THE NUMBERS OF LABELED COLORED AND CHROMATIC TREES

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1. Introduction

The mathematical theory of trees, as first discussed by Cayley in 1857 [3], was concerned in the enumeration aspect with two contrasting cases; the points (or lines) of the trees in question were either all alike or all unlike. For the allied subject of series-parallel electrical networks, R. M. Foster [5] has introduced enumerations by two variables, the number of elements in the network (corresponding to lines of a tree) and the number of these which are marked, with each mark distinct from every other. Two networks which differ only by a permutation of marks are counted as different if the differently marked elements are dissimilar, with dissimilarity as explained in [5] and [2]. The same kind of enumeration is done here for trees, thus including both classical cases in one frame. The trees so marked are called labeled trees.

A second kind of marking is also considered. This is familiar from graph coloring, where each point or line of a graph is given one of c colors. Note that every element (point or line) of a colored tree has some color (if uncolored elements were permitted, they could all be said to be colored with a new color $c+1$) and that in any particular coloring, any number of different colors (up to c) may appear, in contrast to a labeled tree. The enumeration is by number of elements (points or lines) and by number of colors, or by number of elements and by number of distinct colors.

It may be noted that the enumerating functions for colored trees are formally similar to those for unmarked trees, while both differ from those for labeled trees. This formal similarity does not persist when tree colorings are subject to the chromatic condition that adjacent elements (points having a line in common or lines having a point in common) may not have the same color, the third kind of marking considered here. Colored trees satisfying the chromatic condition are here called chromatic.

For each of these three kinds of marking, the trees considered may be rooted or not, oriented or not, and the marked elements may be points or lines; hence here are 24 possible enumerations, all of which are carried out here.

This large amount of material is given a unified exposition by means of a theorem due to Pólya (the Hauptsatz of [8]) for rooted trees, and a combination of the work of Pólya (l.c.) and that of Otter [7] for the free (unrooted) trees. It may be noticed that the labeled cases do not permit the use of Pólya's theorem in its stated form, but the proof given by Pólya is in fact sufficiently general to permit its application also to these cases; this is also the case for the labeled series-parallel networks treated in [2]. It may be noted also that the same procedure flows with equal directness for the various kinds of linear graphs in Harary [5] when these are marked in any of the three ways, but for brevity these results are deferred to a later occasion.

Generating functions are used here, as in other combinatorial settings, for convenience in expressing briefly the relationships of their coefficients, which here are numbers of the various kinds of trees in question. These relationships in principle could have been derived in a strictly finite combinatorial way. In this usage, the variables of the generating functions are to be regarded as indeterminates or tags whose powers identify the coefficients, and formal operations on the generating functions used to express coefficient relations are replacements of basic rules in an underlying algebra of sequences, as in Bell [1].

2. Definitions

The definitions necessary for clarity as to what is being enumerated are given in this section. Because they are mostly well known, they are given with a minimum of explanation.

A tree is a connected linear graph without cycles (or slings), hence of nullity (Zusammenhangszahl: Pólya) zero. Because of the absence of cycles, the number of points in a tree is one greater than the number of lines, and either may be used as a parameter. A rooted tree is a tree in which one point, the root, is distinct from the others; when the root is connected to only one other point, the tree is called planted (Setzbaum: Pólya). An oriented tree is a tree in which every line is directed.

Two points of a tree are adjacent if they have a line in common, two lines if they have a point in common.

A branch of a tree at any point consists of a line to an adjacent point and all the points and lines which may be reached by paths from the given point through this line; the branches at a root are all planted trees. The *weight* of a branch is the number of lines it contains; the weight of a point is the weight of its largest branch. The centroid (Massenzentrum) of a tree is the set of all points of smallest weight; it is well known that this set consists either of a single point, when the tree is said to be (uni)centroidal, or of two adjacent points, when it is said to be bicentroidal. It is also known that bicentroidal trees with an odd number of points are impossible, while for an even number of points, $p = 2q$, the branches at each centroid, excluding the one through the other, have in total q points. These facts will be used in relating the enumeration of trees to that of rooted trees.

