

A HÖLDER ESTIMATE FOR QUASICONFORMAL MAPS BETWEEN SURFACES IN EUCLIDEAN SPACE

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In [2] C. B. Morrey proved a Hölder estimate for quasiconformal mappings in the plane. Such a Hölder estimate was a fundamental development in the theory of quasiconformal mappings, and had very important applications to partial differential equations. L. Nirenberg in [3] made significant simplifications and improvements to Morrey's work (in particular, the restriction that the mappings involved be 1-1 was removed), and he was consequently able to develop a rather complete theory for second order elliptic equation with 2 independent variables.

In Theorem (2.2) of the present paper we obtain a Hölder estimate which is analogous to that obtained by Nirenberg in [3] but which is applicable to quasiconformal mappings between *surfaces* in Euclidean space. The methods used in the proof are quite analogous to those of [3], although there are of course some technical difficulties to be overcome because of the more general setting adopted here.

In § 3 and § 4 we discuss applications to graphs with quasiconformal Gauss map. In this case Theorem (2.2) gives a Hölder estimate for the unit normal of the graph. One rather striking consequence is given in Theorem (4.1), which establishes the linearity of any $C^2(\mathbf{R}^2)$ function having a graph with quasiconformal Gauss map. This result includes as a special case the classical theorem of Bernstein concerning $C^2(\mathbf{R}^2)$ solutions of the minimal surface equation, and the analogous theorem of Jenkins [1] for a special class of variational equations. There are also in § 3 and § 4 a number of other results for graphs with quasiconformal Gauss map, including some gradient estimates and a global estimate of Hölder continuity. § 4 concludes with an application to the minimal surface system.

One of the main reasons for studying graphs satisfying the condition that the Gauss map is quasiconformal (or (Λ_1, Λ_2) -quasiconformal in the sense of (1.8) below) is that such

a condition must automatically be satisfied by the graph of a solution of any equation of mean curvature type (see (1.9) (ii) below). However we here only briefly discuss the application of the results of § 3 and § 4 to such equations; a more complete discussion will appear in [7].

§ 1. Terminology

M , N will denote oriented 2-dimensional C^2 submanifolds of \mathbf{R}^n , \mathbf{R}^m respectively, $n, m \geq 2$. Given $X \in M^{(1)}$ and $Y \in N$ we let $T_X(M)$, $T_Y(N)$ denote the tangent spaces (considered as subspaces of \mathbf{R}^n and \mathbf{R}^m) of M at X and N at Y respectively. δ will denote the gradient operator on M ; that is, if $h \in C^1(M)$, then

$$\delta h(X) = (\delta_1 h(X), \dots, \delta_n h(X)) \in T_X(M)$$

is defined by

$$(1.1) \quad \delta_i h(X) = \sum_{j=1}^n \tilde{g}^{ij}(X) D_j \tilde{h}(X),$$

where \tilde{h} is any C^1 function defined in a neighbourhood of M with $\tilde{h}|_M = h$; and where $(\tilde{g}^{ij}(X))$ is the matrix of the orthogonal projection of \mathbf{R}^n onto $T_X(M)$.

We note that of course the definition (1.1) is independent of the particular C^1 extension \tilde{h} of h that one chooses to use. We note also that in the special case $n=3$ we can represent $\tilde{g}^{ij}(X)$ explicitly in terms of the unit normal $\nu(X) = (\nu_1(X), \nu_2(X), \nu_3(X))$ of M at X according to the formula

$$(1.2) \quad \tilde{g}^{ij}(X) = \delta_{ij} - \nu_i(X) \nu_j(X), \quad i, j = 1, 2, 3.$$

η , θ will denote area forms for M , N respectively; that is, η and θ are C^1 differential 2-forms on M and N respectively such that

$$\int_A \eta = \text{area}(A), \quad \int_B \theta = \text{area}(B)$$

whenever $A \subset M$ and $B \subset N$ are Borel subsets of finite area.

