A CHARACTERIZATION OF DOUGLAS SUBALGEBRAS

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1. Introduction

Let L^∞ be the complex Banach algebra of bounded Lebesgue measurable functions on the unit circle ∂D in the complex plane. The norm in L^∞ is the essential supremum over ∂D. Via radial limits, the algebra H^∞ of bounded analytic functions on the unit disc D forms a closed subalgebra of L^∞. This paper studies the closed subalgebras B of L^∞ properly containing H^∞. For such an algebra B we let B_I denote the closed algebra generated by H^∞ and the complex conjugates of those inner functions which are invertible in the algebra B. (An inner function is an H^∞ function unimodular on ∂D). It is clear that $B_I \subset B$. R. G. Douglas [4] has conjectured that $B = B_I$ for all B, and consequently algebras of the form B_I are called Douglas algebras.

A discussion of the Douglas problem and a survey of related work can be found in [11]. In particular, it is noted in [11] that the maximal ideal space $\mathfrak{M}(B)$ of B can be identified with a closed subset of $\mathfrak{M}(H^\infty)$, and when B is a Douglas algebra, $\mathfrak{M}(B)$ completely determines B. This means that if the Douglas question has an affirmative answer then distinct algebras B has distinct maximal ideal spaces. That the latter assertion is true when one of the algebras is a Douglas algebra is the main result of this paper. We prove that if B and B_1 are closed subalgebras of L^∞ containing H^∞, if $\mathfrak{M}(B) = \mathfrak{M}(B_1)$ and if B is a Douglas algebra, then $B = B_1$. Using this theorem, D. E. Marshall [9] has answered the Douglas question affirmatively.

Using functions of bounded mean oscillation, D. Sarason [13] had proved the theorem above in the special case when B is generated by H^∞ and the space of continuous functions on ∂D. By similar means, S. Axler [1], T. Weight [15] and the author [3] had verified the theorem for some other specific Douglas algebras.

Section 2 contains some preliminary definitions and lemmas. The more technical aspects of the proof are in section 3 and the main theorem is proved in section 4. Some

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readers may perfer reading section 4 before sections 2 and 3. In section 5 we describe the largest C^*-algebra contained in a Douglas algebra.

The proof of our main result follows a pattern from Sarason's paper [12]. The proof of Theorem 6 below uses techniques from C. Fefferman and E. M. Stein [6]. I would like to express my warm thanks to Professor D. Sarason for giving invaluable aid, and to Professors R. G. Douglas and A. Shields for very helpful discussions. I am also grateful to Professor J. Garnett for re-organizing the paper, improving the English and giving a simplified proof of Lemma 2 below.

2. Preliminaries

For an integrable function $f(t)$ on ∂D, denote the harmonic extension of f to D by

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, \theta - t)\, f(t)\, dt$$

where $P(r, t) = (1-r^2)/(1-2r\cos t + r^2)$ is the Poisson kernel. Let $\nabla f(re^{i\theta}) = (\partial f/\partial x(re^{i\theta}), \partial f/\partial y(re^{i\theta}))$, and $|\nabla f(re^{i\theta})|^2 = |\partial f/\partial x(re^{i\theta})|^2 + |\partial f/\partial y(re^{i\theta})|^2$. Our first lemma is a Littlewood-Paley identity.

LEMMA 1. *If $f, g \in L^2$ and at least one of $f(0)$ and $g(0)$ vanishes, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\, g(e^{it})\, dt = \frac{1}{\pi}\int_D \nabla f(re^{i\theta}) \cdot \nabla g(re^{i\theta})\, r \log\frac{1}{r}\, dr\, d\theta.$$

This lemma follows from the Parseval identity after expressing the gradients in polar coordinates. The corresponding result for the upper half plane is in [14, p. 83].

The second lemma can be proved using methods in [6] but it also follows from an invariant formulation of Lemma 1. For $z_0 = r_0 e^{i\theta_0} \in D$, let $(S(\theta_0, r_0) = \{re^{i\theta}\colon |\theta - \theta_0| \leqslant 4(1-r_0), r_0 \leqslant r < 1\}$.

