TO REVERSE A MARKOV PROCESS

BY

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Owing to the symmetry with respect to past and present in the definition of the Markov property, this property is preserved if the direction of time is reversed in a process, but the temporal homogeneity is in general not. Now a reversal preserving the latter is of great interest because many analytic and stochastic properties of a process seem to possess an inner duality and deeper insights into its structure are gained if one can trace the paths backwards as well as forwards, as in human history. Such is for instance the case with Brownian motion where the symmetry of the Green's function and the consequent reversibility plays a leading role. Such is also the case of Markov chains where for instance the basic notion of first entrance has an essential counterpart in last exit, a harder but often more powerful tool. Indeed there are many results in the general theory of Markov processes which would be evident from a reverse point of view but are not easy to apprehend directly.

The question of reversal has of course been considered by many authors.(2) One early line of attack (see e.g., [16]) hinged on finding a stationary distribution for the process; once such a distribution is found it is relatively easy to calculate the transition probabilities of the stationary process reversed in time. A more general approach is to reverse the process $\{X_t\}$ from a random time α to get a process $\tilde{X}_t = X_{\alpha-t}$. Hunt [8] considered such a reversal from last exit times in a discrete parameter Markov chain. Chung [4] observed that this could be done with more dispatch from the life time of a continuous parameter minimal chain. Going to a general state space, Ikeda, Nagasawa and Sato [10] considered reversal from the life time of certain processes. This was extended by Nagasawa [15], who reversed more general types of processes from L-times, natural generalizations of last exit times, and later by Kunita and T. Watanabe [11]. An assumption common to

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these papers is the existence of semigroups or resolvents in duality. Some of the results in this direction have been neatly summarized in [2], [3], [14], [17].

Our approach here is quite different in that, having defined the reverse process $\tilde{X}_t$ as above with α the life time of X_t, we derive the existence of a reverse transition function by showing that the reverse process is indeed a homogeneous Markov process. Our assumptions all bear on the original process, never going beyond those for a Hunt (or standard) process minus the quasi-left continuity. Our fullest result states that we can always reverse such a process from its life time whenever finite to obtain a "moderately strong" homogeneous Markov process, and we give an explicit construction of its transition function. Finally, the restriction to life time will turn out to be only an apparent one, because any such reversal time can be shown to be necessarily the life time of a subprocess. This last important point, requiring compactifications of the state space in its proof, will however not be proved in this paper and will be published later by the second-named author.

Our method takes off from the case of reversal of a minimal Markov chain mentioned earlier (see also [5]). The interesting thing is that this method, which is apparently limited to the special situation of a discrete state space there, can be adapted to the general setting by a natural stretching-out of the life time which renders the smoothness needed for analytic manipulations. The stretching-out is finally removed by probabilistic considerations based on the notion of "essential limit" leading to an "almost fine topology". This notion seems to combine the advantages of separability and shift-invariance and may well turn out to be an essential tool in similar investigations. However, we content ourselves with these remarks here without amplifying them.

1. The finite dimensional distributions of the reverse process

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ be a locally compact separable metric space and its Borel field. Let $\{X_t, t \geqslant 0\}$ be a homogeneous Markov process with respect to the increasing family of Borel subfields $\{\mathcal{F}_t, t \geqslant 0\}$ of $\mathcal{F}$ and taking values in E; $\mu_t(B)$ and $P_t(x, B)$, $t \geqslant 0$, $x \in E$, $B \in \mathcal{E}$, respectively its absolute distribution and transition function. This means the following:

(i) for each t and x, $B \to P_t(x, B)$ is a probability measure on $\mathcal{E}$;

(ii) for each t and B, $x \to P_t(x, B)$ is in $\mathcal{E}$;

(iii) for each s, t, x and B, we have

$$P_{s+t}(x, B) = \int_E P_s(x, dy)\, P_t(y, B);$$

(iv) for each s, t and B in $\mathcal{E}$, we have with probability one:

$$P\{X_{s+t} \in B \mid \mathcal{F}_s\} = P_t(X_s, B).$$

The terminology and notation used above is roughly the same as in [2; p. 14]. Condition (iii), the Chapman–Kolmogorov equation, may be dispensed with; if so we shall qualify the transition function as one "in the loose sense". We need this generalization below. Observe that condition (iv), the Markov property, implies with (i) and (ii) the loose version of (iii) as follows. For each r, s and t, we have with probability one:

$$P_{s+t}(X_r, B) = \int_E P_s(X_r, dy)\, P_t(y, B). \tag{1.1}$$

This often suffices instead of (iii).

Furthermore, we shall assume that the Borel fields $\{\mathcal{F}_t\}$ are augmented with all sets of probability zero. Phrases such as "almost surely" and "for a.e. ω" will mean "for all ω except a set $N \in \mathcal{F}$ with $P(N)=0$". Our first basic hypothesis is that all sample paths of the process X are right continuous. Only later in § 6 will we add the hypothesis that they have also left limits everywhere and finally that X is strongly Markovian. It is of great importance to remember that we are dealing with a fixed process with given initial distribution, and not a family of processes with arbitrary initial values as is customary in Hunt's theory. Thus, convenient notation such as P^x and E^x will not be used.

An "optional time" T is a random variable such that for each t, $\{T<t\} \in \mathcal{F}_t$. The Borel field of sets Λ in $\mathcal{F}$ such that $\Lambda \cap \{T<t\} \in \mathcal{F}_t$ for each t is denoted by $\mathcal{F}_{T+}$. If "$<$" is replaced by "$\leqslant$" in both occurrences above, T will be called "strictly optional" and $\mathcal{F}_{T+}$ replaced by $\mathcal{F}_T$.

Let Δ be an "absorbing state" in E, namely one with the property that if a path ever takes the value Δ it will remain there from then on. There may be more than one such state but one has been singled out. Put

$$\alpha(\omega) = \inf \{t > 0 \colon X_t(\omega) = \Delta\},$$

where, as later in all such definitions, the inf is taken to be $+\infty$ when the set in the braces is empty. It is easy to see that α is an optional time, to be called the "life time" of the process. We shall be concerned with reversing the process from such a life time whenever it is finite. Observe that this situation obtains if our process is obtained as a subprocess by "killing" a bigger one in some appropriate manner.

For each x, $t \to P_t(x, \Delta)$ is a distribution function to be denoted by $L(x, t)$. [If the process starting at x were defined, this would be the distribution of its life time.] Our method of reversal relies, *au préalable*, on the following assumptions:

(H1) for each $x \neq \Delta$, $L(x, t)$ is an absolutely continuous function of t with density function $l_t(x)$;

(H2) for each $x \neq \Delta$, $t \to l_t(x)$ is equi-continuous on $(0, \infty)$ with respect to x.

Note that condition (H2) coupled with the fact that $\int_0^\infty l_t(x)\,dt \leqslant 1$ implies that $l_t(x)$ is uniformly bounded on compact subsets of $(0, \infty)$. These conditions seem strong but we shall show later that *both can be entirely removed* if X is assumed to be strongly Markovian (Theorem 6.4). Even without this assumption, their removal will still leave us meaningful and tangible results (Theorem 4.1).

We begin with two lemmas. Throughout the paper we shall use popular concise notation such as $P_s f(x) = \int_E P_s(x, dy) f(y)$.

LEMMA 1.1. *For each $x \neq \Delta$ and $s \geqslant 0, t > 0$ we have*

$$l_{s+t}(x) = P_s l_t(x).$$

Proof. We have if $0 < u < v$,

$$\int_u^v l_{s+r}(x)\,dr = L(x, s+v) - L(x, s+u) = \int_E P_s(x, dy) \int_u^v l_r(y)\,dr = \int_u^v P_s l_r(x)\,dr,$$

where the second equation is a consequence of the Chapman–Kolmogorov equation. It follows that for each fixed $x \neq \Delta$ and $s \geqslant 0$,

$$P_s l_r(x) = l_{s+r}(x)$$

holds for almost all r (Lebesgue measure). Now (H2) implies that both members above are continuous in r, hence the equation holds for all $r > 0$.

LEMMA 1.2. *For each $s > 0$, $t > 0$, and sequence $t_n \downarrow t$, $\lim_{n\to\infty} l_s(X_{t_n})$ exists almost surely; it is equal to $l_s(X_t)$ almost surely provided that for each t, $\mathfrak{F}_t = \mathfrak{F}_{t+}$.*

Proof. Let $0 < t' - t < s$; then by Lemma 1.1,

$$l_s(x) = P_{t'-t}\, l_{s-t'+t}(x).$$

It follows by the Markov property that a.s.

$$l_s(X_{t_n}) = E\{l_{s-t'+t_n}(X_{t'}) \mid \mathfrak{F}_{t_n}\}. \tag{1.2}$$

If $t_n \downarrow t$, then $l_{s-t'+t_n}(X_{t'}) \to l_{s-t'+t}(X_{t'})$ by (H2) and consequently the right member above converges a.s. to $E\{l_{s-t'+t}(X_{t'}) \mid \mathfrak{F}_{t+}\}$. This last step is a case of a useful remark due to Hunt [9], which will be referred to later as Hunt's lemma:

Suppose that the sequence of random variables $\{X_n\}$ converges dominatedly to X_∞ and the sequence of Borel fields $\{\mathcal{F}_n\}$ is monotone with limit $\mathcal{F}_\infty$. Then

$$\lim_n E\{X_n \mid \mathcal{F}_n\} = E\{X_\infty \mid \mathcal{F}_\infty\}.$$

Remark. Equation (1.2) above remains true even if the transition function of X is in the loose sense, as follows easily from (1.1). Thus Lemma 1.2 remains in force, and condition (iii) may be omitted since it will not be needed again.