(1.3) *Remark.* We can always take a C^1 2-form ζ on M to be the restriction to M of a C^1 form $\tilde{\zeta}$ defined in a neighbourhood of $M \subset \mathbf{R}^n$, so that $\tilde{\zeta}(X) \in \wedge^2(\mathbf{R}^n)$ for each $X \in M$. Thus in case $n=3$, we can write

$$\zeta(\tilde{X}) = \zeta_1(X) dx_2 \wedge dx_3 + \zeta_2(X) dx_1 \wedge dx_3 + \zeta_3(X) dx_1 \wedge dx_2,$$

(¹) We will use $X = (x_1, \dots, x_n)$ to denote points in M ; the symbol x will be reserved to denote points $(x_1, x_2) \in \mathbf{R}^2$.

where $\zeta_1, \zeta_2, \zeta_3$ are C^1 in some neighbourhood of M . Using the notation $\ast\tilde{\zeta}(X) = (\zeta_1(X), -\zeta_2(X), \zeta_3(X))$ (\ast is the usual linear isometry of $\Lambda^2(\mathbf{R}^3)$ onto \mathbf{R}^3) we then have

$$\int_A \zeta = \int_A \nu \cdot (\ast\tilde{\zeta}) d\mathcal{H}^2, \quad A \subset M,$$

where ν is the appropriately oriented unit normal for M and \mathcal{H}^2 denotes 2-dimensional Hausdorff measure in \mathbf{R}^3 . In particular, we see that ζ is an area form for M if and only if $(\zeta_1(X), -\zeta_2(X), \zeta_3(X))$ is a unit normal for M at each point $X \in M$. Thus there is no difficulty in recognizing an area form in case $n=3$. (Of course one can give an analogous, but not quite so convenient, characterization of area forms for arbitrary n .)

Our basic assumption concerning N is that there is a 1-form $\omega(X) = \sum_{i=1}^m \omega_i(X) dx_i$ which is C^2 in a neighbourhood of N and such that

$$(1.4) \quad d\omega_N = \theta, \quad \sup_N \left\{ \sum_{i=1}^m \omega_i^2 \right\}^{1/2} + \sup_N \left\{ \sum_{i,j=1}^m (D_j \omega_i)^2 \right\}^{1/2} \leq \Lambda_0 < \infty.$$

Here Λ_0 is a constant and ω_N denotes the restriction of ω to N ; henceforth we will not distinguish notationally between ω and ω_N .

(1.5) *Examples.* (i) If N is an open ball of radius R and centre 0 in \mathbf{R}^2 , we can take $\omega = -\frac{1}{2}x_2 dx_1 + \frac{1}{2}x_1 dx_2$ and $\Lambda_0 = R + 1$.

(ii) If N is the upper hemisphere S^2 of the unit sphere $S^2 \subset \mathbf{R}^3$, we can take $\omega = (-x_2/(1+x_3))dx_1 + (x_1/(1+x_3))dx_2 + 0dx_3$ and $\Lambda_0 = 4$. One can easily check this by directly computing $d\omega$ and using the relation $\sum_{i=1}^3 x_i^2 = 1$ on S^2 ; to check that $d\omega$ is an area form for S^2_+ it is convenient to use the characterization of area forms given in Remark (1.3) above. (Alternatively one obtains $d\omega$ as an area form by using an elementary computation involving example (i) above and stereographic projection of S^2_+ into \mathbf{R}^2 .)

(iii) More generally, we can let N be the surface obtained from a compact surface $L \subset \mathbf{R}^m$ by deleting a compact neighbourhood of an arbitrary chosen point $y_0 \in L$. There will then always exist ω as in (1.4) because the 2-dimensional de Rham cohomology group $H^2(L \sim \{y_0\})$ is zero. (And this of course guarantees that *any* 2-form ζ on $L \sim \{y_0\}$ can be written in the form $d\omega$ for some 1-form ω on $L \sim \{y_0\}$.) To check that $H^2(L \sim \{y_0\}) = 0$ we first note that de Rahm's theorem gives an isomorphism $H^2(L \sim \{y_0\}) \cong H^2(L \sim \{y_0\}, \mathbf{R})$, where $H^2(L \sim \{y_0\}, \mathbf{R})$ denotes the 2-dimensional singular cohomology group with real coefficients. Next we note the duality isomorphism $H^2(L \sim \{y_0\}, \mathbf{R}) \cong \text{Hom}(H_2(L \sim \{y_0\}), \mathbf{R})$, where $H_2(L \sim \{y_0\})$ denotes the 2-dimensional singular homology group with integer coefficients. Finally we note that $H_2(L \sim \{y_0\}) = 0$. This follows from the exactness of the homology

sequence for the pair $(L, L \sim \{y_0\})$, together with the fact that the inclusion map $(L, \phi) \subset (L, L \sim \{y_0\})$ induces an isomorphism $H_2(L) \cong H_2(L, L \sim \{y_0\})$ (see [9]).