LEMMA 2. *Let $\varepsilon > 0$, $|z_0| = r_0 \geqslant 1/2$. If $f \in L^\infty$, $\|f\|_\infty \leqslant 1$ and $|f(z_0)| > 1-\varepsilon$, then*

$$\iint_{S(\theta_0, r_0)} (1-r)\,|\nabla f|^2 r\, dr\, d\theta \leqslant C_1\, \varepsilon(1-r_0)$$

where C_1 is independent of ε and r_0.

Proof. Let $w = (z - z_0)/(1 - \bar{z}_0 z) = se^{i\varphi}$. On $s = 1$, $z_0 = r_0 e^{i\theta_0}$, $d\varphi = P(r_0, \theta - \theta_0)\, d\theta$. Let $f(z) = F(w) = (2\pi)^{-1}\int_{-\pi}^{\pi} P(s, t - \varphi)\, F(t)\, dt$. Then

$$\begin{aligned}(2\pi)^{-1}\int |F(e^{i\varphi}) - F(0)|^2 d\varphi &= (2\pi)^{-1}\int P(r_0, \theta - \theta_0)\, |f(e^{i\theta}) - f(z_0)|^2 d\theta \\ &= (2\pi)^{-1}\int P(r_0, \theta - \theta_0)\, |f(e^{i\theta})|^2 d\theta - |f(z_0)|^2 < 2\varepsilon.\end{aligned}$$

Hence by Lemma 1,

$$\frac{1}{\pi}\int_D |\nabla F(w)|^2 \log\frac{1}{|w|}\, s\, ds\, d\varphi < 2\varepsilon.$$

Now $1-|w|^2=(1-|z_0|^2)(1-|z|^2)/|1-\bar{z}_0 z|^2$ and when $z\in S(\theta_0, r_0)$, $|1-\bar{z}_0 z|\leqslant c_1(1-|z_0|)$ for some constant c_1, for all z. Thus for $re^{i\theta}\in S(\theta_0, r_0)$ we have

$$\frac{1-r}{1-r_0}\leqslant c_2(1-|w|^2)\leqslant c_3\log\frac{1}{|w|} \quad \text{for some constants } c_2, c_3.$$

Because $|\nabla F(w)|^2 s\, ds\, d\varphi = |\nabla f(z)|^2 r\, dr\, d\theta$, we have

$$\iint_{S(\theta_0, r_0)} (1-r)|\nabla f(z)|^2 r\, dr\, d\theta < c_3(1-r_0)\iint_D |\nabla F(w)|^2 \log\frac{1}{|w|}\, s\, ds\, d\varphi \leqslant C_1(1-r_0)\,\varepsilon.$$

We thus complete the proof.

If I is an arc on ∂D with center e^{it} and length $|I|=2\delta$, we let

$$R(I)=\{re^{i\theta}: |\theta-t|\leqslant\delta,\ 1-\delta\leqslant r<1\}.$$

A finite positive measure μ on D is called a Carleson measure if there exists a constant c such that $\mu(R(I))<c|I|$ for all subarcs I of ∂D.

LEMMA 3. (Carleson [2]). *Let μ be a Carleson measure on D such that $\mu(R(I))<c|I|$ for all subarcs I of ∂D. Then for $1<p<\infty$*

$$\int_D |f(z)|^p\, d\mu(z) < CA_p\|f\|_p^p$$

for all $f\in L^p(\partial D)$, where the constant A_p depends only on p.

Following an argument in [14, p. 236] one can easily prove Lemma 3 using maximal functions.

For an arc $I\subset\partial D$, let $f_I=|I|^{-1}\int_I f(t)\,dt$ be the average of a function f over I. For $f\in L^1(\partial D)$, define

$$\|f\|_* = \sup_{|I|\leqslant 2\pi}\frac{1}{|I|}\int_I |f-f_I|\,dt.$$

We say f has bounded mean oscillation, $f\in BMO$, if $\|f\|_*<\infty$. Functions in BMO can be related to Carleson measures by the following

LEMMA 4. (Fefferman and Stein). *For $f\in L^1(\partial D)$, the following conditions are equivalent:*

(i) $f\in BMO$

(ii) *If $d\mu=(1-r)|\nabla f(re^{i\theta})|^2 r\, dr\, d\theta$, then μ is a Carleson measure.*

Furthermore if

$$c = \sup_{|I| \leqslant 2\pi} \frac{\mu(R(I))}{|I|},$$

then there is a constant A_1 *such that*

$$\frac{c}{A_1} < \|f\|_*^2 \leqslant A_1 c.$$

Lemma 4 is proved in [6] for the case of upper half spaces. The proof there can easily be adapted to the present case using Lemmas 1 and 3.

