OSCILLATION AND DISCONJUGACY FOR LINEAR DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC COEFFICIENTS[1]

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1. Introduction and résumé of results

Equations of the form

$$(r(x)y')' + K(x)y = 0, \tag{1}$$

where $r(x) > 0$ and $K(x)$ are real continuous functions on $-\infty < x < \infty$, are classified, by the behavior of their real solutions, as $(+)$-oscillatory or non-oscillatory. In the first instance one non-trivial (not identically zero), and thereby every, solution vanishes at arbitrarily large abscissas; in the second instance every non-trivial solution is non-vanishing for sufficiently large abscissas. A special instance of non-oscillation is the disconjugate case in which every (non-trivial) solution has at most one zero on $-\infty < x < \infty$. It is known that an equation of the form (1) is disconjugate if and only if there is a solution which is everywhere positive.

Our principal interest concerns the situation where $r(x) \equiv 1$ and $K(x) = -a + bp(x)$. Here (a,b) are real parameters and $p(x)$ is a real almost periodic function. We shall note, in this case, that non-oscillation and disconjugacy are coincident. Also we shall find that the domain D in the (a,b)-parameter plane, for which the corresponding equations are disconjugate, is closed and convex.

We generalize the theory of Hill's equation (in which $p(x)$ is periodic) but, of course, without using the Floquet representation, which is not applicable here. For example, interior to the disconjugacy domain D there is a basis of solutions each of which has an almost periodic logarithmic derivative. For the boundary of D the analysis is more com-

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plicate since here we can display an example of an appropriate equation having no (non-trivial) bounded solution—from the analogy with Hill's equation one might expect an almost periodic solution in such a case.

Finally we investigate the effect on D of perturbations in the function $p(x)$.

2. The disconjugacy domain for equations with almost periodic coefficients

DEFINITION. *The disconjugacy domain D of*

$$(r(x)y')' + (-a + bp(x))y = 0, \tag{2}$$

$r(x) > 0$ and $p(x)$ being real continuous functions on $-\infty < x < \infty$, is the subset of the real (a, b)-plane wherein the corresponding equations are disconjugate.

We shall picture the b-axis horizontally and the a-axis vertically. Of course, D depends on the particular functions $r(x) > 0$ and $p(x)$, but D always contains the half-axis $a \geq 0$, $b = 0$. The set of points (a, b) for which the corresponding equations (2) are oscillatory is the oscillation domain O and, when $r(x) \equiv 1$, this always contains the half-axis $a < 0$, $b = 0$.

We shall be interested in the disconjugacy domain D, particularly for $r(x) \equiv 1$, and for various functions $p(x)$. Below are several examples indicating some of the possibilities for the form of D.

Example 1. $\lim_{x\to\infty} p(x) = +\infty$, $\lim_{x\to-\infty} p(x) = -\infty$, e.g. $p(x) = x$. Here D is the half-axis $a \geq 0$, $b = 0$.

Example 2. $p(x) \equiv 0$. Here D is the half-plane $a \geq 0$.

Example 3. $p(x)$ unbounded above (below). Here D contains no ray in $b < 0$ (in $b > 0$). However D can contain an interior, e.g., for $p(x) = x \sin(x^2)$, D contains the parabolic region $a \geq b^2/4$.

Example 4. $\sup_{-\infty<x<\infty} p(x) = M > 0$, $\inf_{-\infty<x<\infty} p(x) = m < 0$. Here the largest sector contained in D is bounded by the rays $a = Mb$, $a \geq 0$ and $a = mb$, $a \geq 0$. If, say, $M > 0$ and $m > 0$, then the largest sector contained in D is bounded by the rays $a = Mb \geq 0$ and $a = mb \leq 0$.

Example 5. $p(x) = \sin \log_+ |x|$. Here D is exactly the sector in which $|b| \leq a \geq 0$.

In studying the linear equation (1) one often utilizes the associated Riccati equation

$$u' + u^2/r(x) + K(x) = 0. \tag{3}$$

For a solution $y(x)$ of (1), non-vanishing on some interval I, $u(x) = y'(x)r(x)/y(x)$ is a solution of (3) on I. Moreover, every solution of (3) can be so obtained. We shall be primarily interested in the case where $r(x) \equiv 1$ and $|K(x)| < M^2$, for some real bound M^2. Then a solution $u(x)$ of the Riccati equation can be defined for all x if and only if $|u(x)| < M$ everywhere.

Also, in case $r(x) \equiv 1$ and $|K(x)| < M^2$, the bounded solutions of the Riccati equation fill a closed band which is either empty, a single solution curve, or a homeomorph of $0 \leq y \leq 1$, $-\infty < x < \infty$ in the plane. This band is non-empty if and only if the corresponding linear equation (1) is disconjugate. The band is closed, that is the extremal upper and lower solutions are easily seen to be bounded since otherwise each nearby solution $u(x)$ would somewhere satisfy $|u(x_0)| > M$. But then $|u(x)|$ would grow more rapidly to infinity then an unbounded solution of $u' = -u^2 + M^2$ and so $u(x)$ would not exist for all real x.

LEMMA 1. *Let $r(x) > 0$ and $K(x)$ be real almost periodic functions. If the equation*

$$(r(x)y')' + K(x)y = 0 \tag{1}$$

is not disconjugate, then it is oscillatory at both $\pm\infty$.

Proof. Suppose a non-trivial solution $y(x)$ of (1) vanishes at two distinct points $x = \alpha$ and $x = \beta$. For each $\varepsilon > 0$ there are arbitrarily large ε -almost periods, say τ_n, of $r(x)$ and $K(x)$. Consider the translated equations

$$(r(x + \tau_n)y')' + K(x + \tau_n)y = 0 \tag{1_n}$$

with solutions $y_n(x)$ which assume the same initial values at $x = \alpha$ as does $y(x)$. Also there is a solution $Y_n(x)$ of (1) for which $Y_n(x + \tau_n) = y_n(x)$.

For each $\xi > 0$ there exists an $\varepsilon > 0$ such that $y_n(x)$ vanishes on $\beta - \xi < x < \beta + \xi$. Then $Y_n(x)$ vanishes at $x = \alpha + \tau_n$ and also near $\beta + \tau_n$. Hence every solution of (1) must vanish on $\alpha + \tau_n \leq x \leq \beta + \tau_n + \xi$. Since the translation numbers τ_n are arbitrarily large (or small), (1) is oscillatory. Q.E.D.

LEMMA 2. *Let the real continuous functions $K_n(x)$, $r_n(x) > 0$, $n = 1, 2, \ldots$, on $-\infty < x < \infty$, converge uniformly on each compact interval to $K(x)$ and $r(x) > 0$, respectively. If each equation*

$$(r_n(x)y')' + K_n(x)y = 0 \tag{1_n}$$

is disconjugate, then so is

$$(r(x)y')' + K(x)y = 0. \tag{1}$$

Proof. Suppose that a (non-trivial) solution $y(x)$ of (1) has two distinct zeros, x_0 and x_1. Consider the solutions $y_n(x)$ of (1_n) with initial data $y_n(x_0) = 0$, $y_n'(x_0) = y'(x_0)$. Let L be a compact interval containing x_0 and x_1 in its interior. For sufficiently large n, $|r_n(x) - r(x)|$, $|K_n(x) - K(x)|$, and $|y_n(x) - y(x)|$ are smaller than any prescribed

$\varepsilon > 0$ for $x \in L$. Therefore $y_n(x)$ vanishes near x_1. But this contradicts the hypothesis that (1_n) is disconjugate. Thus (1) is necessarily disconjugate. Q.E.D.

LEMMA 3. *Let $r_i(x) > 0$, $K_i(x)$ be real continuous functions on $-\infty < x < \infty$ and such that*

$$(r_i(x)y')' + K_i(x)y = 0 \qquad (i = 1, 2) \tag{1_i}$$

are non-oscillatory at $x = +\infty$. Then for each t on $0 \le t \le 1$ the equation

$$[(tr_1(x) + (1-t)r_2(x))y']' + [tK_1(x) + (1-t)K_2(x)]y = 0 \tag{1_t}$$

is also non-oscillatory at $x = +\infty$.