We now consider a C^1 mapping

$$\varphi = (\varphi_1, \dots, \varphi_m): M \rightarrow N.$$

In order to formulate the concept of quasiconformality for φ we need to introduce some terminology. Firstly, for $X \in M$ we let

$$\delta\varphi(X): T_X(M) \rightarrow T_{\varphi(X)}(N)$$

denote the linear map between tangent spaces induced by φ . We note that the matrix $(\delta_i \varphi_j(X))$ represents $\delta\varphi(X)$ in the sense that if $v = (v_1, \dots, v_n) \in T_X(M)$, $w = (w_1, \dots, w_m) \in T_{\varphi(X)}(N)$ and $w = \delta\varphi(X)(v)$, then

$$w_j = \sum_{i=1}^n \delta_i \varphi_j(X) v_i, \quad j = 1, \dots, m.$$

(Here $\delta_i \varphi_j(X)$ is defined by (1.1)). The adjoint transformation $(\delta\varphi(X))^*$ is represented in a similar way by the transposed matrix $(\delta_j \varphi_i(X))$. We define

$$|\delta\varphi(X)| = \left\{ \sum_{i=1}^n \sum_{j=1}^m (\delta_i \varphi_j(X))^2 \right\}^{1/2};$$

thus $|\delta\varphi(X)|$ is just the inner product norm $\{\text{trace}((\delta\varphi(X))^* \delta\varphi(X))\}^{\dagger}$. Next, we let $J\varphi(X)$ denote the signed area magnification factor of φ computed relative to the given area forms η, θ . That is, letting

$$\Lambda^2(\delta\varphi(X)): \Lambda^2(T_{\varphi(X)}(N)) \rightarrow \Lambda^2(T_X(M))$$

be the linear map of 2-forms induced by $\delta\varphi(X)$, we define the real number $J\varphi(X)$ by

$$(1.7) \quad \Lambda^2(\delta\varphi(X))d\omega(\varphi(X)) = J\varphi(X)\eta(X), \quad X \in M.$$

Notice that this makes sense as a definition for $J\varphi(X)$ because $\Lambda^2(T_X(M))$ and $\Lambda^2(T_{\varphi(X)}(N))$ are 1-dimensional vector spaces spanned by the unit vectors $\eta(X)$ and $d\omega(\varphi(X))$ respectively. Notice also that $|J\varphi(X)| = \|\Lambda^2(\delta\varphi(X))\|$. In fact,

$$J\varphi(X) = \pm \|\Lambda^2(\delta\varphi(X))\|,$$

with $+$ or $-$ according as φ preserves or reverses orientation at X .

(1.8) *Definition.* We say φ is (Λ_1, Λ_2) -quasiconformal on M if Λ_1, Λ_2 are constants with $\Lambda_2 \geq 0$, and if

$$|\delta\varphi(X)|^2 \leq \Lambda_1 J\varphi(X) + \Lambda_2$$

at each point $X \in M$.⁽²⁾

The geometric interpretation of this condition is well known:

$$\delta\varphi(X): T_X(M) \rightarrow T_{\varphi(X)}(N)$$

maps the unit circle of $T_X(M)$ onto an ellipse with semi-axes a and b , $a \geq b$, in $T_{\varphi(X)}(N)$, and

$$|\delta\varphi(X)|^2 = a^2 + b^2, \quad |J\varphi(X)| = ab.$$

Thus the definition (1.8), with $\Lambda_2 = 0$, implies

$$a^2 + b^2 \leq |\Lambda_1| ab,$$

which implies $|\Lambda_1| \geq 2$ and

$$a \leq \left(\frac{|\Lambda_1|}{2} + \sqrt{\frac{\Lambda_1^2}{4} - 1} \right) b.$$

Furthermore, (1.8) can hold with $|\Lambda_1| = 2$ if and only if $a = b$; that is, either $\delta\varphi(X) = 0$ or $\delta\varphi(X)$ takes circles into circles. This latter property holds if and only if φ is *conformal* at X .

In case $\Lambda_2 \neq 0$ a similar interpretation holds if $a^2 + b^2$ is sufficiently large relative to Λ_2 ; an important point however is that in this case condition (1.8) imposes no restriction on the mapping φ at points X where $|\delta\varphi(X)|$ is sufficiently small relative to Λ_2 .