3. A distance estimate

Throughout this section we fix an non-constant inner function $b(z) \in H^\infty$ and we set, for $0 < \delta < 1$,

$$G_\delta = \{z \in D: |b(z)| \geqslant 1 - \delta\}.$$

For convenience we assume $G_\delta \subset \{1/2 \leqslant |z| < 1\}$.

LEMMA 5. *Let* $0 < \varepsilon, \delta < 1$. *If* $f \in L^\infty$, $\|f\|_\infty \leqslant 1$ *and* $|f(z)| \geqslant 1 - \varepsilon$ *on* G_δ, *then the measure* μ *defined by*

$$d\mu = \chi_{G_\delta}(z)(1-r)|\nabla f(z)|^2 r\, dr\, d\theta$$

satisfies

$$\sup_I \frac{\mu(R(I))}{|I|} \leqslant C_1 \varepsilon,$$

where C_1 *is the constant in Lemma* 2.

Proof. Let I be some arc on ∂D. By Lemma 2 it suffices to find points $r_j e^{i\theta_j}$ in G_δ such that $G_\delta \cap R(I) \subset \bigcup_j S(\theta_j, r_j)$ and such that $\Sigma(1 - r_j) \leqslant |I|$.

For $n = 0, 1, 2, \ldots$ and $1 \leqslant k \leqslant 2^n$, let $\{I_{n,k}\}$ be the partition of I into closed arcs of length $|I_{n,k}| = 2^{-n}|I|$. Let $T(I_{n,k}) = \{z \in R(I_{n,k}); 1 - |z| \geqslant 2^{-n-2}|I|\}$ be the top half of $R(I_{n,k})$. We select a subfamily $\mathfrak{J}$ of $\{I_{n,k}\}$ by the rule $I_j \in \mathfrak{J}$ if I_j is a maximal arc among those $I_{n,k}$ for which $T(I_{n,k}) \cap G_\delta \neq \emptyset$. Then $G_\delta \cap R(I) \subset \bigcup_{\mathfrak{J}} R(I_j)$ and the arcs in $\mathfrak{J}$ have pairwise disjoint interiors.

For $I_j \in \mathfrak{J}$ choose $r_j e^{i\theta_j} \in T(I_j) \cap G_\delta$ with smallest modulus r_j. Then $G_\delta \cap R(I_j) \subset S(\theta_j, r_j)$ and $1 - r_j \leqslant |I_j|$. Hence $G_\delta \cap R(I) \subset \bigcup_j S(\theta_j, r_j)$ and $\Sigma(1 - r_j) \leqslant \Sigma |I_j| \leqslant |I|$.

Now consider a function f with the following property:

(P_1) $f \in L^\infty$ and there exist ε and δ, $0<\varepsilon$, $\delta<1$, such that the measure μ_δ defined by $d\mu_\delta = \chi_{G_\delta}(z)(1-r)|\nabla f|^2 r\,dr\,d\theta$ satisfies $\sup_I \mu_\delta(R(I))/|I| \leqslant \varepsilon$.

For example, a function satisfying the hypothesis of Lemma 5 has property (P_1).

THEOREM 6. *There is a constant C such that if f has property* (P_1) *then*

$$\limsup_{n\to\infty} d(fb^n, H^\infty) \leqslant C\,\varepsilon^{1/2}.$$

Proof. Since L^∞/H^∞ is the dual of $H_0^1 = \{g \in H^1;\ g(0)=0\}$ we have

$$d(fb^n, H^\infty) = \sup\left\{\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\,b^n(e^{i\theta})\,g(e^{i\theta})\,d\theta\right| : g \in H_0^1,\ \|g\|_1 \leqslant 1\right\}. \tag{1}$$

By a density argument we can assume $g \in H^\infty$. Moreover, if u is the Blaschke factor of g and $k = g/u$, then $g = k + k(u-1)$ where neither k nor $k(u-1)$ has zeros in D. Thus in estimating $d(fb^n, H^\infty)$ using (1), we can assume $g \in H^\infty$ and $g = h^2$, $h \in H^\infty$, $\|h\|_2 \leqslant 1$. Finally, replacing f by $af + c$ with $|a| \leqslant 1$ does not harm property (P_1), so that we can assume $\|f\|_\infty \leqslant 1$ and $f(0)=0$.