Two trees are *isomorphic* if there is a one to one correspondence between their points which preserves adjacency. Two rooted trees are isomorphic if they are isomorphic as trees and the correspondence leaves the root unchanged. Points or lines carried into each other by an isomorphism are called similar. Two labeled trees are isomorphic if they are isomorphic as unlabeled trees and the elements labeled (points or lines) either remain unchanged or are changed to similar elements. Similar definitions hold for colored and chromatic trees as well as for the corresponding oriented varietes.

As in the classical case of Cayley, all enumerations are of nonisomorphic trees (of the various kinds cited above).

Note that because labeling is done with distinct marks, every isomorphism of a labeled rooted tree carries each labeled branch at the root either into itself or into one of its isomorphic images, and never into another labeled branch. This is to say that two branches of the same labeled tree can be isomorphic only when each has no labels.

3. Pólya's Theorem

For brevity of statement, the theorem will be stated in a limited two variable form which satisfies present needs. Also Pólya's geometrical terminology will be abandoned. A few preliminary remarks are necessary.

The theorem concerns the relations of two enumerating generating functions, for brevity here called enumerators.

The first of these, $S(x, y)$, is the enumerator of a store of objects according to their rank or size with respect to two given characteristics e.g. the labeled trees enumerated by number of points and by number of labels. This enumerator in general is of the form $S(x, y) = \Sigma S_{ij} f_i(x) g_j(y)$

with the sets of functions $(f_i(x))$, $(g_j(y))$ linearly independent. The form of these functions is dictated by the given characteristics. The proper choice for labeled trees, as will appear, is $f_n(x) = x^n$, $g_m(y) = y^m/m!$.

The second enumerator, $T(x, y)$, is the enumerator, with respect to the same characteristics as that of the store, of the inequivalent selections of the objects n at a time and in order, with each object chosen independently. Thus $T(x, y)$ is of the form $T(x, y) = \Sigma T_{ij} f_i(x) g_j(y)$.

This leaves undefined equivalence of selections and composition of characteristics. The rules for these are as follows. The first refers to the order of selection and must be preassigned. Two selections are equivalent if there is a permutation of a group G which sends one into the other. The group G is specified by its *cycle index*

$$H(t_1, t_2, \ldots, t_n) = \frac{1}{h} \Sigma h_{i_1 i_2 \ldots i_n} t_1^{i_1} t_2^{i_2} \ldots t_n^{i_n}$$

with $i_1 + 2i_2 + \cdots + n i_n = n$, h the order of G and $h_{i_1 i_2 \ldots i_n}$ the number of permutations of G having i_1 cycles of length one, i_2 of length two and so on. Note that $H_n(1, 1, \ldots, 1) = 1$, and that if all orders of selection are distinct $H_n = t_1^n$ while if all are equivalent H_n is the cycle index of the symmetric group, which I write $C_n(t_1, t_2, \ldots, t_n)/n!$ and take

$$1 + \sum_1 C_n(t_1, t_2, \ldots, t_n)/n! = \exp(t_1 + t_2/2 + \cdots + t_n/n + \cdots), \tag{1}$$

which can be easily verified from the explicit expression of the C_n (Pólya [8], p. 162).

The second, the composition of characteristics, applies in the case of a fixed order of selection, and is taken as defined by the *product* rule, as in Ford and Uhlenbeck [4]; namely, the enumerator for two objects together with both orders of selection distinct and hence with cycle index $H_2 = t_1^2$, say $T_2(x, y)$, must satisfy $T_2(x, y) = S^2(x, y)$. This rule and the nature of the characteristics of the objects enumerated determines the functions $f_i(x)$ and $g_j(y)$ in the enumerator, $S(x, y)$.

The theorem may now be stated as follows:

THEOREM (Pólya): *If objects are chosen independently from a store of objects having enumerator* $S = S(x, y)$ *which satisfies the product rule, and if the order equi-*

valence of choices of n is specified by the cycle index $H_n(t_1, t_2, \dots, t_n)$, then the distinct (inequivalent) choices of n have enumerator $T_n(x, y) = H_n(S_1, S_2, \dots, S_n)$ with S_k the enumerator for choices of k identical objects ($k = 1, 2, \dots, S_1 = S$).