(1.9) *Examples.* (i) A classical example considered by Morrey [2] and Nirenberg [3] involves equations

$$\sum_{i,j=1}^2 a_{ij}(x) D_{ij}u = b(x)$$

on a domain $\Omega \subset \mathbb{R}^2$, with conditions

$$|\xi|^2 \leq \sum_{i,j=1}^2 a_{ij}(x) \xi_i \xi_j \leq \lambda_1 |\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^2$$

$$|b(x)| \leq \lambda_2, \quad x \in \Omega.$$

Provided that $\sup_{\Omega} |Du| < \infty$, we can define M, N, φ by $M = \Omega$, $N = \{x \in \mathbb{R}^2: |x| < \sup_{\Omega} |Du|\}$ (see example (1.5)(i)) and

$$\varphi = Du: M \rightarrow N.$$

⁽²⁾ Notice that we do *not* require φ to be 1-1.

In this case we have $J(\varphi) = (D_{11}u)(D_{22}u) - (D_{12}u)^2$, $|\delta\varphi|^2 \equiv |D\varphi|^2 = \sum_{i,j=1}^2 (D_{ij}u)^2$, and φ is (Λ_1, Λ_2) -quasiconformal with $\Lambda_1 = -2\lambda_1$, $\Lambda_2 = \lambda_1\lambda_2^2$. To prove this last assertion we choose coordinates which diagonalize $(D_{ij}u(x))$ at a given point $x = x_0$; in these coordinates the equation takes the form

$$\alpha_1 D_{11}\tilde{u} + \alpha_2 D_{22}\tilde{u} = \beta,$$

where

$$1 \leq \alpha_i \leq \lambda_i, \quad i = 1, 2, \quad |\beta| \leq \lambda_2.$$

Squaring and dividing by $\alpha_1\alpha_2$ then gives

$$\frac{1}{\lambda_1} ((D_{11}\tilde{u})^2 + (D_{22}\tilde{u})^2) \leq -2(D_{11}\tilde{u})(D_{22}\tilde{u}) + \lambda_2^2.$$

In the original coordinates, this gives

$$\sum_{i,j=1}^2 (D_{ij}u)^2 \leq -2\lambda_1((D_{11}u)(D_{22}u) - (D_{12}u)^2) + \lambda_1\lambda_2^2$$

as asserted.

(ii) Another important example of a quasiconformal map arises by considering the equations of mean curvature type; that is, any equation of the form

$$\sum_{i,j=1}^2 a_{ij}(x, u, Du) D_{ij}u = b(x, u, Du), \quad x \in \Omega,$$

where the following conditions (see [7] for a discussion) are satisfied:

$$(a) \quad \sum_{i,j=1}^2 g^{ij}\xi_i\xi_j \leq \sum_{i,j=1}^2 a_{ij}(x, u, Du)\xi_i\xi_j \leq \lambda_1 \sum_{i,j=1}^2 g^{ij}\xi_i\xi_j$$

where

$$g^{ij} = \delta_{ij} - \nu_i\nu_j, \quad \nu_i = -D_i u / \sqrt{1 + |Du|^2},$$

$$(b) \quad |b(x, u, Du)| \leq \lambda_2 \sqrt{1 + |Du|^2}.$$

It is shown in [7] that (a), (b) imply that the graph $M = \{X = (x_1, x_2, x_3) : x_3 = u(x_1, x_2)\}$ has principal curvatures κ_1, κ_2 which satisfy, at each point of M , an equation of the form

$$\alpha_1\kappa_1 + \alpha_2\kappa_2 = \beta,$$

where

$$1 \leq \alpha_i \leq \lambda_i, \quad i = 1, 2, \quad |\beta| \leq \lambda_2.$$

Squaring, we obtain

$$\frac{\alpha_1}{\alpha_2} \kappa_1^2 + \frac{\alpha_2}{\alpha_1} \kappa_2^2 = -2\kappa_1 \kappa_2 + \frac{\beta}{\alpha_1 \alpha_2},$$

so that

$$(1.10) \quad \kappa_1^2 + \kappa_2^2 \leq \Lambda_1 \kappa_1 \kappa_2 + \Lambda_2,$$

where

$$\Lambda_1 = -2\lambda_1, \quad \Lambda_2 = \lambda_1 \lambda.$$

We now let $N = S_+^2$ (see example (1.5) (ii)) and we let $\varphi: M \rightarrow N$ be the Gauss map ν , defined by setting $\nu(X)$ equal to the upward unit normal of M at X ; that is,

$$\nu(X) = (-Du(x), 1) / \sqrt{1 + |Du(x)|^2}, \quad X = (x, u(x)), \quad x \in \Omega.$$

Then, as is well known,

$$J\nu = K \equiv \kappa_1 \kappa_2 \quad (= \text{Gauss curvature of } M).$$

(This is easily checked by working with a ‘‘principal coordinate system at X ’’: that is, a coordinate system with origin at X and with coordinate axes in the directions $e_1(X)$, $e_2(X)$, $\nu(X)$, where $e_1(X)$, $e_2(X)$ are principal directions of M at X .)

