CONTRIBUTIONS TO HARMONIC ANALYSIS

In Memory of the School of Analysis of H. Hahn, E. Helly, J. Radon, at the University of Vienna

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1. The problem

Wiener's approximation theorem was the starting point of many developments in harmonic analysis. Carleman, in his proof of the theorem [1], introduced a new method which is of considerable generality and leads to the formulation and solution of new approximation problems. These problems are of the following type:

In the space[2] $L^1(G)$ of integrable functions on a locally compact abelian group G a closed linear subspace I is given which is invariant under "translations", i.e., which contains with a function $f_0(x)$ also all functions $f_0(ax)$, for arbitrary $a \in G$.

It is required to find, for a given function $f(x)$ in $L^1(G)$, the number

$$\inf_{f_0 \in I} \int |f(x) - f_0(x)|\, dx$$

which indicates how closely $f(x)$ may be approximated, in the metric of $L^1(G)$, by means of the functions belonging to I. Using geometrical language, this number is called the *distance* of $f(x)$ from the linear subspace I and denoted by dist $\{f, I\}$.

As is well known, this distance is the norm in the quotient-space $L^1(G)/I$ (cf. [3], Theorem 22.11.4; since $L^1(G)$ is a (commutative) Banach algebra, with convolution as multiplication, and I an ideal in $L^1(G)$, $L^1(G)/I$ is actually a quotient-algebra). The exact calculation of the distance makes it possible to determine explicitly the structure of $L^1(G)/I$.

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[1] The author wishes to acknowledge with thanks the opportunities for research accorded to him at the University of Reading where this paper was written during the tenure of a temporary lectureship in 1955–56.

[2] Notation and terminology are as usual; cf., e.g., [4]. In particular, dx denotes the Haar measure, and integration extends over the whole group G unless otherwise specified.

This was carried out before for two classes of invariant subspaces (cf. [4], Theorem 1.3 and [5]). In the present paper the distance is obtained in a more general case which includes the previous ones; the study of the corresponding quotient-space will be left for a future communication.

2. General method of solution

If I is an arbitrary closed linear subspace of $L^1(G)$, not necessarily invariant under translations, and $f(x)$ any given function in $L^1(G)$ then dist $\{f, I\}$ may be found as follows (cf. [1], Chap. III and [4], pp. 402–403):

If there is no bounded, measurable function $\varphi(x)$ satisfying the conditions

$$\int f_0(x)\overline{\varphi(x)}\,dx = 0 \quad \textit{for all} \quad f_0 \in I, \tag{1}$$

$$\int f(x)\overline{\varphi(x)}\,dx = 1, \tag{2}$$

then $f \in I$.

If there are bounded, measurable functions $\varphi(x)$ satisfying (1) *and* (2), *then*

$$\operatorname{dist}\{f, I\} = 1/\inf \|\varphi\|_\infty,$$

the greatest lower bound being taken for all such functions φ.

Condition (1) is expressed by saying that φ is *orthogonal to I.*

We assert that *if I is invariant then already the continuous, or even the uniformly continuous functions orthogonal to I suffice for calculating the distance*, i.e., we may replace "measurable" by "continuous" or even "uniformly continuous". This will be important for the applications.

To prove this assertion we show that if there is a bounded, measurable function $\psi(x)$ satisfying (1) and (2) then there is also a bounded, (uniformly) continuous function $\varphi(x)$ which satisfies them and which is such that $\|\varphi\|_\infty$ exceeds $\|\psi\|_\infty$ by as little as we please.

First, the fact that I is an invariant subspace implies that $\psi(y^{-1}x)$ is orthogonal to I for each (fixed) $y \in G$. It follows that for any $u(x) \in L^1(G)$ the function

$$\varphi_u(x) = \int u(y)\,\psi(y^{-1}x)\,dy$$

is orthogonal to I. Moreover, $\varphi_u(x)$ is uniformly continuous[1] and $\|\varphi_u\|_\infty \le \|\psi\|_\infty \cdot \|u\|_1$.