Proof. Let $u_i(x) = r_i(x)\, y_i'(x)/y_i(x)$, where $y_i(x)$ is a solution of (1_i), positive for $x > x_0$. Then

$$u_i' + u_i^2/r_i(x) + K_i(x) = 0 \qquad (i = 1, 2) \tag{3_i}$$

for $x > x_0$. Consider $u_t(x) = tu_1(x) + (1-t)\, u_2(x)$. Then we compute

$$u_t' + u_t^2/[tr_1 + (1-t)r_2] + [tK_1 + (1-t)K_2] = \frac{-t(1-t)(r_2u_1 - r_1u_2)^2}{r_1r_2[tr_1 + (1-t)r_2]}.$$

Thus there is a solution, continuous for $x > x_0$, of

$$u' + u^2/[tr_1 + (1-t)r_2] + [tK_1 + (1-t)K_2] = 0.$$

Therefore (1_t) is non-oscillatory at $x = +\infty$. Q.E.D.

One is often interested in the disconjugacy domain of the translates of equation (1) or even of limit translates. If $K(x)$ is almost periodic and x_n are real numbers such that $\lim_{n\to\infty} K(x + x_n) = K^*(x)$ uniformly on $-\infty < x < \infty$, then one states that $K^*(x)$ is in the hull $H\{K(x)\}$ generated by $K(x)$. It is known that $K^*(x)$ is almost periodic and $K(x) \in H\{K^*(x)\}$, cf. [5, p. 73].

THEOREM 1. *Let $r(x) > 0$ and $p(x)$ be real almost periodic functions. Then the disconjugacy domain D of*

$$(r(x)y')' + (-a + bp(x))y = 0 \tag{2}$$

is a closed convex subset of the (a,b)-plane. Furthermore, if for real translates x_n, $n = 1, 2, \ldots$, $\lim_{n\to\infty} r(x + x_n) = r^*(x) > 0$ and $\lim_{n\to\infty} p(x + x_n) = p^*(x)$ uniformly on $-\infty < x < \infty$, then $D^* = D$, where D^* is the disconjugacy domain of

$$(r^*(x)y')' + (-a + bp^*(x))y = 0. \tag{2^*}$$

Proof. By the above three lemmas, D is closed, convex and its complement is the oscillation domain O. To show $D^* = D$ we need only show that $D \subset D^*$ and then the conclusion follows from symmetry.

For a fixed $(a,b) \in D$ each solution of (2) and of

$$(r(x + x_n)y')' + (-a + bp(x + x_n))y = 0 \tag{2_n}$$

is disconjugate. Suppose there is a non-trivial oscillatory solution $y^*(x)$ of (2^*). Then, for sufficiently large integers n, the solution of (2_n) having the same initial values as $y^*(x)$ must have more than one zero and thus be oscillatory. But this contradicts the disconjugacy of (2_n). Therefore $D \subset D^*$. Q.E.D.

COROLLARY. *The oscillation domain* O *of* (2) *is open, connected and its complement in the* (a,b)*-plane is* D.

Proof. Since D and O are complementary, O is open. If $(a_0, b_0) \in O$, then, by the Sturm-Picone comparison theorem, so is $(a_0 - \xi, b_0) \in O$ for each $\xi > 0$. Moreover, O contains the sector $a < -|b| \sup\limits_{-\infty < x < \infty} p(x)$. Therefore O is connected. Q.E.D.

We next proceed to a detailed study of the form of D. To simplify the analysis we treat only the case $r(x) \equiv 1$ and we often make the convention that $p(x)$ has mean zero.

For the equation (1) Leighton [9] gives the following criterion for oscillation:

$$\int_{x_0}^{\infty} dx/r(x) = \infty \quad \text{and} \quad \int_{x_0}^{\infty} K(x)\,dx = \infty.$$

From this, if $r(x) \equiv 1$ and $p(x)$ is real almost periodic with zero mean, it follows that D lies in the half-plane $a \geq 0$, for the linear equation

$$y'' + (-a + bp(x))y = 0. \tag{L}$$

By a refinement of Leighton's test one can show that D, excepting the origin, lies in $a > 0$.

THEOREM 2. *Let* $p(x) \not\equiv 0$ *be a real almost periodic function of mean zero. Then for*

$$y'' + (-a + bp(x))y = 0, \tag{L}$$

D, *excepting the origin, lies in* $a > 0$

Proof. We need only show that for $a = 0$, $b \neq 0$ the corresponding equation (L) is oscillatory. For the first case assume $\int_0^x p(s)\,ds$ is unbounded. Then one can replace $p(x)$ by a function in $H\{p(x)\}$, which we shall still denote as $p(x)$, for which either $\limsup\limits_{x \to \infty} b \int_0^x p(s)\,ds = \infty$ or $\limsup\limits_{x \to -\infty} b \int_0^x p(s)\,ds = \infty$, [4, p. 48]. Then from a known result [11, p. 138], (L) is oscillatory.

For the second case assume that $\int_0^x p(s)\,ds$ is almost periodic. Let $v(x)$ be the almost periodic function with mean zero and such that $v'(x) = p(x)$. Then there exists a number x_0 for which $v(x_0) = 0$ and $v(x) = \int_{x_0}^{x} p(s)ds$. Suppose that the Riccati equation

$$u' + u^2 + bp(x) = 0, \qquad b \neq 0 \tag{R}$$

has a bounded solution. Now if for every $p^*(x) \in H\{p(x)\}$, the corresponding Riccati equation (R^*) had a unique bounded solution, then it is easy to show (cf. Theorem 9 and remark after Theorem 16) that this solution $u(x)$ is almost periodic. But this contradicts the fact that the mean of $u(x)^2$ is clearly zero. Thus we replace $p(x)$ by some function in $Hp\{(x)\}$, still called $p(x)$, for which there are two bounded solutions, $u_1(x)$ and $u_2(x)$.

Now
$$u_i(x) = u_i(x_0) - \int_{x_0}^{x} u_i(s)^2 ds - b\int_{x_0}^{x} p(s)ds.$$

Let
$$\lim_{x\to+\infty} [u_i(x_0) - \int_{x_0}^{x} u_i(s)^2 ds] = \omega_i$$

and
$$\lim_{x\to-\infty} [u_i(x_0) - \int_{x_0}^{x} u_i(s)^2 ds] = \alpha_i, \qquad i = 1, 2.$$

It can be shown, cf. Theorem 14, that $\liminf_{x\to\pm\infty} |u_1(x) - u_2(x)| = 0$ and so $\alpha_1 = \alpha_2 = \alpha$, $\omega_1 = \omega_2 = \omega$. Either $\alpha > 0$ or $\omega < 0$. Say $\omega < 0$ and the other case is similar. Then $u_i(x) < \omega/2 - b\int_{x_0}^{x} p(s)ds = z(x)$ for large $x > k$. But $z(x)$ is an almost periodic function with negative mean and so $0 < \exp\int_{k}^{x} u_i(s)ds < \exp\int_{k}^{x} z(s)ds < K$, for some bound K, when $x > k$. Therefore the linear equation (L) has a basis of bounded solutions on a half-axis and thus (L) is oscillatory. But this contradicts the assumption that (R) has a bounded solution and so (L) is oscillatory. Q.E.D.

THEOREM 3. *The disconjugacy domain D of*

$$y'' + (-a + bp(x))y = 0, \tag{L}$$

where $p(x) \not\equiv 0$ is real almost periodic with mean zero, contains the sector bounded by the rays $a = b \sup_{-\infty<x<\infty} p(x) > 0$ and $a = b \inf_{-\infty<x<\infty} p(x) > 0$. This is the largest sector belonging to D in that no other rays are in D. The boundary of D is a continuous curve $a(b)$ which is strictly monotone decreasing on $b \leq 0$ and increasing on $b \geq 0$.

Proof. Consider a ray $a = kb > 0$. Then for such parameter values (L) becomes

$$y'' + b(-k + p(x))y = 0. \tag{L_k}$$

Thus in $b > 0$ the rays for which $-k + \sup_{-\infty < x < \infty} p(x) \leq 0$ yield disconjugate equations. In $b < 0$, the rays which lie in D correspond to $-k + \inf_{-\infty < x < \infty} p(x) \geq 0$. For any other ray, in $a > 0$, $b(-k + p(x))$ becomes positive for x on some interval. Thus for large $|b|$, the corresponding equation is oscillatory and such rays do not belong to D.

That the boundary of D is a monotone continuous curve $a(b)$ follows from the fact that D is convex and contains a sector. Q.E.D.

COROLLARY. *$a(b) \sim Mb$ for $b \to +\infty_2$; $a(b) \sim mb$ for $b \to -\infty$.*

If $p(x)$ assumes its supremum M or its infimum m at some point, then one can further estimate $a(b)$.