With these assumptions we have by Lemma 1,

$$\frac{1}{2\pi}\int f(e^{i\theta})\,b^n(e^{i\theta})\,g(e^{i\theta})\,d\theta = \frac{1}{\pi}\iint_D \nabla f \cdot \nabla(b^n g)\, r \log\frac{1}{r}\,dr\,d\theta. \tag{2}$$

Since b^n and g are analytic functions, we have $(b^n g)(z) = b^n(z) g(z)$ so that $\nabla(b^n g) = b^n \nabla g + g \nabla b^n$ on D.

We now estimate as follows:

$$\begin{aligned}
&\left|\frac{1}{\pi}\iint_D \nabla f \cdot (b^n \nabla g)\, r \log\frac{1}{r}\,dr\,d\theta\right| \\
&\quad \leqslant \frac{1}{\pi}\iint_D |b^n|\,|\nabla f|\,|\nabla g|\, r \log\frac{1}{r}\,dr\,d\theta \\
&\quad = \frac{\sqrt{2}}{\pi}\iint_D |b^n|\,|\nabla f|\,2|h|\,|h'|\, r \log\frac{1}{r}\,dr\,d\theta \\
&\quad \leqslant \sqrt{2}\left(\frac{1}{\pi}\iint_D |b^{2n}|\,|\nabla f|^2 |h|^2 r \log\frac{1}{r}\,dr\,d\theta\right)^{1/2}\left(\frac{4}{\pi}\iint_D |h'|^2 r \log\frac{1}{r}\,dr\,d\theta\right)^{1/2}.
\end{aligned}$$

By Lemma 1 the second factor is

$$\left(\frac{4}{\pi}\int_{-\pi}^{\pi} |h - h(0)|^2\,d\theta\right)^{1/2} \leqslant (8\|g\|_1)^{1/2}. \tag{3}$$

To estimate the first factor write

$$\frac{1}{\pi}\iint_D |b^{2n}|\,|\nabla f|^2 |h|^2 r \log\frac{1}{r}\,dr\,d\theta = \int_{G_\delta} + \int_{D\setminus G_\delta} = S_1 + S_2.$$

Since $G_\delta \subset \{|z| \geqslant 1/2\}$ we have $\log 1/r \leqslant c(1-r)$ on G_δ. Using (P_1) and Lemma 3 we then have

$$S_1 \leqslant cA_2\varepsilon\|h\|_2^2 \leqslant cA_2\varepsilon\|g\|_1. \tag{4}$$

Also

$$S_2 \leqslant (1-\delta)^{2n}\frac{1}{\pi}\iint_D |\nabla f|^2|h|^2 r\log\frac{1}{r}\,dr\,d\theta.$$

Since $\|f\|_* \leqslant 2\|f\|_\infty \leqslant 2$, Lemmas 3 and 4 give

$$S_2 \leqslant (1-\delta)^{2n} 8A_1A_2\|g\|_1. \tag{5}$$

Combining (3), (4) and (5) gives

$$\left|\frac{1}{\pi}\iint_D \nabla f\cdot b^n \nabla g r\log\frac{1}{r}\,dr\,d\theta\right| \leqslant C(\varepsilon^{1/2}+(1-\delta)^n)\|g\|_1 \tag{6}$$

for a universal constant C.

We now estimate

$$\frac{1}{\pi}\cdot\iint \nabla f\cdot g\nabla b^n r\log\frac{1}{r}\,dr\,d\theta = \int_{G_\delta} + \int_{D\setminus G_\delta} = S_3 + S_4.$$

Write

$$|S_3| \leqslant \left(\frac{1}{\pi}\iint_{G_\delta}|\nabla f|^2|h|^2 r\log\frac{1}{r}\,dr\,d\theta\right)^{1/2}\left(\frac{1}{\pi}\iint_{G_\delta}|\nabla b^n|^2|h|^2 r\log\frac{1}{r}\,dr\,d\theta\right)^{1/2}.$$