[1] Indeed, $\varphi_u(x) = \int u(xy)\,\psi(y^{-1})\,dy$ and $\int |u(zxy) - u(xy)|\,dy = \int |u(zy) - u(y)|\,dy < \varepsilon$ for $z \in U_\varepsilon$ (cf. [8], p. 41).

Secondly, we may write

$$\int f(x)\,\overline{\psi(y^{-1}x)}\,d\,x = 1+\varepsilon(y),$$

where $|\varepsilon(y)|<\varepsilon$ for $y \in U_\varepsilon$, for the left-hand side is a continuous (even uniformly continuous) function of y, by an argument similar to that used before. Now let $u_\varepsilon(x)$ be a real, non-negative function in $L^1(G)$ vanishing outside the neighbourhood U_ε and such that $\int u_\varepsilon(x)\,dx = 1$. Then

$$\int u_\varepsilon(y)\,d\,y \int f(x)\,\overline{\psi(y^{-1}x)}\,d\,x = 1+\eta,$$

where $|\eta|<\varepsilon$. Thus, letting

$$\varphi_\varepsilon(x) = \frac{1}{1+\bar{\eta}} \int u_\varepsilon(y)\,\psi(y^{-1}x)\,d\,y,$$

we have a uniformly continuous function φ_ε satisfying conditions (1) and (2); moreover,

$$\|\varphi_\varepsilon\|_\infty \le \frac{1}{1-\varepsilon}\|\psi\|_\infty,$$

which completes the proof.

Remark. Assertion and proof are valid for general locally compact groups if we replace "invariant" by "left invariant" and "uniformly continuous" by "left uniformly continuous" throughout.

3. A theorem on bounded, continuous functions and some applications

Let $\hat{G}$ be the dual group of the locally compact abelian group G. The closed subgroups of G and $\hat{G}$ are in one-to-one correspondence, in such a way that if $g \subset G$ and $\Gamma \subset \hat{G}$ are corresponding subgroups then the dual group of g is $\hat{G}/\Gamma$ and the dual of Γ is G/g (cf. [8], pp. 108–109).

For a bounded, measurable function $\varphi(x)$ on G, the *spectrum* is defined as the (closed) set of all elements of the dual group for which the Fourier transform of every function $f(x) \in L^1(G)$ satisfying

$$\int f(y\,x)\,\overline{\varphi(x)}\,d\,x = 0 \quad \text{for all} \quad y \in G$$

vanishes (this is equivalent to the usual definition, [2], pp. 128–130).

If $\varphi(x)$ is given, we may consider $\varphi(xs)$, for fixed $x \in G$, as a function of s on a (closed) subgroup $g \subset G$. The spectrum of $\varphi(x)$ is in $\hat{G}$, while $\varphi(xs)$, as a function of $s \in g$, has its spectrum in $\hat{G}/\Gamma$ where Γ is the subgroup of $\hat{G}$ corresponding to g. The relation between the two spectra, for *continuous* φ, is as follows:

THEOREM 1. *Let G be a locally compact abelian group, g a closed subgroup of G and Γ the subgroup of $\hat{G}$ corresponding to g. Let $\varphi(x)$ be a bounded, continuous function on G and $\Omega_\varphi \subset \hat{G}$ the spectrum of φ.*

Then $\varphi(xs)$, considered as a function of s on the subgroup g ($x \in G$ being fixed), has a spectrum which is contained in the closure of the image of Ω_φ resulting from the homomorphism $\hat{G} \to \hat{G}/\Gamma$.

This theorem is the basis of the paper.

Let λ' be an arbitrary element of $\hat{G}/\Gamma$ outside the closure of the image of Ω_φ; we have to show that λ' is not in the spectrum of $\varphi(xs)$.