Note that the product rule itself becomes a special case of the theorem, when $H_n = t_1^n$.

This differs from Pólya, firstly in that the product rule is taken as one of the assumptions of the theorem which makes it possible to use the more general kind of enumerators which apply to the labeled cases, secondly in that the conclusion is stated "a stage earlier". The choices of k invariant for a cycle of length k are exclusively of like objects and if

$$S(x, y) = \Sigma S_{ij} x^i y^j,$$

then $S_k = S(x^k, y^k)$, which is Pólya's conclusion. Pólya's proof applies to this more general case with slight changes of formulation only. As noted above, a different form of enumerator is required for labeled trees.

4. Rooted Trees with Point Labels

Take $r_{p,m}$ as the number of (nonisomorphic) rooted trees with p points, m of which are labeled with distinct labels, in the way first described above. Because the labels are distinct, the enumerator which satisfies the product rule must be taken in the form

$$\left.\begin{aligned} r(x, y) &= \sum r_{p,m} x^p y^m/m! \\ &= \sum_{p=1} x^p \sum_{m=0}^{p} r_{p,m} y^m/m! = \sum x^p r_p(y) \end{aligned}\right\} \tag{2}$$

since the assignment of labels to two trees together involves a binomial coefficient which is properly accounted for by form (2) in a manner familiar from the generating function for permutations of objects of general specification.

Take $r_{p,m}(n)$ as the corresponding number when there are n branches at the root, so that

$$r_{p,m} = \sum_{n=1}^{p-1} r_{p,m}(n). \tag{3}$$

Now apply Pólya's theorem to the enumeration of $r_{p,m}(n)$ or what in the same thing, to the determination of the enumerator:

$$r_n(x, y) = \Sigma r_{p,m}(n) x^p y^m/m!$$

The store is the collection of planted trees which may be branches at the root, hence has enumerator $r(x, y)$. In the form (2), $r(x, y)$ satisfies the product rule, as already ensured. The cycle index is $C_n(t_1, t_2, \ldots, t_n)/n!$, since there is complete symmetry in the n branches.

The choices of k which remain invariant under a cyclic permutation of length k are exclusively of like objects, but by the remark at the end of Section 2, no two labeled branches at the root can be alike; hence $S_k = r(x^k, 0)$.

The root may be labeled or unlabeled, hence contributes $x(1+y)$ to the enumeration. By this remark and the theorem

$$r_n(x, y) = x(1+y)\, C_n(r(x, y),\ r(x^2, 0), \ldots, r(x^n, 0))/n!$$

and finally, by (3) and (1),

$$r(x, y) = \sum_{n=1} r_n(x, y) = x(1+y) \exp\,(r(x, y) + r(x^2)/2 + \cdots + r(x^k)/k + \cdots), \tag{4}$$

where for brevity $r(x^k, 0) \equiv r(x^k)$. Noting that

$$r(x, 0) = r(x) = x \exp\,(r(x)) + r(x^2)/2 + \cdots + r(x^k)/k + \cdots), \tag{5}$$

which is a well-known result (Pólya [8], equation 1′, p. 149) and hence a verification, equation (4) may be given the symmetrical form

$$r(x, y) \exp r(x) = (1+y)\, r(x) \exp r(x, y). \tag{4 a}$$

Equation (4) constitutes the complete enumeration of nonisomorphic rooted trees with point labels, and to emphasize its importance is summarized in

THEOREM 1. *The numbers r_{pm} of rooted trees with p points, m of which have distinct labels, are completely determined by the enumerator identity* (4) *with*

$$r(x, y) = \Sigma\, r_{pm}\, x^p y^p/m!$$

For ease of evalution of these numbers it is helpful to develop some consequences of equation (4).

