INDUCED REPRESENTATIONS OF LOCALLY COMPACT GROUPS

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1. Introduction

In this and in a subsequent paper we study locally convex spaces which are modules over a topological algebra. These are introduced as the appropriate setting for the study of two separate but related problems. The first of these is the subject of this paper and will be described in more detail below, the second concerns the multiplier problem which arose in classical Fourier analysis and has since been studied in various settings by a number of authors. The second paper is an attempt to unify some of these results.

The subject of this paper concerns the analysis of representations of a locally compact group in terms of representations of its subgroups. This problem has a long history. For finite groups it was considered by Frobenius in his development of induced characters and induced representations. Later work has been done by many people. Amongst these Mackey in [8], succeeded in obtaining a rather complete solution to this problem. Central to his work were three theorems which he has called the subgroup theorem, the tensor product theorem and the intertwining number theorem. This last theorem easily yields the classical Frobenius reciprocity theorem. Mackey then generalized this work to unitary representations on separable Hilbert spaces of locally compact groups having a countable basis for the open sets. This is the substance of [9] and [10]. Here the theorems take a quite different form since decompositions into irreducible representations in the sense of direct sums need not exist, and the notion of direct integral decompositions must be employed. This latter is the cause of many measure theoretical difficulties. It appears impossible to extend these results to representations acting in more general spaces. Indeed very difficult problems arise in the attempt to generalize direct integral decompositions to Hilbert spaces which are not separable. Thus if one wishes to establish analogues for the theorems of Mackey for representations in locally convex spaces, one cannot hope for results that are as meaningful as those of Mackey since one cannot reduce the study of representations to the study of irreducible representations.

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There are as pointed out by Mackey in [11; Sect. 8] a number of reasons for studying representations of locally compact groups in Banach spaces, and even in more general spaces. In fact once one begins the study of induced representations in Banach spaces (as is done in [11; Sect. 8] and in [14]), one is quickly lead to consider more general spaces as we shall now show.

In [14] M. A. Rieffel has obtained a version of the Frobenius reciprocity theorem for Banach space representations. His theorem may be stated in the following way. Let Γ be a locally compact group and Δ an open (and therefore closed) subgroup of Γ. Let $\mathcal{A}$ be the category of Γ-modules and $\mathcal{B}$ the category of Δ-modules (see [14] for the definitions). The restriction functor $E \to E_\Delta$ which assigns to each Γ-module the corresponding Δ-module has an adjoint $E \to {}^\Gamma E$ and a coadjoint $E \to E^\Gamma$; that is there are Γ-modules E^Γ and ${}^\Gamma E$ such that

$$\mathrm{Hom}_\Gamma ({}^\Gamma E, F) \cong \mathrm{Hom}_\Delta (E, F_\Delta)$$

and

$$\mathrm{Hom}_\Gamma (F, E^\Gamma) \cong \mathrm{Hom}_\Delta (F_\Delta, E).$$

If Δ is a closed subgroup which is not open, then the restriction functor has a coadjoint but not an adjoint [14; Theorem 7.1]. This raises the possibility that if we consider representations in more general spaces, then we may be able to "find" an adjoint for the restriction functor. That this is in fact the case is one of the main results of this paper.

Note that for finite groups Rieffel's theorem yields the classical Frobenius reciprocity theorem. However Rieffel's theorem, as well as similar theorems obtained by Moore [13], Kleppner [7] are quite different than Mackey's theorems for (infinite) locally compact groups, since if E is irreducible $\mathrm{Hom}_\Gamma (E, F) \neq (0)$ only if F has E as a discrete irreducible component (see [11] Appendix).

In this paper we consider representations acting on locally convex spaces. We begin by introducing locally convex modules and tensor products of these. Using these as tools we develop a theory of induced representations of locally convex algebras and locally compact groups, which includes a Frobenius reciprocity theorem. These results are then applied to the study of linear systems representations which were introduced by Mackey in [11; § 8]. In establishing a Frobenius reciprocity theorem for these, use is made of the fact that for locally convex modules the restriction functor has both an adjoint and coadjoint. It would appear from our work that while we have only made a start on the problem of analyzing linear systems representations of a locally compact group in terms of linear systems representations of its subgroups, further work should be profitable.

The paper is organized as follows. Section 2 introduces locally convex modules and tensor products of these. This section contains a number of our results which are used in later sections. A number of our results can be worded in terms of representations of locally convex algebras and this is done in section 3. Section 4 begins the study of representations of locally compact groups on locally convex spaces. When dealing with representations of locally compact groups, various notions of continuity arise. In section 4 we investigate these in terms of the representation module. Separately continuous representations are reduced to the study in section 3, and in section 5 we give the main theorems for these (Theorems 6, 7 and 8). Section 6 establishes a representation theorem for E^{Γ}. Section 7 contains preliminary results for section 8 which is concerned with showing that in the case of unitary representations our induced modules are Naimark-related to the induced representations studied by Mackey. In section 9 we present a (not very satisfactory) representation theorem for ${}^{\Gamma}E$. Section 10 is concerned with linear systems representations. The final section of the paper extends the results of section 5 to continuous representations.

A word about part of the notation used throughout the paper is in order. When dealing as we do with categories of topological spaces it is sometimes possible to define a Hom functor from the given category to the category of sets and a hom functor from the category into itself which "forgets" to the Hom functor. The former will be denoted by "Hom" and the latter by "hom".

Many of our results on locally convex modules are of course similar to results in homological algebra. Thus those parts of the proof which are purely algebraic are frequently left to the reader, and we usually worry only about the topological aspects of the proof. We use [1] and [12] as standard references for algebraic and categorical results.

One further convention,—all topological spaces are Hausdorff.

2. Locally convex modules

Let E, F be topological vector spaces. $L(E, F)$ is the vector space of continuous linear mappings of E into F. We write $L(E)$ in place of $L(E, E)$. A topological algebra A is a linear associative algebra over the complex field $\mathbf{C}$ which is a topological vector space in which the maps $a \to ab$ and $a \to ba$ are continuous for each $b \in A$. A locally convex algebra is a topological algebra which is a locally convex space.

Definition. Let A be a topological algebra. A locally convex left A-module E is a locally convex space which is a left A-module such that the map $(a, x) \to ax$ of $A \times E \to E$ also satisfies:

LM 1. For each $a \in A$, the map $x \to ax$ is in $L(E)$.

LM 2. For each $x \in E$, the map $a \to ax$ is in $L(A, E)$.

A locally convex right A-module is defined in the analogous fashion.

Let A be a topological algebra, E a locally convex right A-module, F a locally convex left A-module, and G a locally convex space. Recall that a bilinear map f of $E \times F$ into G is called A-balanced if $f(xa, y) = f(x, ay)$ for any $a \in A$, $x \in E$, $y \in F$. $B(E, F, G)$ is the set of all A-balanced bilinear maps. We shall write $B(E, F)$ in place of $B(E, F, \mathbf{C})$.

For each pair $(x, y) \in E \times F$, the map $f \to f(x, u)$ is a linear form on $B(E, F)$ and hence is an element $x \otimes y$ of the algebraic dual $B(E, F)^*$. The map χ: $(x, y) \to x \otimes y$ for $E \times F$ into $B(E, F)^*$ is bilinear and A-balanced. The linear span of $\chi(E \times F)$ in $B(E, F)^*$ is called the tensor product of E and F and is written $E \otimes_A F$.

We intend to study various topologies on $E \otimes_A F$. To do this we require a number of definitions.

Let $\mathfrak{S}$ (resp. $\mathfrak{T}$) be a family of bounded subsets of E (resp. F), and recall that a bilinear map f of $E \times F$ into G is said to be $\mathfrak{S}$-hypocontinuous if f is separately continuous and if for each $S \in \mathfrak{S}$ and each neighbourhood W of 0 in G, there is a neighbourhood V of 0 in F such that $f(S \times V) \subset W$. One defines $\mathfrak{T}$-hypocontinuity in the obvious analogous manner. Let $B^{\mathfrak{S}, \mathfrak{T}}(E, F, G)$ be the set of all A-balanced $(\mathfrak{S}, \mathfrak{T})$-hypocontinuous maps of $E \times F$ into G. Let H be a subset of $B^{\mathfrak{S}, \mathfrak{T}}(E, F, G)$. If for each neighbourhood W of 0 in G, and each S in G, there is a neighbourhood V of 0 in F such that $f(S \times V) \subset W$ for all f in H, then H is said to be $\mathfrak{S}$-equihypocontinuous. $\mathfrak{T}$-equihypocontinuous and $(\mathfrak{S}, \mathfrak{T})$-equihypocontinuous sets are defined in a similar way.

Let m be a linear map of $E \otimes_A F$ into G, then $m \circ \chi$ is a mapping of $E \times F$ into G which is A-balanced. Moreover the map Φ: $m \to m \circ \chi$ is an algebraic isomorphism of the space of all linear maps of $E \otimes_A F$ into G onto the space of all A-balanced bilinear maps of $E \times F$ into G.

THEOREM 1. *Let $\mathfrak{S}$ (resp. $\mathfrak{T}$) be a set of bounded subsets of E (resp. F). There exists on $E \otimes_A F$ a unique topology $\tau = \tau(\mathfrak{S}, \mathfrak{T})$ such that for each locally convex space G, the isomorphism Φ maps $L((E \otimes_A F)_\tau, G)$ onto $B^{\mathfrak{S}, \mathfrak{T}}(E, F, G)$. Moreover, a subset H of $L((E \otimes_A F)_\tau, G)$ is equicontinuous if and only if $\Phi(H)$ is an $(\mathfrak{S}, \mathfrak{T})$-equihypocontinuous subset of $B^{\mathfrak{S}, \mathfrak{T}}(E, F, G)$.*

The proof is analogous to the case $A = \mathbf{C}$ (the complex field). For an indication of the proof in this case see [6].

The topology $\tau(\mathfrak{S}, \mathfrak{T})$ is the topology of uniform convergence on the $(\mathfrak{S}, \mathfrak{T})$-equihypocontinuous subsets of $B(E, F)$.

Definition. Let A and B be topological algebras. A locally convex (A, B)-bimodule E is a locally convex left A-module which is also a locally convex right B-module and satisfies

$$a(xb) = (ax)b \quad \text{for } a \in A,\ b \in B \text{ and } x \in E.$$

PROPOSITION 1. *Let E be a locally convex (B, A)-bimodule, and F a locally convex left A-module. Let $\mathfrak{S}$ (resp. $\mathfrak{T}$) be a set of bounded subsets of E (resp. F) and suppose that for $b \in B$, $b \neq 0$, $S \in \mathfrak{S}$ implies $bS \in \mathfrak{S}$. Then $(E \otimes_A F)_\tau$ is a locally convex left B-module.*

Proof. For $b \in B$, the map $(x, y) \to bx \otimes y$ is easily seen to be *A-balanced*, $(\mathfrak{S}, \mathfrak{T})$-hypocontinuous, and bilinear. By Theorem 1, there is a continuous linear map φ_b of $(E \otimes_A F)_\tau$ into itself such that $\varphi_b(x \otimes y) = bx \otimes y$. We define $b(\Sigma x_i \otimes y_i) = \varphi_b(\Sigma x_i \otimes y_i)$ and it follows that $(E \otimes_A F)_\tau$ is a left B-module. To complete the proof we show LM 2 is satisfied. Let $u = \Sigma_{i=1}^n x_i \otimes y_i$ and let H be an $(\mathfrak{S}, \mathfrak{T})$-equihypocontinuous subset of $B(E, F)$. There is a neighbourhod W of 0 in E such that

$$|f(W, y_i)| \leqslant 1/n \quad \text{for } i = 1, 2, \ldots, m \text{ and } f \text{ in } H.$$

There is a neighbourhood V of 0 in B such that $Vx_i \subset W$ for $i = 1, 2, \ldots, n$. It follows that $Vu \subset H^0$. This completes the proof.

Definition. Let E and F be locally convex left A-modules. $\mathrm{Hom}_A(E, F)$ is the set of continuous A-module homomorphisms $E \to F$. Let $\mathfrak{S}$ be a set of bounded subsets of E. Then $\mathrm{Hom}_A^{\mathfrak{S}}(E, F)$ is $\mathrm{Hom}_A(E, F)$ with the topology of uniform convergence on subsets in $\mathfrak{S}$.

