it is possible to bring it back to life by a further Cohen extension—for example, adding a Cohen real ([3]) to a model $\mathcal{N}$ forces all sets of reals in $\mathcal{N}$ to have strong measure zero in the extension.(1) When we iteratively add $\mathcal{J}$-reals, however, it turns out that once a strong measure zero set is killed, it stays dead.

For $s, t \in (\omega)^{<\omega}$, let $s \leqslant t (s < t)$ mean that s is an initial segment (proper initial segment) of t. The partial ordering $\mathcal{J}$ will consist of certain subtrees of $(\omega)^{<\omega}$; $T \subseteq (\omega)^{<\omega}$ will be a member of $\mathcal{J}$ if and only if $\forall t \in T \; \forall s \leqslant t \; s \in T$, and there exists a member of T, called s_T, such that

(i) $\forall s \in T \; s \leqslant s_T$ or $s_T \leqslant s$, and

(ii) $\forall s \in T$ if $s_T \leqslant s$ then $\mathrm{Card}\{n: s^\frown n \in T\} = \aleph_0$.

(1) Thus Cohen reals and $\mathcal{J}$-generic reals are at opposite ends of the spectrum from this point of view. In between are Sacks reals and Solovay reals; upon forcing with one of these reals, a ground model set will have strong measure zero in the extension if and only if it had strong measure zero in the ground model. $\mathcal{J}$-generic reals have the closest similarity in this context with Mathias reals ([13])—see below.

Order $\mathcal{J}$ by $S \leqslant T$ (T extends S) *iff* $T \subseteq S$. A generic set $G_{\mathcal{J}} \subseteq \mathcal{J}$ canonically creates a generic real $g_{\mathcal{J}}$: $\omega \to \omega$, namely $g_{\mathcal{J}} = \bigcup_{T \in G_{\mathcal{J}}} s_T$. A standard genericity argument shows that $\mathfrak{M}[G_{\mathcal{J}}] = \mathfrak{M}[g_{\mathcal{J}}]$.

Notation: o is $(\omega)^{<\omega}$, the least element of $\mathcal{J}$. For $T \in \mathcal{J}$, $s \in T$, $T_s = \{t \in T : s \leqslant t \text{ or } t \leqslant s\}$, a member of $\mathcal{J}$. Fix an enumeration of $(\omega)^{<\omega}$, $x_0, x_1, \ldots, x_n, \ldots$ $(n<\omega)$, such that (a) $x_i < x_j \Rightarrow i<j$, and (b) $x_i = s^\frown m$, $x_j = s^\frown n$, and $m<n \Rightarrow i<j$. This yields, for each $T \in \mathcal{J}$, an enumeration $T\langle 0\rangle, T\langle 1\rangle, \ldots, T\langle n\rangle, \ldots$ $(n<\omega)$ of $\{s \in T : s \geqslant s_T\}$ (via the natural isomorphism of that set with $(\omega)^{<\omega}$); we will use the notation $T\langle 0\rangle$ in place of s_T from now on. For S, $T \in \mathcal{J}$, define

$$S \leqslant^n T$$

to mean that

(a) $S \leqslant T$

(b) $S\langle i\rangle = T\langle i\rangle$, all $i \leqslant n$.

The set $\{T\langle 0\rangle, \ldots, T\langle n\rangle\}$ determines a family of $n+1$ subtrees of T whose union is T, namely, for each $i \leqslant n$, the subtree S_i (with $S_i \in \mathcal{J}$ and with $S_i\langle 0\rangle = T\langle i\rangle$) is obtained by taking the union of all T_t's such that t is an immediate successor of $T\langle i\rangle$ in T and such that t is not a $T\langle j\rangle$ for any $j \leqslant n$. The set $\{S_0, S_1, \ldots, S_n\}$ will be referred to as the set of components of T at stage n; note that they form a maximal set of incompatibles extending T.

LEMMA 1. *Let $m<\omega$ and $T_i \in \mathcal{J}$ $(m \leqslant i < \omega)$ satisfy $T_i <^i T_{i+1}$. Let $T_w = \bigcup_{m \leqslant i} \{T_i\langle 0\rangle, T_i\langle 1\rangle, \ldots, T_i\langle i\rangle\} \bigcup \{s;\ s \leqslant T_m\langle 0\rangle\}$. Then T_ω is the unique $T \in \mathcal{J}$ such that $\forall i \geqslant m \quad T_i \geqslant^i T$.*

Proof. Clear.

LEMMA 2. *Suppose $T \in \mathcal{J}$, $k<\omega$ and φ_n, $n \leqslant k$ are sentences of the forcing language such that $T \Vdash \bigvee_{n \leqslant k} \varphi_n$. Then for each $i<\omega$ there is a T' with $T \leqslant^i T'$ and an $I \subseteq \{0, 1, \ldots, k\}$ of power $\leqslant i+1$ such that $T' \Vdash \bigvee_{n \in I} \varphi_n$.*

Proof. In the case $i=0$ this says that if $T \Vdash \bigvee_{n \leqslant k} \varphi_n$ then for some T' with $T \leqslant^0 T'$, $\exists n T' \Vdash \varphi_n$. We first prove this fact. Suppose it were false. We will construct an $S \in \mathcal{J}$ with $T \leqslant^0 S$ such that for each $s \in S$

$$\neg \exists T' \exists n T_s \leqslant^0 T' \quad \text{and} \quad T' \Vdash \varphi_n. \tag{1}$$

This will be a contradiction since $T \leqslant S$ and S could not be extended to force any φ_n.

By the assumption, we may begin by putting each $t \leqslant T\langle 0\rangle$ into S. Having put $t \in S$

with $t \geqslant T\langle 0\rangle$ we then put a sequence s, with $s \in T$ and s of the form $t^\frown m$, into S if and only if (1) holds. The inductive construction of S will be done if there are always infinitely many such s's. If that failed for t, then there would be an infinite $A \subseteq \omega$ with $m \in A \Rightarrow t^\frown m \in T$, and an $n \leqslant k$ and trees T^m, $m \in A$, with $T^m \leqslant^0 T_{t^\frown m}$ and $T^m \Vdash \varphi_n$. But then $T_t \leqslant^0 \bigcup_{m\in A} T^m$ and $\bigcup_{m\in A} T_m \Vdash \varphi_n$, contradicting $t \in S$.

This proves the case $i=0$ of the lemma. The case $i>0$ follows by applying the 0 case to each of the stage i components of T.