Take a closed neighbourhood $\hat{U}'$ of λ', with the same property as λ', and a function $f(s) \in L^1(g)$ such that its Fourier transform

$$\hat{f}(\hat{x}') = \int_g f(s)\,\overline{(s, \hat{x}')}\,ds \qquad (\hat{x}' \in \hat{G}/\Gamma)$$

vanishes off $\hat{U}'$ and $\hat{f}(\lambda') \neq 0$; $\hat{x}'$ being the image of $\hat{x} \in \hat{G}$, $\hat{f}(\hat{x}')$ is also a periodic function $\hat{f}(\hat{x})$ on $\hat{G}$ (constant on each coset of Γ).

For arbitrary $h(x) \in L^1(G)$, the function

$$f_1(x) = \int_g f(s)\,h(s^{-1}x)\,ds \tag{3}$$

is in $L^1(G)$ and has the Fourier transform

$$\hat{f}_1(\hat{x}) = \hat{f}(\hat{x}) \cdot \hat{h}(\hat{x}) \qquad (\hat{x} \in \hat{G}).$$

Thus $\hat{f}_1(\hat{x})$ vanishes on an open set containing Ω_φ, namely the inverse image of the complement of $\hat{U}'$. Hence by a theorem of Godement [2], Théorème C, a corollary of Wiener's theorem,

$$\int f_1(yx)\,\overline{\varphi(x)}\,dx = 0 \quad (y \in G).$$

Letting $y = e$, replacing $f_1(x)$ by (3) and changing the order of integration twice, we get

$$\int h(x)\,[\int_g f(s)\,\overline{\varphi(xs)}\,ds]\,dx = 0$$

Since $h(x) \in L^1(G)$ is arbitrary, it follows that

$$\int_g f(s)\,\overline{\varphi(xs)}\,ds = 0 \tag{4}$$

almost everywhere on G. Now the left-hand side is a continuous function of x (since $\varphi(x)$ is a bounded function, uniformly continuous on any compact set) and hence (4) holds for all $x \in G$. Thus λ' is not in the spectrum of $\varphi(xs)$ which proves the theorem.[1]

The next two theorems are applications of Theorem 1.

THEOREM 2. *Let $\varphi(x)$ be a bounded, continuous function on G such that its spectrum is contained in a closed subgroup Γ of the dual group $\hat{G}$.*

Then $\varphi(x)$ is periodic with respect to the subgroup $g \in G$ corresponding to Γ.

According to Theorem 1 the spectrum of $\varphi(xs)$, as a function of $s \in g$, contains at most one element, namely the neutral element, of $\hat{G}/\Gamma$. It follows, for every (fixed) $x \in G$, that $\varphi(xs)$ is constant on g (cf. [4], p. 422) and for $s = e$ we have $\varphi(xs) = \varphi(x)$ which proves the theorem.[2]

THEOREM 3. *Let Γ be a closed subgroup of $\hat{G}$ and suppose that Λ' is a closed, denumerable subset of $\hat{G}/\Gamma$ consisting of independent elements. Let Ω be the inverse image of Λ' in $\hat{G}$, and Λ any representative system (mod. Γ) of Ω.*[3]

Then every bounded, uniformly continuous function $\varphi(x)$ with spectrum in Ω has the form

$$\varphi(x) = \sum_{\lambda \in \Lambda} \varphi_\lambda(x)\,(x, \lambda). \tag{5}$$

The "coefficients" $\varphi_\lambda(x)$ are uniformly continuous functions, periodic with respect to the subgroup $g \subset G$ corresponding to Γ, and

$$\sum_{\lambda \in \Lambda} |\varphi_\lambda(x)| \leq \|\varphi\|_\infty. \tag{6}$$

If, in particular, Γ is a discrete subgroup of $\hat{G}$, then every bounded, uniformly continuous function with spectrum in Ω is almost periodic.