Denoting partial derivatives by the usual suffix notation, equation (4 a) has as immediate consequenses:

$$a(x)\, r_x(x, y) = (1+y)\, r_y(x, y) = r/(1-r) \tag{6}$$

with $r = r(x, y)$ and $\qquad a(x) = r(x)/r'(x)\,(1 - r(x)),$

the prime denoting a derivative. Since $r(x)$ is a power series with integral coefficients so is $r(x)/(1-r(x))$ which is equal to $r(x)+r^2(x)+\cdots$. It then follows by recurrence from its definition that $a(x)$ is also; indeed by direct calculation

$$a(x)=\Sigma\, a_n x^n = x - x^3 - x^4 - 2x^5 + x^6 - 3x^7 + 4x^8 - x^9 + x^{10} + \cdots.$$

Notice that the first half of (6) corresponds to the recurrence

$$(1+y)\, r_p'(y) = \sum_{k=1}^{p} k\, r_k(y)\, a_{p+1-k} \tag{7}$$

with the prime denoting a derivative and $r_p(y)$ defined by the last form of (2); assuming $r_p \equiv r_p(0)$ computed independently (by equation (5) e.g.) this seems to be the simplest computing formula.

For concreteness it may be noticed that

$$\begin{aligned} r(x, y) = {} & x(1+y) + x^2(1+2y+2y^2/2) + x^3(2+5y+9y^2/2+9y^3/6) + \\ & + x^4(4+13y+34y^2/2+64y^3/6+64y^4/24) + \\ & + x^5(9+35y+119y^2/2+326y^3/6+625y^4/24+625y^5/120) + \cdots. \end{aligned}$$

These numerical results are consistent with $r_{nn}=n^{n-1}$, which is a result due to Cayley. It is interesting to see how (4 a) leads to its proof. Make the substitution $x=x$; $xy=z$ in (2) and (4 a); (2) may be rewritten

$$r(x, z) = R_0(z) + x R_1(z) + \cdots + x^p R_p(z) + \cdots \tag{8}$$

$$R_p(z) = \sum_{m=0}^{\infty} r_{p+m,\, m}\, z^m/m!$$

and (4 a) becomes

$$r(x, z) \exp r(x) = (x+z)\,(r(x)/x) \exp r(x, z). \tag{9}$$

Since $r(x)/x = 1 + x + \cdots$, it follows from (9) that

$$r(0, z) = R_0(z) = x \exp R_0(z), \tag{10}$$

which is Pólya's equation (l.c. (2.37) p. 200) from which he establishes Cayley's result.

For completeness it may be noted that

$$x\, a(x)\, r_x(x, z) = (x^2 + xz - z\, a(x))\, r_z(x, z), \tag{11}$$

which implies a recurrence in $R_p(z)$ and its derivatives. An instance of this is

$$R_1(z) = R_0'(z),$$

which verifies the identity $r_{n,\,n-1} = r_{nn}$ appearing in the numerical results and otherwise evident by a simple argument.

Turning now to the colored case, the number of colors c may be regarded as fixed; the enumerator of rooted trees with colored points is taken as

$$q\,(x;\,c) = x\,q_1\,(c) + x^2\,q_2\,(c) + \cdots \tag{12}$$

with c as parameter. Note that $q\,(x;\,1) = r\,(x)$ in the notation appearing above.

Consider the colored rooted trees with n branches at the root. The store enumerator is that of planted trees, hence is $q\,(x;\,c)$, and $S_k\,(x) = q\,(x^k;\,c)$. The cycle index, as above, is $C_n\,(t_1,\,t_2,\,\ldots,\,t_n)/n!$ The root may be colored in c ways. Hence by the theorem and (1)

$$q\,(x;\,c) = x\,c\,\exp\,(q\,(x;\,c) + q\,(x^2;\,c)/2 + \cdots\,q\,(x^k;\,c)/k + \cdots). \tag{13}$$

It may be helpful to summarize this result also in

THEOREM 2. *The numbers $q_p\,(c)$ of rooted trees with p points, each of which may be colored with any of c colors, are completely determined by the enumerator identity* (13) *with*

$$q\,(x;\,c) = \Sigma\,x^p\,q_p\,(c).$$

It may be noted for numerical evaluations of the coefficients that

$$x\,q_x\,(x;\,c) = q\,(x;\,c)\,[1 + x\,q_x\,(x;\,c) + x^2\,q_x\,(x^2;\,c) + \cdots + x^k\,q_x\,(x^k;\,c) + \cdots]. \tag{14}$$

The recurrence obtained from this may be used to evaluate the functions $q_n\,(c)$ as polynomials in c in the manner familiar from the evaluation of the coefficients r_n of $r\,(x)$.