PROPOSITION 2. *Let $m: E \to F$ and $n: G \to H$ be continuous A-module homomorphisms where E and F are locally convex right A-modules and G and H are locally convex left A-modules. Let $\mathfrak{S}$ (resp. $\mathfrak{T}$) be a set of bounded subsets of E (resp. F), and suppose that for each $S \in \mathfrak{S}$, $m(S) \in \mathfrak{T}$. Then there is a unique continuous linear map $m \otimes n$ of $(E \otimes_A G)_{\tau(\mathfrak{S})}$ into $(F \otimes_A H)_{\tau(\mathfrak{T})}$ such that*

$$(m \otimes n)(x \otimes y) = m(x) \otimes n(y).$$

If, in addition, E and F are locally convex (B, A)-bimodules and m is also a B-module homomorphism then $m \otimes n$ is a B-module homomorphism.

Proof. The map $(x, y) \to m(x) \otimes n(y)$ is A-balanced and $\mathfrak{S}$-hypocontinuous so the existence and uniqueness of the continuous linear map $m \otimes n$ is immediate from Theorem 1. The proof of the second statement is an easy calculation.

PROPOSITION 3. *Let E be a locally convex (A, B)-bimodule and F a locally convex left A-module. Let $\mathfrak{S}$ be a set of bounded subsets of E, covering E and such that $b \in B$ and $S \in \mathfrak{S}$ imply $Sb \in \mathfrak{S}$. If the map $(b, x) \to xb$ is $\mathfrak{S}$-hypocontinuous then $\mathrm{Hom}_A^{\mathfrak{S}}(E, F)$ is a locally convex left B-module.*

Proof. For $m \in \mathrm{Hom}_A^{\mathfrak{S}}(E, F)$ and b in B, define bm by $bm(x) = m(xb)$. Then bm is an A-module homomorphism and the continuity of the map $x \to xb$ implies that bm is continuous. To show that LM 1 holds, let $b \in B$ and let V be a 0-neighbourhood in $\mathrm{Hom}_A^{\mathfrak{S}}(E, F)$ of the form $\{m: m(S) \subset W\}$ where S is in $\mathfrak{S}$ and W is a 0-neighbourhood in F. Let $U = \{m: m(Sb) \subset W\}$, then U is a 0-neighbourhood in $\mathrm{Hom}_A^{\mathfrak{S}}(E, F)$ and $bU \subset V$. To show that LM 2 holds, let $m \in \mathrm{Hom}_A(E, F)$ and V, S and W be as above. There is a 0-neighbourhood U in E such that $m(U) \subset W$. Since the map $(b, x) \to xb$ is $\mathfrak{S}$-hypocontinuous, there is a 0-neighbourhood V' in B such that $SV' \subset U$. It follows that $V'm \subset V$ and this completes the proof.

PROPOSITION 4. *Let E be a locally convex (B, A)-bimodule, $\mathfrak{S}$ a set of bounded subsets of E and F and G locally convex left B-modules. For $n \in \mathrm{Hom}_B(F, G)$ define a map n_*: $\mathrm{Hom}_B^{\mathfrak{S}}(E, F) \to \mathrm{Hom}_B^{\mathfrak{S}}(E, G)$ by $n_*(m) = n \circ m$. Then n_* is a continuous A-module homomorphism.*

Proof. It is clear that n_* is an A-module homomorphism. To show continuity let U be a 0-neighbourhood in $\mathrm{Hom}_B(E, G)$ of the form $\{m: m(S) \subset W\}$ where S is in $\mathfrak{S}$ and W is a 0-neighbourhood in G. There is a 0-neighbourhood W' in F such that $n(W') \subset W$. Then $V = \{m: m(S) \subset W'\}$ is a 0-neighbourhood in $\mathrm{Hom}_B(E, F)$ and $n_*(V) \subset U$.

Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the category of locally convex left A-modules (resp. B-modules) and continuous A-module (resp. B-module) homomorphisms. Let E be a locally convex (B, A)-bimodule and $\mathfrak{S}$ a set of subsets of E such that for $b \in B$, $a \in A$ and $S \in \mathfrak{S}$; $bS \in \mathfrak{S}$ and $Sa \in \mathfrak{S}$. Suppose also that the map $(a, x) \to xa$ is $\mathfrak{S}$-hypocontinuous. Propositions 1–4 tell us that we can define functors $E \otimes_A -: \mathcal{A} \to \mathcal{B}$ and $\mathrm{hom}_B(E, -): \mathcal{B} \to \mathcal{A}$ as follows. For objects F, G in $\mathcal{A}$ and $m \in \mathrm{Hom}_A(F, G)$ define

$$(E \otimes_A -)(F) = (E \otimes_A F)_{\tau(\mathfrak{S})}$$

and

$$(E \otimes_A -)(m) = I_E \otimes m,$$

where I_E is the identity map $E \to E$.

For objects H, K in $\mathcal{B}$ and $n \in \mathrm{Hom}_B(H, K)$ define

$$\mathrm{hom}_B(E, -)(H) = \mathrm{Hom}_B^{\mathfrak{S}}(E, H) \quad \text{and} \quad \mathrm{hom}_B(E, -)(n) = n_*.$$

THEOREM 2. *$E \otimes_A -$ is the adjoint of $\mathrm{hom}_B(E, -)$. That is: for each locally convex left A-module F and for each locally convex left B-module G, there is a natural isomorphism*

$$\varphi_{FG}: \mathrm{Hom}_B(E \otimes_A F, G) \cong \mathrm{Hom}_A(F, \mathrm{hom}_B(E, G))$$

Proof. For $m \in \mathrm{Hom}_B(E \otimes_A F, G)$, define $\varphi_{FG} m$ by $\varphi_{FG} m(y)(x) = m(x \otimes y)$, $x \in E$, $y \in F$. Let χ be the canonical map $E \times F \to E \otimes_A F$. Then $\varphi_{FG} m(y)(x) = m \circ \chi(x, y)$ and $m \circ \chi$ is

$\mathfrak{S}$-hypocontinuous (Theorem 1). It follows easily that $\varphi_{FG}m(y) \in \hom_B(E, G)$. To show that $\varphi_{FG}m$ is continuous let U be a 0-neighbourhood in $\hom_B(E, G)$ of the form $U = \{u: u(S) \subset W\}$ where $S \in \mathfrak{S}$ and W is a 0-neighbourhood in G. Since $m \circ \chi$ is $\mathfrak{S}$-hypocontinuous there is 0-neighbourhood V in F such that $m \circ \chi(S \times V) \subset W$. Then $\varphi_{FG}m(V) \subset U$, which shows that $\varphi_{FG}m$ is continuous.

Let $m' \in \operatorname{Hom}_A(F, \hom_B(E, G))$, and define a bilinear map $m^{\#}: E \times F \to G$ by $m^{\#}(x, y) = m'(y)(x)$. It is easy to see that $m^{\#}$ is A-balanced. The separate continuity of $m^{\#}$ follows from the continuity of m'. Let $S \in \mathfrak{S}$ and V a 0-neighbourhood in G. Since m' is continuous, there is a 0-neighbourhood W in F such that $m'(W) \subset \{u: u(S) \subset V\}$. It follows that $m^{\#}(S \times W) \subset V$ so that $m^{\#}$ is $\mathfrak{S}$-hypocontinuous. By Theorem 1 there is an $m \in L(E \otimes_A F, G)$ such that $m \circ \chi = m^{\#}$. It follows easily that $\varphi_{FG}m = m'$ and that $m \in \operatorname{Hom}_B(E \otimes_A F, G)$. The remainder of the proof is the same as the algebraic case.

Let A be a locally convex algebra with a unit u and consider A as a locally convex (B, A)-bimodule where B is a unitary subalgebra of A. Let $\mathfrak{S}$ be a set of bounded subsets of A and suppose that for $b \in B$, $a \in A$, and $S \in \mathfrak{S}$ we have $bS \in \mathfrak{S}$ and $Sa \in \mathfrak{S}$. Suppose also that the map $(a, c) \to ca$ of $A \times V$ into A is $\mathfrak{S}$-hypocontinuous. Under these hypotheses we have:

PROPOSITION 5. *Let E be a locally convex left A-module and suppose that the map $(a, x) \to ax$ is an $\mathfrak{S}$-hypocontinuous map $A \times E \to E$. There is a bicontinuous B-module isomorphism φ_E: $A \otimes_A E \to E$.*

Proof. Let φ_E be the unique continuous linear map such that $\varphi_E(a, x) = ax$ (Theorem 1). It follows as in the algebraic case that φ_E is an isomorphism. To show that φ_E is bicontinuous we show that if $\varphi_E'(H)$ is an equicontinuous subset of $(A \otimes_A E)'$ then H is an equicontinuous subset of E'. (E' is the dual of E and φ_E' the transpose of φ_E). Let χ be the canonical map $A \times E \to A \otimes_A E$, then $\{\varphi_E'(h) \circ \chi: h \in H\}$ is an $\mathfrak{S}$-equihypocontinuous subset of $B(A, E)$. Let u be the unit of A, then there is a 0-neighbourhood V in E such that for $x \in V$, $h \in H$ we have

$$|h(x)| = |\varphi_E'(h) \circ \chi(u, x)| \leq 1.$$

This completes the proof.

In the next proposition we consider A as a locally convex (A, B)-bimodule.

PROPOSITION 6. *Let E be as in Proposition 5. There is a bicontinuous B-module isomorphism φ_E: $E \to \hom_A(A, E)$.*

Proof. For $x \in E$ let $\varphi_E(x)$ be defined by

$$\varphi_E(x)(a) = ax.$$

Since the map $(a, x) \to ax$ is $\mathfrak{S}$-hypocontinuous we may conclude that $\varphi_E(x)$ is continuous. It is easily seen that $\varphi_E(x) \in \hom_A(A, E)$ and that φ_E is a B-module homomorphism. Let V be a 0-neighbourhood in $\hom_A(A, E)$ of the form $V = \{m: m(S) \subset W\}$ where W is a 0-neighbourhood in E and $S \in \mathfrak{S}$. There is then a 0-neighbourhood U in E such that $a \in S$ and $x \in U$ imply $ax \in W$. Thus $\varphi_E(U) \subset W$. Now define $\Psi_E: \hom_A(A, E) \to E$ by

$$\Psi_E(m) = m(u),$$

where u is the unit of A. Since the $\mathfrak{S}$-topology of $\hom_A(A, E)$ is finer than the topology of pointwise convergence, Ψ_E is continuous. It is straightforward to conclude that Ψ_E is a B-module homomorphism and the $\varphi_E \circ \Psi_E$ and $\Psi_E \circ \varphi_E$ are identity maps. This completes the proof.

We now wish to prove an associativity theorem for our tensor products. In this we take $\mathfrak{S}$ to be the set of finite subsets so the topology on our tensor products is the topology of uniform convergence on the separately equicontinuous subsets of bilinear forms.

PROPOSITION 7. *Let E be a locally convex (A, B)-bimodule, F a locally convex right A-module, and G a locally convex left B-module. Then there is a unique natural bicontinuous isomorphism*

$$\varphi_{FG}: (F \otimes_A E) \otimes_B G \cong F \otimes_A (E \otimes_B G)$$

such that $\varphi_{FG}((x \otimes y) \otimes z) = x \otimes (y \otimes z)$.

Proof. For each $z \in G$, $n_z: y \to y \otimes z$ is a continuous A-module homomorphism of $E \to E \otimes_B G$. Put $n_z = 1_F \otimes n_z$, then m_z is a continuous linear map of $F \otimes_A E$ into $F \otimes_A (E \otimes_B G)$ (Proposition 2). The map $(x, z) \to m_z(x)$ of $(F \otimes_A E) \times G$ into $F \otimes_A (E \otimes_B G)$ is A-balanced bilinear and separately continuous, thus by Theorem 1 there is a unique continuous linear map

$$\varphi_{FG}: (F \otimes_A E) \otimes_B G \to F \otimes_A (E \otimes_B G)$$

such that $\varphi_{FG}(x \otimes z) = m_z(x)$ which immediately gives $\varphi_{FG}((x \otimes y) \otimes z) = x \otimes (y \otimes z)$. In an analogous manner one can define a continuous linear map

$$\Psi_{FG}: F \otimes_A (E \otimes_B G) \to (F \otimes_A E) \otimes_B G$$

such that $\Psi_{FG}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z$. It follows that $\varphi_{FG} \circ \Psi_{FG}$ and $\Psi_{FG} \circ \varphi_{FG}$ are identity maps. The remainder of the proof is straightforward.