Furthermore (and again one can easily check this by working with a principal coordinate system at X)

$$|\delta\nu|^2 = \kappa_1^2 + \kappa_2^2.$$

Thus the inequality (1.10) above asserts that the Gauss map ν is (Λ_1, Λ_2) -quasiconformal with $\Lambda_1 = -2\lambda_1$, $\Lambda_2 = \lambda_1 \lambda_2^2$.

Thus the main Hölder continuity result we are to obtain below (Theorem (2.2)) will apply to the gradient map $x \rightarrow Du(x)$, $x \in \Omega$, in the case of uniformly elliptic equations (as in (i)) and to the Gauss map $X \rightarrow \nu(X)$, $X \in \text{graph}(u)$, in the case of equations of mean curvature type. In the former case one obtains the classical estimate of Morrey-Nirenberg concerning Hölder continuity of first derivatives for uniformly elliptic equations; in the latter case we obtain a new Hölder continuity result for the unit normal of the graph of the solution of an equation of mean curvature type. (See the remarks at the beginning of § 4 below and the reference [7] for further discussion and applications.)

We conclude this section with some notations concerning the subsets obtained by intersecting the surface M with an n -dimensional ball. We write

$$S_\varrho(X_1) = \{X \in M: |X - X_1| < \varrho\}$$

whenever $X_1 \in M$ and $\varrho > 0$. $X_0 \in M$ and $R > 0$ will be such that

$$(\bar{M} \sim M) \cap \{X \in \mathbb{R}^n: |X - X_0| \leq R\} = \emptyset$$

(here \bar{M} denotes the closure of M taken in \mathbf{R}^n), so that $\bar{S}_R(X_0)$ is a compact subset of M . Λ_3 will denote a constant such that

$$(1.11) \quad (3R/4)^{-2} |S_{3R/4}(X_0)| \leq \Lambda_3.$$

Here and subsequently we let $|S_\varrho(X_1)|$ denote the 2-dimensional Hausdorff measure of $S_\varrho(X_1)$.

In the important special case when M is a graph with (Λ_1, Λ_2) -quasiconformal Gauss map, we will show in § 3 that Λ_3 can be chosen to depend only on Λ_1 and $\Lambda_2 R^2$.

It will be proved in the appendix that

$$(1.12) \quad \sigma^{-2} |S_\sigma(X_1)| \leq 40 \left\{ \varrho^{-2} |S_\varrho(X_1)| + \int_{S_\varrho(X_1)} H^2 dA \right\}$$

for any $X_1 \in S_R(X_0)$ and any σ, ϱ with $0 < \sigma \leq \varrho < R - |X_1 - X_0|$. (Here H denotes the mean curvature vector of M .)

§ 2. The Hölder estimate

The main Hölder continuity result (Theorem (2.2) below) will be obtained as a consequence of estimates for the *Dirichlet integral* corresponding to the map $\varphi: M \rightarrow N$ (cf. the original method of Morrey [2].) For a given $X_1 \in S_{R/2}(X_0)$ and $\varrho \in (0, R/2)$, the Dirichlet integral is denoted $\mathcal{D}(X_1, \varrho)$, and is defined by

$$\mathcal{D}(X_1, \varrho) = \int_{S_\varrho(X_1)} |\delta\varphi|^2 dA.$$

Before deriving the estimates for these integrals, some preliminary remarks are needed. We are going to adopt the standard terminology that if ζ is a k -form on N ($k=1, 2$) then $\varphi^*\zeta$ denotes the “pulled-back” k -form on M , defined by

$$(\varphi^*\zeta)(X) = \mathbf{A}^k(\delta\varphi(X))\zeta(\varphi(X)), \quad X \in M.$$