The *potential measure* G is defined as follows: for each $A \in \mathcal{E}$:

$$G(A) = \int_0^\infty \mu_t(A)\,dt = E\left\{\int_0^\infty 1_A(X_t)\,dt\right\}.$$

Since the process need not be transient, G may not be a Radon measure. However, we shall presently prove a certain finiteness for it. For each $s > 0$, define the measure K_s on $\mathcal{E}$ by

$$K_s(A) = \int_0^\infty \mu_t[1_A l_s]\,dt.$$

We have by Fubini's theorem

$$K_s(E) = \int_0^\infty \mu_0[P_t l_s]\,dt = \mu_0\left[\int_0^\infty l_{t+s}\,dt\right] = \mu_0\left[\int_s^\infty l_t\,dt\right] = P\{s < \alpha < \infty\} \leqslant 1.$$

Hence if $f \in \mathcal{E}$ and f is dominated by l_s for some s, then $Gf < \infty$; in particular G is σ-finite on $\bigcup_{s>0}\{x: l_s(x) > 0\}$ and so another application of Fubini's theorem yields

$$K_s(A) = \int_A G(dx)\,l_s(x). \tag{1.3}$$

Now we define the *reverse process* $\tilde{X} = \{\tilde{X}_t, t > 0\}$ as follows. Adjoin a new point $\tilde{\Delta}$ to E, where $\tilde{\Delta} \notin E$ and $\tilde{\Delta}$ is isolated in $E \cup \tilde{\Delta}$: put

$$\tilde{X}_t = \begin{cases} X_{\alpha-t} & \text{if } 0 < t \leqslant \alpha < \infty; \\ \tilde{\Delta} & \text{if } \alpha < \infty, t > \alpha; \\ \tilde{\Delta} & \text{if } \alpha = \infty, t > 0. \end{cases} \tag{1.4}$$

The sample paths of $\tilde{X}$ are therefore just those of X with t reversed in direction, apart from trivial completions; hence they are left continuous. Furthermore, $\tilde{X}$ never takes the value Δ and it takes the value $\tilde{\Delta}$ wherever it is not in E. Hence when we specify its absolute distributions and transition probabilities we may confine ourselves to subsets of E, as we do in the theorem below.

THEOREM 1.1. *Under hypotheses* (H1) *and* (H2), *the absolute distribution of* $\tilde{X}_s$ *is* K_s *given by* (1.3), *the joint distribution of* $\tilde{X}_s$ *and* $\tilde{X}_t$, $0 < s \leqslant t$, *is given by*

$$P\{\tilde{X}_s \in A, \tilde{X}_t \in B\} = \int_B G(dx) \int_A P_{t-s}(x, dy)\, l_s(y), \tag{1.5}$$

where $A \in \mathcal{E}$, $B \in \mathcal{E}$. *More generally, if* $0 < t_1 < t_2 < \ldots < t_n$ *and* $A_1, \ldots, A_n$ *all belong to* $\mathcal{E}$ *we have*

$$P\{\tilde{X}_{t_j} \in A_j, 1 \leqslant j \leqslant n\} = \int_{A_n} G(dx_n) \int_{A_{n-1}} P_{t_n - t_{n-1}}(x_n, dx_{n-1}) \ldots \int_{A_1} P_{t_2 - t_1}(x_2, dx_1)\, l_{t_1}(x_1). \tag{1.6}$$

Note: As remarked following Lemma 1.2, the P_t's may be transition functions in the loose sense.

Proof. We shall prove only (1.5) which contains the main argument; the proof of (1.6) requires no new argument while the assertion about absolute distributions follows from (1.5) if we take $t = s$ and $B = A$ there.

Let C_K^+ denote the class of positive continuous functions on $E \cup \tilde{\Delta}$ with compact support and vanishing at Δ and $\tilde{\Delta}$. It is sufficient to prove that for each f and g in C_K^+, we have

$$E\{f(\tilde{X}_s)\, g(\tilde{X}_t)\} = G[gP_{t-s}(f l_s)]. \tag{1.7}$$

We do this by a discrete approximation. Set

$$\alpha_n = [2^n \alpha + 1]\, 2^{-n},$$

where $[2^n\alpha + 1]$ is the greatest integer $\leqslant 2^n \alpha + 1$; then $\alpha_n > \alpha$ and $\alpha_n \downarrow \alpha$ as $n \to \infty$. Since X has right continuous paths, the left number of (1.7) is equal to the limit of

$$E\{f \circ X(\alpha_n - s) \cdot g \circ X(\alpha_n - t)\} \tag{1.8}$$

as $n \to \infty$, where "$\circ$" denotes composition of functions. For each integer N, write (1.8) as

$$\sum_{2^n t \leqslant k \leqslant 2^n N} E\{f \circ X(k 2^{-n} - s) \cdot g \circ X(k 2^{-n} - t); \alpha_n = k 2^{-n}\} \\ + E\{f \circ X(\alpha_n - s) \cdot g \circ X(\alpha_n - t); N < \alpha < \infty\}. \tag{1.9}$$

The second term tends to zero as $N \to \infty$ uniformly in n. Observing that

$$E\{\alpha_n = k 2^{-n} \,|\, \mathcal{F}_{k2^{-n}-s}\} = \int_{s-2^{-n}}^{s} l_r \circ X(k 2^{-n} - s)\, dr,$$

we write the kth term of the sum in (1.0) as

$$E\{f\circ X(k2^{-n}-s)\cdot g\circ X(k2^{-n}-t)\cdot \int_{s-2^{-n}}^{s} l_r\circ X(k2^{-n}-s)\,dr\}$$

$$=\int_E \mu_{k2^{-n}-t}(dx)\,g(x)\int_E P_{t-s}(x,dy)\,f(y)\,\{2^{-n}l_s(y)+\int_{s-2^{-n}}^{s}[l_r(y)-l_s(y)]\,dr\}$$

$$=2^{-n}\mu_{k2^{-n}-t}[gP_{t-s}(fl_s)]+F_{kn},$$

where

$$F_{kn}=\int_E \mu_{k2^{-n}-t}(dx)\,g(x)\int_E P_{t-s}(x,dy)\,f(y)\int_{s-2^{-n}}^{s}[l_r(y)-l_s(y)]\,dr.$$

We have,

$$\sum_{2^n t\leqslant k\leqslant 2^n N} F_{kn}\leqslant 2^n(N-t)\,\|f\|\,\|g\|2^{-n}\sup_{y\in E}\sup_{|r-s|\leqslant 2^{-n}}|l_r(y)-l_s(y)|$$

which converges to zero as $n\to\infty$ by (H2), for each N. It remains to evaluate the limit as $n\to\infty$ of

$$\sum_{2^n t\leqslant k\leqslant 2^n N} 2^{-n}\mu_{k2^{-n}-t}[gP_{t-s}(fl_s)]. \tag{1.10}$$

Consider the function

$$u\to\mu_u[gP_{t-s}(fl_s)]=E\{g(X_u)\,f(X_{u+t-s})\,l_s(X_{u+t-s})\}. \tag{1.11}$$

Clearly $u\to g(X_u)f(X_{u+t-s})$ is right continuous. Since $l_s(x)$ is bounded in x by (H2), it follows from Lemma 1.2 and Lebesgue's bounded convergence theorem that the function in (1.11) has a right limit everywhere. Hence it is integrable in the Riemann sense and consequently the limit of (1.10) is the Riemann (ergo Lebesgue) integral

$$\int_0^N \mu_u[gP_{t-s}(fl_s)]\,du.$$

Letting $N\to\infty$ we obtain the right member of (1.7), which is finite by the remarks preceding (1.3), q.e.d.

2. The transition function of the reverse process

We prove in this section that the reverse process is temporally homogeneous and exhibit a loose transition function $\tilde{P}_t(y,A)$ for it. If such a function exists, it must be the Radon–Nikodym derivative

$$\frac{P\{\tilde{X}_s\in dy,\tilde{X}_{s+t}\in A\}}{P\{\tilde{X}_s\in dy\}}.$$

The problem is to define this measurably in y and simultaneously for all A in $\mathcal{E}$. Doob [6] has given a similar procedure in connection with conditional probability distributions in the wide sense which has been extended by Blackwell [1] to a more general space. We shall indicate a simpler argument using the functional approach.

Define the function h on E by

$$h(x) = \int_0^\infty e^{-s} l_s(x)\, ds.$$

We have by Lemma 1.1,

$$P_t h = \int_0^\infty e^{-s} P_t l_s ds = e^t \int_t^\infty e^{-s} l_s ds \leqslant e^t h,$$

from which it follows that h is 1-excessive with respect to (P_t). Furthermore, $h(x)=0$ if and only if $l_s(x)=0$ for all s by the continuity of $s \to l_s(x)$. Next, recalling (1.3), we define the measure K on $\mathcal{E}$ by

$$K(A) = \int_0^\infty e^{-s} K_s(A)\, ds = \int_A G(dx)\, h(x). \tag{2.1}$$

Since $K_s(E) \leqslant 1$ for each s we have $K(E) \leqslant 1$.