These polynomials in their turn determine the Newton series

$$q_n\,(c) = Q_{n1}\,c + Q_{n2}\binom{c}{2} + \cdots\,Q_{nn}\binom{c}{n}, \tag{15}$$

which is of interest because the coefficients Q_{nk} are the numbers of rooted trees with n points and k specified colors. Note that $Q_{n1} = r_{n0} = r_n$, $Q_{nn} = r_{nn}$. The first few results are

$$q_1 = c \qquad\qquad q_3 = 2\,c + 10\binom{c}{2} + 9\binom{c}{3}$$

$$q_2 = c + 2\binom{c}{2} \qquad q_4 = 4\,c + 44\binom{c}{2} + 102\binom{c}{3} + 64\binom{c}{4}.$$

For the chromatic case, take $p(x; c)$ for the enumerator, with c again a parameter, and consider again the rooted trees with n branches at the root. The root again may have any one of c colors but none of the (planted) trees at the root may have the root color on the point adjacent to the root. The number of rooted trees with any given root color is the same as that with any other given color. Hence the store enumerator is $((c-1)/c)\ p(x; c)$ and it follows at once that

$$p(x; c) = x c \exp \frac{c-1}{c}(p(x; c) + p(x^2; c)/2 + \cdots + p(x^k; c)/k + \cdots) \tag{16}$$

and, for the recurrence relations,

$$x p_x(x; c) = p(x; c)\left[1 + \frac{c-1}{c}(x p_x(x; c) + x^2 p_x(x^2; c) + \cdots x^k p_x(x^k; c) + \cdots)\right].$$

The first few values of the Newton series are

$$p_1 = c \qquad p_3 = 4\binom{c}{2} + 9\binom{c}{3}$$

$$p_2 = 2\binom{c}{2} \qquad p_4 = 8\binom{c}{2} + 54\binom{c}{3} + 64\binom{c}{4}.$$

Notice that in a notation corresponding to (15), $P_{n2} = 2 r_{n0}\ (n > 1)$, $P_{nn} = r_{nn}$, which serve as verifications.

4. Rooted Trees with Line Labels

The procedure of course is the same as above, and only points of difference will be noticed.

For the labeled case, take $r^*(x, y)$ as the enumerator by number of lines and number of labels with $r^*(x, y) = 1 + x r_1^*(y) + \cdots$. The store for the theorem now has enumerator $x(1+y)\ r^*(x, y)$ since a line at the root may be labeled or not, and connected to any line labeled rooted tree. But by shifting point labels to lines from the outer points in, it is clear that the store enumerator is also $r(x, y)$. Hence the essential relation is

$$x(1+y)\ r^*(x, y) = r(x, y) \tag{17}$$

which of course implies that $r_n(y)$ has a factor $(1+y)$.

For the colored case, take $q^*(x; c)$ as the enumerator with $q^*(x; c) = 1 + x q_1^*(c) + x^2 q_2^*(c) + \cdots$; then, just as above

$$x c q^*(x; c) = q(x; c). \tag{18}$$

The chromatic condition introduces essential differences. Take $p^*(x; c)$ as the enumerator for rooted chromatic trees with n lines at the root with $p^*(x; c) = 1 + x p_1^*(c) + \cdots$, and c again a parameter. With n branches at the root, no two of the lines at the root of these branches, which are stems of planted trees, may have the same color since they have the root as common point. Take $g_n(c)$ as the number of chromatic planted trees with c line colors, $n+1$ lines, and a given color on the stem. Then the store enumerator in the theorem is