THEOREM 4. *Let $p(x) \not\equiv 0$ be a real almost periodic function of mean zero and $p(x) \in C^{(2)}$. If $p(x_0) = M = \sup_{-\infty < x < \infty} p(x)$ (or if $p(x_0) = m = \inf_{-\infty < x < \infty} p(x)$) then the domain D for the equation (L) has a boundary*

$$a(b) = Mb - O(b^{\frac{1}{2}}) \quad \text{for} \quad b > 0$$

(or

$$a(b) = mb - O(b^{\frac{1}{2}}) \quad \text{for} \quad b < 0).$$

Proof. Let $p(x_0) = M$. Then for each $h < p''(x_0)$ there is an $\varepsilon > 0$ such that $p(x) \geq M + \frac{1}{2}h(x - x_0)^2$ on $|x - x_0| < \varepsilon$. Take ε small and define k by $\frac{1}{4}(M - k)/h = -\varepsilon^2$. Consider the rays $a = kb > 0$ for k just less than M. Along such rays we have

$$y'' + b(-k + p(x))y = 0, \quad b > 0. \tag{L_k}$$

For these equations, whenever $|x - x_0| \leq \varepsilon$

$$b(-k + p(x)) \geq b\left(-k + M + \frac{h}{2}(x - x_0)^2\right) \geq \frac{7b}{8}(M - k).$$

Now if $(7b/8)(M - k) \geq -h\pi^2/(M - k)$ on an interval of length $[-(M - k)/h]^{\frac{1}{2}}$, our equation (L) is oscillatory. Thus the curve

$$\frac{7b}{8}(M - k) = -h\pi^2/(M - k), \quad a = kb$$

lies in the oscillatory region of (L). Thus $a(b)$ lies above $Mb - 2\pi\sqrt{-2hb/7}$ for sufficiently large $b > 0$. A similar calculation holds for $b < 0$. Q.E.D.

If one assumes that $p(x)$ has a higher order of "flatness" at its maximum then still sharper asymptotic estimates for $a(b)$ are possible. We indicate the results only in the extreme case where $p(x) = M$ (or $p(x) = m$) on an interval.

THEOREM 5. *Let* $p(x) \not\equiv 0$ *be real almost periodic of mean zero. Assume* $p(x) = M = \sup_{-\infty<x<\infty} p(x)$ *(or* $p(x) = m = \inf_{-\infty<x<\infty} p(x)$*) for* x *on an interval. Then, for* (L), $a(b) = Mb - O(1)$ *for* $b > 0$ *(or* $a(b) = mb - O(1)$ *for* $b < 0$*). Thus the boundary curve* $a(b)$ *is asymptotic to a line of slope* M *(or slope* m*).*

Proof. Assume $p(x) = M$ on $|x - x_0| \leq \varepsilon$. On the ray $a = kb$, $b > 0$, we have

$$y'' + b(-k + p(x))y = 0. \tag{L_k}$$

But $b(-k + p(x)) = b(-k + M)$ on $|x - x_0| \leq \varepsilon$. If $b(-k + M) \geq \pi^2/4\varepsilon^2$, (L) is oscillatory. Thus for $a = Mb - \pi^2/4\varepsilon^2$, large $b > 0$, we have oscillation. Because of its convexity, $a(b)$ is asymptotic to a line of slope M between $a = Mb$ and $a = Mb - \pi^2/4\varepsilon^2$. The case for $p(x) = m$ is similar. Q.E.D.

We now show that, in most important cases, the boundary of D is tangent to the b-axis at the origin. Thereby D contains the maximal sector properly in its interior.

THEOREM 6. *Let* $p(x)$ *be real almost periodic and* $\int_0^x p(s)\,ds$ *also almost periodic. Then the domain* D *of equation* (L) *has a boundary* $a(b)$ *which is tangent to the* b*-axis at the origin.*

Proof. Let $y = \exp(\sqrt{a}\,x)z$, where $y(x)$ is a solution of

$$y'' + (-a + bp(x))y = 0, \qquad a > 0. \tag{L}$$

Then one computes

$$(\exp(2\sqrt{a}\,x)\,z')' + (\exp(2\sqrt{a}\,x)\,bp(x))z = 0.$$

Now a non-oscillation test of Moore [12] states that equation (1) is non-oscillatory on $0 < x < \infty$ in case $w(x) = \int_0^x K(t) \int_t^\infty \frac{ds}{r(s)}\,dt$ has an oscillation ≤ 1 on $0 \leq x < \infty$. For our above equation

$$w(x) = \frac{b}{2\sqrt{a}} \int_0^x p(t)\,dt.$$

Let $w =$ oscillation $\int_0^x p(t)\,dt$. Then the parabola $(b/2\sqrt{a})w = 1$ or $a = (w^2/4)b^2$ lies in D. Therefore $a'(b)$ exists at $b = 0$ and there equals zero. Q.E.D.

COROLLARY. *In case* $\int_0^x p(s)$ *is almost periodic, the domain* D *of* (L) *contains the maximal sector (except for the origin) in its interior. Also on the boundary curve* $a(b)$ *of* D, $(-a + bp(x))$ *is somewhere positive.*

Proof. Since $a'(0) = 0$, the convex domain D contains the rays $a = mb$, $a = Mb$, for $b > 0$, in its interior. Thus D contains the rays $a = Mb - \varepsilon$, $a = mb - \varepsilon$ for some $\varepsilon > 0$ when $|b|$ is large. Since $(-a + bp(x))$ is arbitrarily near to zero at points of the extremal rays, $(-a + bp(x))$ becomes positive in D. Q.E.D.

Finally we show that in many important cases D is symmetric in the a-axis.

THEOREM 7. *Let $p(x)$ be real almost periodic with Fourier frequencies $\{\lambda_n\}$. If the $\{\lambda_n\}$ are rationally independent, then for the boundary curve $a(b)$ of D for the equation (L), we have the symmetry*

$$a(b) = a(-b).$$

Proof. Let $p(x)$ have Fourier series $\sum_{n=-\infty}^{\infty} \alpha_n e^{i\lambda_n x}$ where $\alpha_n = \bar{\alpha}_{-n}$. Within the hull $H\{p(x)\}$, convergence of the Fourier coefficients of translates $p(x + h_m)$ implies uniform convergence on $-\infty < x < \infty$. Now one can select a translate h_m so that $|\lambda_n h_n - \pi| < (\frac{1}{2})^m$ (mod 2π) for $n = 1, 2, \ldots, m$. This is possible since the frequencies are linearly independent over the rationals. But then we can define a sequence of translates $p(x + h_m)$ whose Fourier coefficients converge to those of $-p(x)$. Thus $-p(x) \in H\{p(x)\}$ and D is symmetric in the a-axis. Q.E.D.

3. Interior of the disconjugacy domain

We shall now consider the differential equations

$$y'' + (-a + bp(x))y = 0 \tag{L}$$

for (a,b) in the interior of the disconjugacy domain D. Then the associated Riccati equation

$$u' + u^2 + (-a + bp(x)) = 0 \tag{R}$$

has solutions which are defined and bounded on $-\infty < x < \infty$.

THEOREM 8. *Let*

$$u' + u^2 + K(x) + \varepsilon = 0 \tag{R_ε}$$

have bounded solutions on $-\infty < x < \infty$ for all small ε, $e > \varepsilon \geq 0$, where $K(x)$ is a real continuous bounded function on $-\infty < x < \infty$, $|K(x)| < M^2$. Then for each such ε there are infinitely many bounded solutions of R_ε and these fill a closed bounded domain B_ε in the (x,u)-plane. Also B_{ε_2} lies interior to B_{ε_1} if $\varepsilon_1 < \varepsilon_2$. The upper and lower bounded solutions $u_u(x)$ and $u_L(x)$, respectively, of R_0 are separated, that is,

$$\inf_{-\infty < x < \infty} |u_u(x) - u_L(x)| > \frac{e^2}{M(32M^2 + 8e)} > 0.$$

Furthermore for any two bounded solutions $u_1(x)$, $u_2(x)$ of R_0 which are not both the extreme solutions, we have

$$\lim |u_1(x) - u_2(x)| = 0 \quad as \quad x \to +\infty \quad or \quad as \quad x \to -\infty.$$

Proof. In the (x, u)-plane each Riccati equation R_ε defines a slope field. If $\varepsilon_2 > \varepsilon_1 \geq 0$ then the slope of the solution curve of R_{ε_2} is more negative than that of the solution curve of R_{ε_1}, through each point (x, u). If R_{ε_1} had just one bounded solution on $-\infty < x < \infty$, then it is easily seen that R_{ε_2} would have no such bounded solutions. Thus each R_ε has a band B_ε of bounded solutions. Each band B_ε is closed (by the argument just preceding Lemma 1 of Section 2) and so has an upper and a lower solution curve for the corresponding equation R_ε.