Now for each t we define a function Π_t on product Borel sets of $E \times E$ by

$$\Pi_t(A, B) = \int_A G(dx) \int_B P_t(x, dy)\, h(y). \tag{2.2}$$

It follows from (2.1) that

$$\Pi_t(A, E) \leqslant \int_A G(dx)\, e^t h(x) \leqslant e^t K(A); \tag{2.3}$$

on the other hand, since $GP_t \leqslant G$, we have

$$\Pi_t(E, B) \leqslant \int_B G(dy)\, h(y) = K(B). \tag{2.4}$$

Consequently $\Pi_t(A, \cdot)$ and $\Pi_t(\cdot, B)$ are both measures which are absolutely continuous with respect to K $(<< K)$.

THEOREM 2.1. *The reverse process $\{\tilde{X}_t, t > 0\}$ is a homogeneous Markov process taking values in $E \cup \tilde{\Delta}$, with a version of the Radon–Nikodym derivative*

$$\frac{\Pi_t(A, dy)}{K(dy)} = \tilde{P}_t(y, A), \quad t \geqslant 0,$$

as its transition function in the loose sense.

Note: $\tilde{P}_0(y, A) = \varepsilon_A(y)$.

Proof. Let D_0 be a countable dense subset of C_K. Let D be the smallest class of functions on E containing D_0 and the constant 1 which is closed under addition and multiplication by a rational number. D is countable and contains all rational constants. Fix $t>0$; for each f in D and B in $\mathcal{E}$, we put

$$\Pi_t(f, B) = \int_E f(x)\,\Pi_t(dx, B).$$

As a signed measure $\Pi_t(f,\cdot) << K(\cdot)$ by (2.4). Let $L(f, y)$ denote a version of the Radon–Nikodym derivative $\Pi_t(f, dy)/K(dy)$ such that $y \to L(f, y)$ is in $\mathcal{E}$ for each $f \in D$. There is a set Z in $\mathcal{E}$ with $K(Z)=0$, such that if $y \in E-Z$, then

(a) $L(f, y) \geq 0$ if $f \in D$, $f \geq 0$;
(b) $L(0, y)=0$;
(c) $L(cf, y) = cL(f, y)$ if $f \in D$ and $cf \in D$ where c is real;
(d) $L(f+g, y) = L(f, y) + L(g, y)$ if $f \in D$, $g \in D$;
(e) $|L(f, y)| \leq \|f\|$ if $f \in D$.

The proofs of these assertions are all trivial. E.g., to show (c), we write for each $B \in \mathcal{E}$,

$$\int_B L(cf, y)\, K(dy) = \Pi_t(cf, B) = c\Pi_t(f, B) = \int_B cL(f, y)\, K(dy)$$

and take B to be $\{y\colon L(cf, y) > cL(f, y)\}$ or $\{y\colon L(cf, y) < cL(f, y)\}$. It follows that the relation in (c) holds for each pair f and cf in D, for K-a.e. y. Since D is countable, this establishes (c).

For $y \in E-Z$, and $f \in C_K$, we define

$$L(f, y) = \lim_n L(f_n, y), \tag{2.5}$$

where $\{f_n\}$ is any sequence in D which converges to f in norm. It follows from (e) that the limit above exists and does not depend on the choice of the sequence. It is trivial to verify that $L(\cdot, y)$ so extended to C_K is a linear functional over the real coefficient field with norm ≤ 1. To see that it is positive, let $f \in C_K$, $f \geq 0$; then for every rational $\varepsilon > 0$, we have $f+\varepsilon \geq \varepsilon$. Hence if $\|f_n - f\| \to 0$, then $f_n + \varepsilon \geq 0$ for sufficiently large n. It follows from (a) above and (2.5) that $L(f+\varepsilon, y) \geq 0$, and hence, by linearity and (c), that $L(f, y) \geq 0$.

Thus the linear functional $L(\cdot, y)$ defined in (2.5) is a Radon measure on $\mathcal{E}$ with total mass ≤ 1. We now put for $y \in E-Z$ and $A \in \mathcal{E}$:

$$\hat{P}_t(y, A) = L(A, y),$$

$$\hat{P}_t(y, \tilde{\Delta}) = 1 - L(E, y);$$

for $y \in Z$ and $A \in \mathcal{E}$: $\tilde{P}_t(y, A) = \varepsilon_{\{y\}}(A)$;

finally $\tilde{P}_t(\tilde{\Delta}, \{\tilde{\Delta}\}) = 1$.

Then for $y \in E \cup \{\tilde{\Delta}\}$, $A \to \tilde{P}_t(y, A)$ is a probability measure on $\mathcal{E}_{\tilde{\Delta}}$, the Borel field generated by $\mathcal{E}$ and $\tilde{\Delta}$; $y \to \tilde{P}_t(y, A)$ is in $\mathcal{E}_{\tilde{\Delta}}$ for each $A \in \mathcal{E}$; and we have

$$\Pi_t(A, B) = \int_B \tilde{P}_t(y, A)\, K(dy). \tag{2.6}$$

Thus $\tilde{P}_t(y, A)$ will be a transition function in the loose sense for $\tilde{X}$ provided we can verify the relation corresponding to (iv) at the beginning of § 1, namely that we have with probability one:

$$P\{\tilde{X}_{s+t} \in A \mid \tilde{\mathfrak{F}}_s\} = \tilde{P}_t(\tilde{X}_s, A); \tag{2.7}$$

where for each $t > 0$, $\tilde{\mathfrak{F}}_s$ is the Borel field generated by $\{\tilde{X}_r, 0 < r \leqslant s\}$ and augmented with all sets of probability zero.

Equivalently, we may verify that the finite-dimensional distributions of $\tilde{X}$ obtained in Theorem 1.1 can be written in the proper form by means of K_s and $\tilde{P}_t$ as shown below. We begin with the following lemma which embodies the *duality relation* mentioned in the introduction.

LEMMA 2.1. *For every positive g in $\mathcal{E} \times \mathcal{E}$ such that g vanishes on the set $E \times \{y: h(y) = 0\}$, we have*

$$\int_E G(dx) \int_E P_t(x, dy)\, g(x, y) = \int_E G(dy) \int_E \tilde{P}_t(y, dx)\, g(x, y). \tag{2.8}$$

Proof. If $g(x, y) = 1_A(x)\, 1_B(y)\, h(y)$, $A \in \mathcal{E}$, $B \in \mathcal{E}$, then the left member of (2.8) is just

$$\int_A G(dx) \int_B P_t(x, dy)\, h(y) = \Pi_t(A, B) = \int_B \tilde{P}_t(y, A)\, K(dy)$$

by (2.6), which reduces to

$$\int_B \tilde{P}_t(y, A)\, h(y)\, G(dy) = \int_E G(dy) \int_E \tilde{P}_t(y, dx)\, 1_A(x)\, 1_B(y)\, h(y).$$

Hence (2.8) is true for g of the specified form, and so is true for all positive g of the form fh, where $f \in \mathcal{E} \times \mathcal{E}$, by a familiar monotone class argument. Now it is trivial that this coincides with the class of g stated in the lemma, q.e.d.

Returning to the proof of Theorem 2.1, let us define for $0 < t_1 < \ldots < t_n$ and arbitrary $x_1, \ldots, x_n$ in E:

$$k_1(x_1) = l_{t_1}(x_1),$$

$$k_n(x_n) = \int_{A_{n-1}} P_{t_n-t_{n-1}}(x_n, dx_{n-1}) \dots \int_{A_1} P_{t_2-t_1}(x_2, dx_1)\, l_{t_1}(x_1), \quad n \geq 2.$$

It follows from Lemma 1.1 that

$$k_n(x_n) \leq P_{t_n-t_{n-1}} \dots P_{t_2-t_1} l_{t_1}(x_n) = l_{t_n}(x_n)$$

so that k_n vanishes where h does. We have therefore by repeated application of Lemma 2.1 to the right member of (1.6):

$$\int_{A_n} G(dx_n) \int_{A_{n-1}} P_{t_n-t_{n-1}}(x_n, dx_{n-1})\, k_{n-1}(x_{n-1})$$

$$= \int_{A_{n-1}} G(dx_{n-1})\, k_{n-1}(x_{n-1})\, \tilde{P}_{t_n-t_{n-1}}(x_{n-1}, A_n)$$

$$= \int_{A_{n-1}} G(dx_{n-1}) \int_{A_{n-2}} P_{t_{n-1}-t_{n-2}}(x_{n-1}, dx_{n-2})\, k_{n-2}(x_{n-2})\, \tilde{P}_{t_n-t_{n-1}}(x_{n-1}, A_n)$$

$$= \int_{A_{n-2}} G(dx_{n-2})\, k_{n-2}(x_{n-2}) \int_{A_{n-1}} \tilde{P}_{t_{n-1}-t_{n-2}}(x_{n-2}, dx_{n-1})\, \tilde{P}_{t_n-t_{n-1}}(x_{n-1}, A_n)$$

$$= \dots = \int_{A_1} G(dx_1)\, l_{t_1}(x_1) \int_{A_2} \tilde{P}_{t_2-t_1}(x_1, dx_2)$$

$$\dots \int_{A_{n-1}} \tilde{P}_{t_{n-1}-t_{n-2}}(x_{n-2}, dx_{n-1})\, \tilde{P}_{t_n-t_{n-1}}(x_{n-1}, A_n).$$

Comparing this with Theorem 1.1 we see that $\tilde{X}$ has indeed $\tilde{P}_t$ as a version of its transition function and Theorem 2.1 is completely proved.