LEMMA 3. *Suppose $T \in \mathcal{J}$, $T \Vdash \dot{a} \in \mathcal{M}$. Then there is a countable A in $\mathcal{M}$ and a T' with $T \leqslant^0 T'$ such that $T' \Vdash \dot{a} \in A$.*

Proof. If not, then we may construct, similarly to the case $i=0$ of Lemma 2, an S with $T \leqslant^0 S$ such that for each $s \in S$,

$$\neg \exists T' \exists A \; A \in \mathcal{M},\ A \text{ countable in } \mathcal{M},\ T_s \leqslant^0 T',\ T' \Vdash \dot{a} \in A. \tag{2}$$

(for the induction use that if (2) holds for $s \in T$ with $s \geqslant T\langle 0\rangle$, then it must hold for infinitely many immediate successors of s in T). But S could not be extended to force $\dot{a}=a$, for any $a \in \mathcal{M}$, a contradiction.

LEMMA 4. *Suppose $T \in \mathcal{J}$, $T \Vdash \dot{A}$ is a countable subset of $\mathcal{M}$. Then for each $n<\omega$ there is a countable A in $\mathcal{M}$ and a T' with $T \leqslant^n T'$ such that $T' \Vdash \dot{A} \subseteq A$.*

Proof. Case $n=0$. Let $\dot{A} = \{\dot{a}_0, \dot{a}_1, \ldots, \dot{a}_i, \ldots\}$. Construct T' by an appropriate ω stage fusion argument, arranging at step i, by applications of Lemma 3, that the possible values of $\dot{a}_i$ will be countably bounded by T'.

For the case $n>0$, apply the case $n=0$ of the lemma to each of the stage n components of T.

Thus $\mathcal{J}$ preserves cofinalities (over a ground model of $2^{\aleph_0}=\aleph_1$).

The ordering $\mathcal{P}_{\omega_2}$ for iteratively adding ω_2 $\mathcal{J}$-generic reals is defined as follows. We inductively define $\mathcal{P}_\alpha$, for $1 \leqslant \alpha \leqslant \omega_2$ (ordered by $\leqslant_\alpha$, with least element o_α, and forcing relation $\Vdash_\alpha$) where $p \in \mathcal{P}_\alpha \Rightarrow p$ is a function whose domain is α, and where G_α is a name for the generic set over $\mathcal{P}_\alpha$, by:

(i) $\mathcal{P}_1$ (isomorphic to $\mathcal{J}$) is the set of all functions $p\colon 1 \to \mathcal{J}$, where $p \leqslant_1 q$ iff $p(0) \leqslant q(0)$.

(ii) $\mathcal{P}_{\alpha+1} = \mathcal{P}_\alpha \bar{\otimes} \mathcal{J}^{\mathcal{M}[G_\alpha]} = \{p\colon \operatorname{dom} p = \alpha+1,\ p \restriction \alpha \in \mathcal{P}_\alpha$, and $p(\alpha)$ is a canonical term in the forcing language of $\mathcal{P}_\alpha$ for a member of $\mathcal{J}$ in $\mathcal{M}[G_\alpha]\}$, ordered by

$$p \leqslant_{\alpha+1} q \text{ iff } p\restriction\alpha \leqslant_\alpha q\restriction\alpha \quad\text{and}\quad q\restriction\alpha \Vdash_\alpha p(\alpha) \leqslant q(\alpha).$$

(iii) For α a limit ordinal, $\mathcal{P}_\alpha$ is the set of all p with dom $p=\alpha$ such that

(a) if $1 \leqslant \beta < \alpha$ then $p \upharpoonright \beta \in \mathcal{P}_\beta$, and

(b) for all but countably many β with $1 \leqslant \beta < \alpha$, $\Vdash_\beta p(\beta)=0$.[1]

The subscripts in $\leqslant_\alpha$, 0_α, and $\Vdash_\alpha$ will be dropped when context permits.

If $1 \leqslant \alpha < \beta$, $\mathcal{P}^{\alpha\beta}$ is the set of all functions f with domain $[\alpha, \beta)$ such that $o_\alpha \cup f \in \mathcal{P}_\beta$. Order $\mathcal{P}^{\alpha\beta}$ in $\mathcal{M}[G_\alpha]$ by

$$f \leqslant g \Leftrightarrow \exists p \in G_\alpha \quad p \cup f \leqslant p \cup g.$$

Standard facts about iterated forcing yield the following things. If $\alpha < \beta$ then $\mathcal{P}_\alpha$ is canonically embedded in $\mathcal{P}_\beta$, and $G_\alpha = \{p \upharpoonright \alpha : p \in G_\beta\}$. $\mathcal{M}[G_\beta] = \mathcal{M}[G_\alpha][G^{\alpha\beta}]$, where $G^{\alpha\beta}$ is $\mathcal{M}[G_\alpha]$-generic over $\mathcal{P}^{\alpha\beta}$.

Write $\mathcal{P}^\alpha$, G^α for $\mathcal{P}^{\alpha\omega_2}$, $G^{\alpha\omega_2}$. We will show later that in $\mathcal{M}[G_\alpha]$, forcing with $\mathcal{P}^\alpha$ is equivalent to forcing with $(\mathcal{P}_{\omega_2})^{\mathcal{M}[G_\alpha]}$.

If $p \in \mathcal{P}_\alpha$ or $p \in \mathcal{P}^{\alpha\beta}$, let support p be the countable set in $\mathcal{M}$, $\{\beta \in \text{dom } p : \not\Vdash p(\beta)=0\}$.

Suppose $1 \leqslant \beta \leqslant \omega_2$, F is a finite subset of β, and $n < \omega$. Then define, for $p, q \in \mathcal{P}_\beta$,

$$p \leqslant_F^n q$$

to mean that

(a) $p \leqslant q$

(b) for all $\alpha \in F$ $q \upharpoonright \alpha \Vdash p(\alpha) \leqslant^n q(\alpha)$.

Fusion arguments will be done in accord with the following lemma.