If $\varepsilon_2 > \varepsilon_1 \geq 0$, then each solution curve of R_{ε_2} which intersects either extremal solution of R_{ε_1} must become unbounded. Thus B_{ε_2} lies interior to B_{ε_1}.

Let $u_\varepsilon(x)$ be the solution of R_ε, $\varepsilon > 0$, through $(x_0, u_u(x_0))$ for some x_0. Then on $x > x_0$ $u_u(x) > u_\varepsilon(x)$ and, since $u_\varepsilon(x)$ lies above the lower edge of B_ε, $u_\varepsilon(x) > u_L(x)$. Now

$$u_u(x) - u_\varepsilon(x) = \int_{x_0}^{x} [-u_u(t)^2 + u_\varepsilon(t)^2 + \varepsilon] dt$$

and

$$u_u(x) - u_\varepsilon(x) \leq (2M^2 + \varepsilon)(x - x_0) \quad \text{for} \quad x \geq x_0.$$

But then

$$u_u(x) - u_\varepsilon(x) = \varepsilon(x - x_0) + Q(x),$$

where

$$Q(x) = \int_{x_0}^{x} [u_\varepsilon(t) - u_u(t)][u_\varepsilon(t) + u_u(t)] dt.$$

Hence

$$|Q(x)| \leq (2M^2 + \varepsilon) 2M \int_{x_0}^{x} (t - x_0) dt$$

or

$$|Q(x)| \leq M(2M^2 + \varepsilon)(x - x_0)^2.$$

An elementary calculation shows that

$$u_u(x_0 + h) - u_\varepsilon(x_0 + h) \geq \eta > 0,$$

where

$$h = \frac{\varepsilon}{2M(2M^2 + \varepsilon)} \quad \text{and} \quad \eta = h\varepsilon - M(2M^2 + \varepsilon)h^2.$$

Thus $u_u(x_0 + h) - u_L(x_0 + h) > \eta$. But x_0 is arbitrary and so $u_u(x) - u_L(x) \geq \eta$ on $-\infty < x < \infty$. In the statement of the theorem we take $\varepsilon = e/2$.

Let $u_1(x)$ and $u_2(x)$ be two bounded solutions of R_0 and say that $u_1(x) < u_2(x)$ and that $u_1(x)$ is not the lowest bounded solution of R_0 (the other cases are similar). We shall show that

$$\lim_{x \to \infty} |u_u(x) - u_1(x)| = 0.$$

Let $u_\infty(x)$ be a solution of R_0 lying below the band of bounded solutions. Then

$$\frac{u_1(x) - u_u(x)}{u_1(x) - u_L(x)} = \lambda \frac{u_\infty(x) - u_u(x)}{u_\infty(x) - u_L(x)},$$

for a constant λ. Now at $x = 0$, choose $u_\infty(0)$ so near to $u_L(0)$ that $|\lambda|$ is determined smaller than a prescribed positive number ξ. Now for each number $-N^2$ there is an abscissa $\bar{x}$ such that $u_\infty(x) < -N^2$ wherever $u_\infty(x)$ is defined for $x \geq \bar{x}$. Thus one can choose N^2 so large that, for $x \geq \bar{x}$, $[u_\infty(x) - u_u(x)]/[u_\infty(x) - u_L(x)]$ is arbitrarily near $+1$. But this means that the ratio $|[u_1(x) - u_u(x)]/[u_1(x) - u_L(x)]|$ becomes smaller than ξ. Thus $\liminf\limits_{x \to \infty} |u_u(x) - u_1(x)| = 0$.

We complete the proof by showing that the ratio $[u_\infty(x) - u_u(x)]/[u_\infty(x) - u_L(x)]$ is monotonely decreasing. But the derivative of this ratio is easily found to be

$$[u_u(x) - u_\infty(x)]\,[u_u(x) - u_L(x)]/[u_\infty(x) - u_L(x)] < 0.$$

Therefore the ratio $[u_u(x) - u_1(x)]/[u_1(x) - u_L(x)]$ decreases monotonely to zero as $x \to \infty$. Thus $\lim\limits_{x \to \infty} |u_u(x) - u_1(x)| = 0$. Q.E.D.

We can now prove an important result concerning almost periodic solutions for the Riccati equation (R) interior to the domain of disconjugacy. We follow the method introduced by Favard [5, Ch. 3]. Similar results are obtained in [1] and [7].

THEOREM 9. *Let $u_1(x)$ and $u_2(x)$ be bounded solutions of*

$$u' + u^2 + K(x) = 0, \tag{R}$$

where $K(x)$ is a real almost periodic function. If $u_1(x) - u_2(x) > \delta > 0$ on $-\infty < x < \infty$, then $u_1(x)$ and $u_2(x)$ are almost periodic and the modules of their frequencies are contained in the module of $K(x)$.

Proof. Since $u_1(x) - u_2(x) > \delta > 0$, $u_1(x)$ is the upper bounded solution and $u_2(x)$ is the lower bounded solution of (R). Further we make the notational simplification $\inf\limits_{-\infty < x < \infty} |u_1(x) - u_2(x)| = \delta$.

Consider a sequence of real numbers $\{h_n\}$ and corresponding translates of $u_1(x)$, that is, $u_1(x + h_n)$. We shall show that, for some subsequence, $u_1(x + h_n)$ converges uniformly on $-\infty < x < \infty$. This shows that $u_1(x)$ is almost periodic and a similar argument would hold for $u_2(x)$.

Since the numbers $u_1(0+h_n)$ and $u_2(0+h_n)$ are bounded sets, extract a subsequence (again called h_n) for which $u_1(0+h_n)\to\alpha$, $u_2(0+h_n)\to\beta$. Also one can require $K(x+h_n)\to K^*(x)$ uniformly on $-\infty<x<\infty$.

Let $u_1^*(x)$ and $u_2^*(x)$ be the solutions of

$$u'+u^2+K^*(x)=0 \tag{R^*}$$

with initial conditions $u_1^*(0)=\alpha$, $u_2^*(0)=\beta$. Then, by the continuous dependence of the solutions of a differential equation upon the coefficients, $\lim\limits_{n\to\infty} u_1(x+h_n)=u_1^*(x)$ and $\lim\limits_{n\to\infty} u_2(x+h_n)=u_2^*(x)$ where the convergence is uniform on each compact interval.

Now $\inf\limits_{-\infty<x<\infty} |u_1(x+h_n)-u_2(x+h_n)|=\delta$ for each h_n. If $|u_1^*(x_0)-u_2^*(x_0)|<\delta$ at some point x_0, then for large n, $|u_1(x_0+h_n)-u_2(x_0+h_n)|<\delta$, which is false. Thus $\inf\limits_{-\infty<x<\infty} |u_1^*(x)-u_2^*(x)|\geq\delta>0$. Therefore $u_1^*(x)$ is the upper bounded solution of (R^*) and $u_2^*(x)$ is the lower bounded solution of (R^*).

We need to show that a subsequence of $u_1(x+h_n)$ is Cauchy in the metric space of real, bounded, continuous functions $C_B(-\infty,\infty)$. Suppose the contrary. Then there exists $\varepsilon>0$ such that: for each N_1 there are integers $n_1>N_1$, $m_1>N_1$ and some x_1 at which $|u_1(x_1+h_{n_1})-u_1(x_1+h_{m_1})|>\varepsilon$. Choose a sequence $N_k\to\infty$ and corresponding integers n_k, $m_k>N_k$ and numbers x_k at which $|u_1(x_1+h_{h_k})-u_1(x_1+h_{m_k})|>\varepsilon$.

Extract a subsequence k_i (again called k) for which $u_1(x_k+h_{n_k})\to\tilde{\alpha}$ and also $u_1(x_k+h_{m_k})\to\alpha\neq\tilde{\alpha}$. But consider the translates $u_1(x+x_k+h_{n_k})$ and $u_1(x+x_k+h_{m_k})$ and again extract a subsequence k_i (again called k) so that $K(x+x_k+h_{h_k})\to\tilde{K}^*(x)$ and $K(x+x_k+h_{m_k})\to\hat{K}^*(x)$. Then the upper bounded solutions of the corresponding equations $(\tilde{R}^*)$ and $(\hat{R}^*)$ are $\tilde{u}_1^*(x)$ and $\hat{u}_1^*(x)$, respectively, with $\tilde{u}_1^*(0)=\tilde{\alpha}$ and $\hat{u}_1^*(0)=\hat{\alpha}$.

But $|K((x+x_k)+h_{n_k})-K((x+x_k)+h_{m_k})|<\eta$ for any prescribed $\eta>0$ and sufficiently large k. Therefore $\tilde{K}^*(x)=\hat{K}^*(x)$ and $(\tilde{R}^*)$ is the same as $(\hat{R}^*)$. But then $\tilde{u}^*(x)=\hat{u}^*(x)$ and $\tilde{\alpha}=\hat{\alpha}$ which is a contradiction.