By Theorem 1 the spectrum of $\varphi(xs)$, as a function of $s \in g$, is contained in Λ'. Hence it follows from the lemma in [5] that

$$\varphi(xs) = \sum_{\lambda' \in \Lambda'} a_{\lambda'}(x) \cdot (s, \lambda') \tag{7}$$

and from Kronecker's theorem (cf. loc. cit.) that

$$\sum_{\lambda' \in \Lambda'} |a_{\lambda'}(x)| = \sup_{s \in g} |\varphi(xs)| \leq \|\varphi\|_\infty. \tag{8}$$

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[1] The author is obliged to the referee who gave a proof both simpler and more general than the original one which needlessly restricted $\varphi(x)$ to be uniformly continuous. The proof above is but a slight modification of the proof of the referee.

[2] Theorem 2 was originally stated only for uniformly continuous functions. The uniformity of the continuity is not required, however, as pointed out by the referee (cf. the preceding footnote).

[3] I.e., a subset of Ω which contains exactly one equivalent element (mod. Γ) to every element of Ω. For the definition of independent elements of an abelian group, cf. [5].

The "coefficients" $a_{\lambda'}(x)$ are uniformly continuous functions of x: $a_{\lambda'}(x)$ is the mean value of $\varphi(xs)\cdot\overline{(s, \lambda')}$ over the subgroup g and thus

$$|a_{\lambda'}(x_1) - a_{\lambda'}(x_2)| \leq \sup_{s \in g} |\varphi(x_1 s) - \varphi(x_2 s)| \leq \varepsilon$$

for $x_1 \cdot x_2^{-1} \in U_\varepsilon$, by the uniform continuity of $\varphi(x)$.

Another property of $a_{\lambda'}(x)$ is obtained by substituting xt $(t \in g)$ for x in (7):

$$\varphi(xts) = \sum_{\lambda' \in \Lambda'} a_{\lambda'}(xt) \cdot (s, \lambda').$$

Moreover, replacing s by ts in (7) we have

$$\varphi(xts) = \sum_{\lambda' \in \Lambda'} a_{\lambda'}(x) \cdot (t, \lambda') \cdot (s, \lambda').$$

Since the coefficients of (s, λ') are uniquely determined, it follows that

$$a_{\lambda'}(xt) = a_{\lambda'}(x) \cdot (t, \lambda').$$

Now we use the representative system Λ mentioned in the statement of the theorem. There is a one-to-one correspondence between Λ and Λ', so we may write $a_\lambda(x)$ instead of $a_{\lambda'}(x)$, λ being the element of Λ corresponding to $\lambda' \in \Lambda'$. We define now functions $\varphi_\lambda(x)(\lambda \in \Lambda)$ by the relation

$$a_\lambda(x) = \varphi_\lambda(x) \cdot (x, \lambda).$$

These functions depend in an obvious way on the choice of the representative system Λ.

Each $\varphi_\lambda(x)$ is uniformly continuous, and periodic with respect to g, i.e., $\varphi_\lambda(xs) = \varphi_\lambda(x)$ $(s \in g)$. Assertion (5) of the theorem follows from (7) for $s = e$, while (8) implies assertion (6) or more precisely

$$\Sigma |\varphi_\lambda(x)| = \sup_{s \in g} |\varphi(xs)|,$$

where the summation extends over all $\lambda \in \Lambda$.

To prove the last part of the theorem we observe that if $\varphi(x)$ is a bounded, uniformly continuous function of x, then so is $\sup_{s \in g} |\varphi(xs)|$ which is periodic with respect to g and hence a (uniformly) continuous function on G/g. Thus the sum of the series $\Sigma|\varphi_\lambda(x)|$ is continuous. If now Γ is discrete then G/g is compact and it follows from Dini's theorem that $\Sigma|\varphi_\lambda(x)|$ converges uniformly on G/g, and thus on G itself. But then $\Sigma\varphi_\lambda(x)\cdot(x, \lambda)$ converges uniformly on G; hence its sum is almost periodic.