We close this section with a result which will be used later. Let A be a locally convex $\sim$-algebra, i.e. A is a locally convex algebra having a continuous map $a \to a^{\sim}$ satisfying

$(a^{\sim})^{\sim} = a$; $(\alpha a + \beta b)^{\sim} = \bar{\alpha} a^{\sim} + \bar{\beta} b^{\sim}$; and $(ab)^{\sim} = b^{\sim} a^{\sim}$. Let E be a locally convex left A-module. For $x' \in E'$ and $a \in A$ we define ax' by $ax'(x) = x'(a^{\sim} x)$. It follows that E' is a left A-module which we write as E^c. Let $\mathfrak{S}$ be a set of bounded subsets of E such that $\bigcup \mathfrak{S} = E$. Let E^c be E^c with the topology of uniform convergence on sets in $\mathfrak{S}$.

PROPOSITION 8. *If $S \in \mathfrak{S}$ implies $aS \in \mathfrak{S}$ for each a in A, and if the map $(a, x) \to ax$ is $\mathfrak{S}$-hypocontinuous, then E^c is a locally convex left A-module.*

Proof. Let $S \in \mathfrak{S}$, and $a \in A$; then $a^{\sim} S \in \mathfrak{S}$ and $a(a^{\sim} S)^0 \subset S^0$ so that the map $x' \to ax'$ is continuous. Let $x' \in E^c$, then $\{x'\}^0$ is a 0-neighbourhood in E so there is a 0-neighbourhood V in A such that $VS \subset \{x'\}^0$. Since the map $a \to a^{\sim}$ is continuous there is a 0-neighbourhood W in A such that $W^{\sim} \subset V$. For $a \in W$, and $x \in S$ we have $|x'(a^{\sim} x)| \leq 1$. This means that $Wx' \subset S^0$ so that $a \to ax'$ is a continuous map.

3. Induced representations of locally convex algebras

In this section, A is a locally convex algebra having a unit u, and B is a unitary subalgebra of A.

Let E be a locally convex left A-module, then by restricting the map $(a, x) \to ax$ to $B \times E$, E is a locally convex left B-module which we shall write as E_B. In this way we obtain a functor from the category of locally convex left A-modules to the category of locally convex left B-modules which we shall call the restriction functor. We apply the results of section 2 to show that the restriction functor has both an adjoint and a coadjoint. The adjoint is the functor $A \otimes_B -$ and the coadjoint is $\hom_B(A, -)$, defined in § 2. (Here we take $\mathfrak{S}$ to be the set of finite subsets of A, and we consider A as an (A, B)-bimodule in the first case, and as an (B, A)-bimodule in the second).

THEOREM 3. *The functor $A \otimes_B -$ is the adjoint of the restriction functor, and the functor $\hom_B(A, -)$ is the coadjoint.*

Proof. By Theorem 2, we have

$$\mathrm{Hom}_A(A \otimes_B E, F) \cong \mathrm{Hom}_B(E, \hom_A(A, F)).$$

By Proposition 6, $\hom_A(A, F)$ is topologically isomorphic to F_B and it is easy to see that this isomorphism is natural. This shows that $A \otimes_B -$ is the adjoint of the restriction functor. Again by Theorem 2, we have

$$\mathrm{Hom}_B(A \otimes_A E, F) \cong \mathrm{Hom}_A(E, \hom_B(A, F)).$$

By Proposition 5, $A\otimes_A E$ is topologically isomorphic to E_B and it is seen that this isomorphism is natural. Thus $\hom_B(A, -)$ is the coadjoint of the restriction functor.

In future we shall write ${}^A E$ in place of $A\otimes_B E$ and E^A in place of $\hom_B(A, E)$. In this notation the results of Theorem 3 can be written

$$\mathrm{Hom}_A({}^A E, F) \cong \mathrm{Hom}_B(E, F_B)$$

and
$$\mathrm{Hom}_A(F, E^A) \cong \mathrm{Hom}_B(F_B, E).$$

COROLLARY. *Let A be a locally convex algebra and B, C unitary subalgebras of A. Let E be a locally convex left B-module and F a locally convex left C-module. Then*

$$\mathrm{Hom}_C(({}^A E)_C, F) \cong \mathrm{Hom}_B(E, (F^A)_B).$$

This next theorem concerns induction in stages.

THEOREM 4. *Let C be a unitary subalgebra of B. Then*

$${}^A E \cong {}^A({}^B E) \quad \text{and} \quad E^A \cong (E^B)^A$$

for any locally convex left C-module E, and the isomorphism is natural.

Proof. Viewing B as a locally convex (B, C)-module we have

$$\begin{aligned} {}^A({}^B E) &= A\otimes_B(B\otimes_C E) \\ &\cong (A\otimes_B B)\otimes_C E \qquad \text{(Proposition 7)} \\ &\cong A\otimes_C E \qquad \text{(Proposition 5)} \end{aligned}$$

and this proves the first assertion. For the second assertion define a map

$$\varphi_E\colon E^A = \hom_C(A, E) \to \hom_B(A, \hom_C(B, E)) = (E^B)^A$$

by
$$\varphi_E m(a)(b) = m(ba) \quad m\in E^A,\ a\in A,\ b\in B.$$

Since multiplication is continuous it is immediate that $\varphi_E m(a)\in\hom_C(B, E)$. To show that $\varphi_E m$ is continuous let U be a 0-neighbourhood in $\hom_C(B, E)$ of the form $U=\{u\colon u(b)\in W\}$ where $b\in B$ and W is a 0-neighbourhood of E. There is then a 0-neighbourhood V in A such that $m(bV)\subset W$. Then $\varphi_E m(V)\subset U$ which shows that $\varphi_E m$ is continuous. We now show that φ_E is continuous. Let U be a 0-neighbourhood in $(E^B)^A$ of the form $\{u\colon u(a)(b)\in W\}$ where W is a 0-neighbourhood in E and $a\in A$, $b\in B$. Let $V=\{m\in E^A\colon m(ba)\in W\}$, then V is a 0-neighbourhood in E^A and $\varphi_E(V)\subset U$.

Now define a map $\qquad \Psi_E\colon (E^B)^A \to E^A$

by
$$\Psi_E m(a) = m(a)(u), \quad m \in (E^B)^A, \; a \in A,$$

and u is the unit of A. It is clear that the map is well defined. Let U be a 0-neighbourhood in E^A of the form $V = \{u: u(a) \in W\}$ where W is a 0-neighbourhood in E. Then $V = \{m \in (E^B)^A: m(a)(u) \in W\}$ is a 0-neighbourhood in $(E^B)^A$ and $\Psi_E(V) \subset U$. Thus Ψ_E is continuous. Observe that $\Psi_E \circ \varphi_E$ and $\varphi_E \circ \Psi_E$ are identity maps and the remainder of the proof is algebraic.

Let A be a locally convex $\tilde{}$-algebra and E a locally convex left A-module. The contragradient module E^c of E is the module E^c (see § 2) given the $\sigma(E', E)$-topology. By Proposition 8, E^c is a locally convex left A-module.

Proposition 9. *Let A be a locally convex $\tilde{}$-algebra having a unit, and B a unitary $\tilde{}$-subalgebra of A. Considering A as a left B-module via the map $(b, a) \to ab^{\sim}$, and as a right B-module via the map $(a, b) \to ab$, we have for any locally convex left B-module E*

$$(E^c)^A \cong ({}^A E)^c$$

the isomorphism being natural and bicontinuous.

Proof. For $a \in A$, $x \in E$ and $m' \in (A \otimes_B E)^c$, put

$$\Phi_E m'(a)(x) = m'(a \otimes x).$$

It follows by methods analogous to the proof of Theorem 3 that Φ_E is a natural algebraic isomorphism of $(A \otimes_B E)^c$ onto $\hom_B(A, E^c)$. To show that Φ_E is continuous let V be a 0-neighbourhood in hom (A, E^c) of the form

$$V = \{m: m(S_0) \subset (S_1)^0\},$$

where $S_0 \subset A$ and $S_1 \subset E$ are finite sets. Then $S = S_0 \otimes S_1$ is a finite set and $\Phi_E(S^0) \subset V$. To show that Φ_E is open, let S be a finite set in $A \otimes_B E$, then there are finite sets S_0, S_1 such that
$$S \subset S_2 = \{\Sigma a_i \otimes x_i: a_i \in S_0, \; x_i \in S_1\}.$$

For $u \in S$ let $r(u)$ be the minimal number of summands $a_i \otimes x_i$ such that $u = \Sigma a_i \otimes x_i$. Let $n = \sup \{r(u): u \in S\}$. Since S is finite, so is n, moreover

$$[n(S_0 \otimes S_1)]^0 \subset S^0$$

and it follows that

$$\{m: m(nS_0) \subset S_1^0\} \subset \Phi_E(S^0)$$

so that Φ_E is open.

Definition. Let A be a locally convex $\tilde{}$-algebra and let E, F be locally convex left A-modules. An intertwining form for E and F is a bilinear form f on $E \times F$ such that

$$f(ax, y) = f(x, a^{\sim} y).$$

Let $J(E, F)$ be the set of all separately continuous intertwining forms on $E \times F$. For $f \in J(E, F)$ let m: $F \to E^c$ be defined by

$$\langle x, my \rangle = f(x, y).$$

It follows easily that m is a continuous module homomorphism. Conversely every $m \in \mathrm{Hom}_A(F, E^c)$ is of this form. It then follows that

$$J(E, F) \cong \mathrm{Hom}_A(F, E^c).$$

THEOREM 5. *Let A be a locally convex $\tilde{}$-algebra with unit u, and B a unitary $\tilde{}$-subalgebra of A. If E is a locally convex left B-module and F a locally convex left A-module, then*

$$J({}^{A}E, F) \cong J(E, F_B)$$

Proof. This is immediate from Theorem 3 and Proposition 9.

4. Representation modules

Definition. Let Γ be a locally compact group and E a locally convex space. A linear representation π (i.e. a homomorphism of Γ into a multiplicative group in $L(E)$) is said to be

(a) continuous if the map $(\gamma, x) \to \pi(\gamma)x$ of $\Gamma \times E$ into E is continuous

(b) separately continuous if for each x in E the map $\gamma \to \pi(\gamma)x$ is continuous

(c) weakly continuous if the map $\gamma \to \langle \pi(\gamma)x, x' \rangle$ is continuous for each x in E and x' in E'.

Note that if E is barreled then (a) and (b) are equivalent [4; Chapitre 8, § 2, Proposition 1] and if E is a Banach space then (a), (b) and (c) are equivalent [Anonymous].

In the sequel we will be concerned only with continuous and separately continuous representations. Note that the study of separately continuous representations on locally convex spaces includes the study of weakly continuous representations.

Let $C(\Gamma)$ be the space of continuous complex-valued functions on Γ with the topology of uniform convergence on compact subsets of Γ. Let $M_c(\Gamma) = C(\Gamma)'$; $M_c(\Gamma)$ is the space of regular Borel measures on Γ having compact support. Throughout the following we shall take $M_c(\Gamma)$ with the topology of uniform convergence on the compact subsets of $C(\Gamma)$.

Let E be a locally convex space and suppose that for any compact subset K of E, the closed convex hull of K is compact. If π is a continuous representation of Γ on E then E can be given the structure of a locally convex left $M_c(\Gamma)$-module in the following way. For μ in $M_c(\Gamma)$ and x in E, define μx in E'^* by

$$\langle \mu x, x' \rangle = \int \langle \pi(\gamma)\, x, x' \rangle \, d\mu\,(\gamma) \qquad (*)$$

then $\mu x \in E$ and the map $(\mu, x) \to \mu x$ is hypocontinuous relative to the equicontinuous subsets of $M_c(\Gamma)$ and the compact subsets of E [4; Chapitre 6, § 1, Remarque 2 following Proposition 14 and Proposition 16]. This motivates the following.

Definition. A locally convex left $M_c(\Gamma)$-module E is called a locally convex continuous Γ-module if the map $(\mu, x) \to \mu x$ is hypocontinuous relative to the equicontinuous subsets of $M_c(\Gamma)$.

We will show later that the map $(\mu, x) \to \mu x$ is also hypocontinuous with respect to the compact subsets of $M_c(\Gamma)$.

A locally convex left $M_c(\Gamma)$-module will be called a locally convex Γ-module. We shall write $\hom_\Gamma$ in place of $\hom_{M_c(\Gamma)}$, and a similar convention applies to tensor products etc.

For $\gamma \in \Gamma$, let ε_γ be the Dirac measure at γ; i.e. $\varepsilon_\gamma(f) = f(\gamma)$ for f in $C(\Gamma)$, and let Γ^ε be the subset of $M_c(\Gamma)$ of all Dirac measures. It is known [4; Chapitre 6, § 1, Remarques 1 following Proposition 14] that the map $\gamma \to \varepsilon_\gamma$ is a homeomorphism of Γ onto Γ^ε and that Γ^ε is total in $M_c(\Gamma)$. Thus if E is a locally convex Γ-module, then the continuity of the map $\mu \to \mu x$ of $M_c(\Gamma)$ into E implies that the formula $(*)$ holds [*ibid.*]. We now show the connections between locally convex (continuous) Γ-modules and separately continuous (continuous) representations having "integrated forms".