Therefore $u_1(x+h_{n_k})\to u_1^*(x)$ uniformly on $-\infty<x<\infty$ and $u_1(x)$ is almost periodic. Furthermore for each sequence $\{h_n\}$ with $K(x+h_n)\to K^*(x)$ there is a subsequence h_{n_i} such that $u_1(x+h_{n_i})\to u_1^*(x)$ with uniform convergence on $-\infty<x<\infty$. Therefore we actually must have $u_1(x+h_n)\to u_1^*(x)$ uniformly on $-\infty<x<\infty$. Thus the module of frequencies of $u_1(x)$ is contained in that of $K(x)$. Q.E.D.

Using Theorem 8, one can easily see that there are no other almost periodic solutions than $u_1(x)$ and $u_2(x)$.

We now relate our results directly to the linear differential equation (L) and describe a distinguished solution basis for this equation.

THEOREM 10. *Let*

$$y'' + (-a + b\,p(x))y = 0, \tag{L}$$

with $p(x)$ real almost periodic, belong to the interior of the disconjugacy domain D. Then there is a solution basis of the form

$$y_u(x) = e^{\alpha x} \exp \int_0^x \phi_u(t)\,dt$$

$$y_L(x) = e^{-\alpha x} \exp \int_0^x \phi_L(t)\,dt,$$

where $\alpha > 0$ and $\phi_u(x)$, $\phi_L(x)$ are almost periodic functions of mean zero. Also 2α is mean of the width of the band of bounded solutions of the associated Riccati equation and $\int_0^x [\phi_u(t) + \phi_L(t)]\,dt$ is almost periodic. Thus $y_u(x)y_L(x)$ and its reciprocal are almost periodic.

Proof. Let $u_u(x)$ and $u_L(x)$ be the upper and lower almost periodic solutions of the Riccati equation. Then define

$$y_u(x) = \exp \int_0^x u_u(t)\,dt \quad \text{and} \quad y_L(x) = \exp \int_0^x u_L(t)\,dt.$$

These are clearly linearly independent and thus form the required basis.

Now $u_u(x) = \alpha_u + \phi_u(x)$, $u_L(x) = \alpha_L + \phi_L(x)$ where $\phi_u(x)$, $\phi_L(x)$ are almost periodic with mean zero. The width of the band B of bounded solutions of the Riccati equation is

$$\Delta(x) = u_u(x) - u_L(x) = \frac{W}{y_u(x)y_L(x)},$$

where the Wronskian $W = y_u'(x)y_L(x) - y_L'(x)y_u(x)$ is a non-zero constant. Therefore

$$0 < c < y_u(x)y_L(x) < C$$

for bounds c, C and for all x.

However, $y_u(x)\,y_L(x) = \exp\,\{(\alpha_u + \alpha_L)\,x + \int_0^x [\phi_u(t) + \phi_L(t)]\,dt\}$. Since $y_u(x)y_L(x)$ is bounded as indicated, $\alpha_u = -\alpha_L = \alpha$ and the almost periodic function $\Delta(x)$ has a mean of $2\alpha \neq 0$. Thus $y_u(x)y_L(x)$ and its reciprocal are almost periodic. But this means that $\int_0^x [\phi_u(t) + \phi_L(t)]dt$ is bounded and thus almost periodic. Q.E.D.

COROLLARY 1. *$y_u(x)$ (or $y_L(x)$) is the unique (up to a constant factor) solution of (L) bounded on a negative (or positive) half-axis. Also if the product of two solutions of (L) is bounded, then the factors are (up to constant multiples) $y_u(x)$* amd *$y_L(x)$.*

If $\int_0^x \phi_u(t)dt$ and $\int_0^x \phi_L(t)dt$ are bounded, then they are each almost periodic. In this case the canonical solution basis has the form

$$y_u(x) = \Psi_u(x)e^{\alpha x},\ y_L(x) = \Psi_L(x)e^{-\alpha x},$$

where $\Psi_u(x)$ and $\Psi_L(x)$ are almost periodic functions. Moreover one shows easily that both the integrals $\int_0^x \phi_u(t)dt$, $\int_0^x \phi_L(t)dt$ are bounded if one of them is bounded.

COROLLARY 2. *There is a solution basis for (L) of the form $\phi(x)\exp\int_0^x dt/\phi(t)^2$, $\phi(x)\exp\int_0^x -dt/\phi(t)^2$, where $\phi(x)$ is almost periodic and $0 < c < \phi(x) < C$.*

Proof. Define $z(x)$ by $y(x) = \sqrt{y_u(x)y_L(x)}\,z = \phi(x)z$ where $y(x)$ is a solution of $(L.)$ Then we compute

$$(\phi(x)^2 z')' - \frac{W}{4\phi(x)^2} z = 0,$$

where we can take the Wronskian $W = y_u y'_L - y_L y'_u = 2$. But this equation can be solved explicitly to yield solutions $\exp\int_0^x \pm dt/\phi(t)^2$. Q.E.D.

We next explicitly list certain data on the means of the almost periodic functions described above.

COROLLARY 3. mean $[u_u(x) - u_L(x)] = 2\alpha$

mean $u_u(x)^2 =$ mean $u_L(x)^2$

mean $\phi_u(x)^2 =$ mean $\phi_L(x)^2$

mean $\phi(x)^{-2} = \alpha$.

If mean $p(x) = 0$, then $\alpha^2 +$ mean $\phi_i(x)^2 = a$ and thus [mean $\phi_i(x)]^2 \le a - \alpha^2$, $i = L$ or u.

Proof. In the theorem we showed that mean $[u_u(x) - u_L(x)] =$ mean $\Delta(x) = 2\alpha$. Now mean $u_u(x)^2 =$ mean $[\alpha^2 + \alpha\phi_u(x) + \phi_u(x)^2]$ and mean $u_L(x)^2 =$ mean $[\alpha^2 + \alpha\phi_L(x) + \phi_L(x)^2]$. Then since mean $\phi_u(x) =$ mean $\phi_L(x) = 0$, mean $[u_u(x)^2 - u_L(x)^2] =$ mean $[\phi_u(x)^2 - \phi_L(x)^2]$. But $u_u(x)^2 - u_L(x)^2 = u'_L(x) - u'_u(x)$ which has a zero mean.

Now $\phi(x)^{-2} = \Delta(x)/W = \Delta(x)/2$ and this has a mean of α.

Finally from the Riccati equation

$$\phi_i' + (\pm \alpha + \phi_i)^2 + (-a + bp(x)) = 0.$$

Taking means one obtains $\alpha^2 + \text{mean}\ \phi_i(x)^2 - a = 0$. Since $[\text{mean}\ \phi_i(x)]^2 \leq \text{mean}\ \phi_i(x)^2$ one also has $[\text{mean}\ \phi_i(x)]^2 \leq a - \alpha^2$. Q.E.D.

For the equation (L) interior to D we say that the numbers $(\alpha, -\alpha)$ appearing in the canonical solutions of Theorem 10 are the characteristic exponents [10] of (L). They satisfy the usual definition of characteristic exponents in that

$$\limsup_{x\to\infty} \frac{1}{x} \log |y_u(x)| = \alpha.$$

and

$$\limsup_{x\to\infty} \frac{1}{x} \log |y_L(x)| = -\alpha.$$

Using this definition for the characteristic exponents we shall later show that they are zero when (L) belongs to the boundary of D.

For the classical case of Hill's equation, the solutions are exponential functions multiplied by periodic functions. The following example shows that the difficulties which are mentioned in the above theorem and corollaries do actually arise.

Consider the equation

$$y'' + (-a + bp(x))y = 0, \tag{L}$$

where $a = \alpha^2 - \text{mean}\ \chi(x)^2$, $b = -1$, and $p(x) = 2\alpha\chi(x) + \chi(x)^2 + \chi'(x) - \text{mean}\ \chi(x)^2$. Then taking $\alpha > 0$ and $\chi(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{x}{n}\right)$, $p(x)$ is an almost periodic function with mean zero and (L) has a solution

$$y(x) = \exp\left\{\alpha x + \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{x}{n}\right)\right\} = \exp\left\{\alpha x + \int_0^x \chi(t)\,dt\right\}.$$

Since the characteristic exponent of $y(x)$ is $\alpha > 0$, $\left(\lim_{x\to\infty} \frac{1}{x} \int_0^x \chi(t)\,dt = 0\right)$, the equation (L) belongs to the interior of D, cf. Theorem 15. Then it is easy to see that there is no solution basis of the form $\exp\{(-1)^i\alpha x + \pi_1(x)\}$ where $\pi_i(x)$ are almost periodic, $i = 1, 2$. For one would then require that $y(x) = c \exp(\alpha x + \pi_1(x))$, for $c \neq 0$, and $\pi_1(x) = \int_0^x \chi(t)\,dt$ which is not bounded. It is unknown whether such a difficulty can occur if $p(x)$ is, say, a trigonometric polynomial.