Remark 1. Theorem 3 holds in a somewhat more general form: the set Λ' may contain, besides independent elements, also the neutral element of $\hat{G}/\Gamma$, and need only be reducible instead of denumerable. This is due to the fact that the lemma in [5] holds in a corresponding more general form as may readily be shown. On each compact subset of G/g at most countably many "coefficients" $\varphi_\lambda(x)$ can assume values different from zero.

Remark 2. If in Theorem 3 Λ' is assumed to be discrete, then every bounded, continuous function with spectrum in Ω is of the form (5), the coefficients $\varphi_\lambda(x)$ being then also continuous. For, if $f(s) \in L^1(g)$ is such that $\hat{f}(\hat{x}')$ vanishes outside a neighbourhood of $\lambda' \in \Lambda'$ which contains no other point of Λ', then by (7)

$$\int_g f(s)\,\varphi(s^{-1}x)\,ds = \hat{f}(\lambda') \cdot a_{\lambda'}(x).$$

Now the left-hand side is a continuous function of x (cf. p. 256); as we may suppose $\hat{f}(\lambda') \neq 0$, the assertion follows.

4. On certain invariant subspaces

The *cospectrum* of an invariant subspace $I \subset L^1(G)$ is defined as the (closed) set of all those elements of the dual group $\hat{G}$ for which the Fourier transforms of all functions in I are zero [7]. If a bounded, measurable function is orthogonal to I its spectrum is contained in the cospectrum of I.

The preceding theorems yield results about some classes of invariant subspaces.

THEOREM 4. *Let I be a closed, invariant subspace of $L^1(G)$ such that the cospectrum of I is a subgroup Γ of $\hat{G}$.*

Then I consists of all functions in $L^1(G)$ the Fourier transforms of which vanish on Γ.

As shown in §2, a function $f \in L^1(G)$ will belong to I if there is no bounded, continuous function $\varphi(x)$ orthogonal to I which satisfies (2).

Now Theorem 2 is applicable to φ; it follows that $\varphi(x)$ is a function on G/g. Hence (2) may be written[1]

$$\int_{G/g} dx'\,\overline{\varphi(x')} \int_g f(xs)\,ds = 1.$$

Thus
$$\|\varphi\|_\infty \cdot \int_{G/g} \left| \int_g f(xs)\,ds \right| dx' \geq 1.$$

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[1] We denote the Haar measure on G/g by dx'; the measures are assumed to be so normalized that

$$\int_{G/g} dx' \int_g h(xs)\,ds = \int h(x)\,dx \qquad (h \in L^1(G)).$$

It follows at once that there can be no bounded, continuous function $\varphi(x)$ satisfying conditions (1) and (2) if

$$\int_{G/g} \left| \int_g f(xs)\, ds \right| dx' = 0,$$

i.e., if the function $$f'(x') = \int_g f(xs)\, ds$$

vanishes almost everywhere on G/g. This will happen if (and only if) the Fourier transform of $f'(x')$ vanishes identically on Γ, the dual group of G/g. This Fourier transform is just the restriction of that of $f(x)$ to the subgroup $\Gamma \subset \hat{G}$. Hence, if the Fourier transform of $f(x)$ vanishes on Γ then $f(x)$ belongs to I (and, of course, conversely).

The distance of an arbitrary function $f(x) \in L^1(G)$ from I is precisely

$$\|f'\|_1 = \int_{G/g} \left| \int_g f(xs)\, ds \right| dx'$$

as may be shown by the method of §2. This result was proved already in [4], Theorem 1.3, with the assumption that I is given, in advance, as the subspace of *all* functions in $L^1(G)$ the Fourier transform of which vanishes on the subgroup $\Gamma \subset \hat{G}$.