$$x g(x; c) = x[1 + x g_1(c) + x^2 g_2(c) + \cdots]. \tag{19}$$

Since each planted tree at the root has a different stem color symmetry is lost; instead the cycle index is t_1^n and since n colors for the stems may be chosen from c in $\binom{c}{n}$ ways

$$p_n^*(x; c) = \binom{c}{n} x^n g^n(x; c). \tag{20}$$

Hence

$$p^*(x; c) = 1 + \sum_{n=1} p_n^*(x; c) = [1 + x g(x; c)]^c. \tag{21}$$

On the other hand $g(x; c)$ may be enumerated in terms of $p^*(x; c)$, since a planted tree is formed by adding a stem to a rooted tree; thus exactly as above

$$\left.\begin{aligned} g(x; c) &= 1 + \Sigma \frac{c-n}{c} p_n^*(x; c) \\ &= [1 + x g(x; c)]^{c-1}. \end{aligned}\right\} \tag{22}$$

The factor $(c-n)/c = \binom{c-1}{n} \Big/ \binom{c}{n}$ is required because none of the n lines joined to the stem may have the stem color.

Equations (21) and (22) completely determine both enumerators $p^*(x; c)$ and $g(x; c)$; thus e.g. $g(x; 1) = 1$ and $p^*(x, 1) = 1 + x$, while $g(x; 2) = 1 + x g(x; 2) = (1-x)^{-1}$ and $p^*(x; 2) = g^2(x; 2) = (1-x)^{-2}$.

To determine the polynomials $p_n^*(c)$ and their Newton series defined as in (15), the following development is helpful. First by differentiation of (22) it follows that

$$g_x(x; c) = \left[c - 1 + \binom{c-2}{2} x \frac{\partial}{\partial x}\right] g^2(x; c) \tag{23}$$

and from (21) and (22)

$$p^*(x; c) = [1 + x g(x; c)] g(x: c). \tag{24}$$

These lead to the recurrences

$$\left.\begin{aligned} 2\,n\,g_n(c) &= [(n+1)\,c - 2\,n]\,g_{n-1,2}(c) \\ 2\,n\,p_n^*(c) &= (n+1)\,c\,g_{n-1,2}(c) \end{aligned}\right\} \tag{25}$$

with $g_{n2}(c)$ defined by $\qquad g^2(x;\,c) = 1 + \sum_{n=1} x^n\,g_{n2}(c)$

or by its consequence

$$g_{n2}(c) = g_{n-1}(c)\,g_1(c) + g_{n-2}(c)\,g_2(c) + \cdots + g_1(c)\,g_{n-1}(c).$$

These lead to a serial computation, the first few results of which in Newton series form are

$$\begin{aligned} &p_0^* = 1 && p_3^* = 4\binom{c}{2} + 16\binom{c}{3} \\ &p_1^* = c && p_4^* = 5\binom{c}{2} + 75\binom{c}{3} + 125\binom{c}{4} \\ &p_2^* = 3\binom{c}{2} && p_5^* = 6\binom{c}{2} + 279\binom{c}{3} + 1296\binom{c}{4} + 1296\binom{c}{5}. \end{aligned}$$

5. Trees

For unlabeled trees, enumerated by number of points by $t(x) = t_1\,x + t_2\,x^2 + \cdots$, it is known (Otter [7]) that

$$t(x) = r(x) - \frac{1}{2}\,r^2(x) + \frac{1}{2}\,r(x^2) \tag{26}$$

where $r(x)$ is the corresponding enumerator for rooted trees. Pólya [8] has proved an equivalent, though less compact, result by a procedure which is easily adapted to the marked cases.

Briefly this consists of dividing the enumeration by considering separately the centroidal and bicentroidal trees defined above, and relating these enumerations to those for rooted trees, following the obvious suggestion of the pictures of these trees. As a remainder for the reader, note again that bicentroidal trees with an odd number of points are impossible, while for an even number $(p = 2\,q)$ the branches at each centroid, excluding the one through the other, have in total q points; the picture is of two rooted trees joined by a line. The centroidal trees with p points have branches at the centroid having at most $[(p-1)/2]$ points.