Lemma. *Let*

$$y'' + (-a + bp(x))y = 0. \tag{L}$$

with $p(x)$ a real almost periodic function, be in the interior of D. Then the characteristic exponent α, and the quantities $\delta = \inf_{-\infty < x < \infty} |u_u(x) - u_L(x)|$ (for the upper and lower almost periodic solutions of the associated Riccati equation), and $\mu = \sup_{-\infty < x < \infty} |u_u(x) - u_L(x)|$ are continuous, strictly increasing functions along each line $b =$ const., as a increases away from the boundary of D.

Proof. Let (a_0, b_0) lie in the interior of D and let $u_u(x)$, $u_L(x)$ be the upper and lower almost periodic solutions of the associated Riccati equation. Then $\Delta(x) = u_u(x) - u_L(x)$ has a mean of $2\alpha_0$ where α_0 is the corresponding characteristic exponent. By Theorem 8, α, δ, and μ are strictly increasing along the line $b =$ const. and we next show that they are continuous functions.

Consider the almost periodic functions

$$w_1(x) = t u_u(x) + (1 - t) u_L(x)$$

and

$$w_2(x) = (1 - t) u_u(x) + t u_L(x) \text{ for } 0 < t < 1.$$

Then a calculation shows that both of these almost periodic functions are solutions of the equation

$$w' + w^2 + [-a_0 + b_0 p(x) + t(1 - t)(u_u(x) - u_L(x))^2] = 0.$$

Also $w_1(x) - w_2(x) = (2t - 1)\,[u_u(x) - u_L(x)]$. Then the width $\Delta_w(x)$ of the band of bounded solutions of

$$w' + w^2 + [-a_0 + b_0 p(x) + c] = 0,$$

where $0 < c \leq t(1 - t) \inf_{-\infty < x < \infty} (u_u - u_L)^2$, satisfies

$$\Delta_w(x) \geq (2t - 1)\,[u_u(x) - u_L(x)] = (2t - 1)\Delta(x).$$

For a prescribed $\varepsilon > 0$ choose t so near 1 that $\Delta(x) - \Delta_w(x) < \varepsilon$ whenever $c > 0$ is sufficiently small. Therefore α, δ, and μ are continuous from below along $b =$ const. But within a fixed neighborhood of (a_0, b_0), c can be chosen independently of (a', b_0) so that the band width of

$$w' + w^2 + [-a' + b_0 p(x)] = 0$$

and that of

$$w' + w^2 + [-a' + c + b_0 p(x)] = 0$$

differ by less than ε. Therefore α, δ, and μ are continuous along $b = b_0$. Q.E.D.

THEOREM 11. *Let*

$$y'' + (-a + b p(x)) y = 0, \tag{L}$$

for a real almost periodic $p(x)$, belong to the interior of D. On the interior of D the real functions $\alpha(a, b)$ and $\delta(a, b)$ are continuous. Let D_{α_0} and E_{δ_0} be the subsets of the interior of D wherein $\alpha \geq \alpha_0$ and $\delta \geq \delta_0$, respectively. Then each D_α and E_δ is a convex set which is relatively closed in the interior of D. Also D_{α_1} (or E_{δ_1}) is properly contained in the interior of D_{α_2} (or E_{δ_2}) whenever $\alpha_2 < \alpha_1$ (or $\delta_2 < \delta_1$). On the relative boundary of D_α (of E_δ) the characteristic exponent (the $\inf\limits_{-\infty<x<\infty} |u_u(x) - u_L(x)|$) is exactly α (or δ).

Proof. Let (a_1, b_1) and (a_2, b_2) lie in D_α (or lie in E_δ). Let $u_u^{(1)}(x)$, $u_u^{(2)}(x)$ be the upper almost periodic solutions and $u_L^{(1)}(x)$, $u_L^{(2)}(x)$ be the lower almost periodic solutions of the corresponding Riccati equations. Consider $w_u(x) = t\,u_u^{(1)}(x) + (1-t)\,u_u^{(2)}(x)$ and $w_L(x) = t u_L^{(1)}(x) + (1-t) u_L^{(2)}(x)$ for $0 < t < 1$. A computation shows that $w_u(x)$ satisfies

$$w' + w^2 + \{-[ta_1 + (1-t)a_2] + [tb_1 + (1-t)b_2]p(x))\} + t(1-t)[u_u^{(1)}(x) - u_u^{(2)}(x)]^2 = 0.$$

Thus the upper almost periodic solution of

$$w' + w^2 + \{-[ta_1 + (1-t)a_2] + [tb_1 + (1-t)b_2]p(x)\} = 0 \qquad (R_t)$$

lies above $w_u(x)$. Similarly the lower almost periodic solution of the last equation (R_t) lies below $w_L(x)$. Thus the band width Δ_t of (R_t) satisfies

$$\Delta_t \geq t\Delta^{(1)}(x) + (1-t)\Delta^{(2)}(x).$$

Thus E_δ is convex. Since mean $\frac{1}{2}\Delta_t = \alpha_t$, the characteristic exponent corresponding to (R_t), D_α is convex.

By the lemma, the characteristic exponent corresponding to a boundary point of D_α, interior to D, is exactly α. Also on the relative boundary of E_δ, $\inf\limits_{-\infty<x<\infty} |u_u(x) - u_L(x)| = \delta$. Thus both D_α and E_δ are relatively closed in the interior of D. The enclosure relations mentioned in the theorem are then elementary.

Finally we verify that $\alpha(a, b)$ and $\delta(a, b)$ are continuous on the interior of D. But this follows easily from the enclosure and convexity conditions. Q.E.D.

One can further describe D_α by noting that it must contain the sector bounded by the rays

$$a = \alpha^2 + b \inf_{-\infty<x<\infty} p(x)$$

and

$$a = \alpha^2 + b \sup_{-\infty<x<\infty} p(x).$$

Similarly E_δ must contain the sector bounded by the rays

$$a = \delta^2 + b \inf_{-\infty<x<\infty} p(x)$$

and

$$a = \delta^2 + b \sup_{-\infty<x<\infty} p(x).$$

Also each domain D_α and E_δ is invariant under translations, and even limits of such, of the equation (L).

Later, cf. Theorem 14, we shall show that $\delta \to 0$ near the boundary of D and thus that each E_δ lies interior to D. However, although it is very likely true, we have not been able to show that $\alpha \to 0$ near the boundary of D.

We conclude this section with a result for a forced or non-homogeneous differential equation.

THEOREM 12. *Let*

$$y'' + (-a + bp(x))y = 0, \tag{L}$$

where $p(x)$ and $p'(x)$ are real almost periodic functions, lie interior to D. If $f(x)$ is almost periodic, then the equation

$$y'' + (-a + bp(x))y = f(x) \tag{F}$$

has a unique bounded solution and this is almost periodic.

Proof. Since the homogeneous equations (L^*) have no (non-trivial) bounded solutions, (F) has at most one bounded solution. Moreover, by Favard's theory [5, Ch. 3], if there is a bounded solution $y(x)$ for which $y'(x)$ is also bounded, then $y(x)$ is almost periodic.

A solution basis for the homogeneous equation is $\phi(x)\exp\int_0^x \pm\, dt/\phi(t)^2$ where $\phi(x)$ is almost periodic and $0 < c < \phi(x) < C$. Construct the Green's function

$$G(x, \xi) = \frac{\phi(x)\phi(\xi)}{2} \exp\left\{-\left|\int_\xi^x \frac{dt}{\phi(t)^2}\right|\right\}.$$

Consider the solution of (F) given by

$$y(x) = \int_{-\infty}^{\infty} G(x, \xi) f(\xi)\, d\xi.$$

The integral exists since $f(x)$ is bounded and $G(x, \xi)$ has an exponential decrease at both $\pm\infty$.

From the definition of $\phi(x)$, cf. Corollary 2 of Theorem 10, we see that $\phi'(x)$ and $\phi''(x)$ are also almost periodic. Then one can differentiate the expression for $y(x)$ to see that $y(x)$ is a solution of (F).

Since $y(x)$ and $y''(x)$ are bounded, so is $y'(x)$ bounded, as is required. Q.E.D.