Lemma 5. *Suppose p_n: $n < \omega$ are members of $\mathcal{P}_\beta$, and F_n: $n < \omega$ is an increasing chain of finite sets such that $\bigcup_{n<\omega} F_n = \bigcup_{n<\omega}$ support p_n, and for each n, $p_n \leqslant_{F_n}^n p_{n+1}$. Then there is a (unique up to equivalence in the ordering on $\mathcal{P}_\beta$) $p_\omega \in \mathcal{P}_\beta$ such that for each $n < \omega$, $p_n \leqslant_{F_n}^n p_\omega$.*

Proof. For $\alpha < \beta$, define $p_\omega(\alpha)$ to be a term in the language of $\mathcal{P}_\alpha$ representing the downwards closure of the set of all $p_n(\alpha)\langle i \rangle$'s such that $\alpha \in F_n$ and $i \leqslant n$. We show by induction on γ, $1 \leqslant \gamma \leqslant \beta$, that

$$p_\omega \upharpoonright \gamma \in \mathcal{P}_\gamma$$

and that for each n,

$$p_n \upharpoonright \gamma \leqslant_{F_n \cap \gamma}^n p_\omega \upharpoonright \gamma.$$

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[1] This clause, when stated in the Boolean algebraic terminology of [25], becomes: B_λ is the direct limit of $\langle B_\beta : \beta < \lambda \rangle$ when $cf(\lambda) > \omega$, and is the inverse limit of $\langle B_\beta : \beta < \lambda \rangle$ when $cf(\lambda) = \omega$.

(a) $\gamma = 1$. This case follows from Lemma 1.

(b) $\gamma = \delta + 1$, $1 \leqslant \delta$. Then for all n, $p_n \restriction \delta \leqslant p_\omega \restriction \delta$ by induction; thus $p_\omega \restriction \delta$ forces that the term $p_\omega(\delta)$ stands for the element of $\mathcal{J}$ constructed from $\langle p_i(\delta) \colon \delta \in F_i \rangle$ in the manner of Lemma 1. It follows that for all n, $p_n \restriction \delta + 1 \leqslant^{n}_{F_n \cap (\delta+1)} p_\omega \restriction \delta + 1$, as desired.

(c) γ a limit ordinal. This case follows from the induction hypothesis.

An important difference between iterated forcing and side by side forcing is the following. Suppose $1 \leqslant \alpha < \beta$, $q' \in \mathcal{P}_\alpha$, and Q is a maximal antichain of $\{q \in \mathcal{P}_\alpha \colon q \text{ compatible with } q'\}$. Suppose that for each $q \in Q$, p_q is a condition in $\mathcal{P}_\beta$ satisfying $p_q \restriction \alpha = q$. Then there is a $p \in \mathcal{P}_\beta$ which is the disjunction (greatest lower bound) of $\{p_q \colon q \in Q\}$. Namely, $p \restriction \alpha = q'$, and for γ with $\alpha \leqslant \gamma < \beta$, $p(\gamma)$ is a term corresponding to the conjunction of the implications $q \Rightarrow p_q(\gamma)$ for $q \in Q$. If $p' \in \mathcal{P}_\beta$ and $p' \leqslant^{i}_{F} p_q$, all $q \in Q$, then $p' \leqslant^{i}_{F} p$.

Suppose $p \in \mathcal{P}_\beta$, $\{r_1, r_2, \ldots, r_i\} \subseteq \omega$, and $n < \omega$ and $F \subseteq \beta$ are fixed by context with $F = \{\alpha_1, \alpha_2, \ldots, \alpha_i\}$, $\alpha_1 < \alpha_2 < \ldots < \alpha_i$. Then define $p^{r_1, \ldots, r_i}$, a member of $\mathcal{P}_\beta$ extending p, to be equal to p on coordinates not in F, and to satisfy, for $1 \leqslant s \leqslant i$,

$$p^{r_1, \ldots, r_i}(\alpha_s) \text{ is the } r_s^{\text{th}} \text{ stage } n \text{ component of } p(\alpha_s).$$

For $n < \omega$, the set of all $p^{r_1, \ldots, r_i}$'s with each $r_i \leqslant n$ is a maximal antichain above p. Namely, if $p \leqslant p_1$, then there are numbers $r_1, \ldots, r_i$ and a $p_2 \geqslant p_1$ such that for each $j \leqslant i$,

$p_2 \restriction \alpha_j \Vdash$ length $p_2(\alpha_j)\langle 0 \rangle > n +$ length $p(\alpha_j)\langle 0 \rangle$ and $p_2(\alpha_j)\langle 0 \rangle$ belongs to the r_jth stage n component of $p(\alpha_j)$.

Then p_2 extends $p^{r_1, \ldots, r_i}$.

Suppose $p \in \mathcal{P}_\beta$, $F = \{\alpha_1, \ldots, \alpha_i\}$, $\alpha_1 < \ldots < \alpha_i < \beta$, and $n < \omega$. Suppose $q \in \mathcal{P}_\beta$ satisfies $p^{r_1, \ldots, r_i} \leqslant^{0}_{F} q$ (where each $r_j \leqslant n$ and $p^{r_1, \ldots, r_i}$ is taken with respect to F and n). Then the amalgamation of p and q is a $p' \in \mathcal{P}_\beta$ such that

$$p \leqslant^{n}_{F} p'$$
$$q \leqslant^{0}_{F} (p')^{r_1, \ldots, r_i}$$

defined as follows. Suppose $\alpha < \beta$ and $p' \restriction \alpha$ has been defined.

(a) $\alpha = \alpha_j$. Let

$$p'(\alpha) = (p(\alpha) - (p(\alpha))^{r_j}) \cup q(\alpha).$$

(we may assume without loss of generality that the term $q(\alpha)$ is such that $p \restriction \alpha \Vdash (p(\alpha))^{r_j} \leqslant q(\alpha)$).

(b) $\alpha \notin F$. If $\alpha < \alpha_1$, let $p'(\alpha) = q(\alpha)$. If $\alpha > \alpha_1$, let α_m be the maximum element of F less than α, and let $p'(\alpha)$ be the conjunction of

$$(p' \restriction \alpha)^{r_1, \ldots, r_m} \Rightarrow q(a),$$

and

$$\neg (p' \restriction \alpha)^{r_1, \ldots, r_m} \Rightarrow p(\alpha).$$

It is seen that p' satisfies the two properties above.

LEMMA 6. *Let* $1 \leqslant \beta \leqslant \omega_2$, $p \in \mathcal{P}_\beta$, *and let* $F = \{\alpha_1, \alpha_2, \ldots, \alpha_i\}$ *be a finite subset of* β *with* $\alpha_1 < \alpha_2 < \ldots < \alpha_i$. *Let* $n < \omega$. *Then*

(i) *If* $k < \omega$ *and* $p \Vdash \bigvee\!\!\bigvee_{j<k} \varphi_j$, *then there is an* $I \subseteq \{0, 1, \ldots, k\}$ *with* Card $I \leqslant (n+1)^i$ *and a* p' *such that* $p \leqslant^n_F p'$ *and* $p' \Vdash \bigvee\!\!\bigvee_{j \in I} \varphi_j$.