Definition. A linear representation π of a locally compact group Γ on a locally convex space E is said to have an integrated form if for each $\mu \in M_c(\Gamma)$ there is a linear map $x \to \mu x$ of E into itself such that for $x \in E$, $x' \in E'$

$$\langle \mu x, x' \rangle = \int_\Gamma \langle \pi(\gamma)\, x, x' \rangle \, d\mu(\gamma).$$

PROPOSITION 10. *Let π be a separately continuous representation of Γ in a locally convex space E and suppose that π has an integrated form, and that the map $x \to \mu x$ is continuous for each $\mu \in M_c(\Gamma)$. Then E is a locally convex Γ-module.*

Proof. We need to show that the map $\mu \to \mu x$ is continuous for each x in E. Give $L(E)$ the topology of simple convergence, then π is a continuous map of Γ into $L(E)$. For $z' \in L(E)'$ and $\mu \in M_c(\Gamma)$ let $\int \pi \, d\mu$ be the element of $L(E)'^*$ defined by

$$\langle z', \int \pi d\mu \rangle = \int_\Gamma z' \circ \pi d\mu.$$

We first show that $\int \pi d\mu \in L(E)$. Now $E \otimes_C E'$ is algebraically isomorphic to $L(E)'$ via the map $x \otimes x' \to z'$ where for $m \in L(E)$, $\langle m, z' \rangle = \langle mx, x' \rangle$ [3; Chapitre IV, § 2, No. 9, cor. de la prop. 11]. Thus

$$\langle z', \int \pi d\mu \rangle = \int z' \circ \pi d\mu = \int \langle \pi(\gamma) x, x' \rangle \, d\mu = \langle \mu x, x' \rangle.$$

Therefore $\int \pi d\mu$ is the map $x \to \mu x$ and it follows that $\int \pi d\mu \in L(E)$. The proof is now completed by Remarque 2 following Proposition 14 of [4; Chapitre 6, § 1].

A stronger assertion for continuous representations will follow from the next proposition.

PROPOSITION 11. *Let E be a locally convex Γ-module. The following conditions on E are equivalent.*

(a) *E is a locally convex continuous Γ-module.*

(b) *The map $(\varepsilon_\gamma, x) \to \varepsilon_\gamma x$ of $\Gamma^\varepsilon \times E$ into E is continuous.*

(c) *For each compact subset K of Γ, the set of maps $\{x \to \varepsilon_\gamma x \colon \gamma \in K\}$ is an equicontinuous subset of $L(E)$.*

Proof. Suppose (a) holds and let K be any compact subset of Γ, then $K^\varepsilon = \{\varepsilon_\gamma \colon \gamma \in K\}$ is an equicontinuous subset of $M_c(\Gamma)$ and the map $(\varepsilon_\gamma, x) \to \varepsilon_\gamma x$ is continuous as a map of $K^\varepsilon \times E$ into E. Since Γ^ε is homeomorphic to Γ and Γ is locally compact this proves (b).

For each 0-neighbourhood V in E, and each γ in K, there is by (b) a neighbourhood V_γ in Γ^ε and a 0-neighbourhood W_γ in E such that $\varepsilon_{\gamma'} \in V_\gamma$ and $x \in W_\gamma$ implies $\varepsilon_{\gamma'} x \in V$. Cover $K^\varepsilon = \{\varepsilon_\gamma \colon \gamma \in K\}$ by a finite number of neighbourhoods V_{γ_i} and let $W = \bigcap_{i=1}^n W_{\gamma_i}$. Then $\varepsilon_\gamma x \in V$ whenever $\gamma \in K$ and $x \in W$. This shows (b) implies (c).

Let V be a convex circled closed 0-neighbourhood in E and let H be an equicontinuous subset of $M_c(\Gamma)$; we may suppose $H = C^0$ where $C = \{f \in C(\Gamma) \colon |f(\gamma)| \leq 1, \gamma \in K\}$ and K is a compact subset of Γ. Now (c) implies that $\{x \to \varepsilon_\gamma x \colon \gamma \in K\}$ is equicontinuous, so there is a 0-neighbourhood W in E such that $\gamma \in K$ and $x \in W$ imply $\varepsilon_\gamma x \in V$. Thus if $x \in W$ and $x' \in V^0$, then $\gamma \to \langle \varepsilon_\gamma x, x' \rangle$ is in C, and consequently for $\mu \in H$

$$|\langle \mu x, x' \rangle| = \left| \int \langle \varepsilon_\gamma x, x' \rangle \, d\mu(\gamma) \right| \leq 1.$$

This shows that (c) implies (a). The proof is complete.

COROLLARY. *Let π be a continuous representation of Γ on a locally convex space E and suppose that π has an integrated form. Then E is a locally convex continuous Γ-module.*

Proof. This is an immediate consequence of the above Proposition and No. 7, of Chapitre 6, § 1 of [4].

PROPOSITION 12. *Let E be a locally convex continuous Γ-module. Then the map $(\mu, x) \to \mu x$ of $M_c(\Gamma) \times E$ into E is hypocontinuous relative to the compact subsets of E.*

Proof. Let C be a compact subset of E and V a convex circled closed 0-neighbourhood in E; it is sufficient to find a relatively compact subset $L \subset C(\Gamma)$ such that $\mu \in L^0$, $x \in C$ and $x' \in V^0$ imply $|\langle \mu x, x' \rangle| \leq 1$. For this we show that

$$L = \{\gamma \to \langle \varepsilon_\gamma x, x' \rangle : x \in C,\ x' \in V^0\}$$

is relatively compact in $C(\Gamma)$. Given $\gamma \in \Gamma$, $\{\varepsilon_\gamma x : x \in C\}$ is a compact subset of E, hence absorbed by V and consequently

$$L(\gamma) = \{\langle \varepsilon_\gamma x, x' \rangle : x \in C,\ x' \in V^0\}$$

is relatively compact subset of $\mathbf{C}$. Thus by Ascoli's theorem, to show L is relatively compact it is sufficient to show that L is equicontinuous. For any $\gamma \in \Gamma$ let $\pi(\gamma)$ be the map $x \to \varepsilon_\gamma x$. Then by (c) of Proposition 11, for any compact set $K \subset \Gamma$, $\{\pi(\gamma): \gamma \in K\}$ is an equicontinuous subset of $L(E)$. By the definition of a locally convex module the map $\gamma \to \pi(\gamma)$ is continuous when one gives $L(E)$ the topology of simple convergence. Since this topology coincides with the topology of compact convergence on equicontinuous sets [3: Chapitre 3, § 3, Proposition 5] the local compactness of Γ implies, that $\gamma \to \pi(\gamma)$ is continuous into $L(E)$ given this latter topology. Thus given $\gamma_0 \in \Gamma$ and $\delta > 0$ there is a neighbourhood U of γ_0 such that $\gamma \in U$ and $x \in C$ implies

$$\varepsilon_\gamma x - \varepsilon_{\gamma_0} x \in \delta V.$$

Thus $\gamma \in U$, $x \in C$ and $x' \in V^0$ imply

$$|\langle \varepsilon_\gamma x, x' \rangle - \langle \varepsilon_{\gamma_0} x, x' \rangle| \leq \delta$$

so that L is equicontinuous. This proves the Proposition since

$$\int \langle \varepsilon_\gamma x, x' \rangle \, d\mu(\gamma) = \langle \mu x, x' \rangle.$$

COROLLARY. *Let f be any Γ-balanced bilinear form on $M_c(\Gamma) \times E$ which is hypocontinuous relative to the equicontinuous subsets of $M_c(\Gamma)$. Then f is hypocontinuous relative to the compact subsets of E.*

Proof. Since f is separately continuous there is a 0-neighbourhood V in E such that $x \in V$ implies $|f(\varepsilon, x)| \leqslant 1$. Given any compact $C \subset E$, there is by the above proposition a 0-neighbourhood W in $M_c(\Gamma)$ such that $WC \subset V$. Thus $\mu \in W$ and $x \in C$ imply

$$|f(\mu, x) = |f(\varepsilon, \mu x)| \leqslant 1.$$

We now turn our attention towards $C(\Gamma)$ and $M_c(\Gamma)$. We show that if Δ, Δ' are closed subgroups of Γ, then $M_c(\Gamma)$ is a locally convex, continuous (Δ, Δ')-bimodule. We begin with some preliminary results.

For f in $C(\Gamma)$ and γ in Γ, we define f_γ and ${}_\gamma f$ by $f_\gamma(\gamma') = f(\gamma'\gamma)$ and ${}_\gamma f(\gamma') = f(\gamma\gamma')$. It is clear that f_γ and ${}_\gamma f$ are in $C(\Gamma)$.

LEMMA 1. *The maps* $(\gamma, f) \to {}_\gamma f$ *and* $(\gamma, f) \to f_\gamma$ *are continuous maps of* $\Gamma \times C(\Gamma)$ *into* $C(\Gamma)$.

Proof. This is an immediate consequence of [4; Chapitre 8, § 2, Lemma 3].

For $f \in C(\Gamma)$ and $\mu \in M_c(\Gamma)$ we define a function $\bar{\mu}(f)$ on Γ by $\bar{\mu}(f)(\gamma) = \mu({}_\gamma f)$.

LEMMA 2. *Let* C *be a compact subset of* $C(\Gamma)$ *and* $\mu \in M_c(\Gamma)$, *then* $\{\bar{\mu}(f): f \in C\}$ *is a compact subset of* $C(\Gamma)$.

Proof. First note that it follows from Lemma 1 that $\bar{\mu}(f)$ is continuous. To show that $\{\bar{\mu}(f): f \in C\}$ is compact we show that the map $f \to \bar{\mu}(f)$ is continuous. Let $K = \operatorname{Supp}(\mu)$, given $\varepsilon > 0$ and a compact set K_0, there is by Lemma 1 for each $\gamma' \in K \cup K_0$, a neighbourhood $V_{\gamma'}$ of γ', and a neighbourhood $W_{\gamma'}$ of f such that for $\gamma \in V_{\gamma'}$ and $g \in W_{\gamma'}$ we have

$$|{}_{\gamma'} f(\gamma'') - {}_\gamma g(\gamma'')| \leqslant \varepsilon/2\|\mu\|$$

for all $\gamma'' \in K_0 \cup K$. It follows that $g \in W_{\gamma'}$ and $\gamma \in V_{\gamma'}$ imply

$$|\bar{\mu}(f)(\gamma') - \bar{\mu}(g)(\gamma)| \leqslant \varepsilon/2.$$

Since $K \cup K_0$ is compact there is a finite set $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that $K \cup K_0 \subset \bigcup_{i=1}^n V_{\gamma_i}$. Let $W = \bigcap_{i=1}^n W_{\gamma_i}$. For $\gamma \in K_0$ and $g \in W$ we have

$$|\bar{\mu}(f)(\gamma) - \bar{\mu}(g)(\gamma)| \leqslant \varepsilon.$$

This completes the proof.

For $f \in C(T)$, let f^* be the function defined by $f^*(\gamma) = f(\gamma^{-1})^-$ ($^-$ means complex conjugate). It is clear that $f^* \in C(\Gamma)$. For $\mu \in M_c(\Gamma)$, define $\mu^\sim$ by $\mu^\sim(f) = \mu(f^*)^-$. The map $\mu \to \mu^\sim$ is an involution in $M_c(\Gamma)$.

PROPOSITION 14. $M_c(\Gamma)$ *is a locally convex algebra with a continuous involution* $\mu^\sim \to \mu$.

Proof. We first show that the map $\nu \to \nu * \mu$ is continuous. Let C be a compact subset of $C(\Gamma)$ and let $C' = \{\bar{\mu}(f): f \in C\}$. By Lemma 2, C' is compact and $C'^0 * \mu \subset C^0$ since $\nu * \mu(f) = \nu(\bar{\mu}(f))$. To show that the map $\nu \to \mu * \nu$ is continuous it suffices to show that the map $\mu \to \mu^{\sim}$ is continuous since $\nu \to \mu * \nu$ can be written as $\nu \to \nu^{\sim} \to \nu^{\sim} * \mu^{\sim} = (\mu * \nu)^{\sim} \to \mu * \nu$. To show that $\mu \to \mu^{\sim}$ is continuous, we show that if C is a compact subset of $C(\Gamma)$ then $C^* = \{f^*: f \in C\}$ is compact. For this it is enough to show that the map $f \to f^*$ is continuous. Given $f \in C(\Gamma)$, $\varepsilon > 0$ and a compact set $K \subset \Gamma$, K^{-1} is compact, and $|f(\gamma)| \leqslant \varepsilon$ for all $\gamma \in K^{-1}$ implies $|f^*(\gamma)| \leqslant \varepsilon$.