3. A regularity property of the reverse transition function

We shall show that an arbitrary collection of versions of the Radon–Nikodym derivatives $\{\tilde{P}_t, t > 0\}$ obtained in Theorem 2.1 has certain regularity properties and use these to construct a "standard modification" that is *vaguely left continuous* in t. This results from the fact that $\tilde{P}_t(y, A)$ is the loose-sense transition function of a homogeneous Markov process whose sample paths are all left continuous, and will be stated in this general form, using the notation X_t and P_t instead of $\tilde{X}_t$ and $\tilde{P}_t$.

From now on we write R for $[0, \infty)$ and Q for an arbitrary countable dense subset of R. To alleviate the notation *we shall reserve in this section the letters r and r' to denote members of Q. Thus, for instance, $r \to t$ means $r \in Q$ and $r \to t$.* The notation $s \to t+$ means

$s>t$ and $s\to t$, similarly $s\to t-$ means $s<t$ and $s\to t$. If X_t is a Markov process with absolute distributions μ_t, a set Z in the completion of $\mathcal{E}$ with respect to all $\{\mu_t,\ t>0\}$ such that $\mu_t(Z)=0$ for all $t>0$ will be called "insignificant".

THEOREM 3.1. *Suppose $\{X_t,\ t>0\}$ is a homogeneous Markov process taking values in E and having left continuous sample paths. Suppose $P_t(x, B)$ is its transition function in the loose sense and $\mu_t(B)$ its absolute distribution. Then the following two assertions are true.*

(a) *For each Q there is an insignificant set Z such that for every $x\notin Z$ and $f\in C_K$, we have*

$$\forall t>0:\ \lim_{r\to t-} P_r f(x) \quad \textit{exists}.$$

(b) *For each $t>0$, there is an insignificant set Z_t such that for every $x\notin Z_t$ and $f\in C_K$, we have*

$$\lim_{r\to t-} P_r f(x) = P_t f(x).$$

Remark. There is an obvious analogue if X has right continuous paths.

Proof of (a). Let $\varepsilon>0$, $f\in C_K$ and put

$$H=\{(t,x):\ \underline{\lim}_{s\to t-} P_s f(x) < \overline{\lim}_{s\to t-} P_s f(x)-\varepsilon\}. \tag{3.1}$$

If Π denotes the projection of $R\times E$, we have

$$\Pi(H)=\{x:\ \exists t>0:\ \underline{\lim}_{s\to t-} P_s f(x) < \overline{\lim}_{s\to t-} P_s f(x)-\varepsilon\}. \tag{3.2}$$

To prove (a) it is sufficient to show that $\Pi(H)$ is insignificant and this will be done by a capacity argument due to P. A. Meyer [12]. We sketch the set-up below; note that a "$\mathcal{k}$-analytic" set below is an "analytic" or "Souslin" set in the classical sense.

Let $\mathcal{B}$ be the Borel field of R, $\mathcal{C}$ the class of compact sets of R, $\mathcal{k}$ the class of compact sets of E. It is easy to see that $H\in\mathcal{B}\times\mathcal{E}$ because Q is countable (cf. e.g., [5; pp. 161–2]), hence $\Pi(H)$ is $\mathcal{k}$-analytic and so measurable with respect to the completed measure μ_s.

LEMMA 3.1. *If $s>0$, there exists $L\in\mathcal{E}$, such that $L\subset\Pi(H)$ with $\mu_s(L)=\mu_s(\Pi(H))$, and a strictly positive $\mathcal{E}$-measurable function τ defined on L whose graph*

$$\{(x,\tau(x)):\ x\in L\}$$

is contained in H.

This is a particular case of Meyer's theorem but can be proved quickly as follows. For every subset A of $R\times E$ define

$$\varphi(H) = \mu_s^*(\Pi(H)),$$

where μ_s^* is the outer measure induced by μ_s. Then φ is a capacity and H is analytic, both with respect to the class of compact sets of the product space $R \times E$. Hence H is φ-capacitable and there is a compact subset K_1 of H such that $\varphi(K_1) > \varphi(H)/2$. Now define τ_1 on $L_1 = \Pi(K_1)$ by

$$\tau_1(x) = \inf \{t\colon (t, x) \in K_1\}.$$

The compactness of K_1 implies, first that $(x, \tau_1(x)) \in K_1$ and second that for each real c, the set $\{x\colon \tau_1(x) \leqslant c\}$ is closed so that τ_1 is $\mathcal{E}$-measurable, indeed lower semi-continuous. [We owe Professor Wendell Fleming the last remark which replaces a longer argument.] If we choose an increasing sequence of compact K_n with $\varphi(K_n) \uparrow \varphi(H)$ and define the corresponding L_n and τ_n as above we see that the set $L = \bigcup_n L_n$ and the function τ such that $\tau(x) = \tau_n(x)$ for $x \in L_n - L_{n-1}$ (with $L_0 = \varnothing$) satisfy the requirements, q.e.d.

It follows from the lemma that for every $x \in L$, we have

$$\varliminf_{r \to \tau(x)-} P_r f(x) < \varlimsup_{r \to \tau(x)-} P_r f(x) - \varepsilon. \tag{3.3}$$

Hence if we define two subsets of Q as follows:

$$\Gamma_1(x) = \left\{ r \in Q : P_r f(x) > \varlimsup_{r' \to \tau(x)-} P_{r'} f(x) - \frac{\varepsilon}{3} \right\},$$

$$\Gamma_2(x) = \left\{ r \in Q : P_r f(x) < \varliminf_{r' \to \tau(x)-} P_{r'} f(x) + \frac{\varepsilon}{3} \right\};$$

then for every $x \in L$, $\tau(x)$ is an accumulation point from the left of both $\Gamma_1(x)$ and $\Gamma_2(x)$, namely that for every $\delta > 0$, we have $(\tau(x) - \delta, \tau(x)) \cap \Gamma_i(x) \neq \varnothing$, $i = 1, 2$. It follows from this that for either i we can construct $\mathcal{E}$-measurable functions σ_n on L, taking values in $\Gamma_i(x)$, and such that $\sigma_n(x) \to \tau(x)-$ for all x in L. This is a familiar construction of which a more elaborate form will be stated in § 6. Assuming this, we are ready to prove (a). Let $\{\sigma_n'\}$ and $\{\sigma_n''\}$ be the $\{\sigma_n\}$ just mentioned corresponding to Γ_1 and Γ_2 respectively, and let $\{\tau_n\}$ be the alternating sequence $\{\sigma_1', \sigma_1'', \sigma_2', \sigma_2'', \ldots\}$. We have then for every $x \in L$:

$$P_{\sigma_n'(x)} f(x) > P_{\sigma_n''(x)} f(x) - \frac{\varepsilon}{3}. \tag{3.4}$$

Now consider the equation

$$\int_L \mu_s(dx)\, P_{\tau_n(x)} f(x) = E\{X_s \in L;\ f \circ X(s + \tau_n(X_s))\}, \tag{3.5}$$

which is a consequence of the Markov property of X since τ_n is countably-valued. The right member of (3.5) converges as $n \to \infty$ to the limit obtained by replacing τ_n with τ there, since X has left continuous paths. But by (3.4) the left member cannot converge unless $\mu_s(L)=0$. This must then be true and so $\mu_s(\Pi(H))=0$ by Lemma 3.1. Since s is arbitrary, $\Pi(H)$ is insignificant. Writing $H_f(\varepsilon)$ for this H, letting f run through a countable set D dense in C_K, and setting $Z=\bigcup_{f\in D}\bigcup_{n=1}^{\infty}H_f(n^{-1})$, we obtain (a).

The proof of (b) is similar but simpler. For a fixed $t>0$, consider

$$H_t=\left\{x: \varliminf_{r\to t-} P_r f(x)<P_t f(x)-\varepsilon\right\}.$$

$$\Gamma_t(x)=\left\{r\in Q: P_r f(x)<P_t f(x)-\frac{\varepsilon}{2}\right\}.$$

Then $H_t\in\mathcal{E}$ (no capacity argument is needed here), and there exist $\mathcal{E}$-measurable functions τ_n defined on H_t, taking values in $\Gamma_t(x)$, and increasing to t as $n\to\infty$. It follows that

$$\int_{H_t}\mu_s(dx)\left(P_t f(x)-\frac{\varepsilon}{2}\right)\geqslant\int_{H_t}\mu_s(dx)P_{\tau_n(x)}f(x)$$
$$=E\{X_s\in H_t; f\circ X(s+\tau_n(X_s))\}\to E\{X_s\in H_t; f\circ X(s+t)\}=\int_{H_t}\mu_s(dx)\,P_t f(x).$$

Hence $\mu_s(H_t)=0$. Together with a symmetric argument on the upper limit, this establishes (b).

THEOREM 3.2. *Under the hypotheses of Theorem 3.1, there exists a transition function $P_t^*(x,B)$ in the loose sense for the process X such that for each $f\in C_K$ we have*

$$\forall t>0;\ \lim_{s\to t-} P_s^* f=P_t^* f.$$

This means: for each $x, t\to P_t^*(x,\cdot)$ is vaguely left continuous as measures. We shall write a vague limit in this sense as "v lim" below.