4. Boundary of the disconjugacy domain

We first give a criterion for distinguishing between the interior and the boundary of D by the behavior of the solutions of (L).

THEOREM 13. *Let*

$$y'' + (-a + bp(x))y = 0, \tag{L}$$

where $p(x)$ is real almost periodic, be disconjugate. Then (L) belongs to the interior of D if and only if there is a solution basis $y_u(x) y_L(x)$ of (L) for which $0 < y_u(x) y_L(x) < C$.

Proof. If (L) lies interior to D, then by Theorem 10 there is a solution basis of the required type.

Conversely let $y_0(x)$, $y_1(x)$ be a basis whose product is bounded, as above. Then the functions $u_0(x) = y_0'(x)/y_0(x)$ and $u_1(x) = y_1'(x)/y_1(x)$ are solutions of the associated Riccati equation

$$u' + u^2 + (-a + bp(x)) = 0. \tag{R}$$

Let us write $u_1(x) > u_0(x)$ and $u_1(x) - u_0(x) \geq 2/C$. Consider the function $w(x) = \frac{1}{2}[u_1(x) - u_0(x)]$. A computation yields

$$w' + w^2 + (-a + bp(x)) + \tfrac{1}{4}[u_1(x) - u_0(x)]^2 = 0.$$

Thus the equation $$w' + w^2 + (-a + bp(x)) + 1/C^2 = 0$$

has a bounded solution. Therefore $$y'' + (-a + bp(x) + 1/C^2)y = 0$$

is disconjugate and (L) lies interior to D. Q.E.D.

COROLLARY. *In the interior of D no (non-trivial) solution of the corresponding linear differential equation (L) has a square which is bounded. However, if on the boundary of D, the product of two positive solutions of (L) is bounded (even only positive with a bounded product on a half-axis), then the two solutions are linearly dependent.*

We can now complete the discussion of $\delta(a, b)$ which was begun in Theorem 11.

THEOREM 14. *Let*

$$y'' + (-a + bp(x))y = 0, \tag{L}$$

where $p(x)$ is real almost periodic, belong to D. Then on the boundary of D

$$\liminf_{x \to \pm\infty} |u_u(x) - u_L(x)| = 0$$

for any two (possibly coincident) bounded solutions of the associated Riccati equation (R). Therefore $\delta(a, b) = 0$ on the boundary of D and furthermore δ is continuous on all of D.

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Proof. Suppose the point (a_0, b_0) yields a point interior to D but $(a_0 - \varepsilon, b_0)$ for a certain $\varepsilon > 0$ yields an oscillatory equation (L). If, for (a_0, b_0), the width of the band for the associated Riccati equation is $\Delta(x) \geq 2\sqrt{\varepsilon}$, then the point $(a_0 - \varepsilon, b_0)$ lies interior to D, as follows from the calculation occuring in the proof of Theorem 13. Therefore $\Delta(x) < 2\sqrt{\varepsilon}$ for some x and $\delta \to 0$ near the boundary of D.

Now for (a, b) on the boundary of D, the band for the associated Riccati equation must lie interior to the band for $(a + \varepsilon, b)$, $\varepsilon > 0$. But for the width $\Delta_\varepsilon(x)$ of this band one has $\liminf_{x \to \pm\infty} \Delta_\varepsilon(x) = \delta(a + \varepsilon, b)$. Since $\delta(a + \varepsilon, b) \to 0$ as $\varepsilon \to 0$, as we have the desired result. Q.E.D.

We next turn to the problem of computing the characteristic exponent α on the boundary of D.

THEOREM 15. *Let*

$$u' + u^2 + (-a + bp(x)) = 0, \tag{R}$$

for real almost periodic $p(x)$, correspond to the boundary of D. If a solution of (R) is almost periodic, it must have zero mean. Also there can be at most one almost periodic solution of (R).

Proof. Let $u(x)$ be an almost periodic solution of (R) and suppose mean $u(x) = \alpha > 0$. Consider the auxillary equation

$$z' - 2u(x)z = 1.$$

Then a solution is $z(x) = -\int_x^\infty [\exp \int_x^t - 2u(s)ds]dt$, which is bounded on a right half-axis, say $x > 0$. Define $w(x)$ by $z(x) = -w(x)\exp \int_0^x u(t)dt$. An easy computation shows that $w(x)$ is a positive solution of

$$y'' + (-a + bp(x))y = 0. \tag{L}$$

But then the product of two positive solutions of (L) is bounded on a right half-axis and, by the Corollary to Theorem 13, they must be linearly dependent solutions. But this is impossible since $\int_0^x u(s)ds \to +\infty$ as $x \to \infty$ and so $w(x) \to 0$ as $x \to \infty$. From this contradiction we conclude that mean $u(x) \leq 0$.

A similar contradiction arises from the supposition that mean $u(x) < 0$. Here one uses the function

$$z_1(x) = \int_{-\infty}^x [\exp \int_t^x 2u(s)ds]dt$$

which satifies the same auxilliary equation, and which is bounded on a left half-axis. Again define $w_1(x)$ by $z_1(x) = w_1(x) \exp \int_0^x u(t)dt$ and observe, that $w_1(x)$ is a positive solution of (L). Then the same reasoning as in the earlier case shows that mean $u(x) = 0$.

If there were two almost periodic solutions of (R) then these differences would be a positive almost periodic function of zero mean and this is impossible. Q.E.D.

COROLLARY. *Let $y(x)$ be a positive solution of (L), for the boundary of D, such that $u(x) = y'(x)/y(x)$ is almost periodic. Then the characteristic exponent* $\alpha = \limsup_{x\to\infty} (1/x) \log |y(x)| = 0$.

The following example shows that an equation (L) on the boundary of D need not have a (non-trivial) bounded solution. Our example does have a solution $y(x) > 0$ such that $y'(x)/y(x)$ is almost periodic. Whether or not there is always such a solution in this situation is unknown. We shall have further comments on this dilemma later.

Consider $y'' + [-\chi(x)^2 - \chi'(x)]\, y = 0$ where $\chi(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{x}{n}\right)$. Here take $a =$ mean $\chi(x)^2$, $b = 1$, $p(x) = -\chi(x)^2 - \chi'(x) + \text{mean}\,\chi(x)^2$. The equation is disconjugate since a positive solution is $y(x) = \exp \int_0^x \chi(t)dt$. Since $\limsup_{x\to\infty} (1/x) \log y(x) = \text{mean}\,\chi(x) = 0$, the equation belongs to the boundary of D. We also note that $\limsup_{x\to\infty} y(x) = +\infty$, $\liminf_{x\to\infty} y(x) = 0$.

If $Y(x)$ were a bounded solution, say $Y(x) > 0$ for $x > x_0$, then $0 < Y(x) < cy(x)$ for $x > x_0$ and a constant c. But then $\liminf_{x\to\infty} W(x) = 0$ where $W(x)$ is the Wronskian of $Y(x)$ and $y(x)$. However, $W(x)$ is constant. Thus there are no (non-trivial) bounded solutions.

The next theorem shows that one can always obtain a bounded solution for (L) on the boundary of D, merely by translating to a limit equation (L^*).

THEOREM 16. *Let*

$$y'' + (-a + bp(x))y = 0, \tag{L}$$

where $p(x)$ and $p'(x)$ are real almost periodic functions, belong to the boundary of D. Then there exists $p^*(x) \in H\{p(x)\}$ such that

$$y'' + (-a + bp^*(x))y = 0 \tag{L^*}$$

has a positive bounded solution.

Proof. For each $\varepsilon > 0$ a solution basis $y_1(x), y_2(x)$ of $y'' + (-a + bp(x) - \varepsilon)y = 0$ yields a general solution

$$c_1 y_1(x)^2 + c_2 y_1(x) y_2(x) + c_3 y_2(x)^2$$

of

$$w''' + 4(-a + bp(x) - \varepsilon)w' + 2bp'(x)w = 0.$$

Then there is just one positive solution $w_\varepsilon(x) = y_1(x)y_2(x)$ which is bounded on $-\infty < x < \infty$ and such that $\sup_{-\infty < x < \infty} w_\varepsilon(x) = 1$. An easy calculation, based on the explicit form of $y_1(x)$ and $y_2(x)$ (cf. Theorem 10), yields that $|w_\varepsilon''(x)| < M_2$ and so $|w_\varepsilon'(x)| < M_1$ where the bounds M_1 and M_2 depend only on $(-a + bp(x))$ and not on ε.