PROPOSITION 15. *Let Δ be a closed subgroup of Γ. The subspace $M_c(\Gamma, \Delta)$ of $M_c(\Gamma)$ of measures whose support is in Δ is closed in $M_c(\Gamma)$.*

Proof. It is sufficient to show that $M_c(\Gamma, \Delta)$ is $\sigma(M_c(\Gamma), C(\Gamma))$-closed. For $f \in C(\Gamma)$, let $A_f = \{\mu \in M_c(\Gamma): \mu(f) = 0\}$. Then A_f is $\sigma(M_c(\Gamma), C(\Gamma))$-closed and $M_c(\Gamma, \Delta) = \bigcap \{A_f: \operatorname{Supp}(f) \cap \Delta = \Phi\}$.

Let Δ, Δ' be closed subgroups of Γ. Then Δ, Δ' are locally compact groups and we can identify $M_c(\Delta)$ and $M_c(\Delta')$ with closed subalgebras of $M_c(\Gamma)$. Propositions 14 and 15 then yield that $M_c(\Gamma)$ is a locally convex (Δ, Δ')-bimodule. More is true.

PROPOSITION 16. *Let Δ, Δ' be closed subgroups of Γ. Then $M_c(\Gamma)$ is a locally convex continuous (Δ, Δ')-bimodule.*

Proof. By the above remarks, and Proposition 11, it is sufficient to show that for each compact subset K of Δ, the set of maps $\{\mu \to \varepsilon_\gamma * \mu: \gamma \in K\}$, is equicontinuous; and for each compact subset K' of Δ', the set of maps $\{\mu \to \mu * \varepsilon_\gamma: \gamma \in K'\}$ is equicontinuous. Let C be a compact subset of $C(\Gamma)$, then $KC = \{{}_\gamma f: \gamma \in K, f \in C\}$ and $CK' = \{f_\gamma: \gamma \in K', f \in C\}$ are compact by Lemma 1. Since $\varepsilon_\gamma * (KC)^0 \subset C^0$ for all $\gamma \in K$ and $(CK')^0 * \varepsilon_\gamma \subset C^0$ for all $\gamma \in K'$, the proof is complete.

5. Induced separately continuous representations

Let Γ be a locally compact group and Δ a closed subgroup of Γ. Then $M_c(\Delta)$ is a unitary subalgebra of $M_c(\Gamma)$, so we can immediately apply the results of § 3 to the categories of locally convex Γ-modules and locally convex Δ-modules.

THEOREM 6. *Let Δ be a closed subgroup of the locally compact group Γ then the restriction functor from the category of locally convex Γ-modules to the category of locally convex Δ-modules has an adjoint and coadjoint. The adjoint is the functor $M_c(\Gamma) \otimes_\Delta -$ and the coadjoint is the functor $\hom_\Delta(M_c(\Gamma), -)$.*

Proof. This is a consequence of Theorem 3 and Proposition 14.

COROLLARY. *Let* Δ, Δ' *be closed subgroups of* Γ. *Then*

$$\mathrm{Hom}_{\Delta'}\,(({}^{\Gamma}E)_{\Delta'}, F) \cong \mathrm{Hom}_{\Delta}\,(E, (F^{\Gamma})_{\Delta})$$

for any locally convex Δ'*-module* F *and any locally convex* Δ*-module* E.

THEOREM 7. *Let* Δ' *be a closed subgroup of* Δ. *Then*

$${}^{\Gamma}E \cong {}^{\Gamma}({}^{\Delta}E) \quad and \quad E^{\Gamma} \cong (E^{\Delta})^{\Gamma}$$

for any locally convex Δ'*-module* E.

In terms of intertwining forms, we have the following:

THEOREM 8. *Let* Δ *be a closed subgroup of* Γ. *For any locally convex* Δ*-module* E *and any locally convex* Γ*-module* F *we have*

$$J({}^{\Gamma}E, F) \cong J(E, F_{\Delta}).$$

Proof. This is immediate from Theorem 5 and Proposition 14.

6. The representation of E^{Γ}

Having established our principal results for locally convex Γ-modules, we devote this section to the representation of the locally convex Γ-module E^{Γ}. Our purpose in doing this is to display the connection of our results with the more classical results in this area.

Let Δ be a closed subgroup of the locally compact group Γ and let E be a locally convex Δ-module. Let $C(\Gamma, \Delta, E)$ be the space of continuous maps f of Γ into E which satisfy

$$f(\delta\gamma) = \delta[f(\gamma)] \quad \text{for } \delta \in \Delta \text{ and } \gamma \in \Gamma.$$

We define a topology on $C(\Gamma, \Delta, E)$ as follows. For each 0-neighbourhood V in E and each μ in $M_c(\Gamma)$, let $p_{\mu, V}(f) = \sup\{|\int\langle f(\gamma), x'\rangle d\mu(\gamma)|: x' \in V^0\}$ and we give $C(\Gamma, \Delta, E)$ the topology generated by these seminorms. Observe that if $\int f d\mu \in E$ for each $f \in C(\Gamma, \Delta, E)$ then this topology is the coarsest such that the map $f \to \int f d\mu$ is continuous for each μ in $M_c(\Gamma)$.

In order to define a Γ-module structure on E we shall have to impose some restriction on E.

Definition. A locally convex space E is said to satisfy condition (K) if the closed convex hull of each compact subset of E is compact.

Note that if E is complete, or quasicomplete, then E satisfies condition (K).

Whenever E satisfies condition (K) then for each $f \in C(\Gamma, \Delta, E)$ and each $\mu \in M_c(\Gamma)$ we define a map $\mu f: \Gamma \to E$ by

$$\langle \mu f(\gamma), x' \rangle = \int_\Gamma \langle f(\gamma\gamma'), x' \rangle \, d\mu(\gamma).$$

[4; Chapitre 6, § 1, Proposition 8]. (In Bourbaki's notation $\mu f(\gamma) = \int_\gamma f \, d\mu$.)

PROPOSITION 17. *Let Δ be a closed subgroup of the locally compact group Γ, and let E be a locally convex Δ-module which satisfies condition* (K). *For $f \in C(\Gamma, \Delta, E)$ and $\mu \in M_c(\Gamma)$, $\mu f \in C(\Gamma, \Delta, E)$, and $C(\Gamma, \Delta, E)$ is a locally convex Γ-module.*

Proof. We first show μf is continuous whenever $f \in C(\Gamma, \Delta, E)$ and $\mu \in M_c(\Gamma)$. Let $K = \operatorname{Supp}(\mu)$. Given a convex circled 0-neighbourhood U in E and a γ_0 in Γ, the continuity of f_γ gives for each γ in K a neighbourhood W_γ of the unit ε in Γ such that $\gamma' \in \gamma_0 W_\gamma$ implies

$$f(\gamma'\gamma) - f(\gamma_0\gamma) \in (1/2\|\mu\|)\, U.$$

There are symmetric neighbourhoods V_γ such that $V_\gamma V_\gamma \subset W_\gamma$. Since K is compact there is a finite subset $\gamma_1, \gamma_2, \ldots, \gamma_n$ of K such that

$$K \subset \bigcup_{i=1}^{n} V_{\gamma_i}\gamma_i.$$

Let $V = \bigcap V_{\gamma_i}$. Let $\gamma' \in \gamma_0 V$ and $\gamma \in K$. Then there is an i such that $\gamma \in V_{\gamma_i}\gamma_i$ and hence

$$\gamma'\gamma\gamma_i^{-1} \in \gamma_0 V V_{\gamma_i} \subset \gamma_0 W_{\gamma_i}.$$

It follows that

$$f(\gamma'\gamma) - f(\gamma_0\gamma_i) \in (1/2\|\mu\|)\, U.$$

Moreover,

$$\gamma_0\gamma \in \gamma_0 V_{\gamma_i}\gamma_i \subset \gamma_0 W_{\gamma_i}\gamma_i$$

so that

$$f(\gamma_0\gamma) - f(\gamma_0\gamma_i) \in (1/2\|\mu\|)\, U.$$

It follows that for $\gamma' \in \gamma_0 V$ and any $\gamma \in K$ we have

$$f(\gamma'\gamma) - f(\gamma_0\gamma) \in (1/\|\mu\|)\, U.$$

Then for $x' \in U^0$, $\gamma \in \gamma_0 V$ we have

$$|\langle \mu f(\gamma) - \mu f(\gamma_0), x' \rangle| \leqslant \int |\langle f(\gamma\gamma') - f(\gamma_0\gamma'), x' \rangle| \, d|\mu|(\gamma') \leqslant 1.$$

Hence $\mu f(\gamma) \in \mu f(\gamma_0) + U$. This shows that μf is continuous. To show that μf is in $C(\Gamma, \Delta, E)$ let $\delta \in \Delta$ and $\gamma \in \Gamma$. Then

$$\langle \mu f(\delta\gamma), x' \rangle = \int_\Gamma \langle f(\delta\gamma\gamma'), x' \rangle \, d\mu(\gamma') = \int \langle f(\gamma\gamma'), \delta^{-1}x' \rangle \, d\mu(\gamma')$$
$$= \langle \mu f(\gamma), \delta^{-1}x' \rangle = \langle \delta[\mu f(\gamma)], x' \rangle.$$

To show that the maps $f \to \mu f$ and $\mu \to \mu f$ are continuous, note that

$$\int_\Gamma \langle \mu f(\gamma), x' \rangle \, d\nu(\gamma) = \int_\Gamma \langle f(\gamma), x' \rangle \, d\nu * \mu(\gamma). \tag{$*$}$$

It follows immediately from $(*)$ that the map $f \to \mu f$ is continuous. To show that the map $\mu \to \mu f$ is continuous it suffices in virtue of $(*)$ and of the continuity of the map $\mu \to \nu * \mu$ [Proposition 14], to show that for any given 0-neighbourhood V in E, the set of maps $H = \{\gamma \to \langle f(\gamma), x' \rangle: \ x' \in V^0\}$ is relatively compact in $C(\Gamma)$. Clearly for each $\gamma, H(\gamma) = \{\langle f(\gamma), x' \rangle: x \in V^0\}$ is relatively compact since V is absorbing. Since f is continuous it follows that H is equicontinuous. Thus by Ascoli's theorem H is relatively compact. The remainder of the proof is straightforward.

By the above Proposition we can define a functor $\mathfrak{f}$ from the category of locally convex Δ-modules satisfying condition (K) to the category of locally convex Γ-modules by

$$\mathfrak{f}(E) = C(\Gamma, \Delta, E) \quad \text{and} \quad \mathfrak{f}(m)(f) = m \circ f$$

for $m \in \hom_\Delta(E, F)$ and $f \in C(\Gamma, \Delta, E)$. The functor $\hom_\Delta(M_c(\Gamma), -)$ can be considered as a functor between these categories. For each locally convex Δ-module E satisfying condition (K), define a map φ_E: $E^\Gamma \to C(\Gamma, \Delta, E)$ by $\varphi_E m(\gamma) = m(\varepsilon_\gamma)$ for $\gamma \in \Gamma$.

THEOREM 9. *Let Δ be a closed subgroup of the locally compact group Γ. Let E be a locally convex Δ-module satisfying condition* (K). *Then the map φ_E:*

$$\varphi_E: E^\Gamma \cong C(\Gamma, \Delta, E)$$

is a natural bicontinuous isomorphism.

Proof. We first demonstrate the continuity of φ_E. Let $p_{\mu, V}$ be a seminorm on $C(\Gamma, \Delta, E)$, then $W = \{m: m(\mu) \subset V\}$ is a 0-neighbourhood in E^Γ and $m \in W$ implies $p_{\mu, V}(\varphi_E m) \leq 1$. Now define Ψ_E:

$$\Psi_E: C(\Gamma, \Delta, E) \to E^\Gamma$$

by $\Psi_E f(\mu) = \mu f(\varepsilon)$. The continuity of the maps $f \to \mu f$ and $f \to f(\varepsilon)$ imply that Ψ_E is continuous. Straightforward calculations yield that φ_E and Ψ_E are Γ-module homomorphisms and that $\varphi_E \circ \Psi_E$ and $\Psi_E \circ \varphi_E$ are identity maps. It is straightforward to show that φ_E is natural.

7. Density theorems

We present in this section some results which we shall need in the sequel.