Proof. In view of (a) of the preceding theorem, we may define

$$\forall t>0, x\notin Z:\quad P_t^*(x,\cdot)=\mathrm{v}\lim_{r\to t-} P_r(x,\cdot)$$

$$\forall t>0, x\in Z:\quad P_t^*(x,\cdot)=\delta_x(\cdot)=P_0^*(x,\cdot).$$

For each $f\in C_K, x\to P_t^* f(x)$ is in $\mathcal{E}$. By (b) of the theorem, we have for every s almost surely

$$P_t^*(X_s,f)=P_t(X_s,f),$$

and consequently P_t^* as well as P_t serves as a transition function in the loose sense. Finally, from $P_t^* f = \lim_{r \to t-} P_r f$ ($r \in Q$!) it follows that $t \to P_t^* f$ is left continuous, q.e.d.

Applying Theorem 3.2 to the reverse process $\tilde{X}$ in Theorem 2.1, we conclude that its transition function $\tilde{P}_t(y, \cdot)$ may be modified to be vaguely left continuous in t for each y, as defined above.

4. The removal of assumptions (H1) and (H2)

The preceding results were proved under (H1) and (H2). If the life time α of the given process does not satisfy these conditions, it will now be shown in what sense the results may be carried over. Roughly speaking, they remain true provided that "reversed time" be liberally interpreted as beginning at some fictitious (but by no means nebulous) origin. Or else if this is not allowed, then the results are still true provided that an exceptional set of reversed time of zero Lebesgue measure be ignored. Finally we shall show in § 6 that all fiction or exception may be dropped provided that the given (forward) process is assumed to be strongly Markovian instead of merely Markovian as we do now. This however lies deeper.

For the present a little device suffices: one simply extends the life time from α to α^* by adding exponentially distributed holding times until the distribution of α^*, being the convolution of that of α with smooth densities, achieves the kind of good behavior required by (H1) and (H2). In fact, this device will make the distribution of α^* as smooth as one may wish as a function of t, but only mildly so as a function of x. This will be sufficient since we need only a certain uniformity with respect to x in (H2). Now we can reverse the prolonged process from the new life time α^* by the preceding theorems. The true reversal from α will then appear as the portion of the reversed prolonged process starting from $\alpha^* - \alpha$, which is an optional time for it. Hence if the last-mentioned process is moderately strongly Markovian – as we shall prove in § 6 – the true reverse process will behave in like fashion.

Let β_i, $i = 1, 2, 3$, be three random variables independent of one another and of the Borel field generated by $\{X_t, t \geq 0\}$, and having the common distribution with density $\lambda e^{-\lambda t} dt$, $\lambda > 0$. Adjoin three distinct new points Δ_i, $i = 1, 2, 3$ to E and define the prolonged process as follows:

$$Y_t = \begin{cases} X_t & \text{if } t < \alpha, \\ \Delta_1 & \text{if } \alpha \leq t < \alpha + \beta_1, \\ \Delta_2 & \text{if } \alpha + \beta_1 \leq t < \alpha + \beta_1 + \beta_2, \\ \Delta_3 & \text{if } \alpha + \beta_1 + \beta_2 \leq t < \alpha + \beta_1 + \beta_2 + \beta_3, \\ \Delta & \text{if } t \geq \alpha + \beta_1 + \beta_2 + \beta_3. \end{cases}$$

We shall regard λ as fixed, put

$$\beta = \beta_1 + \beta_2 + \beta_3, \qquad \alpha^* = \alpha + \beta,$$

and denote the density of β by

$$b(t) = 2^{-1}\lambda^3 t^2 e^{-\lambda t}, \quad t \geq 0.$$

Thus the distribution of α^* is given by

$$L^*(x, t) = \int_0^t L(x, s)\, b(t-s)\, ds$$

with the density
$$l^*(x, t) = \int_0^t L(x, s)\, b'(t-s)\, ds.$$

Since
$$|l^*(x, t)| \leq \int_0^t |b'(s)|\, ds \leq \int_0^\infty 2^{-1}\lambda^3 |2t - t^2| e^{-\lambda t}\, dt < \infty,$$

and
$$\left|\frac{d}{dt} l^*(x, t)\right| \leq \int_0^t |b''(s)|\, ds \leq \int_0^\infty 2^{-1}\lambda^3 |2 - 4\lambda t + \lambda^2 t^2| e^{-\lambda t} dt < \infty,$$

$l^*(x, t)$ is uniformly bounded in all x and t and $t \to l^*(x, t)$ has a derivative bounded uniformly in x and t and hence is equi-continuous in t with respect to all x. This means (H1) and (H2) hold.

The parameter λ will play no role below, but let us remark that as $\lambda \to \infty$, $Y_t \to X_t$ for all t almost surely. Now we define the reverse process to Y from α^* just as we did the reverse to X from α in (1.4):

$$\tilde{Y}_t = \begin{cases} Y_{\alpha^*-t} & \text{if } 0 < t \leq \alpha^* < \infty, \\ \tilde{\Delta} & \text{if } \alpha^* < \infty, t > \alpha^*, \\ \tilde{\Delta} & \text{if } \alpha^* = \infty, t > 0. \end{cases}$$

Then $\tilde{Y}_t = \tilde{X}_{t-\beta}$ if $t > \beta$. Since Y satisfies (H1) and (H2), $\tilde{Y}$ satisfies the conclusions of Theorems 1.1 and 2.1.

The independence of $\alpha^* - \alpha$ and $\{X_t, t \geq 0\}$ should be formalized by considering the product measure space $(\Omega \times R, \mathcal{F} \times \mathcal{B}, P \times \nu)$ where ν is the measure with density b on R. If we regard the Y process as defined on this space and write $\hat{\omega} = (\omega, \omega')$, $Y(t, \hat{\omega}) = X(t, \omega)$ if $t < \alpha(\omega')$, etc., then the following lemma is not only obvious but even true (it may be false otherwise).

LEMMA 4.1. *Let f_j, $1 \leq j \leq n$, be bounded, $\mathcal{E}$-measurable functions vanishing at Δ_i, $i = 1, 2, 3$. Then for each $t_0 < t_1$:*

$$E\left\{\beta < t_0; \prod_{j=1}^n f_j(\tilde{Y}_{t_j})\right\} = \int_0^{t_0} E\left\{\prod_{j=1}^n f_j(\tilde{X}_{t_j - s})\right\} b(s)\, ds. \tag{4.1}$$

We now state and prove the result accruing from Theorem 2.1 after the removal of (H1) and (H2).

THEOREM 4.1. *Let $\{X_t, t \geq 0\}$ be a homogeneous Markov process with right continuous paths and life time α. Let $\{\tilde{X}_t, t>0\}$ be the reverse process defined by* (1.1), *and $\tilde{\mu}_t(A) = P\{\tilde{X}_t \in A\}$ for $A \in \mathcal{E}$. Then there exists $\tilde{P}_t(x, A)$, $t \geq 0$, $x \in E$, $A \in \mathcal{E}$, satisfying conditions* (i) *and* (ii) *for a transition function given at the beginning of § 1, such that $t \to \tilde{P}_t(x, \cdot)$ is vaguely left continuous for each x, with the following property. Given $0 \leq t_1 < \ldots < t_n$ and $A_0, A_1, \ldots, A_n$ in $\mathcal{E}$, we have for almost every* (*Lebesgue*) *t_0 in $(0, t)$:*

$$P\{\tilde{X}_{t_j} \in A_j, 0 \leq j \leq n\} = \int_{A_0} \tilde{\mu}_{t_0}(dx_0) \int_{A_1} \tilde{P}_{t_1 - t_0}(x_0, dx_1) \ldots \int_{A_n} \tilde{P}_{t_n - t_{n-1}}(x_{n-1}, dx_n). \tag{4.2}$$

Remark. The proof will show how to calculate $\tilde{\mu}_t$ and $\tilde{P}_t$.

Proof. Let f_j be as in Lemma 4.1. Since $\tilde{Y}$ satisfies the conclusions of Theorems 2.1 and 3.1, let P_t^* be its transition function in the loose sense having the stated regularity property. We have then for $t < t_0$:

$$E\left\{\beta < t; \prod_{j=1}^{n} f_j(\tilde{Y}_{t_j})\right\} = E\{\beta < t; (f_0 \varphi)(\tilde{Y}_{t_0})\}, \tag{4.3}$$

where

$$\varphi(x) = \int_E P^*_{t_1 - t_0}(x, dx_1) f_1(x_1) \int_E P^*_{t_2 - t_1}(x_1, dx_2) f_2(x_2) \ldots \int_E P^*_{t_n - t_{n-1}}(x_{n-1}, dx_n) f_n(x_n).$$

Using Lemma 3.1 in both members of (4.3), we obtain

$$\int_0^t ds\, b(s) E\left\{\prod_{j=0}^{n} f_j(\tilde{X}_{t_j - s})\right\} = \int_0^t ds\, b(s)\, E\left\{(f_0 \varphi)(\tilde{X}_{t_0 - s})\right\}. \tag{4.4}$$

This being true for all $t < t_0$, and $b(s) > 0$ for $s > 0$, we conclude that the two expectations in (4.4) are equal for almost all $s < t_0$. Since t_0 is arbitrary, it follows that given $t_1 < \ldots < t_n$, we have

$$E\left\{f_0(\tilde{X}_t) \prod_{j=1}^{n} f_j(\tilde{X}_{t_j})\right\} = E\{(f_0 \varphi)(\tilde{X}_t)\}$$

for almost all $t < t_0$. This implies (4.2). Note that the loose-sense transition for $\tilde{X}$ may be taken to be that of $\tilde{Y}$ for any $\lambda > 0$, and that its absolute distribution $\tilde{\mu}_t$ is determined by the equation below, valid for $t > 0$, $A \in \mathcal{E}$;

$$\int_0^t \mu_{t-s}(A)\, b(s)\, ds = P\{\tilde{Y}_t \in A\} = \int_A G(dx)\, l^*(x, t).$$

We end this section with some examples to illustrate the possibilities and limitations:

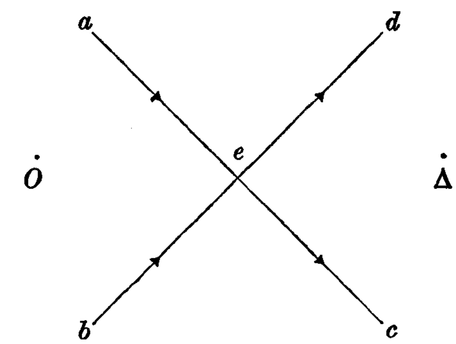

Fig. 1.