Since $\|b^n\|_* \leqslant 2$, these two factors can be bounded as were S_1 and S_2 so that

$$|S_3| \leqslant 4\frac{A_1}{\pi}\varepsilon^{1/2}A_2\|g\|_1. \tag{7}$$

For S_4 we again use the Schwartz inequality to get

$$|S_4| \leqslant \left(\iint_{D\setminus G_\delta}|\nabla f|^2|h|^2 r\log\frac{1}{r}\,dr\,d\theta\right)^{1/2}\left(\iint_{D\setminus G_\delta}|\nabla b^n|^2|h|^2 r\log\frac{1}{r}\,dr\,d\theta\right)^{1/2}.$$

As with the estimate for S_2, the first factor is dominated by $(8A_1A_2\|g\|_1)^{1/2}$, and since $|\nabla b^n| \leqslant n(1-\delta)^{n-1}|\nabla b|$ on $D\setminus G_\delta$, the second factor does not exceed $n(1-\delta)^{n-1}(8A_1A_2\|g\|_1)^{1/2}$. Combining our bound for S_4 with (7) gives

$$\left|\frac{1}{\pi}\iint_D \nabla f\cdot g\nabla b^n r\log\frac{1}{r}\,dr\,d\theta\right| \leqslant C_3(\varepsilon^{1/2}+n(1-\delta)^{n-1})\|g\|_1.$$

for a universal constant C_3.

With (6) and (2) this inequality implies

$$\left|\frac{1}{2\pi}\int f(e^{i\theta})\,b^n(e^{i\theta})\,g(e^{i\theta})\,d\theta\right| \leqslant C(\varepsilon^{1/2}+n(1-\delta)^{n-1})\|g\|_1$$

whenever $g \in H^\infty$ has no zeros. By (1) and our remarks about g immediately following (1) we have

$$d(fb^n, H^\infty) \leqslant 3C(\varepsilon^{1/2} + n(1-\delta)^{n-1}),$$

and this proves the theorem.

4. A characterization of Douglas algebras

Before proving the main theorem we must make some observations about maximal ideal spaces. Further details are in [11]. Because H^∞ is a logmodular subalgebra of L^∞ [8], each $\varphi \in \mathcal{M}(H^\infty)$ has a unique representing measure m_φ supported on $\mathcal{M}(L^\infty)$. For any $f \in L^\infty$ we can define $\hat{f}(\varphi) = \int f dm_\varphi$ and by the uniqueness of m_φ, $\hat{f}$ is continuous on $\mathcal{M}(H^\infty)$. Of course, if for all $g \in H^\infty$, $\varphi(g) = g(z)$ with $z \in D$, then $\hat{f}(\varphi) = f(z)$ for $f \in L^\infty$. If $H^\infty \subset B \subset L^\infty$, then $\mathcal{M}(B) = \{\varphi \in \mathcal{M}(H^\infty)\colon \hat{f}(\varphi)\hat{g}(\varphi) = (fg)^\wedge(\varphi)$ for all $f, g \in B\}$. If $f \in (L^\infty)^{-1}$ (i.e. f is an invertible element of L^∞) and if $|f| = 1$ a.e., then we denote $f^{-1} = \bar{f}$. If B is a Douglas algebra, then $\mathcal{M}(B) = \{\varphi\colon |\varphi(b)| = 1$ whenever b is inner and $\bar{b} \in B\}$ (c.f. [11], [4]).

THEOREM 7. *If B and B_1 are closed subalgebras of L^∞ containing H^∞, if $\mathcal{M}(B) = \mathcal{M}(B_1)$ and if B is a Douglas algebra, then $B = B_1$.*

Proof. That $B \subset B_1$ is not difficult. It reduces to showing that $\bar{b} \in B_1$, whenever b is an inner function invertible in B. But since $\mathcal{M}(B) = \mathcal{M}(B_1)$, b has no zeros on $\mathcal{M}(B_1)$ and as $b \in H^\infty \subset B_1$, b is invertible in B_1. Hence $\bar{b} = b^{-1}$ is in B_1.

To prove $B_1 \subset B$ suppose B is generated by H^∞ and a family $\{\bar{b}_\lambda\}$ of conjugates of inner functions. For any finite set F of the index set $\{\lambda\}$, let $b_F = \Pi_F b_\lambda$, and let B_F be the algebra generated by H^∞ and $\bar{b}_F$. Clearly $\bar{b}_\lambda \in B_F$ if $\lambda \in F$. Write $G_\delta(b_F) = \{z \in D\colon |b_F(z)| \geqslant 1-\delta\}$, $0 < \delta < 1$.