Let Γ be a locally compact group and Δ a closed subgroup of Γ. Let λ be the right invariant Haar measure on Δ. For a locally convex Δ-module E, let $\mathcal{K}(\Gamma, E)$ be the space of continuous functions $\Gamma \to E$ which have compact support taken with the inductive topology [4: Chapitre 3, § 1]. We write $\mathcal{K}(\Gamma)$ in place of $\mathcal{K}(\Gamma, \mathbf{C})$. Let $\mathcal{K}(\Gamma, \Delta, E)$ be the subset of $C(\Gamma, \Delta, E)$ consisting of functions f whose support is contained in the saturant of some compact subset K of Γ (i.e. Supp $(f) \subset \Delta K$).

PROPOSITION 18. *Let $f \in \mathcal{K}(\Gamma, E)$, and define $f^b(\gamma) \in E'^*$ by*

$$\langle f^b(\gamma), x' \rangle = \int_\Delta \langle \delta^{-1} f(\delta\gamma), x' \rangle \, d\lambda(\delta).$$

If E satisfies condition (K), then $f^b(\gamma) \in E$. If in addition E is a locally convex continuous Δ-module then $f^b \in C(\Gamma, \Delta, E)$, and f^b vanishes outside ΔS where $S = \mathrm{Supp}\,(f)$.

Proof. If E satisfies condition (K) then $f^b(\gamma) \in E$ by [4: Chapitre 3, § 3, Proposition 7] since the map $\delta \to \delta^{-1} f(\delta\gamma)$ is in $\mathcal{K}(\Delta, E)$. To show that f^b is continuous let U be any convex circled closed 0-neighbourhood of E, and let V' be a compact symmetric neighbourhood of the unit of Γ. Given $\gamma_0 \in \Gamma$ if $\gamma \in \gamma_0 V'$ and $f(\delta\gamma) \neq 0$. then

$$\delta \in \delta\gamma^{-1} \cap \Delta \subset S V' \gamma_0^{-1} \cap \Delta \subset (S \cap \Delta\gamma_0 V') V' \gamma_0^{-1} = K$$

and K is compact since K is the product of two compact subsets of Γ. By Proposition 11 (c) of § 4 there is a 0-neighbourhood W in E such that

$$(\Delta \cap K)^{-1} W \subset (1/\lambda(K \cap \Delta))\, U.$$

Since f is uniformly continuous there is a compact neighbourhood $V \subset V'$ of the identity of Γ such that $\gamma \in \gamma_0 V$ and $\delta \in \Delta$ imply

$$f(\delta\gamma) - f(\delta\gamma_0) \in W$$

and thus $x' \in U^0$, $\gamma \in \gamma_0 V$ imply

$$|\langle f^b(\gamma) - f^b(\gamma_0), x' \rangle| \leqslant \int_\Delta |\langle \delta^{-1} [f(\delta\gamma) - f(\delta\gamma_0)], x' \rangle| \, d\lambda(\delta) \leqslant 1.$$

Therefore $f^b(\gamma) - f^b(\gamma_0) \in U$. The remainder of the proof is straightforward.

In the preceding we needed to know that E was a continuous Δ-module in order to conclude that f^b was continuous. For certain other Δ-modules we can also make this conclusion.

PROPOSITION 19. *Let E be a barreled locally convex Δ-module. Then $f \in \mathcal{K}(\Gamma, E^c)$ implies $f^b \in C(\Gamma, \Delta, E^c)$.*

Proof. First note that E^c is quasicomplete so that $f^b(\gamma) \in E^c$. Let $x \in E$ be given. Let $U = \{x\}^0$ and let V', S, γ_0 and K be as in Proposition 18. There is a $c > 0$ such that

$$(\Delta \cap K)\, x \subset c\{x\}^{00}.$$

Now choose $V \subset V'$ such that $\gamma \in \gamma_0^{-1} V$ and $\delta \in \Delta$ imply

$$f(\delta\gamma) - f(\delta\gamma_0) \in (1/c\, \lambda(K))\, \{x\}^0$$

Then

$$\int_\Delta |\langle \delta x, f(\delta\gamma) - f(\delta\gamma_0)\rangle|\, d\lambda(\delta) \leqslant 1$$

which demonstrates the continuity of f^b.

PROPOSITION 20. *Let E be a locally convex Δ-module and suppose that $f \in \mathcal{K}(\Gamma, E)$ implies $f^b \in C(\Gamma, \Delta, E)$. Then $\mathcal{K}(\Gamma, \Delta, E)$ is the image of $\mathcal{K}(\Gamma, E)$ by the map $f \to f^b$.*

Proof. Let $g \in \mathcal{K}(\Gamma, \Delta, E)$ and choose K such that Supp $(g) \subset \Delta K$. Let $u \in \mathcal{K}(\Gamma)$ be such that $u(\gamma) \geqslant 0$ and $u(\gamma) > 0$ for $\gamma \in K$. Let u^I be defined by

$$u^I(\gamma) = \int_\Delta u(\delta_\gamma)\, d\lambda(\delta)$$

then u^I is continuous by Proposition 18 and $u^I(\delta\gamma) = u^I(\gamma)$. Define $f \in \mathcal{K}(\Gamma, E)$ by

$$f(\gamma) = u(\gamma)\, g(\gamma)/u^I(\gamma)$$

for $\gamma \in \Delta K$ and $f(\gamma) = 0$ if $\gamma \in \Gamma \diagdown \Delta K$. (Observe that $\gamma \in \Delta K$ implies $u^I(\gamma) > 0$.) Now:

$$f^b(\gamma) = \int_\Delta u(\delta\gamma)\, \delta^{-1} g(\delta\gamma)/u^I(\gamma)\, d\lambda(\delta) = g(\gamma) \int u(\delta\gamma)/u^I(\gamma)\, d\lambda(\delta) = g(\gamma).$$

This proves the Proposition.

PROPOSITION 21. *Let E be a locally convex Δ-module. Then $\mathcal{K}(\Gamma, \Delta, E)$ is dense in $C(\Gamma, \Delta, E)$.*

Proof. Given $f \in C(\Gamma, \Delta, E)$, and a seminorm $p_{\mu, V}$, choose a $u \in \mathcal{K}(\Gamma)$ with $u \geqslant 0$, $u(\gamma) = 1$ for $\gamma \in$ Supp (μ). Then $u^I f \in \mathcal{K}(\Gamma, \Delta, E)$ where u^I is as in the proof of the preceding proposition. Then for $\gamma \in$ Supp (μ), $u^I f(\gamma) = f(\gamma)$ and so $p_{\mu, V}(u^I f) = p_{\mu, V}(f)$. This completes the proof.

Remarks. The above propositions show that whenever E is a locally convex Δ-module which is barreled and (quasi-complete, there are lots of functions in $C(\Gamma, \Delta, E)$ and in $C(\Gamma, \Delta, E^c)$. We also note that when E is one-dimensional Proposition 19 is in [4: Chapitre 7, § 2, No. 1] and when E is a Hilbert space and $x \to \delta x$ a unitary map, in Mackey [9: § 3].

8. Mackey's induced representations

The purpose of this section is to make precise the relationship between our induced representation E^Γ and the unitary induced representations of Mackey.

The following definition was proposed by Naimark as an extension of the notion of unitary equivalence to nonunitary representations on Banach spaces. It has been used by Mackey in [11] and Fell in [5].

Definition. Let E and F be locally convex Γ-modules. E and F are said to be Naimark related if there is a module isomorphism from a dense submodule of E onto a dense submodule of F whose graph is closed in $E \times F$.

Let E be a locally convex Γ-module. E is called a unitary Γ-module if E is a Hilbert space and if for each $\gamma \in \Gamma$ the map $x \to \gamma x$ is unitary.

Let Δ be a closed subgroup of the locally compact group Γ and let E be a unitary Δ-module. The induced unitary representation E^U is defined in the following way:

Let ϱ be a continuous positive function on Γ such that

$$\varrho(\delta\gamma) = \frac{\Delta(\delta)}{\Gamma(\delta)} \varrho(\gamma)$$

(here Δ is the modular function of Δ and Γ the modular function of Γ. See [4: Chapitre 7, § 2, Théorème 2] for the existence of such a function).

Let β be the right Haar measure of Δ and λ the right Haar measure of Γ, then the measure

$$\mu = (\varrho \circ \lambda)/\beta$$

is a positive quasi-invariant measure on Γ/Δ [ibid.].

Let E^U be the set of all functions $f: \Gamma \to E$ such that

(i) $\gamma \to \langle f(\gamma), x \rangle$ is a Borel function

(ii) $f(\delta\gamma) = \delta f(\gamma)$

(iii) $\int_{\Gamma/\Delta} \|f(\gamma)\|^2 \, d\mu < \infty.$

(Notice that $\gamma \to \|f(\gamma)\|^2$ is constant on the right cosets of Δ and hence defines a function on Γ/Δ.) Define γf by

$$\gamma f(\gamma') = f(\gamma'\gamma)\sqrt{\frac{\varrho(\gamma'\gamma)}{\varrho(\gamma)}}.$$

Mackey has shown [9] that with these definitions, E^U is a unitary Δ-module.

THEOREM 10. *Let E be a unitary Δ-module, then E^U and E^Γ are Naimark related.*

Proof. Let G be the subspace of E^U consisting of continuous functions. Then $\mathcal{K}(\Gamma, \Delta, E) \subset G$ so by Lemma 3.3 of [9], G is dense in E^U, and moreover it is easily seen that G is a submodule. By Theorem 9, $E^\Gamma \cong C(\Gamma, \Delta, E)$ and thus we can define a map $m: G \to E^\Gamma$ by

$$mf(\gamma) = f(\gamma)\sqrt{P(\gamma)}.$$

It is clear that $\mathcal{K}(\Gamma, \Delta, E) \subset m(G)$ and thus $m(G)$ is a dense submodule of E (Proposition 20). We now show that m is closed. Let (f, g) be in the closure of m. By the Riesz–Fischer Theorem [4, Chapitre 4, 3, No. 4, Corollaire 1] there is a sequence $(f_n) \subset G$ such that $f_n \to f$ in E^U and $f_n(\gamma) \to f(\gamma)$ for all γ outside some set $N \subset \Gamma$ with $\mu(\pi(N)) = 0$ where π is the natural map $\Gamma \to \Gamma/\Delta$. Since $mf_n(\gamma) \to g(\gamma)$ for every γ it follows that $f(\gamma)\sqrt{\varrho(\gamma)} = g(\gamma)$ for all $\gamma \notin N$ and hence that $f = \sqrt{\varrho}$ in E^U and therefore $f \in G$ and $mf = g$. A straightforward calculation shows that m is a module homomorphism and this completes the proof.

9. The representation of ${}^\Gamma E$

Let Γ be a locally compact group and Δ a closed subgroup. Let E be a locally convex Δ-module which is barreled. Using Proposition 9 of § 3 together with the fact that there is a natural continuous isomorphism $F \cong F^{cc}$ where F is a locally convex Γ-module we have:

$$\Psi_E\colon M_c(\Gamma) \otimes_\Delta E \cong (M_c(\Gamma) \otimes_\Delta E)^{cc} \cong [\hom_\Delta (M_c(\Gamma), E^c)]^c \cong [C(\Gamma, \Delta, E^c)]^c.$$

(We use the fact that E barreled implies E' quasicomplete.) Moreover Ψ_E is continuous and

$$\Psi_E(\sum_i \mu_i \otimes x_i)(f) = \sum_i \int \langle x_i, f(\gamma)\rangle \, d\mu_i(\gamma).$$

Let τ be the topology on $[C(\Gamma, \Delta, E^c)]^c$ of uniform convergence on the relatively compact equicontinuous subsets of $C(\Gamma, \Delta, E^c)$.