Thus, letting h be an arbitrary C^1 function on M , and using the definition (1.7) together with the relation

$$\varphi^*d = d\varphi^*,$$

we have

$$(2.1) \quad d(h\varphi^*N) = dh \wedge \varphi^*\omega + hJ\varphi dA,$$

where dA denotes the area form η for M . We also need to note that if $X_1 \in S_R(X_0)$ and if r_{X_1} is the Euclidean distance function defined by

$$(2.2) \quad r_{X_1}(X) = |X - X_1|, \quad X \in \mathbf{R}^n,$$

then, by Sard's theorem, we have that, for almost all $\varrho \in (0, R - |X_1 - X_0|)$, δr_{X_1} vanishes at no point of $\partial S_\varrho(X_1)$. For such values of ϱ we can write

$$(2.3) \quad \partial S_\varrho(X_1) = \bigcup_{j=1}^{N(\varrho)} \Gamma_\varrho^{(j)},$$

where $N(\varrho)$ is a positive integer and $\Gamma_\varrho^{(j)}$, $j=1, \dots, N(\varrho)$, are C^2 Jordan curves in M . Thus, by Stoke's theorem, for almost all $\varrho \in (0, R - |X_1 - X_0|)$ (2.1) will imply

$$(2.4) \quad \int_{S_\varrho(X_1)} hJ\varphi dA = - \int_{S_\varrho(X_1)} dh \wedge \varphi^\# \omega + \sum_{j=1}^{N(\varrho)} \int_{\Gamma_\varrho^{(j)}} h\varphi^\# \omega.$$

(We are assuming that the $\Gamma_\varrho^{(j)}$ are appropriated oriented.) In case h has compact support in $S_\varrho(X_1)$ we can write

$$(2.5) \quad \int_{S_\varrho(X_1)} hJ\varphi dA = - \int_{S_\varrho(X_1)} dh \wedge \varphi^\# \omega,$$

and of course this holds for all $\varrho \in (0, R - |X_1 - X_0|)$.

The following lemma gives a preliminary bound for $\mathcal{D}(X_0, R/2)$.

LEMMA (2.1). *If φ is (Λ_1, Λ_2) -quasiconformal, then*

$$\mathcal{D}(X_0, R/2) \leq c,$$

where c depends only on $\Lambda_0, \Lambda_1, \Lambda_2 R^2$ and Λ_3 .

Proof. We let ψ be a C^1 "cut-off function" satisfying $0 \leq \psi \leq 1$ on M , $\psi \equiv 1$ on $S_{R/2}(X_0)$, $\psi \equiv 0$ outside $S_{3R/4}(X_0)$ and $\sup_M |\delta\psi| \leq 5/R$. (Such a function is obtained by defining $\psi(X) = \gamma(|X - X_1|)$, where γ is a suitably chosen $C^1(\mathbf{R})$ function.)

Since

$$(2.6) \quad \varphi^\# \omega = \sum_{i=1}^m \omega_i \circ \varphi d\varphi_i,$$

we can easily check, by using (1.4), that

$$|(d\psi \wedge \varphi^\# \omega)(X)| \leq \Lambda_0 |\delta\psi(X)| |\delta\varphi(X)| \leq 5R^{-1} \Lambda_0 |\delta\varphi(X)|, \quad X \in M.$$

(Here, on the left, $||$ denotes the usual inner product norm for forms on $T_X(M)$.) Then by using (2.5) with $X_1 = X_0$, $\varrho = R$ and $h = \psi^2$, we easily obtain

$$\left| \int_{S_R(X_0)} \psi^2 J\varphi dA \right| \leq 10R^{-1} \Lambda_0 \int_{S_R(X_0)} \psi |\delta\varphi| dA.$$

The quasiconformal condition (1.8) then implies

$$\int_{S_R(X_0)} \psi^2 |\delta\varphi|^2 dA \leq 10R^{-1}\Lambda_0 |\Lambda_1| \int_{S_R(X_0)} \psi |\delta\varphi| dA + \Lambda_2 \int_{S_R(X_0)} \psi^2 dA.$$

Using the Cauchy inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ and the definition of Λ_3 , we then obtain

$$\int_{S_R(X_0)} \psi^2 |\delta\varphi|^2 dA \leq \frac{1}{2} \int_{S_R(X_0)} \psi^2 |\delta\varphi|^2 dA + (50\Lambda_0^2 \Lambda_1^2 + \Lambda_2 R^2) \Lambda_3.$$

Since $\psi \equiv 1$ on $S_{R/2}(X_0)$, the required inequality then follows (with $c = (100\Lambda_0^2 \Lambda_1^2 + 2\Lambda_2 R^2) \Lambda_3$).