Let $g \in B_1$. Adding a constant, we can assume $g \in B_1^{-1}$. Let $h \in (H^\infty)^{-1}$ satisfy $|h| = |g|$ a.e. and let $f = gh^{-1} \in B_1$. Then $f \in B_1^{-1}$ and $|f| = 1$ a.e. It suffices to prove $f \in B_1$.

Since B is a Douglas algebra, $\mathcal{M}(B) = \bigcap \{\mathcal{M}(B_F)\colon F \subset \{\bar{b}_\lambda\}$, F finite$\}$. Since $|\hat{f}| = 1$ on $\mathcal{M}(B_1) = \mathcal{M}(B)$, compactness implies that for any $\varepsilon > 0$ there is a finite set $F \subset \{\bar{b}_\lambda\}$ such that $|\hat{f}| > 1 - \varepsilon/2$ on $\mathcal{M}(B_F)$. This means $|f(z)| > 1 - \varepsilon$ on some region $G_\delta(b_F)$, $\delta > 0$. Indeed, if there were $z_n \in G_{1/n}(b_F)$ with $|f(z_n)| \leqslant 1 - \varepsilon$, then any cluster point φ of $\{z_n\}$ in $\mathcal{M}(H^\infty)$ would satisfy $|\varphi(b_F)| = 1$ so that $\varphi \in \mathcal{M}(B_F)$. But since $\hat{f}$ is continuous on $\mathcal{M}(H^\infty)$. We would have a contradiction. Decreasing δ, we can assume $G_\delta(b_F) \subset \{|z| > 1/2\}$. From Lemma 5 and Theorem 6 we now have

$$d(f, B) \leqslant d(f, B_F) < d(f, \overline{b_F^n} H^\infty) = d(fb_F^n, H^\infty) < C\varepsilon^{1/2}$$

for suitably large n. Because B is closed this means $f \in B$.

5. A description of the largest C*-algebra contained in a subalgebra

Suppose B is a closed subalgebra of L^∞ properly containing H^∞. The largest C^*-algebra contained in B is the algebra $B \cap \bar{B}$ where $\bar{B}$ denotes the space of complex conjugates of functions in B. The proof of Theorem 7 yields a description of the functions in $B \cap \bar{B}$ when B is a Douglas algebra. In view of the paper [9] this description of $B \cap \bar{B}$ is valid whenever $H^\infty \subset B \subset L^\infty$.

THEOREM 8. *Suppose B is a Douglas algebra. Let $f \in L^\infty$. Then $f \in B \cap \bar{B}$ if and only if f satisfies*

(P_2) *for every $\varepsilon > 0$ there is an inner function $b \in B^{-1}$ and there is δ, $0 < \delta < 1$ such that the measure $d\mu = \chi_{G_\delta(b)}(z)(1-r)\,|\nabla f|^2 r\,dr\,d\theta$ satisfies $\mu(R(I)) \leq \varepsilon |I|$ for all subarcs I of ∂D.*

Proof. Suppose f satisfies (P_2). Then for any $\varepsilon > 0$ there is $b \in B^{-1}$ so that by Theorem 6, $d(f, \bar{b}^n H^\infty) < C\varepsilon^{1/2}$ when n is large. Hence $f \in B$. Since $\bar{f}$ also satisfies (P_2), $f \in B \cap \bar{B}$.

On the other hand, if $f \in B \cap \bar{B}$ and $|f| = 1$, then the proof of Theorem 7 shows that f has (P_2). Being a C^* algebra, $B \cap \bar{B}$ is the closed linear span of the unimodular functions in $B \cap \bar{B}$. And by Lemma 4 and the inequality $\|g\|_* \leq 2\|g\|_\infty$, the space of functions in L^∞ having (P_2) is uniformly closed. Hence each $f \in B \cap \bar{B}$ has (P_2).

In the special case $b = z$, the closed algebra generated by H^∞ and $\bar{z}$ is actually the space $H^\infty + C$ ([7], [11]). Theorem 8 then gives the description from [12] of $(H^\infty + C) \cap \overline{(H^\infty + C)}$ as $VMO \cap L^\infty$.

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