THEOREM 11. *Suppose that E is a barreled locally convex Δ-module. There is a natural bicontinuous isomorphism*

$$\Psi_E\colon M_c(\Gamma) \otimes_\Delta E \cong [C(\Gamma, \Delta, E^c)]^c_\tau.$$

Proof. Let φ be the map $C(\Gamma, \Delta, E^c) \to B(M_c(\Gamma), E)$ defined by

$$\varphi f(\mu, x) = \int \langle x, f(\gamma) \rangle \, d\mu(\gamma).$$

To show that Ψ'_E is bicontinuous it suffices to show that $\varphi(H)$ is a separately equicontinuous of $B(M_c(\Gamma), E)$ if and only if H is relatively compact and equicontinuous. Given $x_1, x_2, \ldots, x_n$ in E, there is a compact subset C in $C(\Gamma)$ such that $\mu \in V = C^0$ and $f \in H$ imply

$$|\varphi f(\mu, x_i)| \leqslant 1, \quad i = 1, 2, \ldots, n.$$

Thus for each $f \in H$ and i, the map $\gamma \to \langle x_i, f(\gamma) \rangle$ is in $V^0 = C^{00}$. Now C^{00} is compact in $C(\Gamma)$ since the closed convex hull of a compact subset of a complete space is compact and this closure is the same for all topologies consistent with the duality $\langle C(\Gamma), M_c(\Gamma) \rangle$. Consequently the set of maps

$$D = \{\gamma \to \langle x_i, f(\gamma) \rangle : f \in H, i = 1, 2, \ldots, n\}$$

is a relatively compact subset of $C(\Gamma)$. By the Ascoli theorem D is equicontinuous so given $\gamma_0 \in \Gamma$ there is a neighbourhood W of γ_0 such that $\gamma \in W$ and $f \in H$ implies

$$f(\gamma) - f(\gamma_0) \in \{x_1, x_2, \ldots, x_n\}^0$$

and we conclude that H is equicontinuous. To show that H is relatively compact, it suffices to show that for each γ_0 in Γ, $H(\gamma_0)$ is relatively compact, since the Ascoli theorem implies that H is relatively compact in $C(\Gamma, \Delta, E^c)$. Let U be a 0-neighbourhood in E such that $x \in U$ implies

$$|\varphi f(\varepsilon_{\gamma_0}, x)| \leqslant 1 \quad \text{for} \quad f \in H.$$

Then $H(\gamma_0) \subset U^0$ and U^0 is equicontinuous as a subset of E^c and therefore relatively compact by the Alaoglu–Bourbaki theorem [3: Chapitre 4, § 2, Proposition 2].

Now suppose that H is relatively compact and equicontinuous. Given μ in $M_c(\Gamma)$, let $K = \text{Supp}(\mu)$. Then $H(K) = \bigcup_{\gamma \in K} H(\gamma)$ is relatively compact [2: Chapitre 10, § 3, Remarques 3, p. 46] and since E is barreled, $H(K)^0 = V$ is a 0-neighbourhood in E. Thus

$$|\varphi f(\mu, x)| \leqslant 1 \quad \text{for } x \in (1/\|\mu\|)\, V \text{ and } f \in H.$$

Now given $x \in E$, since H is relatively compact and equicontinuous it follows that map $\varphi_x \colon H \to C(\Gamma)$ defined by

$$\varphi_x f(\gamma) = \langle x, f(\gamma) \rangle$$

is continuous since the topology of uniform convergence on compact sets coincides with the topology of $C(\Gamma, \Delta, E^c)$ on equicontinuous sets [2: Chapitre 10, § 2, Théorème 1]. Thus $\varphi_x(H)$ is relatively compact and $W=[\varphi_x(H)]^0$ is a 0-neighbourhood in $M_c(\Gamma)$, and

$$|\varphi f(\mu, x)| \leq 1 \quad \text{for } \mu \in U \text{ and } f \in H.$$

10. Linear systems representations

In this section we prove a Frobenius reciprocity theorem for linear systems representations. The following definition is a modification of the definition used by J. M. G. Fell in [5].

Let A be a topological algebra, E an $(A, \mathbf{C})$-bimodule, F a $(\mathbf{C}, A)$-bimodule and let f be an A-balanced bilinear form on $E \times F$. The triple (E, F, f) is called a linear systems representation of A provided that f satisfies

1° $f(x, y) = 0$ for all y in F implies $x = 0$

2° $f(x, y) = 0$ for all x in E implies $y = 0$

3° the map $a \to f(ax, y)$ is continuous for every x in E and y in F.

We shall write $\langle x, y \rangle$ in place of $f(x, y)$ and $\langle E, F \rangle$ in place of the triple (E, F, f).

The category $\mathcal{A}_L$ of linear systems representations of A is defined in the following way. The objects of $\mathcal{A}_L$ are the linear systems representations, and the morphisms of $\mathcal{A}_L$ are maps $m: \langle E, F \rangle \to \langle G, H \rangle$ where m is an $(A, \mathbf{C})$-bimodule homomorphism $E \to G$ such that $m^*(H) \subset F$ (here m^* is the algebraic adjoint and we identify H (resp. F) with a subset of G^* (resp. E^*).

Given a linear systems representation $\langle E, F \rangle$, E_σ $(\sigma = \sigma(E, F))$ is a locally convex space which is a left A-module. The continuity of the map $a \to ax$ is a consequence of 3°. Moreover since $\langle ax, y \rangle = \langle x, ya \rangle$, the map $x \to ax$ has an adjoint so by [3: Chapitre 4, § 4, Proposition 1] it follows that $x \to ax$ is a continuous map of E_σ into itself. Thus E_σ is a locally convex left A-module. Similarly it follows that F_σ $(\sigma = \sigma(F, E))$ is a locally convex right A-module. Now observe that if $\alpha = \langle E, F \rangle$ and $\beta = \langle G, H \rangle$ are linear systems representations of A, and $m: \alpha \to \beta$ a morphism then $m: E_\sigma \to G_\sigma$ is continuous [*ibid.*]. Moreover if $m: E_\sigma \to G_\sigma$ is continuous then m is a morphism of $\mathcal{A}_L$. Thus one can define a functor from $\mathcal{A}_L$ to $\mathcal{A}$ the category of locally convex left A-modules, and this functor is full. Summarizing we have:

PROPOSITION 22. *Let $\langle E, F \rangle$ be a linear systems representation of A. Then E_σ is a locally convex left A-module and the functor $\mathfrak{f}$ taking $\langle E, F \rangle$ to E_σ is full: i.e.* $\mathrm{Hom}_{\mathcal{A}_L}(\alpha, \beta) = \mathrm{Hom}_{\mathcal{A}}(\mathfrak{f}\alpha, \mathfrak{f}\beta)$.

Now given a locally convex left A-module E, one defines the action of A on E' by

$$\langle x, x'a\rangle = \langle ax, x'\rangle$$

and E' is a right A-module and the map $a \to \langle ax, x'\rangle$ is continuous. Thus $\langle E, E'\rangle$ is a linear systems representation of A. Moreover if m: $E \to F$ is continuous, then $m^*(F') \subset E'$ so that m defines a morphism $\langle E, E'\rangle \to \langle F, F'\rangle$.

PROPOSITION 23. *The functor $\mathfrak{H}$ which takes E to $\langle E, E'\rangle$ is the adjoint of the functor $\mathfrak{f}$.*

Proof. Let M and α be objects of $\mathcal{A}$ and $\mathcal{A}_L$ respectively. We shall show

$$\mathrm{Hom}_{\mathcal{A}_L}(\mathfrak{H}M, \alpha) = \mathrm{Hom}_{\mathcal{A}}(M, \mathfrak{f}\alpha).$$

Let $m \in \mathrm{Hom}_{\mathcal{A}_L}(\mathfrak{H}M, \alpha)$, then as we have seen, m: $M_\sigma \to E_\sigma$ ($\alpha = \langle E, F\rangle$) is continuous and hence m: $M \to E_\sigma$ is continuous, since the initial topology of M is finer than $\sigma(M, M')$. Conversely if $m \in \mathrm{Hom}(M, \mathfrak{f}\alpha)$ then m: $M_\sigma \to E_\sigma$ is continuous and this completes the proof.

For the remainder of this section we shall suppose that A is a locally convex algebra having a unit, and that B is a unitary subalgebra of A. $\mathcal{B}_L$ is the category of linear systems representations of B and $\mathcal{B}$ is the category of locally convex left B-modules. In order to use our previous results on inducing representations, we note that if $\langle E, F\rangle$ is a linear systems representation of A, then F_σ is a locally convex left A^*-module where A^* is the algebra "opposite" to A; i.e. A^* has the same linear and topological structure as A, with multiplication defined by $(ab)_{A^*} = (ba)_A$.

Let $\beta = \langle G, H\rangle$ be a linear systems representation of B. We define the induced system ${}^A\beta$ by

$${}^A\beta = \mathfrak{H}(A \otimes_B G_\sigma),$$

where $\mathfrak{H}$ is the functor which takes E to $\langle E, E'\rangle$. Observe that one has

$$({}^AG_\sigma)' \cong (G_\sigma)'^{A^*}$$

(see the proof of Proposition 9 of § 3) so that

$${}^A\beta = \langle {}^AG, H^{A^*}\rangle,$$

where

$$H^{A^*} = \mathrm{Hom}_B(A^*, H_\sigma).$$

If $\alpha = \langle E, F\rangle$ is a linear systems representation of A, then $\alpha_B = \langle E_B, F_B\rangle$ is a linear systems representation of B and the functor which takes to α to α_B is called the restriction functor.

THEOREM 12. *The restriction functor $\alpha \to \alpha_B$ has an adjoint $\beta \to {}^A\beta$. That is, if $\alpha = \langle E, F\rangle$ is a linear systems representation of A and $\beta = \langle G, H\rangle$ is a linear systems representation of B, then there is a natural isomorphism:*

$$\mathrm{Hom}_A\,({}^A\beta,\,\alpha) \cong \mathrm{Hom}_B\,(\beta,\,\alpha_B)$$

Proof. By Proposition 22 and Theorem 3 we have

$$\mathrm{Hom}_B\,(\beta,\,\alpha_B) = \mathrm{Hom}_B\,(G_\sigma,\,(E_\sigma)_\Delta) \cong \mathrm{Hom}_A\,({}^A G_\sigma,\,E_\sigma).$$

To complete the proof we show that

$$\mathrm{Hom}_A({}^A\beta,\,\alpha) \cong \mathrm{Hom}_A({}^A G_\sigma,\,E_\sigma).$$

If $m \in \mathrm{Hom}_A({}^A\beta, \alpha)$ then $m^*: F \to H^{A*} = (G_\sigma)'^{A*} \cong ({}^A G_\sigma)'$ so that $m: {}^A G_\sigma \to E_\sigma$ is continuous. On the other hand, if $m \in \mathrm{Hom}\,({}^A G_\sigma, E_\sigma)$ then

$$m^*((E_\sigma)') = m^*(F) \subset ({}^A G_\sigma)' = H^{A*}$$

so that $m \in \mathrm{Hom}_A\,({}^A\beta,\,\alpha)$. This completes the proof.

For linear systems representations of locally compact groups we leave it to the reader to formulate the appropriate results. Note however that as a consequence of Proposition 10 and [3: Chapitre 4, § 4, Proposition 1] we have the following:

PROPOSITION 24. *Let Γ be a locally compact group, E a $(M_c(\Gamma),\,\mathbf{C})$-bimodule, F a $(\mathbf{C},\,M_c(\Gamma))$-bimodule and $(x, y) \to \langle x, y\rangle$ an $M_c(\Gamma)$-balanced bilinear form on $E \times F$. If the map*

$$\gamma \to \langle \gamma x, y\rangle$$

is continuous for each $(x, y) \in E \times F$ then $\langle E, F\rangle$ is a linear systems representation of $M_c(\Gamma)$.

Thus for locally compact groups, our definition of linear systems representations coincides precisely with that used by Fell in [5].

11. Induced continuous representations

Our purpose in this section is to develop results analogous to those of section 5 for locally convex continuous Γ-modules. We begin by showing that we can define $\mathrm{hom}_\Delta\,(M_c(\Gamma),\,-)$ and $M_c(\Gamma)\otimes_\Delta -$ functors in such a way that $\mathrm{hom}_\Delta\,(M_c(\Gamma),\,E)$ and $M_c(\Gamma)\otimes_\Delta E$ are locally convex continuous Γ-modules whenever E is a locally convex continuous Δ-module. To do this we first establish some technical lemmata which will enable us to use the results of section 2.

LEMMA 3. *Let H be an equicontinuous subset of $M_c(\Gamma)$. Then for $\mu \in M_c(\Gamma)$; $\mu * H$ and $H * \mu$ are equicontinuous.*

Proof. Consider the map $\Phi: C(\Gamma)\to C(\Gamma)$ defined by $\Phi(f)(\gamma)=\mu(f_\gamma)$. We first show that Φ is continuous. By Lemma 1, § 4 the map $(\gamma, f)\to {}_\gamma f$ is continuous as a map $\Gamma\times C(\Gamma)\to C(\Gamma)$. By [4: Chapitre 6, § 1, Proposition 16] the map $(\lambda, f)\to\lambda\cdot f$ where $\lambda\cdot f(\gamma)=\lambda(f_\gamma^\vee)$ is separately continuous since $C(\Gamma)$ is complete. Observe that $\mu\cdot f=\Phi(f)$. Thus Φ is continuous and therefore the adjoint Φ' of Φ maps equicontinuous sets to equicontinuous sets. Thus $\Phi'(H)$ is equicontinuous. Since for ν in $M_c(\Gamma)$ we have

$$\Phi'(\nu)(f)=\nu(\Phi(f))=\iint f(\gamma'\gamma)\,d\mu(\gamma')\,d\nu(\gamma)=\mu*\nu(f)$$

it follows that $\mu*H$ is equicontinuous. A similar argument using the continuity of $(\gamma, f)\to f_\gamma$ shows that $H*\mu$ is equicontinuous.