Example 1. The state space consists of the two diagonals $\overline{ac}$ and $\overline{bd}$ of a square with side length $\sqrt{2}$ and center e, together with two outside points O and Δ. The process starts at O which is a holding point (with density $e^{-t}dt$ for the holding time distribution), then jumps to a or b with probability $\frac{1}{2}$ each. From either point it moves with unit speed along the diagonal until it reaches c or d, and then jumps to the absorbing point Δ. This process is Markovian but not strongly so, as the strong Markov property fails at the hitting time of e. The reverse process is not Markovian at $t=1$, when it is at the state e. Observe that the transition probabilities for the forward process do not satisfy the Chapman–Kolmogorov equation $P_2(x, d) = P_1(x, e)P_1(e, d)$ for both $x=a$ and $x=b$, no matter how $P_1(e, d)$ is defined.

Example 2. This is an elaboration of the preceding example, in which the reverse process is not Markovian at an uncountable set of t (but of measure 0 in accordance with Theorem 4.1). Let f be a nonnegative continuous function on $[0, 1]$, whose set of zeros is the Cantor set. The state space consists of the graphs of f and of $-f$. The process starts at $(0, 0)$ which is a holding point, then follows either the graph of f or the graph of $-f$ with probability $\frac{1}{2}$ each until it reaches $(0, 1)$ which is the absorbing point. This process is Markovian but not strongly so, and the reverse process is not Markovian, for the Markov property fails at all t in the Cantor set.

Example 3. This example shows that even if the forward process is strongly Markovian, the reverse one need not be so. Let the process be the uniform motion on the line starting at -1, moving to the right until it hits O which is a holding point, after which it jumps to Δ. The reverse process is Markovian but not strongly so, since it has continuous paths and yet starts at a holding point.

5. Essential limits

Let $R = [0, \infty)$ and let "measure" below be the Lebesgue measure on R, denoted by m. For an extended real-valued function f on R, we say that "its essential supremum on a measurable set S exceeds c" iff there is a subset of S of strictly positive measure on which $f > c$; the supremum of all such c is the ess sup of f on S, unless the set of c is empty in which case the ess sup is taken to be $-\infty$. Next, e.g.,

$$\text{ess}\limsup_{s\to t+} f(s)$$

is defined as the infimum of the ess sup of f on $(t, t+n^{-1})$ as $n\to\infty$; ess inf and ess lim inf are defined in a similar way. When ess lim $\sup_{s\to t+} f(s)$ and ess lim $\inf_{s\to t+} f(s)$ are equal we say that ess $\lim_{s\to t+} f(s)$ exists and is equal to the common value. We can of course define the latter directly but we need the other concepts below.

Some of the properties of ess $\lim_{s\to t+} f(s)$ are summarized in the next lemma, whose proof is omitted, being elementary analysis.

LEMMA 5.1. *Suppose that for every t in R, $\varphi(t) = \text{ess}\lim_{s\to t+} f(s)$ exists. Then φ is right continuous everywhere, $f=\varphi$ except for a set Z of measure zero, and we have*

$$\forall t: \quad \text{ess}\lim_{s\to t+} f(s) = \lim_{\substack{s\to t+ \\ s\notin Z}} f(s).$$

Finally, we have

$$\forall t: \varphi(t) = \lim_{\lambda\to\infty}\int_0^\infty \lambda e^{-\lambda s} f(t+s)\, ds = \lim_{\delta\downarrow 0}\frac{1}{\delta}\int_t^{t+\delta} f(s)\, ds. \tag{5.1}$$

The next two propositions resemble the main lemmas for separability of a stochastic process due to Doob [6].

LEMMA 5.2. *Suppose that $H\in\mathcal{B}\times\mathfrak{F}$ (the product Borel field of $R\times\Omega$) and put for each $t\in R$:*

$$H(t) = \{\omega: (t,\omega)\in H\}.$$

Let

$$\Delta = \left\{\omega: \int_0^\infty 1_H(t,\omega)\, dt > 0\right\}$$

and let Z be an arbitrary subset of R with $m(Z)=0$. Then there exists a countable dense subset $D=\{t_n, n\geq 1\}$ of R such that $D\cap Z=\emptyset$ and

$$P\{\Delta\,\triangle\,\bigcup_n H(t_n)\} = 0, \tag{5.2}$$

where "$\triangle$" denotes the symmetric difference.

Proof. By Fubini's theorem, $[R\times(\Omega-\Delta)]\cap H$ has $m\times P$ measure zero and there exists $Z'\subset R$ with $m(Z')=0$ such that if $t\notin Z'$ then $P(H(t)\setminus\Delta)=0$. Let

$$T = \{t \in R \colon P(H(t)) > 0\}$$

and consider the class of sets of the form

$$\bigcup_{t \in C} H(t),$$

where C is a countable dense subset of R, disjoint from $Z \cup Z'$. A familiar argument shows that there is a set in the class whose probability is maximal. Call this set Λ and we will show that $P(\Delta \setminus \Lambda) = 0$. Otherwise let $\Delta \setminus \Lambda = \Lambda_0$, $P(\Lambda_0) > 0$. Then $H \cap (R \times \Lambda_0)$ has strictly positive $m \times P$ measure by definition of Δ and Fubini's theorem. Hence by the same theorem there exists some $t \notin Z \cup Z'$ such that $P(H(t) \cap \Lambda_0) > 0$, which contradicts the maximality of Λ since $\Lambda \cup H(t)$ would be in the class above and have a strictly greater probability than Λ. Finally, by the definition of Z' and the choice of C, it is clear that $P(\Lambda \setminus \Delta) = 0$.

THEOREM 5.1. *Let $\{Y_t,\ t \in R\}$ be an extended real-valued Borel measurable stochastic process in $(\Omega, \mathcal{F}, P)$. There exists Ω_0 in $\mathcal{F}$ with $P(\Omega_0) = 1$ and a countable dense set D of R with the following property. For each $\omega \notin \Omega_0$ and every nonempty open interval I of R, we have*

(i) $\operatorname*{ess\,sup}_{t \in I} Y(t, \omega) = \sup_{t \in I \cap D} Y(t, \omega)$

(ii) $\operatorname*{ess\,inf}_{t \in I} Y(t, \omega) = \inf_{t \in I \cap D} Y(t, \omega)$.

Such a set D will be referred to as an "essential limit set for Y".

Proof. For each I with rational endpoints, consider the set

$$\{(t, \omega) \colon t \in I;\ Y(t, \omega) < \operatorname*{ess\,inf}_{s \in I} Y(s, \omega) \ \text{ or } \ Y(t, \omega) > \operatorname*{ess\,sup}_{s \in I} Y(s, \omega)\}.$$

This has $m \times P$ measure zero by Fubini's theorem, hence there is a subset $Z(I)$ of I with measure zero such that if $t \in I - Z(I)$ then for almost every ω:

$$\operatorname*{ess\,inf}_{s \in I} Y(s, \omega) \leqslant Y(t, \omega) \leqslant \operatorname*{ess\,sup}_{s \in I} Y(s, \omega). \tag{5.3}$$

Let Z be the union of $Z(I)$ over all such I. Next, for each rational r, consider

$$H = \{(t, \omega) \colon t \in I;\ Y(t, \omega) > r\}$$

and define Δ corresponding to H as in Lemma 5.2. It follows that there exists a countable dense set $\{t_n, n \geqslant 1\}$, disjoint form Z, such that (5.2) is true. Observe that Δ is the set of ω where $\operatorname{ess\,sup}_{t \in I} Y(t, \omega) > r$, while $\bigcup_n H(t_n)$ is the set of ω where $\sup_n Y(t_n, \omega) > r$.