The next theorem contains the main estimate for $\mathcal{D}(X_1, \rho)$. In the statement of the theorem, and subsequently, Λ_4 denotes a constant such that

$$\int_{S_{R/2}(X_0)} H^2 dA \leq \Lambda_4,$$

where H is the mean-curvature vector of M . (See [4].)

THEOREM (2.1). *If φ is (Λ_1, Λ_2) -quasiconformal, then*

$$\mathcal{D}(X_1, \rho) \leq c(\rho/R)^\alpha$$

for all $X_1 \in S_{R/4}(x_0)$ and all $\rho \in (0, R/4)$, where $c > 0$ and $\alpha \in (0, 1)$ are constants depending only on $\Lambda_0, \Lambda_1, \Lambda_2 R^2, \Lambda_3$ and Λ_4 .

Proof. Since the curves $\Gamma_\rho^{(j)}$ of (2.3) are closed we have $\int \Gamma_\rho^{(j)} d\varphi_i = \int \Gamma_\rho^{(j)} d\varphi_i/ds = 0$. (Here ds denotes integration with respect to arc-length and $d\varphi_i/ds$ denotes directional differentiation in the direction of the appropriate unit tangent T of $\Gamma_\rho^{(j)}$; that is $d\varphi_i/ds = T \cdot \delta\varphi_i$.) Then by (2.6) we have

$$\int_{\Gamma_\rho^{(j)}} \varphi^* \omega \sum_{i=1}^m \int_{\Gamma_\rho^{(j)}} (\omega_i \circ \varphi - \omega_i \circ \varphi(X^{(j)})) \frac{d\varphi_i}{ds} ds,$$

where $X^{(j)}$ denotes an initial point (corresponding to arc-length = 0) of $\Gamma_\rho^{(j)}$. Then using (2.4) with $h \equiv 1$, we obtain

$$(2.7) \quad \left| \int_{S_\rho(X_1)} J\varphi dA \right| = \left| \sum_{j=1}^{N(\rho)} \int_{\Gamma_\rho^{(j)}} \sum_{i=1}^m (\omega_i \circ \varphi - \omega_i \circ \varphi(X^{(j)})) \frac{d\varphi_i}{ds} ds \right| \\ \leq \sum_{j=1}^{N(\rho)} \left\{ \sup_{\Gamma_\rho^{(j)}} |\omega \circ \varphi - \omega \circ \varphi(X^{(j)})| \int_{\Gamma_\rho^{(j)}} |\delta\varphi| ds \right\}.$$

But clearly

$$(2.8) \quad \sup_{\Gamma_\varrho^{(j)}} |\omega \circ \varphi - \omega \circ \varphi(X^{(j)})| \leq \int_{\Gamma_\varrho^{(j)}} \left| \frac{d\omega \circ \varphi}{ds} \right| ds \leq \int_{\Gamma_\varrho^{(j)}} |\delta\omega \circ \varphi| ds.$$

Since

$$|\delta\omega \circ \varphi| \leq \sup_N |D\omega| |\delta\varphi| \leq \Lambda_0 |\delta\varphi|,$$

(2.7) and (2.8) clearly imply

$$(2.9) \quad \begin{aligned} \left| \int_{S_\varrho(X_1)} J\varphi dA \right| &\leq \Lambda_0 \sum_{j=1}^{N(\varrho)} \left\{ \int_{\Gamma_\varrho^{(j)}} |\delta\varphi| ds \right\}^2 \\ &\leq \Lambda_0 \left\{ \sum_{j=1}^{N(\varrho)} \int_{\Gamma_\varrho^{(j)}} |\delta\varphi| ds \right\}^2 = \Lambda_0 \left(\int_{\partial S_\varrho(X_1)} |\delta\varphi| ds \right)^2 \\ &= \Lambda_0 \left(\int_{\partial S_\varrho(X_1)} (|\delta\varphi| |\delta r_{X_1}|^{-1/2}) (|\delta r_{X_1}|^{1/2}) ds \right)^2 \\ &\leq \Lambda_0 \left\{ \int_{\partial S_\varrho(X_1)} |\delta\varphi|^2 |\delta r_{X_1}|^{-1} ds \right\} \left\{ \int_{\partial S_\varrho(X_1)} |\delta r_{X_1}| ds \right\} \\ &= \Lambda_0 \left(\frac{d}{d\varrho} \int_{S_\varrho(X_1)} |\delta\varphi|^2 dA \right) \left(\frac{d}{d\varrho} \int_{S_\varrho(X_1)} |\delta r_{X_1}|^2 dA \right). \end{aligned}$$

Here r_{X_1} is as in (2.2) and in the last equality we have used the differentiated version of the co-area formula:

$$\frac{d}{d\varrho} \int_{S_\varrho(X_1)} h dA = \int_{\partial S_\varrho(X_1)} h |\delta r_{X_1}| ds$$

whenever h is a continuous function on M .