LEMMA 4. *Let H be an equicontinuous subset of $M_c(\Gamma)$ and let K be a compact subset of Γ. Then $KH=\{\varepsilon_\gamma*\mu: \gamma\in K, \mu\in H\}$ and $HK=\{\mu*\varepsilon_\gamma: \gamma\in K, \mu\in H\}$ are equicontinuous subsets of $M_c(\Gamma)$.*

Proof. Let $V=H^0$, then V is a 0-neighbourhood in $C(\Gamma)$. For $\gamma\in K$ there is a neighbourhood V_γ of γ and a 0-neighbourhood W_γ in $C(\Gamma)$ such that $g(V_\gamma\times W_\gamma)\subset V$ where g is the map $g(\gamma, f)=f_\gamma$ (Lemma 1, § 4). Let $\gamma_1, \gamma_2, \dots, \gamma_n\in K$ such that $K\subset\bigcup_{i=1}^n V_{\gamma_i}$ and let $W=\bigcap_{i=1}^n W_{\gamma_i}$. Then $g(K\times W)\subset V$. If $f\in W$ and $\gamma\in K$ then $f_\gamma\in H^0$ so that $\mu\in H$ implies

$$|\varepsilon_\gamma*\mu(f)|=|\mu(f_\gamma)|\leqslant 1.$$

Thus $f\in(KH)^0$ so that KH is equicontinuous. The proof of HK is similar.

PROPOSITION 25. *Let Δ be a closed subgroup of Γ and let E be a locally convex continuous Δ-module. Then $M_c(\Gamma)\otimes_\Delta E$ given the topology of uniform convergence on the subsets of $B(M_c(\Gamma), E)$ that are equihypocontinuous relative to the equicontinuous subsets of $M_c(\Gamma)$ and the compact subsets of E is a locally convex continuous Γ-module.*

Proof. By Proposition 16, § 4, $M_c(\Gamma)$ is a locally convex continuous (Γ, Δ)-bimodule. Lemma 3 shows that the hypotheses of Proposition 1, § 2, are satisfied so we conclude that $M_c(\Gamma)\otimes_\Delta E$ with the given topology is a locally convex L-module. To show that it is a continuous Γ-module we show that condition (c) of Proposition 11, § 4, is satisfied. Let K be a compact subset of Γ and let M be an equihypocontinuous subset of $B(M_c(\Gamma), E)$. To show that the set of maps $\{x\to\varepsilon_\gamma x: \gamma\in K\}$ is an equicontinuous subset of $L(M_c(\Gamma)\otimes_\Delta E)$ it is sufficient to show that KM is equihypocontinuous since $K(KM)^0\subset M^0$. (Here $KM=\{\varepsilon_\gamma f: \gamma\in K, f\in M\}$ and $\varepsilon_\gamma f$ is defined by $\varepsilon_\gamma f(\mu, x)=f(\varepsilon_\gamma\mu, x)$.) Given a compact set N in E there is a compact set C in $C(\Gamma)$ such that $\mu\in C^0$ implies $|f(\mu, x)|\leqslant 1$ for any f in M and

x in N. Since $KC = \{_\gamma g\colon \gamma \in K, g \in V\}$ is compact (a consequence of Lemma 1) and $K(KC)^0 \subset C^0$ it follows that $\mu \in (KC)^0$ implies $|\varepsilon_\gamma f(\mu, x)| \leq 1$ for any f in M and γ in K and x in H. Now given an equicontinuous subset H of $M_c(\Gamma)$, KH is equicontinuous (Lemma 4) so there is a 0-neighbourhood U in E such that x in U and μ in H implies $|\varepsilon_\gamma f(\mu, x) = |f(\varepsilon_\gamma * \mu, x)| \leq 1$ whenever $x \in K$ and $f \in M$. This completes the proof.

PROPOSITION 26. *Let Δ be a closed subgroup of Γ and E a locally convex continuous Δ-module. Then* $\hom_\Delta (M_c(\Gamma), E)$ *given the topology of uniform convergence on the equicontinuous subsets of $M_c(\Gamma)$ is a locally convex continuous Γ-module.*

Proof. By Proposition 16, § 4, $M_c(\Gamma)$ is a locally convex continuous (Δ, Γ)-bimodule, and this together with Lemma 3 proves that $M_c(\Gamma)$ satisfies the hypotheses of Proposition 3, § 2. Thus $\hom_\Delta (M_c(\Gamma), E)$ with the given topology is a locally convex Γ-module. To complete the proof we show that (c) of Proposition 11, § 4, is satisfied. Let K be any compact subset of Γ, H an equicontinuous subset of $M_c(\Gamma)$, V a 0-neighbourhood in E and $W = \{m \in \hom_\Delta (M_c(\Gamma), E)\colon m(H) \subset V\}$. By Lemma 4, HK is equicontinuous so that $V = \{m\colon m(HK) \subset V\}$ is a 0-neighbourhood of $\hom_\Delta (M_c(\Gamma), E)$. Now if $\gamma \in K$ and $m \in U$ then $\varepsilon_\gamma m \in W$, thus the set of maps $\{m \to \varepsilon_\gamma m\colon \gamma \in K\}$ is equicontinuous.

THEOREM 14. *Let Δ and Δ' be closed subgroups of the locally compact group Γ. There is a natural isomorphism Ψ_{EF}*

$$\Psi_{EF}\colon \operatorname{Hom}_{\Delta'}((^\Gamma E)_{\Delta'}, F) \cong \operatorname{Hom}_\Delta(E, (F^\Gamma)_\Delta),$$

where E is a locally convex continuous Δ-module, $^\Gamma E = M_c(\Gamma) \otimes_\Delta E$ and F a locally convex continuous Δ-module, $F^\Gamma = \hom_{\Delta'} (M_c(\Gamma), F)$.

Proof. By looking at the proof of Theorem 2 it is seen that if we define

$$\Psi_{EF} m(x)(\mu) = m(\mu \otimes x)$$

then the only point that requires attention is to show that $m \circ \chi$ is hypocontinuous relative to the compact subsets of E. For this let C be a compact subset of E and W a 0-neighbourhood in F, then

$$V = \{m \in F^\Gamma\colon m(\varepsilon) \in W\}$$

is a 0-neighbourhood in F^Γ. By Proposition 26 there is a 0-neighbourhood U in $M_c(\Gamma)$ such that $U \Psi_{EF} m(C) \subset V$. Thus if $x \in C$ and $\mu \in U$ we have

$$\mu m(\varepsilon \otimes x) \in W$$

and so

$$m \circ \chi(U \times C) \subset W.$$

This completes the proof.

THEOREM 15. *Let Δ be a closed subgroup of the locally compact group Γ; then the restriction functor from the categroy of locally convex continuous Γ-modules to the category of locally convex continuous Δ-modules has an adjoint and a coadjoint. The adjoint is the functor $M_c(\Gamma)\otimes_\Delta -$ and the coadjoint is the functor* $\hom_\Delta(M_c(\Gamma), -)$.

Proof. For the first assertion take $\Delta'=\Gamma$ in Theorem 14. For the second take $\Delta=\Gamma$ and $\Delta'=\Delta$.

THEOREM 16. *Let Δ' be a closed subgroup of Δ. There are natural continuous isomorphisms Ψ and φ,*

$$\Psi_E: {}^\Gamma E \cong {}^\Gamma({}^\Delta E)$$

and

$$\varphi_E: (E^\Delta)^\Gamma \cong E^\Gamma,$$

where E is any locally convex continuous Γ-module.

Proof. We first remark that we know from Theorem 7, that Ψ_E, φ_E defined by

$$\Psi_E(\mu\otimes x)=\mu\otimes\varepsilon\otimes x$$

and

$$\varphi_E m(\mu)=m(\mu)\,(\varepsilon),$$

where $\mu\in M_c(\Gamma)$, $x\in E$ and ε is the unit of $M_c(\Delta)$ are natural isomorphisms, and we now want to demonstrate their continuity. To show that Ψ_E is continuous it suffices by Theorem 1 to show that the bilinear map $\Psi_E\circ\chi$ (where χ: $M_c(\Gamma)\times E\to{}^\Gamma E$ is the natural map) is hypocontinuous. For this let W be a 0-neighbourhood in ${}^\Gamma({}^\Delta E)$, and H an equicontinuous subset of $M_c(\Gamma)$. There is a 0-neighbourhood V in ${}^\Delta E$ such that $H\otimes V\subset W$ and there is a 0-neighbourhood V of E such that $\varepsilon\otimes U\subset V$. It follows that

$$\Psi_E\circ\chi(H\times U)\subset W.$$

Now given a compact set C of E there is a 0-neighbourhood V of $M_c(\Gamma)$ such that $V\otimes\varepsilon\otimes C\subset W$. Thus

$$\Psi_E\circ\chi(V\times C)\subset W.$$

This proves that Ψ_E is continuous.

To show that φ_E is continuous, let W be a 0-neighbourhood in E^Γ of the form

$$W=\{m:\ m(H)\subset V\},$$

where V is a 0-neighbourhood in E and H an equicontinuous subset of $M_c(\Gamma)$. Then

$$U=\{m\in E^\Delta:\ m(\varepsilon)\subset V\}$$

is a 0-neighbourhood in E^Δ and thus

$$X = \{m \in (E^{\Delta})^{\Gamma}: m(H) \subset U\}$$

is a 0-neighbourhood in $(E^{\Delta})^{\Gamma}$ and

$$\varphi_E(X) \subset W.$$

This completes the proof.

PROPOSITION 27. *Let E be a locally convex continuous Γ-module, then E^c given the topology of uniform convergence on the compact subsets of E is a continuous Γ-module.*

Proof. This is an immediate consequence of Proposition 3 of § 2 and Proposition 3 (ii) of [4: Chapitre 8, § 2].

THEOREM 17. *There is a natural continuous isomorphism $\Psi^{\cdot}$*

$$\Psi^{\cdot}_E: ({}^{\Gamma}E)^c \cong (E^c)^{\Gamma},$$

where E is a locally convex continuous Γ-module.

Proof. Define for $u' \in ({}^{\Gamma}E)^c$, $\mu \in M_c(\Gamma)$ and $x \in E$

$$\Psi^{\cdot}_E u'(\mu)(x) = u'(\mu \otimes x).$$

The proof now is similar to that of Theorem 14, and is omitted.

There is a representation theorem for E^{Γ} similar to the theorem of § 6.

PROPOSITION 28. *Let E be a locally convex continuous Δ-module and suppose E satisfies condition* (K). *Then $C(\Gamma, \Delta, E)$ given the topology of uniform convergence on compact sets is a locally convex continuous Γ-module.*

Proof. We know from the results of § 6 that $C(\Gamma, \Delta, E)$ is a $M_c(\Gamma)$-module, so we need only show the continuity properties of the map $(\mu, f) \to \mu f$. The continuity of the map $f \to \mu f$ is an easy consequence of the fact that μ has compact support. We now show that (c) of Proposition 11, § 4, is satisfied. Given a compact set $K \subset \Gamma$ and a 0-neighbourhood V in $C(\Gamma, \Delta, E)$ of the form

$$V = \{f: f(K') \subset U\},$$

where $K' \subset L$ is compact and U is a 0-neighbourhood in E, let

$$W = \{f: f(K'K) \subset U\}$$

then W is a 0-neighbourhood in E and $KW \subset V$. This shows that the set of maps $\{f \to \varepsilon_{\gamma} f: \gamma \in K\}$ is equicontinuous, and thus completes the proof.

As in section 6, we define a map $\varphi_E: E^\Gamma \to C_u(\Gamma, \Delta, E)$ where $C_u(\Gamma, \Delta, E)$ is $C(\Gamma, \Delta, E)$ given the topology of uniform convergence on compact sets.

THEOREM 18. *Let E be a locally convex continuous Δ-module satisfying condition* (K). *Then the map φ*

$$\varphi_E: E^\Gamma \cong C_u(\Gamma, \Delta, E)$$

is a natural bicontinuous isomorphism.

We shall omit the proof—it involves no new ideas.

The results of sections 7, 8, 9 also can be established for the case of continuous modules. We leave this to the ambitious reader.

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