Hence if we denote by D_1 the countable set obtained by uniting the sequences $\{t_n\}$ over all I and r, D_1 is disjoint from Z and we have for almost every ω:

$$\operatorname*{ess\,sup}_{t\in I} Y(t,\omega) \leqslant \sup_{t\in D_1\cap I} Y(t,\omega).$$

Similarly, there is a countable set D_2 disjoint from Z such that for almost every ω:

$$\operatorname*{ess\,inf}_{t\in I} Y(t,\omega) \geqslant \inf_{t\in D_2\cap I} Y(t,\omega).$$

Then if $D = D_1 \cup D_2$, we have for almost every ω and every I:

$$\inf_{t\in D\cap I} Y(t,\omega) \leqslant \operatorname*{ess\,inf}_{t\in I} Y(t,\omega) \leqslant \operatorname*{ess\,sup}_{t\in I} Y(t,\omega) \leqslant \sup_{t\in D\cap I} Y(t,\omega).$$

But since $D\cap Z=\emptyset$, the first and last inequalities above can be reversed by (5.3), proving the theorem.

It will appear in our later applications of Theorem 5.1 to Theorems 6.1 and 6.3 that we shall not need its full strength but merely the existence of a countable dense set D such that if almost all paths have left and right limits along D then they have essential left and right limits. Thus it is sufficient to have the equations in (i) and (ii) above replaced by "$\leqslant$" and "$\geqslant$" respectively. Doob has pointed out that Theorem 5.1 can be circumvented by arguing with separable versions, see the end of proof of Theorem 6.1.

6. The moderately strong Markov property of the reverse process

In this section we assume that the given process X is strongly Markovian relative to right continuous fields $\{\mathfrak{F}_t, t\geqslant 0\}$, whose paths are not only right continuous on $0\leqslant t<\infty$ but also have left limits everywhere on $0<t<\alpha$. Thus for each optional T, $t>0$ and bounded $\mathcal{E}$-measurable f, we have almost surely

$$E\{f(X_{T+t})\,|\,\mathfrak{F}_{T+}\} = P_t(X_T, f).$$

We shall use the "shift operator" θ in the usual way but we remind the reader that we are dealing with a process with a fixed initial distribution and not a family of processes starting at each x.

Let $\{Y_t, t\geqslant 0\}$ be the extended process with lifetime $\alpha^* = \alpha+\beta$ as defined in § 4. Let $\{\tilde{\mathfrak{F}}_t, t>0\}$ be the Borel field generated by the reverse process $\{\tilde{Y}_t, t>0\}$ and $\tilde{P}_t$ its transition function. As we have seen, $\tilde{P}_t$ acts like a transition function of $\tilde{X}$ as well. A random variable (or simply "time") T will be called "reverse-optional" iff for every $t>0$, we have $\{T<t\}\in\tilde{\mathfrak{F}}_t$; it is "strictly" so iff $\{T<t\}$ is replaced by $\{T\leqslant t\}$. This distinction is necessary as the

fields $\tilde{\mathfrak{F}}_t$, unlike $\mathfrak{F}_t$, are not necessarily right continuous. The Borel fields $\tilde{\mathfrak{F}}_{T+}$ and $\tilde{\mathfrak{F}}_T$ are defined in the usual way as in § 1. T is said to be "reverse-predictable" iff there exists a sequence of reverse-optional times $\{T_n\}$ such that $T_n < T$ and $T_n \uparrow T$ almost surely; in this case we have

$$\tilde{\mathfrak{F}}_{T-} = \bigvee_n \tilde{\mathfrak{F}}_{T_n+},$$

where for an increasing family of Borel fields $\{\mathcal{G}_t, t > 0\}$ and an arbitrary random variable T, $\mathcal{G}_{T-}$ is the Borel field generated by the class of sets of the form $\{T > t\} \cap \Lambda$ with $\Lambda \in \mathcal{G}_t$. See [13] for a general discussion of the notion.

We begin with a useful lemma, whose proof is omitted as being intuitively obvious and technically familiar.

LEMMA 6.1. *Let D be a countable dense subset of $R = [0, \infty)$. Let T be an optional time (with respect to $\{\mathfrak{F}_t\}$) with the following property. If $T(\omega) < \infty$ then there is a subset $C(\omega)$ of D such that for each t, the set $\{\omega: t \in C(\omega)\} \in \mathfrak{F}_t$ and for each $\delta > 0$,*

$$(T(\omega),\ T(\omega) + \delta) \cap C(\omega) \neq \emptyset. \tag{6.1}$$

Then there exists a sequence of strictly optional times $\{T_n\}$ such that for each n:

$$T_n(\omega) \in C(\omega), \quad T_n(\omega) > T(\omega)$$

and $T_n \downarrow T$ on $\{T < \infty\}$.

Let T be predictable and (6.1) *be replaced by*

$$(T(\omega) - \delta,\ T(\omega)) \cap C(\omega) \neq \emptyset, \quad \textit{for } 0 < T(\omega) < \infty.$$

Then a similar conclusion is true if "$>$" and "$\downarrow$" are replaced by "$<$" and "$\uparrow$", and $\{T < \infty\}$ by $\{0 < T < \infty\}$.

THEOREM 6.1. *Let D be a countable dense subset of R, $t > 0$ and $f \in C_K$. Then almost surely the path*

$$s \to \tilde{P}_t f \circ \tilde{Y}_s,$$

has left and right limits along D everywhere. In particular, it has left and right essential limits everywhere.

Proof. Let $\{T_k\}$ be a sequence of D-valued strictly reverse-optional times decreasing to a limit T. Notice that on $\{T < \beta\}$, $\tilde{Y}(T_k) \in \{\Delta_1 \cup \Delta_2 \cup \Delta_3\}$ for all large enough k. Since $\tilde{Y}$ is Markovian as proved in § 4, the strong Markov property holds at any discrete strictly reverse-optional time such as T_k, hence

$$E\{f \circ \tilde{Y}(T_k+t);\, T_k > \beta \mid \tilde{\mathfrak{F}}_{T_k}\} = 1_{\{T_k > \beta\}}\, \tilde{P}_t f \circ \tilde{Y}(T_k). \tag{6.2}$$

Since the paths of X have left limits except possibly at α, those of $\tilde{Y}$ have right limits except possibly at β. Thus $f \circ \tilde{Y}(T_k+t)$ converges as $k \to \infty$ on the set $\{T \geqslant \beta\}$, while $\tilde{\mathfrak{F}}_{T_k} \downarrow \wedge_k \tilde{\mathfrak{F}}_{T_k}$. Therefore by Hunt's lemma cited in § 1, the first member of (6.2) converges almost surely. Similarly if $T_k \uparrow$. Writing for short

$$g = \tilde{P}_t f,$$

we have proved that almost surely

$$\lim g \circ \tilde{Y}(T_k) \tag{6.3}$$

exists for any monotone sequence $\{T_k\}$ as specified above. Note that if $T_k \uparrow \infty$, then $\tilde{Y}(T_k) = \tilde{\Delta}$ for all sufficiently large k.

Let $a < b$ and put

$$T' = \inf \{r \in D \colon g(\tilde{Y}_r) < a\},$$

$$T'' = \inf \{r \in D \colon g(\tilde{Y}_r) > b\},$$

where we may suppose that $0 \notin D$. Define inductively

$$S_0 = 0, \quad S_1 = T', \quad S_2 = S_1 + T'' \circ \theta_{S_1},$$

$$S_{2n-1} = S_{2n-2} + T' \circ \theta_{S_{2n-2}}, \quad S_{2n} = S_{2n-1} + T'' \circ \theta_{S_{2n-1}}, \quad n \geqslant 2.$$

These are all reverse-optional times not necessarily D-valued. It is possible that $S_0 = S_2$, but we have $S_n < S_{n+2}$ almost surely for $n \geqslant 1$. For otherwise on the set $\{S_n = S_{n+1} = S_{n+2}\}$ we have

$$\varliminf_{\substack{r \to S_n+ \\ r \in D}} g(\tilde{Y}_r) \leqslant a < b \leqslant \varlimsup_{\substack{r \to S_n+ \\ r \in D}} g(\tilde{Y}_r).$$

By Lemma 6.1 we can then construct D-valued, reverse-optional $\{T_k\}$ such that $T_k \downarrow S_n$ on the set above and

$$g \circ \tilde{Y}(T_k) \leqslant a, \quad g \circ \tilde{Y}(T_{k+1}) \geqslant b, \tag{6.4}$$

contradicting (6.3).

Next, we show that $S_n \to \infty$ almost surely. For on the set $\{S_n \uparrow S < \infty\}$ we can construct as before D-valued, strictly reverse-optional times $\{T_k\}$ such that $T_k \uparrow S$ and (6.4) holds, again contradicting (6.3). The fact that $S_n \uparrow \infty$ almost surely shows that there is no point in R at which the oscillation on the left or on the right of $g \circ \tilde{Y}_s$, $s \in D$, exceeds $b - a$. If we consider all rational pairs $a < b$, we conclude that $s \to g \circ \tilde{Y}_s$ must have left and right limits along D, everywhere in R. Taking D to be the essential limit set for $\tilde{Y}$ in Theorem 5.1, we see that the existence of such limits is equivalent to the existence of left and right essential limits, q.e.d.

Instead of using Theorem 5.1 as in the last sentence above, we may conclude in the following way as suggested by Doob. Let Y' be a separable version of $\tilde{Y}$ with separability set D' such that almost surely $\tilde{Y}(s) = Y'(s)$ for all $s \in D$. Then almost all paths of Y' have left and right limits along D' and consequently by separability have left and right limits without restriction. By Fubini's theorem, almost every path $s \to \tilde{Y}_s(\omega)$ differs from the corresponding path $s \to Y'_s(\omega)$ on a set of s of Lebesgue measure zero. It follows that the former has essential left and right limits.