(ii) *If* $p \Vdash \dot{a} \in \mathcal{M}$ *then there is a countable* A *in* $\mathcal{M}$ *and a* p' *with* $p \leqslant^n_F p'$ *such that* $p' \Vdash \dot{a} \in A$.

(iii) *If* $p \Vdash \dot{A}$ *is a countable subset of* $\mathcal{M}$ *then there is a countable* A *in* $\mathcal{M}$ *and a* p' *with* $p \leqslant^n_F p'$ *such that* $p' \Vdash \dot{A} \subseteq A$.

(iv) *If* $\beta < \delta \leqslant \omega_2$ *and* $p \Vdash \dot{f} \in \mathcal{P}^{\beta\delta}$ *then there is an* $f \in \mathcal{P}^{\beta\delta}$ *and a* p' *with* $p \leqslant^n_F p'$ *such that* $p' \Vdash \dot{f} \equiv f$.

Proof. By induction on β,

(i)(a). $\beta = 1$. This is Lemma 2.

(i)(b). $\beta = \sigma + 1$, $\sigma \geqslant 1$. $\sigma = \alpha_i$ may be assumed. Let $\dot{S}_k$ be a term for the k^{th} stage n component of $p(\sigma)$, each $k \leqslant n$. Since Lemma 2 holds in $\mathcal{M}[G_\sigma]$, there is a term $\dot{S}_k'$ for each $k \leqslant n$ such that

$$p \restriction \sigma \Vdash (\dot{S}_k \leqslant^0 \dot{S}_k' \quad \text{and} \quad \exists j \quad \dot{S}_k' \Vdash \varphi\ ).$$

By the induction hypothesis we may choose

$$p \restriction \sigma \leqslant^n_{F \cap \sigma} q_0 \leqslant^n_{F \cap \sigma} \cdots \leqslant^n_{F \cap \sigma} q_n$$

and sets

$$I_0, I_1, \ldots, I_n$$

such that for $k \leqslant n$, Card $I_k = (n+1)^{i-1}$ and

$$q_k \Vdash \dot{S}_k' \Vdash \bigvee\!\!\bigvee_{j \in I_k} \varphi_j.$$

Let $\dot{S}'$ be a term standing for the union of the $\dot{S}_k'$'s, $I = \bigcup_{k \leqslant n} I_k$. Let

$$p' = q_n {}^\frown \langle \dot{S}' \rangle$$

$p \leqslant^n_F p'$ and $p' \Vdash \bigvee\!\!\bigvee_{j \in I} \varphi_j$, as desired.

(i)(c) β a limit ordinal. Let $\dot{f}$ be a term, in the language of $\mathcal{P}_{\alpha_i+1}$, for a member of $\mathcal{P}^{\alpha_i+1,\beta}$ such that $p \restriction \alpha_i + 1$ forces

$$p \restriction [\alpha_i + 1, \beta) \leqslant \dot{f} \quad \text{and} \quad \exists j \quad \dot{f} \Vdash \varphi_j.$$

Applying part (iv) of the lemma to α_i+1, get a q with $p\restriction\alpha_{i+1}\leqslant^n_F q$ and an $f\in\mathcal{P}^{\alpha_i+1,\beta}$ such that $q\Vdash\dot{f}\equiv f$. Then find, by the induction hypothesis, a q' with $q\leqslant^n_F q'$ and an $I\subseteq\{0,1,\ldots,k\}$ with Card $I\leqslant(n+1)^i$ such that

$$q'\Vdash\exists j\in I\quad f\Vdash\varphi_j.$$

Take

$$p'=q'\cup f.$$

p' has the required properties.

(ii)(a). $\beta=1$. This is Lemma 3.

(ii)(b). $\beta=\sigma+1$, $\sigma\geqslant 1$. Since Lemma 3 holds in $\mathcal{M}[G_\sigma]$ there are terms $\dot{T}$ and $\dot{A}$ in the language of $\mathcal{P}_\sigma$, standing for a member of $(\mathcal{T})^{\mathcal{M}[G_\sigma]}$ and a countable subset of $\mathcal{M}$ lying in $\mathcal{M}[G_\sigma]$, such that $p\restriction\sigma$ forces that $p(\sigma)\leqslant^n\dot{T}$ and that $\dot{T}\Vdash\dot{a}\in\dot{A}$. Applying part (iii) of the lemma to σ, get a $q\in\mathcal{P}_\sigma$ with $p\restriction\sigma\leqslant^n_{F\cap\sigma}q$, and a countable set A in $\mathcal{M}$ such that

$$q\Vdash\dot{A}\subseteq A.$$

Then

$$p'=q^\frown\langle\dot{T}\rangle$$

is as desired.

(ii)(c). β a limit ordinal. Using (iv) at α_{i+1} as in case (i)(c), find a term $\dot{A}$ in the language of $\mathcal{P}_{\alpha_i+1}$ standing for a countable subset of $\mathcal{M}$ lying in $\mathcal{M}[G_{\alpha_i+1}]$, and a q with $p\leqslant^n_F q$ such that

$$q\Vdash\dot{a}\in\dot{A}.$$

Now use (iii) at α_i+1 to get a p' with $q\leqslant^n_F p'$ and a countable A in $\mathcal{M}$ with

$$p'\Vdash\dot{A}\subseteq A.$$

p' is as desired.

(iii) Let $\dot{A}=\{\dot{a}_1,\dot{a}_2,\ldots,\dot{a}_n,\ldots\}$. p' will be constructed by a fusion argument. Namely, start with $p=p_0=p_1=\ldots=p_n$, $F=F_0=F_1=\ldots=F_n$, and build up sequences $p_{n+1},p_{n+2},\ldots$ and $F_{n+1},F_{n+2},\ldots$ and $A_1,A_2,\ldots$ such that

(a) $\forall i\;\; p_i\leqslant^i_{F_i}p_{i+1}$

(b) $\forall i\;\; F_i\subseteq F_{i+1}$, and $\bigcup_{i<\omega}F_i=\bigcup_{i<\omega}$ support p_i

(c) $\forall i>0\;\; A_i$ is a countable set in $\mathcal{M}$ and $p_{i+n}\Vdash\dot{a}_i\in A_i$.

To go from i to $i+1$ satisfying (a) and (c), use (ii). The fact that the support p_i's may increase is handled by choosing the F_i's as the construction proceeds, by a bookkeeping method, in such a way as to make (b) hold. Set $p'=p_\omega$, the condition given by Lemma 5, and let $A=\bigcup_{i<\omega}A_i$. Then p' and A satisfy the requirement of (iii).