Remark 1. Γ, being a closed subgroup, is either a discrete or a perfect subset of $\hat{G}$. In the first case the statement of Theorem 4 is included in a known, and more general, theorem (cf. Theorem 2.2 in [4] and the references there given), but the second case is of quite another nature. Consider the group $R^p = \hat{R}^p$ of translations of p-dimensional euclidean space, for $p \geq 2$. Here the straight lines, planes, and generally the d-dimensional closed linear manifolds $(1 \leq d < p)$ are perfect subsets corresponding to proper subgroups. In this case which belongs to classical analysis, Theorem 4 should be compared with a well-known example, due to L. Schwartz [7], which shows that in $L^1(R^p)$ there exist different closed invariant subspaces having *the same* cospectrum, the surface of a sphere in p-dimensional space, for $p \geq 3$.

Remark 2. Theorem 4 should be compared with a result proved in [6] concerning general locally compact groups. The main problem there is to show that two left invariant subspaces of $L^1(G)$, defined in quite different ways, are actually identical. Now if G is abelian these two subspaces have the same cospectrum which is a subgroup of $\hat{G}$. This is an analogy to Theorem 4, but in the general case treated in [6] no Fourier transforms are available.[1]

[1] A difficulty in [6] should be pointed out: by the lemma on p. 74, loc. cit., the relation $\varphi(x\sigma^{-1}) = \varphi(x)$ holds for every (fixed) $\sigma \in g$ almost everywhere on G. But here the exceptional set of measure zero may depend on σ! The assertion that $\varphi(x)$ is constant on the left cosets of g lacks, therefore, sufficient foundation—it would have to be shown that the exceptional null set may be chosen independently of

THEOREM 5. *Let Γ be a closed subgroup of $\hat{G}$ and suppose that Λ' is a closed, denumerable subset of $\hat{G}/\Gamma$ consisting of independent elements. Let Ω be the inverse image of Λ' in $\hat{G}$, and Λ any representative system (mod. Γ) of Ω, as in Theorem 3. Let I be a closed, invariant subspace of $L^1(G)$ with cospectrum Ω.*

Then I consists of all functions in $L^1(G)$ the Fourier transforms of which vanish on Ω. For arbitrary $f(x) \in L^1(G)$

$$\text{dist}\{f, I\} = \int_{G/g} \sup_{\lambda \in \Lambda} \left| \int_g f(xs)\,\overline{(xs, \lambda)}\,ds \right| dx',$$

where g denotes the subgroup of G corresponding to Γ.

Given $f \in L^1(G)$, we may calculate dist $\{f, I\}$ by means of the uniformly continuous functions $\varphi(x)$ orthogonal to I (§ 2). The spectrum of such a function is contained in Ω so that Theorem 3 is applicable. Thus (2) may be written

$$\int f(x)\,\overline{\{\sum_{\lambda \in \Lambda} \varphi_\lambda(x) \cdot (x, \lambda)\}}\,dx = 1$$

or, since the functions φ_λ are periodic,

$$\int_{G/g} \sum_{\lambda \in \Lambda} \overline{\varphi_\lambda(x')} \{ \int_g f(xs)\,\overline{(xs, \lambda)}\,ds\}\,dx' = 1.$$

Hence by (6)

$$\|\varphi\|_\infty \cdot \int_{G/g} \sup_{\lambda \in \Lambda} \left| \int_g f(xs\,\overline{(xs, \lambda)}\,ds \right| dx' \geq 1. \tag{9}$$

[The function $\sup_{\lambda \in \Lambda} \left| \int_g f(xs)\,\overline{(xs, \lambda)}\,ds \right|$ is in $L^1(G/g)$; in fact, for each λ

$$\left| \int_g f(xs)\,\overline{(xs, \lambda)}\,ds \right| \leq \int_g |f(xs)|\,ds$$

which belongs to $L^1(G/g)$.]