For labeled points take $t(x,\,y)$ as the enumerator; as before

$$t(x,\,y) = x\,t_1(y) + x^2\,t_2(y) + \cdots = \sum x^p \sum_{m=0}^{p} t_{pm}\,y^m/m! \tag{27}$$

Also, for the division into centroidal and bicentroidal trees, write

$$t(x, y) = t'(x, y) + t''(x, y). \tag{28}$$

By the remarks above and by the theorem the bicentroidal trees are completely enumerated by

$$t''_{2q}(y) = \frac{1}{2}[r_q^2(y) + r_q(0]$$

$$t''_{2q+1}(y) = 0.$$

For the centroidal trees, those rooted trees having more than $[(p-1)/2]$ points in any branch at the root must be subtracted from the total and after some simplification it turns out that

$$t'_{2q}(y) = r_{2q}(y) - r_{2q-1}(y)\, r_1(y) - r_{2q-2}(y)\, r_2(y) - \cdots - r_q^2(y)$$

$$t'_{2q+1}(y) = r_{2q+1}(y) - r_{2q}(y)\, r_1(y) - r_{2q-1}(y)\, r_2(y) - \cdots - r_{q+1}(y)\, r_q(y).$$

Summing these on p results in

$$t(x, y) = r(x, y) - \frac{1}{2} r^2(x, y) + \frac{1}{2} r(x^2). \tag{29}$$

For $y = 0$, this is Otter's formula (equation (26) above).

With $a(x)$ as in (6) and a prime denoting a derivative the results corresponding to (6) are

$$\begin{aligned} a(x)\, t_x(x, y) \quad &= r(x, y) + x\, a(x)\, r'(x^2) \\ (1+y)\, t_y(x, y) &= r(x, y) = a(x)\, t_x(x, y) - x\, a(x)\, r'(x^2). \end{aligned} \tag{30}$$

By the last of these $\qquad t_y(x, 0) = r(x),$

which is the same as $t_{p1} = r_p$, i.e. the number of trees with just one label equals the number of rooted trees, another verification.

Finally with $z = xy$ as in (8) and

$$t(x, z) = T_0(z) + x\, T_1(z) + \cdots \tag{31}$$

with $\qquad T_p(z) = \Sigma\, t_{p+m,\, m}\, z^m/m!$

it follows from $\qquad (x+z)\, t_z(x, z) = r(x, z), \qquad (32)$

that $\qquad T'_{n-1}(z) + zT'_n(z) = R_n(z), \qquad (33)$

which entails in particular $z T_0'(z) = R_0(z)$ or passing to coefficients, $n t_{nn} = r_{nn} = n^{n-1}$, so $t_{nn} = n^{n-2}$, another of Cayley's results, and hence a verification.

For trees with colored points there is no essential change from the unlabeled case and if $u(x; c)$ is the enumerator

$$u(x; c) = q(x; c) - \frac{1}{2} q^2(x; c) + \frac{1}{2} q(x^2; c) \tag{34}$$

For point chromatic trees take $v(x; c)$ as the enumerator. The bicentroidal trees are required to have different colors at the two centroids, hence

$$v_{2q}''(c) = \frac{c-1}{c} p_q^2(c).$$

A similar adjustment is required for centroidal trees with the final result that

$$v(x; c) = p(x; c) - \frac{1}{2} \frac{c-1}{c} p^2(x; c) \tag{35}$$

For line marks, the same notational procedure as for rooted trees is followed: a superscript star denotes the line marked case.