(iv) By (iii), there is a p' and a countable $A \subseteq [\beta, \delta)$ in $\mathcal{M}$ such that $p \leqslant^n_F p'$ and

$$p' \Vdash \text{support}\, \dot{f} \subseteq A.$$

If $\gamma \in [\beta, \delta) - A$, let $f(\gamma)$ be a term in the language of $\mathcal{P}_\gamma$ for the 0 element of $(\mathcal{T})^{\mathcal{M}[G_\gamma]}$. If $\gamma \in A$, let $f(\gamma)$ be a term in the language of $\mathcal{P}_\gamma$ such that $\Vdash \dot{f}(\gamma) \equiv f(\gamma)$. Then $f \in \mathcal{P}^{\beta\delta}$ and

$$p' \Vdash \dot{f} \equiv f,$$

as required.

This completes the proof of Lemma 6.

Lemma 6 (iii) implies

LEMMA 7. *For each $\alpha \leqslant \omega_2$, ω_1 is preserved in passage to $\mathcal{M}[G_\alpha]$.*

For the rest of the proof $\mathcal{M}$ is assumed to be a model of $2^{\aleph_0} = \aleph_1$.

To show that for $1 \leqslant \beta \leqslant \omega_2$, $\mathcal{P}_\beta$ has the $\aleph_2$ chain condition, we will define, for $1 \leqslant \beta \leqslant \omega_2$, a $\mathcal{W}_\beta \subseteq \mathcal{P}_\beta$ with $\text{Card}(\mathcal{W}_\beta / \equiv) = \aleph_1$, and show $\mathcal{W}_\beta$ cofinal in $\mathcal{P}_\beta$.

(i) $\mathcal{W}_1 = \mathcal{P}_1$.

(ii) For $\beta \geqslant 1$, $\mathcal{W}_{\beta+1}$ is the set of all $p \in \mathcal{P}_{\beta+1}$ such that $p \upharpoonright \beta \in \mathcal{W}_\beta$ and statements of the form $s = p(\beta)\langle i \rangle$ are decided by countable antichains $\subseteq \mathcal{W}_\beta$ (that is, for $p \in \mathcal{P}_{\beta+1}$, a function h_β: $(\omega)^{<\omega} \times \omega \to \{\text{antichains} \subseteq \{q \in \mathcal{P}_\beta: q \text{ compatible with } p \upharpoonright \beta\}\}$ may be associated with $p(\beta)$ in such a way that

$\Vdash \exists T \in (\mathcal{T})^{\mathcal{M}[G_\beta]} \forall i T\langle i \rangle$ is the unique s with $h_\beta(s, i) \in G_\beta$;

(for $p \in \mathcal{P}_{\beta+1}$ with $p \upharpoonright \beta \in \mathcal{W}_\beta$, $p \in \mathcal{W}_{\beta+1}$ if there is a function h_β as above whose values are countable subsets of $\mathcal{W}_\beta$).

(iii) For β a limit ordinal, $\mathcal{W}_\beta$ is the set of all $p \in \mathcal{P}_\beta$ such that if $1 \leqslant \alpha < \beta$, $p \upharpoonright \alpha \in \mathcal{W}_\alpha$.

LEMMA 8. *If p_n, $n < \omega$, is a sequence of members of $\mathcal{W}_\beta$, and F_n, $n < \omega$, is an increasing sequence of finite subsets of β with $\bigcup_{n<\omega} F_n = \bigcup_{n<\omega}$ support p_n, and for each n, $p_n \leqslant^n_{F_n} p_{n+1}$ then the fusion p_ω of the p_n's, given by Lemma 5, is in $\mathcal{W}_\beta$.*

Proof. $p_\omega(\alpha)\langle i \rangle = p_n(\alpha)\langle i \rangle$ for some/all $n \geqslant i$ with $\alpha \in F_n$. The result follows.

LEMMA 9. *If $p \in \mathcal{P}_\beta$, $n < \omega$, F a finite subset of β, then there is a $p' \in \mathcal{W}_\beta$ with $p \leqslant^n_F p'$.*

Proof. By induction on β

(a) $\beta = 1$. Take $p' = p$.

(b) $\beta = \sigma + 1$, $\sigma \geqslant 1$. By induction, pick $p_0 \in \mathcal{W}_\sigma$ with $p \upharpoonright \sigma \leqslant^n_{F \cap \sigma} p_0$. We will construct members $p_0, p_1, \ldots, p_i, \ldots$ $(i < \omega)$ of $\mathcal{P}_\sigma$ and an increasing sequence $F_0, F_1, \ldots, F_i, \ldots$ of finite subsets of σ, with $F_0 = F_1 = \ldots = F_n = F \cap \sigma$, $p_0 = p_1 = \ldots = p_n$, such that

(i) $\forall i\ p_i \leqslant^i_{F_i} p_{i+1}$

(ii) $\bigcup_{i<\omega} F_i = \bigcup_{i<\omega}$ support p_i

(iii) letting $x_n, x_{n+1}, \ldots, x_{n+k}, \ldots$ enumerate $(\omega)^{<\omega}$, for each $i \geqslant n$ there is a finite maximal antichain A_i of $\{q \in \mathcal{P}_\sigma,\ q$ compatible with $p_{i+1}\}$, with $A_i \subseteq \mathcal{W}_\sigma$, such that for each $q \in A_i$, q decides whether there is a k, and determines k if there is one, such that $x_i = p(\sigma)\langle k\rangle$.

Suppose $i \geqslant n$ and we have constructed $p_0, p_1, \ldots, p_i$ and $F_0, F_1, \ldots, F_i$. Let $a = \text{Card}\ F_i$, $b=(i+1)^a$. Let $\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{b-1}$ be an enumeration of all sequences of the form $\langle r_1, \ldots, r_a\rangle$ where each $r_m \leqslant i$. Let $q_0 = p$. We will define a sequence

$$q_0 \leqslant^i_{F_i} q_1 \leqslant^i_{F_i} \cdots \leqslant^i_{F_i} q_b,$$

and members $q''_0, q''_1, \ldots, q''_{b-1}$ of $\mathcal{W}_\sigma$. Suppose q_j has been defined. Form the condition $q_j^{\mathbf{r}_j}$, taken with respect to F_i and i. By Lemma 6(i) there is a q'_j with $q_j^{\mathbf{r}_j} \leqslant^0_{F_i} q'_j$, which decides for which k, if any, that $x_i = p(\sigma)\langle k\rangle$ (to apply Lemma 6(i), use that there are only finitely many k's such that $\exists T\ x_i = T\langle k\rangle$). By the induction hypothesis there is a $q''_j \in \mathcal{W}_\sigma$ with $q'_j \leqslant^0_{F_i} q''_j$. Let q_{j+1} be the amalgamation of q_j and q''_j. $q_j \leqslant^i_{F_i} q_{j+1}$.