Relation (9) immediately implies that there is no bounded, uniformly continuous function $\varphi(x)$ satisfying (1) and (2) if the integral

$$d \equiv \int_{G/g} \sup_{\lambda \in \Lambda} \left| \int_g f(xs)\,\overline{(xs, \lambda)}\,ds \right| dx'$$

vanishes.[1] This will be the case if (and only if) each of the functions

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σ. Now this difficulty may be entirely avoided by using only continuous functions $\varphi(x)$, in accordance with § 2 above. It was mainly difficulties with null sets, especially in [6] which led to the considerations in § 2.

[1] It should be observed that this integral is independent of the particular representative system Λ used because $\left| \int_g f(xs)\,\overline{(xs, \lambda)}\,ds \right|$ depends only on the coset of Γ to which λ belongs.

$$f'_\lambda(x') = \int_g f(xs)\,\overline{(xs, \lambda)}\,ds \qquad (\lambda \in \Lambda),$$

defined on G/g, vanishes almost everywhere on G/g or, equivalently, if the Fourier transform of each $f'_\lambda(x')$ vanishes identically. The Fourier transform of $f'_\lambda(x')$ coincides with the restriction of the Fourier transform of $f(x)$ to the set $\lambda \cdot \Gamma \subset \Omega$. Hence I contains all functions in $L^1(G)$ the Fourier transforms of which vanish on Ω (and of course no others).

To show that for arbitrary $f \in L^1(G)$ dist $\{f, I\}$ is given by the integral d above, we observe that for $d = 0$ this has just been proved; moreover, by the preceding argument, $f \notin I$ if $d > 0$. Let then $\varphi(x)$ be a bounded, uniformly continuous function satisfying (1) and (2). Then, by (9), inf $\|\varphi\|_\infty \geq 1/d$ and it follows (§ 2) that dist $\{f, I\} \leq d$.

The opposite inequality is easy to establish. If $f(x)$ and $f_0(x)$ are in $L^1(G)$ we have

$$\int |f(x) - f_0(x)|\,dx = \int_{G/g} dx' \int_g |f(xs) - f_0(xs)|\,ds$$

and

$$\int_g |f(xs) - f_0(xs)|\,ds \geq \sup_{\lambda \in \Lambda} \left| \int_g f(xs)\,\overline{(xs, \lambda)}\,ds - \int_g f_0(xs)\,\overline{(xs, \lambda)}\,ds \right|.$$

If now $f_0(x)$ belongs to I, then $\int_g f_0(xs)\,\overline{(xs, \lambda)}\,ds$ vanishes almost everywhere on G/g, for each $\lambda \in \Lambda$. Thus for $f_0 \in I$ and $f \in L^1(G)$

$$\int |f(x) - f_0(x)|\,dx \geq \int_{G/g} \sup_{\lambda \in \Lambda} \left| \int_g f(xs)\,\overline{(xs, \lambda)}\,ds \right| dx',$$

i.e., dist $\{f, I\} \geq d$, which completes the proof.

Theorem 5 may be generalized in the same way as Theorem 3 with regard to the condition concerning the set Λ' (cf. Remark 1 to Theorem 3, p. 258).[1]

In this somewhat generalized form Theorem 5 contains Theorem 4 which corresponds to the case where Λ' contains only the neutral element of $\hat{G}/\Gamma$; moreover, if Γ reduces to the neutral element of $\hat{G}$ one has the result of [5]. Thus Theorem 5 represents the most general case in which the approximation problem, for invariant subspaces I with non-empty cospectrum, has been explicitly solved. The formula for the distance which was obtained has significance in connexion with the quotient-algebra $L^1(G)/I$; this will be studied later.

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[1] If Λ' is not denumerable the formula for the distance is

$$\text{dist}\,\{f, I\} = \sup_{\Delta \subset \Lambda} \int_{G/g} \sup_{\lambda \in \Delta} \left| \int_g f(xs)\,\overline{(xs, \lambda)}\,ds \right| dx',$$

Δ denoting denumerable subsets of Λ.

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