For line labeled trees $t^*(x, y)$ is the enumerator, but for convenience $t_1^*(x, y)$ and $t_2^*(x, y)$ are the enumerators of centroidal and bicentroidal trees, respectively. Remembering that x is the variable for number of lines, the results for bicentroidal trees are

$$t_{2,2q}^*(y) = 0$$
$$t_{2,2q+1}^*(y) = (1+y)\left[(r_q^*(y))^2 + r_q^*(0)\right]/2.$$

Adjusting similarly for centroidal trees, it is found that

$$t^*(x, y) = r^*(x, y) - \frac{1}{2} x(1+y)(r^*(x, y))^2 + \frac{1}{2} x(1+y) r^*(x^2). \tag{36}$$

It is worth noting that

$$\begin{aligned} x(1+y) t^*(x, y) &= r(x, y) - \frac{1}{2} r^2(x, y) + \frac{1}{2}(1+y)^2 r(x^2) \\ &= t(x, y) + (y + y^2/2) r(x^2). \end{aligned} \tag{37}$$

For line colored trees, $u^*(x, c)$ is the enumerator and by an argument like that above

$$u^*(x; c) = q^*(x; c) - \frac{1}{2} x c (q^*(x; c))^2 + \frac{1}{2} x c q^*(x^2; c) \tag{38}$$

$$x\,c\,u^*(x;\,c) = q(x;\,c) - \frac{1}{2}\,q^2(x;\,c) + \frac{c}{2}\,q(x^2;\,c)$$
$$= u(x;\,c) + \frac{c-1}{2}\,q(x^2;\,c). \tag{39}$$

For line chromatic trees, $v^*(x;\,c)$ is the enumerator and

$$v^*(x;\,c) = p^*(x;\,c) - \frac{1}{2}\,x\,c\,g^2(x;\,c) + \frac{1}{2}\,x\,c\,g(x^2;\,c) \tag{40}$$

with $g(x;\,c)$ the enumerator defined by (19).

6. Oriented Rooted and Free Trees

These differ from the trees above only by having each line oriented in one of the two possible directions. As is to be expected the enumerations differ only in minor details, so this section is mainly a compendium. Greek letters are used for the enumerators, ϱ, π, and ν for rooted trees and τ, v, and Φ for trees, with a star as before distinguishing the line labeled cases.

For the point labeled case the results are

$$\varrho(x,\,y) = x(1+y)\exp 2\,(\varrho(x,\,y) + \varrho(x^2)/2 + \cdots + \varrho(x^k)/k + \cdots), \tag{41}$$

$$\tau(x,\,y) = \varrho(x,\,y) - \varrho^2(x,\,y). \tag{42}$$

For the point colored case

$$\pi(x;\,c) = x\,c\exp 2\,(\pi(x;\,c) + \pi(x^2;\,c)/2 + \cdots + \pi(x^k;\,c)/k + \cdots), \tag{43}$$

$$v(x;\,c) = \pi(x;\,c) - \pi^2(x;\,c). \tag{44}$$

For the point chromatic case

$$\nu(x;\,c) = x\,c\exp 2\,\frac{c-1}{c}\,(\nu(x;\,c) + \nu(x^2;\,c)/2 + \cdots), \tag{45}$$

$$\Phi(x;\,c) = \nu(x;\,c) - \frac{c-1}{c}\,\nu^2(x;\,c). \tag{46}$$

For line labels

$$x(1+y)\,\varrho^*(x,\,y) = \varrho(x,\,y), \tag{47}$$

$$x(1+y)\,\tau^*(x,\,y) = \tau(x,\,y). \tag{48}$$

For line colors

$$x\,c\,\pi^*(x;\,c) = \pi(x;\,c), \tag{49}$$

$$x\,c\,v^*(x;\,c) = v(x;\,c). \tag{50}$$

For the line chromatic case

$$\nu^*(x;\,c) = p^*(2x;\,c), \tag{51}$$

$$\Phi^*(x;\,c) = v^*(2x;\,c). \tag{52}$$

The point labeled case has a development like its unoriented correspondent which is omitted to save space; it has been used to obtain the numbers appearing in

$$\varrho(x,\,y) = x(1+y) + x^2(2+4y+4y^2/2) + x^3(7+19y+36y^2/2+36y^3/6) + \\ + x^4(26+94y+264y^2/2+512y^3/6+512y^4/24) + \cdots$$

$$\tau(x,\,y) = x(1+y) + x^2(1+2y+2y^2/2) + x^3(3+7y+12y^2/2+12y^3/6) + \\ + x^4(8+26y+68y^2/2+128y^3/6+128y^4/24) + \cdots.$$

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