Let $p_{i+1} = q_b$. Let $F_{i+1} \supseteq F_i$ be chosen as per a bookkeeping arrangement for making (ii) hold. It is seen that (iii) holds for p_{i+1}, the finite antichain consisting of the q''_j's of the construction.

This completes the definition of the p_i's and F_i's. Let p_ω be the fusion of the p_i's. By the induction hypothesis, pick a $\bar{p} \in \mathcal{W}_\sigma$ with $p_\omega \leqslant^n_{F \cap \sigma} \bar{p}$. Let

$$p' = \bar{p}^\frown\langle p(\sigma)\rangle.$$

We have $p \leqslant^n_F p'$. $p' \in \mathcal{W}_\beta$, namely, an h_σ for $p'(\sigma)$, as in the definition of $\mathcal{W}_\beta$, may be formed by taking $h_\sigma(x_i, k)$ to be the set of q''_j's formed at stage i which force that $x_i = p(\sigma)\langle k\rangle$.

(c) β a limit ordinal. If $cf(\beta) > \omega$ the lemma is clear. Assume $cf(\beta) = \omega$. Let

$$F_0 = \ldots = F_n = F, \quad q_0 = \ldots = q_n \in \mathcal{W}_{\max F_n + 1}$$

satisfy $p \restriction \text{dom}\ q_n \leqslant^n_{F_n} q_n$. Having picked F_j, q_j, and F_{j+1}, let $q_{j+1} \in \mathcal{W}_{\max F_{j+1}+1}$ satisfy

$$p \restriction \text{dom}\ q_{j+1} \leqslant^{j+1}_{F_{j+1}} q^{j+1}$$

and

$$q_j \leqslant^{j+1}_{F_{j+1} \cap \text{dom}\ q_j} q_{j+1} \restriction \text{dom}\ q_j.$$

Choose the F_i's so that $\bigcup_{i<\omega} F_i \supseteq \bigcup_{i<\omega}$ support q_i and so that $\bigcup_{i<\omega} F_i$ is cofinal in β. Then, letting $p_j \in \mathcal{W}_\beta$ be $q_j{}^\frown\langle 0, 0, \ldots, 0, \ldots\rangle$, the p_j's form a fusion sequence. Set

$$p' = p_\omega,$$

the fusion of this sequence. $p' \in \mathcal{W}_\beta$ by Lemma 8, and $p \leqslant^n_F p'$.

This completes the proof of the lemma.

LEMMA 10. (i) For $\beta < \omega_2$, $\mathrm{Card}(\mathcal{W}_\beta / \equiv) = \aleph_1$.

(ii) *For* $\beta \leqslant \omega_2$, $\mathcal{P}_\beta$ *has the* $\aleph_2$ *chain condition.*

(iii) *For* $\beta \leqslant \omega_2$, *cofinalities are preserved in passage from* $\mathfrak{M}$ *to* $\mathfrak{M}[G_\beta]$.

(iv) *For* $\beta < \omega_2$, $\mathfrak{M}[G_\beta] \models 2^{\aleph_0} = \aleph_1$, $\mathfrak{M}[G_{\omega_2}] \models 2^{\aleph_0} = \aleph_2$.

(v) *If* X *is a set of reals in* $\mathfrak{M}[G_{\omega_2}]$ *of power* $\leqslant \aleph_1$, *then* $X \in \mathfrak{M}[G_\alpha]$ *for some* $\alpha < \omega_2$.

Proof. (*i*). By induction on β. At $\beta = 1$ and at β a limit ordinal it is clear since $2^{\aleph_0} = \aleph_1$ holds in $\mathfrak{M}$. If $\beta = \sigma + 1$ the term $p(\sigma)$ of a $p \in \mathcal{W}_\beta$ is determined up to equivalence by the associated function h_σ in the definition of $\mathcal{W}_\beta$; there are only $2^{\aleph_0}$ possible such h_σ's.

(ii). The claim follows from Lemma 9, Lemma 10(i), and the fact that support p is countable, $p \in \mathcal{P}_{\omega_2}$.

(iii). From Lemma 7 and Lemma 10(ii).

(iv). Follows from Lemma 9, Lemma 10(i), and genericity.

(v). Follows from Lemma 10(ii) and Lemma 10(iii), and the countable support property of the conditions.

LEMMA 11. *For* $\beta < \omega_2$, *there is an isomorphism in* $\mathfrak{M}[G_\beta]$ *between* $(\mathcal{P}^\beta / \equiv)$ *and* $(\mathcal{P}_{\omega_2} / \equiv)^{\mathfrak{M}[G_\beta]}$.

Proof. $\omega_2^{\mathfrak{M}} = \omega_2^{\mathfrak{M}}[G_\beta]$ (Lemma 10(iii)). By the nature of the conditions, then, there is a canonical function

$$H\colon \mathcal{P}^\beta \to \mathcal{P}_{\omega_2}^{\mathfrak{M}[G_\beta]},$$

defined in $\mathfrak{M}[G_\beta]$, such that $f \leqslant g \Leftrightarrow H(f) \leqslant H(g)$. We need to show H is onto the equivalence classes of $(\mathcal{P}_{\omega_2})^{\mathfrak{M}[G_\beta]}$. Let $q \in (\mathcal{P}_{\omega_2})^{\mathfrak{M}[G_\beta]}$. Then by Lemma 6(iii) and genericity there is a $p \in G_\beta$ and a countable A in $\mathfrak{M}$ such that

$$p \Vdash \text{support } \dot{q} \subseteq A.$$

It follows that there is an $f \in \mathcal{P}^\beta$ with support $f \subseteq A$ such that in $\mathfrak{M}[G_\beta]$,

$$H(f) \equiv q,$$

as desired.

We will not need to distinguish between these isomorphic partial orderings below.