COROLLARY. *The assertion of the theorem is also true for almost every path*

$$s \to \tilde{P}_t f \circ Y_s.$$

For the essential right limit, e.g., at s, of $\tilde{P}_t f \circ Y_.(\omega)$ is just the essential left limit of $\tilde{P}_t f \circ \tilde{Y}_.(\omega)$ at $\alpha^*(\omega) - s$, since $\tilde{Y}_s = Y_{\alpha^* - s}$ for all $0 < s \leqslant \alpha^*$.

Recalling that (P_t) is the transition function of X, we put

$$R_\lambda = \int_0^\infty e^{-\lambda t} P_t \, dt, \quad \lambda > 0$$

as its resolvent. We shall use this operator only as a familiar way of integral averaging. A set in $\mathcal{E}$ which is hit by X with probability zero will be called "polar".

THEOREM 6.2. *Let g be bounded, $\mathcal{E}$-measurable and suppose that almost surely the path*

$$s \to g(X_s)$$

has essential right limits everywhere. Then the following limit

$$\hat{g}(x) \stackrel{\text{def}}{=} \lim_{\lambda \to \infty} \lambda R_\lambda g(x) \tag{6.5}$$

exists except possibly for a polar set; and we have almost surely

$$\forall\, s \geqslant 0: \ \operatorname*{ess\,lim}_{r \to s+} g(X_r) = \hat{g}(X_s). \tag{6.6}$$

Proof. Put
$$Z_s = \operatorname*{ess\,lim}_{r \to s+} g(X_r);$$

without loss of generality we may suppose that this limit exists everywhere on Ω. The process $(s, \omega) \to Z(s, \omega)$ is measurable since by Theorem 5.1 the essential limit may be replaced by that on a countable set. It is also right continuous and hence well-measurable in the sense of Meyer. Let us complete the definition of $\hat{g}$ by setting it to be a constant greater than an upper bound of g wherever the limit in (6.5) fails to exist. Since $\hat{g} \in \mathcal{E}$, $\hat{g}(X)$ is well-measurable with X, and consequently the set

$$H = \{(s, \omega): Z(s, \omega) \neq \hat{g}(X(s, \omega))\} \tag{6.7}$$

is well-measurable. Let $\Pi(H)$ be its projection on Ω. If $P(\Pi(H)) > 0$, then a theorem by Meyer [12; p. 204] asserts that there exists an optional time T such that $P\{T < \infty\} > 0$ and $(T(\omega), \omega) \in H$ so that $Z_T \neq \hat{g}(X_T)$ on $\{T < \infty\}$. But (almost surely in the third and fifth equations below)

$$\begin{aligned} Z_T &= \operatorname*{ess\,lim}_{s \to T+} g(X_s) = \lim_{\lambda \to \infty} \int_0^\infty \lambda e^{-\lambda t} g(X_{T+t})\, dt \\ &= \lim_{\lambda \to \infty} E\left\{\int_0^\infty \lambda e^{-\lambda t} g(X_{T+t})\, dt \,|\, \mathfrak{F}_{T+}\right\} = \lim_{\lambda \to \infty} \int_0^\infty \lambda e^{-\lambda t} E[g(X_{T+t}) \,|\, \mathfrak{F}_{T+}]\, dt \\ &= \lim_{\lambda \to \infty} \int_0^\infty \lambda e^{-\lambda t} P_t g(X_T)\, dt = \lim_{\lambda \to \infty} \lambda R_\lambda g(X_T) = \hat{g}(X_T), \end{aligned}$$

which is a contradiction. Hence $P\{\Pi(H)\} = 0$ and (6.6) follows. Let A denote the set of x for which the limit in (6.5) fails to exist. Then on $\{T_A < \infty\}$ there exists $s \geq 0$ such that $Z_s < \hat{g}(X_s)$. Thus $\{T_A < \infty\} \subset \Pi(H)$ and A is a polar set.

Recalling that X is an initial portion of Y, we may apply Theorem 6.2 to $g = \tilde{P}_t f$ on account of the Corollary to Theorem 6.1. Thus for each $t > 0$ there exists a polar set A such that for all $f \in C_K$, $x \in E - A$, the following limit

$$\hat{P}_t f(v) \overset{\text{def}}{=} \lim_{\lambda \to \infty} \lambda R_\lambda (\tilde{P}_t f)(x) \tag{6.8}$$

exist. Set $\hat{P}_t f(x) = 0$ if $x \in A$. The operator $\tilde{P}_t$ may be extended to a kernel in the usual way. We state this as follows.

COROLLARY. *For each $t > 0$ and $f \in C_K$, we have almost surely*

$$\forall s \geq 0: \operatorname*{ess\,lim}_{r \to s+} \tilde{P}_t f \circ X_r = \hat{P}_t f \circ X_s. \tag{6.9}$$

In particular, $s \to \hat{P}_t f \circ X_s$ is almost surely right continuous.

The last sentence above would amount to the fine continuity of $x \to \hat{P}_t f(x)$ in the customary set-up where the process X is allowed to start at an arbitrary x.

THEOREM 6.3. *Let T be a reverse-predictable time. Then each $t > 0$ and $f \in C_K$, we have*

$$E\{f \circ \tilde{Y}(T + t) \,|\, \tilde{\mathfrak{F}}_{T-}\} = \hat{P}_t f \circ \tilde{Y}(T). \tag{6.10}$$

Proof. Since $\tilde{Y}_s = Y_{\alpha^*-s}$, it follows from (6.9) extended trivially to Y that we have almost surely

$$\forall s > 0: \operatorname{ess}\lim_{r \to s-} \hat{P}_t f \circ \tilde{Y}_r = \hat{P}_t f \circ \tilde{Y}_s. \tag{6.11}$$

Since T is reverse-predictable, there exists reverse-optional times $\{T_n\}$ such that $T_n < T$, $T_n \uparrow T$ almost surely. It follows from Hunt's lemma and the left continuity of the paths of $\tilde{Y}$ that

$$E\{f \circ \tilde{Y}(T_n + t) \mid \tilde{\mathfrak{F}}_{T_n+}\} \to E\{f \circ \tilde{Y}(T+t) \mid \tilde{\mathfrak{F}}_{T-}\}. \tag{6.12}$$

Let D be an essential limit set for the process $\{P_t f \circ \tilde{Y}_s, s \geqslant 0\}$. By Lemma 6.1 we can find an increasing sequence of reverse-optional times $\{T'_n\}$, D-valued and such that $T_n \leqslant T'_n < T$ for all n. The strong Markov property holds at T'_n since $\tilde{Y}$ is Markovian, so that the left member of (6.12), after T_n is replaced by T'_n, becomes $\hat{P}_t f \circ \tilde{Y}(T'_n)$. By (6.11), the latter converges as $n \to \infty$ to

$$\operatorname{ess}\lim_{s \to T-} \hat{P}_t f \circ \tilde{Y}(s) = \hat{P}_t f \circ \tilde{Y}(T).$$

Thus (6.12) becomes (6.10), q.e.d.

THEOREM 6.4. *The equation* (6.10) *remains true if $\tilde{Y}$ is replaced by $\tilde{X}$ and T is predictable with respect to $\{\mathcal{G}_t, t > 0\}$ where $\mathcal{G}_t$ is the Borel field generated by $\{\tilde{X}_s, 0 < s \leqslant t\}$. In particular $\tilde{X}$ is a homogeneous Markov process with $\{\hat{P}_t, t > 0\}$ as transition function in the loose sense.*

Proof. Recall the β in § 4 such that $\tilde{Y}_{\beta+t} = \tilde{X}_t$, $t > 0$. If T is predictable relative to $\mathcal{G}_t$ as stated, then $\beta + T$ is reverse-predictable. Furthermore, we have

$$\mathcal{G}_{T-} \subset \tilde{\mathfrak{F}}_{(\beta+T)-}. \tag{6.13}$$

To see this we observe first that $\tilde{Y}_{\beta+t} \in \tilde{\mathfrak{F}}_{\beta+t}$ be left continuity of paths, hence $\mathcal{G}_t \subset \tilde{\mathfrak{F}}_{\beta+t}$ and so if $\Lambda \in \mathcal{G}_t$, then for each q, $\{q > \beta + t\} \cap \Lambda \in \tilde{\mathfrak{F}}_q$. Hence for each t,

$$\{T > t\} \cap \Lambda = \bigcup_{q \in Q} [\{\beta + T > q\} \cap \{q > \beta + t\} \cap \Lambda]$$

belongs to $\tilde{\mathfrak{F}}_{(\beta+T)-}$ since each member of the union does, by definition of the field. This proves (6.13) by definition of $\mathcal{G}_{T-}$. Substituting $\beta + T$ for T in (6.10) we obtain

$$E\{f \circ \tilde{X}(T+t) \mid \tilde{\mathfrak{F}}_{(\beta+T)-}\} = \hat{P}_t f \circ \tilde{X}(T);$$

together with (6.13) this implies the first assertion of the theorem. Now take T to be a constant $t_0 > 0$, and observe that as $\mathcal{G}_{t_0-} = \mathcal{G}_{t_0}$ by the left continuity of paths, the resulting equation then implies the second assertion of the theorem.

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