Let ε_n be a sequence of positive numbers decreasing monotonely to zero. Let $w_n(x)$ be the correspondingly defined, positive, bounded solutions of

$$w''' + 4(-a + bp(x) - \varepsilon_n)w' + 2bp'(x)w = 0.$$

Then let x_n be a sequence of points such that $w_n(x_n) \to \alpha_0 \neq 0$, $w_n'(x_n) \to \alpha_1$, $w_y''(x_n) \to \alpha_2$, $p(x + x_n) \to p^*(x)$ and $p'(x + x_n) \to p^{*\prime}(x)$ uniformly on $-\infty < x < \infty$. Let $w^*(x)$ be the solution of

$$w''' + 4(-a + bp^*(x))w' + 2bp^{*\prime}(x)w = 0$$

with the initial data $w^*(0) = \alpha_0$, $w^{*\prime}(0) = \alpha_1$, $w^{*\prime\prime}(0) = \alpha_2$.

Now $w_n(x + x_n) \to w^*(x)$ uniformly on compact intervals. Thus $w^*(x) \leq 1$ on $-\infty < x < \infty$. But $w^*(x)$ is the product of two solutions of (L^*). Thus $w^*(x)$ must be the square of one such solution $y^*(x)$ which is thereby bounded.

Since $w_n'(x + x_n) \to w^{*\prime}(x)$ and $w_n''(x + x_n) \to w^{*\prime\prime}(x)$ uniformly on compact intervals, one must have $w^{*\prime}(x_0) = w^{*\prime\prime}(x_0) = 0$ wherever $w^*(x_0) = 0$. Thus $w^*(x) > 0$ and we take $y^*(x) > 0$. Q.E.D.

COROLLARY. *For the equation (L^*), the associated Riccati equation (R^*) has a unique bounded solution.*

Proof. Let $u^*(x) = y^{*\prime}(x)/y^*(x)$ where $y^*(x) > 0$ is the bounded solution of (L^*). Suppose $\bar{u}(x)$ is a bounded solution of (R^*) and say $\bar{u}(x) < u^*(x)$. Then $\bar{y}(x) = \exp \int_0^x \bar{u}(t)\,dt$ and $0 < \bar{y}(x) < y^*(x)$ for $x > 0$. But if (L^*) has two linearly independent solutions which are bounded on a half-axis, then (L^*) is oscillatory. Q.E.D.

We remark that if for each $p^*(x) \in H\{p(x)\}$ the equation (R^*) on the boundary of D has a unique bounded solution, then this solution is almost periodic and (L^*) has a positive solution whose logarithmic derivative is almost periodic. This remark can be obtained in the manner indicated in the proof of Theorem 9. It is our conjecture that each such (R^*) actually does have a unique bounded solution.

Using the construction of Theorem 16 we can find an equation (L^*) on the boundary of D with a bounded positive solution which is not almost periodic. To do this, let (L) be an equation, on the boundary of D, for which there are no (non-trivial) bounded solutions, cf., previous example. Consider a limit translate (L^*) for which there is a positive bounded solution $y^*(x)$. If $y^*(x)$ were almost periodic, then (L) would have a positive almost periodic solution which is contrary to the construction.

5. Perturbations of $p(x)$.

Consider
$$y'' + (-a + b\,p(x))y = 0, \tag{L}$$

for $p(x)$ real almost periodic, belonging to the boundary of D. Then if $-a + b\,p(x)$ is replaced by $-a + b\,p(x) + \varepsilon(x)$, where $\varepsilon(x) \not\equiv 0$ is a non-negative almost periodic function, the resulting equation is oscillatory. If the band for the associated Riccati equation degenerates to a single bounded solution, as is the case when $p(x)$ is periodic, then $\varepsilon(x)$ need not be almost periodic but merely continuous in order to change (L) from disconjugacy to oscillatory.

We now study the dependence of the disconjugacy domain D upon the choice of the almost periodic function $p(x)$.

THEOREM 17. *Let $p(x)$ be real almost periodic, and $\pi(x) = k\,p(x) + l$, for $k \neq 0$, l real numbers. Then the disconjugacy domain Δ of*

$$y'' + (-\alpha + \beta\,\pi(x))y = 0 \tag{Λ}$$

is the image of the disconjugacy domain D of

$$y'' + (-a + b\,p(x))y = 0 \tag{L}$$

under the map T of the (a,b)-plane onto the (α,β)-plane, where

$$T : (a,\, b) \to (\alpha,\, \beta) = (a + b\,l/k,\, b/k).$$

Proof. The point (α, β) for (Λ) yields the same solutions as does $a = \alpha - \beta\,l$, $b = \beta k$ for (L). Thus a set Δ in the (α, β)-plane describes the same differential equations as the set D in the (a, b)-plane, provided Δ is the image of D under

$$T : (a,\, b) \to (a + b\,l/k,\, b/k).$$

Q.E.D.

COROLLARY. *Let $K(x)$ be a non-constant, real, almost periodic function decomposed as $K(x) = -a_0 + b_0\,p(x)$ and $K(x) = -\alpha_0 + \beta_0\,\pi(x)$. Then the corresponding disconjugacy domains D and Δ for varying (a, b) and (α, β) respectively, are related by the map*

$$T:(a,b)\to(\alpha,\beta)=\left(a+b\frac{\alpha_0-a_0}{b_0},\ \frac{b\beta_0}{b_0}\right).$$

Proof. Here $k=b_0/\beta_0$, $l=(\alpha_0-a_0)/\beta_0$ and we use the formula of the theorem. Q.E.D.

Also one notes that, in the Corollary, if $\pi(x)$ and $p(x)$ have mean zero, b_0 and β_0 are positive, $\sup\limits_{-\infty<x<\infty}\pi(x)=\sup\limits_{-\infty<x<\infty}p(x)$, then $\pi(x)=p(x)$, $\alpha_0=a_0$, $\beta_0=b_0$ and $\Delta=D$.

We shall show that the domain D varies continuously with the choice of the almost periodic function $p(x)$. On the space $\mathfrak{F}$ of real almost periodic functions we utilize the uniform topology defined by the metric $\varrho(p_1(x),p_2(x))=\sup\limits_{-\infty<x<\infty}|p_1(x)-p_2(x)|$. On the space $\mathfrak{S}$ of non-empty closed plane sets we define a topology of uniform convergence on bounded regions by a neighborhood prescription as follows. For $D\in\mathfrak{S}$ define an open neighborhood $U(K,\varepsilon)$ of D depending on an open disc K, centered at the origin, which intersects D and a number $\varepsilon>0$. Let $D_1\in U$ in case D_1 intersects K and

$$\max\{\sup_{P_1\in D_1\cap\bar{K}}\varrho(P_1,D\cap\bar{K}),\ \sup_{P\in D\cap\bar{K}}\varrho(P,D_1\cap\bar{K})\}<\varepsilon.$$

Here ϱ is the distance function in the plane. It is easily verified that a topology on $\mathfrak{S}$ is defined by this open neighborhood system.

Lemma. *Let $\pi(x)\geq p(x)$ for all x,* where $\pi(x)$, $p(x)\in\mathfrak{F}$. *Let the disconjugacy domains D and Δ* of

$$y''+(-a+bp(x))y=0 \tag{L}$$

and

$$y''+(-a+b\pi(x))y=0, \tag{Λ}$$

respectively, have bounding curves $a_L(b)$ and $a_\Lambda(b)$. Then $a_\Lambda(b)\geq a_L(b)$ for $b\geq 0$ and $a_\Lambda(b)\leq a_L(b)$ for $b\leq 0$.

Proof. For $b\geq 0$, $-a+b\pi(x)\geq -a+bp(x)$. Thus, if (L) is oscillatory for some parameter values (a,b), then so is (Λ) oscillatory. Therefore $a_\Lambda(b)\geq a_L(b)$. For $b\leq 0$ the argument is similar. Q.E.D.

Theorem 18. *For each real almost periodic function $p(x)\in\mathfrak{F}$ there is a corresponding disconjugacy domain $D\in\mathfrak{S}$. The map $p(x)\to D$ is continuous.*

Proof. Consider a neighborhood $U(K,\varepsilon)$ of D_0, the disconjugacy domain for $p_0(x)$. There is a number $l>0$ such that for each $l(x)\in\mathfrak{F}$ with $|l(x)|<l$, the domain $D_0+\Delta D$, corresponding to $p_0(x)+l(x)$, has boundary curves which lie between those corresponding to $p_0(x)+l$ and $p_0(x)-l\in\mathfrak{F}$. From Theorem 17 we see that, for l sufficiently small, the boundary curves of $D_0+\Delta D$ are within the required ε-closeness of those of D_0 inside $\bar{K}$. Q.E.D.

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