In the remainder of the proof it is shown that $\mathcal{P}_{\omega_2}$ kills all uncountable strong measure zero sets. Given a set X in the ground model forced by a $p \in \mathcal{P}_{\omega_2}$ to have strong measure zero, a sequence $\dot{\varepsilon}_1, \dot{\varepsilon}_2, ..., \dot{\varepsilon}_n, ...$ will be coded from the generic function g_1 given by G_1. Letting $\langle \dot{I}_1, \dot{I}_2, ..., \dot{I}_n, ... \rangle$ be a term for a sequence of intervals of lengths $\leqslant \dot{\varepsilon}_n$, lying in $\mathcal{M}[G_{\omega_2}]$, the union of which covers X, and letting $\dot{a}_n$ be the center of $\dot{I}_n$, a p' with $p \leqslant^0_{\{0\}} p'$, and finite sets of reals U_x, for all nodes x of $p'(0)$, will be chosen in $\mathcal{M}$ so that if length $x =$ length $p'(0)\langle 0 \rangle + n$, then no extension p'' of p', with $p''(0)\langle 0 \rangle = x$, bounds $\dot{a}_n$ away from all the members of U_x. For each ground model real $v \notin \bigcup_x U_x$, a $p'' \geqslant p'$ can be constructed to force $v \notin \bigcup_{n<\omega} \dot{I}_n$. Such a v thus cannot be in X, so $X \subseteq \bigcup_x U_x$ is a countable set. The stipulation in this argument that $X \in \mathcal{M}$ is then removed by Lemma 10(v) and Lemma 11. The reader might want to check this argument for the case of forcing with $\mathcal{P}_2$, the argument below being like that case with the more general type of fusion.

LEMMA 12. *Let $\dot{a}$ be a term for a real in $\mathcal{M}[G_{\omega_2}]$, $p \in \mathcal{P}_{\omega_2}$, F a finite subset of ω_2. Then there is a p' with $p \leqslant^0_{F \cup \{0\}} p'$ and a real $u \in \mathcal{M}$ such that for every $\varepsilon > 0$, for all but finitely many immediate successors t of $p'(0)\langle 0 \rangle$ in $p'(0)$,*

$$(p'(0))_t {}^\frown p' \upharpoonright [1, \omega_2) \Vdash |\dot{a} - u| < \varepsilon.$$

Proof. Let $t_0, t_1, ..., t_n, ...$ be the immediate successors in $p(0)$ of $p(0)\langle 0 \rangle$.

Fix n. We define an extension of the condition

$$p(0)_{t_n} {}^\frown p \upharpoonright [1, \omega_2).$$

Applying Lemma 6(i) in $\mathcal{M}[G_1]$, there is a term $\dot{f}_n$ such that

$$p(0)_{t_n} \Vdash p \upharpoonright [1, \omega_2) \leqslant^0_{F-\{0\}} \dot{f}_n \text{ and } \dot{f}_n \text{ determines which}$$
$$\text{of the intervals } [0, 1/n), [1/n, 2/n), ..., [(n-1)/n, 1] \text{ contains } \dot{a}.$$

By Lemma 6(iv) there is a q_n with $p(0)_{t_n} \leqslant^0 q_n$ and an $f_n \in \mathcal{P}^1$ such that

$$q_n \Vdash \dot{f}_n \equiv f_n.$$

By Lemma 6(i) there is a q'_n with $q_n \leqslant^0 q'_n$ and an I_n of length $1/n$ such that

$$q'_n \Vdash f_n \Vdash \dot{a} \in I_n.$$

Now pick an infinite $A \subseteq \omega$ such that $\langle I_n : n \in A \rangle$ converges to a real u. Let

$$p_n = q'_n {}^\frown f_n.$$

Define p' to be the disjunction of $\{p_n : n \in A\}$. u and p' satisfy the conditions of the lemma.

LEMMA 13. *Let $\dot{a}$ be a term for a real in $\mathcal{M}[G_{\omega_2}]$, $p \in \mathcal{P}_{\omega_2}$, F a finite subset of $[1, \omega_2)$, and $n<\omega$. Then there is a $p' \in \mathcal{P}_{\omega_2}$ with $p(0) \leqslant^0 p'(0)$, $p \upharpoonright [1, \omega_2) \leqslant^n_F p' \upharpoonright [1, \omega_2)$, and a finite set U of reals in $\mathcal{M}$ such that for each $\varepsilon > 0$, for all but finitely many immediate successors t of $p'(0)\langle 0 \rangle$ in $p'(0)$,*

$$p'(0)_t \frown p' \upharpoonright [1, \omega_2) \Vdash \exists u \in U \, |\dot{a} - u| < \varepsilon.$$

Proof. Let Card $F = i$, $b = (n+1)^i$. Let $\mathbf{r}_0, \ldots, \mathbf{r}_{b-1}$ enumerate the sequences $\langle r_1, \ldots, r_i \rangle$ with each $r_j \leqslant n$. Let $p_0 = p$. We will define a sequence $p_0, p_1, \ldots, p_b$ such that for $j < b$, $p_j(0) \leqslant^0 p_{j+1}(0)$ and $p_j \upharpoonright [1, \omega_2) \leqslant^n_F p_{j+1} \upharpoonright [1, \omega_2)$. Suppose $p_0, \ldots, p_j$ have been defined. Apply Lemma 12 to $p_j^{\mathbf{r}_j}$, getting a $q_j \in \mathcal{P}_{\omega_2}$ and a real u_j such that $p_j^{\mathbf{r}_j} \leqslant^0_{F \cup \{0\}} q_j$ and such that for all $\varepsilon > 0$, for all but finitely many immediate successors t of $q_j(0)\langle 0 \rangle$ in $q_j(0)$,

$$(q_j(0))_t \frown q_j \upharpoonright [1, \omega_2) \Vdash |\dot{a} - u_j| < \varepsilon.$$

Let p_{j+1} be the amalgamation of p_j and q_j.

Let $p' = p_b$, and $U = \{u_0, \ldots, u_{b-1}\}$. Given $\varepsilon > 0$, it is seen that for each immediate successor t of $p'(0)\langle 0 \rangle$ in $p'(0)$ such that

$$\forall j \, (q_j(0))_t \frown q_j \upharpoonright [1, \omega_2) \Vdash |\dot{a} - u_j| < \varepsilon$$

that

$$(p'(0))_t \frown p' \upharpoonright [1, \omega_2) \Vdash \exists j \, |\dot{a} - u_j| < \varepsilon$$

proving the lemma.

LEMMA 14. *Let $\dot{a}_0, \dot{a}_1, \ldots, \dot{a}_j, \ldots$ be terms for reals in $\mathcal{M}[G_{\omega_2}]$, and let $p \in \mathcal{P}_{\omega_2}$. Then there is a p' with $p \leqslant p'$, $p(0) \leqslant^0 p'(0)$, and a finite set of reals U_s for each $s \in p'(0)$, $p'(0)\langle 0 \rangle \leqslant s$, such that for each $\varepsilon > 0$ and each node s of $p'(0)$ with levels $s = $ level $p(0)\langle 0 \rangle + j$, for all but finitely many immediate successors t of s in $p'(0)$,*

$$(p'(0))_t \frown p' \upharpoonright [1, \omega_2) \Vdash \exists u \in U_s \, |\dot{a}_j - u| < \varepsilon.$$

Proof. (Remark: Given F and n, $p \leqslant^n_F p'$ could also be arranged as in the previous lemmas.) The proof is by a fusion argument, modified (for convenience) in the first coordinate. Let $a = $ level $p(0)\langle 0 \rangle$. Let $p_0 = p$. We will construct

$$p_0, p_1, \ldots, p_j, \ldots \quad \text{and} \quad F_0, F_1, \ldots, F_j, \ldots$$

such that $\bigcup_{j<\omega} F_j = \bigcup_{j<\omega}$ support $p_j - \{0\}$, $p_j \leqslant^j_{F_j} p_{j+1}$, and all nodes of $p_j(0)$ having level $\leqslant a + j$ belong to $p_{j+1}(0)$.

Suppose p_j has been defined. Fix a node s of level $a+j$ in $p_j(0)$. Apply Lemma 13 to the condition

$$(p_j(0))_s \frown p_j \restriction [1, \omega_2)$$

getting a q^s and a finite set of reals U_s such that $(p_j(0))_s \frown p_j \restriction [1\ \omega_2) \leqslant^j_{F_j \cup \{0\}} q^s$ and such that for every $\varepsilon > 0$, for all but finitely many immediate successors t of s in $q^s(0)$,

$$(q^s(0))_t \frown q^s \restriction [1, \omega_2) \Vdash \exists u \in U_s |\dot{a}_j - u| < \varepsilon.$$

Let p_{j+1} be the disjunction of $\{q^s: s \text{ on level } a+j \text{ of } p_j(0)\}$. p_{j+1} satisfies the claimed properties. Let F_{j+1} be chosen by the appropriate bookkeeping arrangement.

Let p' be defined by: $p'(0) = \bigcap_j p_j(0)$, $p' \restriction [1, w_2) =$ the fusion of $\{p_j \restriction [1, \omega_2): j < \omega\}$ (which exists under the assumption of $p'(0)$). p' is as desired.

LEMMA 15. *If X is a set of reals in $\mathfrak{M}$ which has strong measure zero in $\mathfrak{M}[G_{\omega_2}]$, then X is countable.*

Proof. Suppose $p \Vdash X$ has strong measure zero. Let length $p(0)\langle 0 \rangle = n$. In $\mathfrak{M}[G_1]$, let g_1 be the generic function: $\omega \to \omega$ given by G_1. Define ε_j for $j \geqslant n$ by $\varepsilon_j = 1/g_1(j+1)$. Let

$$\langle I_n, I_{n+1}, \ldots \rangle$$

be a sequence of intervals of lengths $\varepsilon_n, \varepsilon_{n+1}, \ldots$ such that $X \subseteq \bigcup_j I_j$. Let a_j be the center of I_j.

Back in $\mathfrak{M}$, apply Lemma 14 to $\dot{a}_n, \dot{a}_{n+1}, \ldots$ and p, getting a p' and sets U_s, $s \in p'(0)$, $s \geqslant p'(0)\langle 0 \rangle$ $(= p(0)\langle 0 \rangle)$.

We claim that

$$X \subseteq \bigcup_s U_s,$$

which will give the lemma. Suppose v is a real in $\mathfrak{M}$, $v \notin \bigcup_s U_s$. Then we will find a T with $p'(0) \leqslant^0 T$ such that

$$T \frown p' \restriction [1, \omega_2) \Vdash v \notin \bigcup_j \dot{I}_j,$$

which will prove $v \notin X$. T is defined by induction on the nodes of $p'(0)$; the choice of nodes of level $j+1$ forces that $v \notin I_j$. Suppose t on level $j \geqslant n$ has been put into T. The immediate successors of t in T will be all but finitely many of the immediate successors of t in $p'(0)$. Namely, since $v \notin U_t$, pick an ε with $2\varepsilon < |v-u|$, all $u \in U_t$. Discard finitely many of the immediate successors of t so that, letting q be the set of nodes of $p'(0)$ comparable with any of the remaining immediate successors of t,

$$q \frown p' \restriction [1, \omega_2) \Vdash \exists u \in U_t |\dot{a}_j - u| < \varepsilon.$$

Choose k such that $1/k < \varepsilon$; discard in addition all immediate successors of t whose value at j is less than k. Doing this at each t of level j gives the induction step. The tree T is as desired, giving the lemma.

To prove the theorem, it suffices to show that if X is a set of reals in $\mathcal{M}[G_{\omega_2}]$ of power $\aleph_1$, then X does not have strong measure zero. By Lemma 10(v), $X \in \mathcal{M}[G_\beta]$ for some $\beta < \omega_2$. If X had strong measure zero, we could, in view of Lemma 10 (iv) and Lemma 11, apply Lemma 15 with $\mathcal{M}[G_\beta]$ taken as ground model, concluding that X is countable.

This completes the proof.

We mention an alternative to using $\mathcal{T}$-generic reals in the proof. The forcing conditions for adding a Mathias real ([13]) to a ground model of ZFC are defined as follows. A condition is a function f whose domain is an infinite, coinfinite subset of ω, whose range $\subseteq \{0, 1\}$, with $f^{-1}\{0\}$ finite. The conditions are ordered by function extension. Some time ago, R. Solovay informed the author that he and R. Jensen had proved that adding one Mathias real kills the uncountable strong measure zero sets in the ground model. I recently proved that iteratively adding ω_2 Mathias reals (taking countable supports as in the above proof) to a ground model of $2^{\aleph_0} = \aleph_1$ gives a model of Borel's conjecture.

Let $\mathcal{N}$ be a countable transitive model of ZFC. For which cardinals $\varkappa$, λ, μ in $\mathcal{N}$ is there a cofinality preserving Cohen extension of $\mathcal{N}$ in which $\varkappa$ is the least cardinal such that there are no strong measure zero sets of that power, λ is the least cardinal such that there are no universal measure zero sets of that power, and $2^{\aleph_0} = \mu$?

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