Now by using (1.12) and the identity (A.2) with $h \equiv 1$, it is easily seen that

$$(2.10) \quad \frac{d}{d\varrho} \int_{S_\varrho(X_1)} |\delta r_{X_1}|^2 dA \leq c_1 \varrho,$$

where c_1 depends only on Λ_3 and Λ_4 . Hence, by combining (2.9) and (2.10) we have

$$\left| \int_{S_\varrho(X_1)} J\varphi dA \right| \leq c_1 \Lambda_0 \varrho \frac{d}{d\varrho} \mathcal{D}(X_1, \varrho).$$

The condition (1.8) then implies (after using (1.11), (1.12))

$$\mathcal{D}(X_1, \varrho) \leq c'_1 \left(|\Lambda_1| \Lambda_0 \varrho \frac{d}{d\varrho} \mathcal{D}(X_1, \varrho) + \Lambda_2 \varrho^3 \right)$$

for almost all $\varrho \in (0, R/4)$. If we now define

$$\mathcal{E}(\varrho) = \mathcal{D}(X_1, \varrho) + \Lambda_2 \varrho^2$$

we see that this last inequality implies

$$\mathcal{E}(\varrho) \leq c_2 \varrho \mathcal{E}'(\varrho), \quad \text{a.e. } \varrho \in (0, R/4),$$

where c_2 depends only on $\Lambda_0, \Lambda_1, \Lambda_3$ and Λ_4 . This can be written

$$\frac{d}{d\varrho} \log \mathcal{E}(\varrho) \geq c_2^{-1} \varrho^{-1}, \quad \text{a.e. } \varrho \in (0, R/4).$$

Since $\mathcal{E}(\varrho)$ is increasing in ϱ , we can integrate to obtain

$$\log (\mathcal{E}(\varrho)/\mathcal{E}(R/4)) \leq c_2^{-1} \log (4\varrho/R), \quad \varrho \leq R/4;$$

that is

$$(2.11) \quad \mathcal{E}(\varrho) \leq 4^\alpha \mathcal{E}(R/4) (\varrho/R)^\alpha, \quad \alpha = c_2^{-1}, \varrho \in (0, R/4).$$

Since $S_{R/4}(X_1) \subset S_{R/2}(X_0)$, we must have

$$(2.12) \quad \mathcal{E}(R/4) \leq \mathcal{D}(X_0, R/2) + \Lambda_2 (R/4)^2.$$

The required estimate for $\mathcal{D}(X_1, \varrho)$ now follows from (2.11), (2.12) and Lemma (2.1); note that the exponent α is actually independent of Λ_2 .

We next need an analogue of the Morrey lemma ([2], Lemma 1) for surfaces; this will enable us to deduce a Hölder estimate for φ from Theorem (2.1) (cf. the original method of Morrey [2].)

LEMMA (2.2). *Suppose h is C^1 on M and suppose $K > 0, \beta \in (0, 1)$ are such that*

$$\int_{S_\varrho(X_1)} |\delta h| dA \leq K \varrho (\varrho/R)^\beta$$

for all $X_1 \in S_{R/4}(X_0)$ and all $\varrho \in (0, R/4)$. Then

$$\sup_{x \in S_\varrho^*(X_0)} |h(x) - h(X_0)| \leq cK (\varrho/R)^\beta, \quad \varrho \in (0, R/4),$$

where c depends on Λ_3 and Λ_4 , and where $S_\varrho^*(X_0)$ denotes the component of $S_\varrho(X_0)$ which contains X_0 .

This lemma is proven in the appendix.

We can now finally deduce the Hölder estimate for quasiconformal maps.

THEOREM (2.2). *If φ is (Λ_1, Λ_2) -quasiconformal, then*

$$\sup_{X \in S_\varrho^*(X_0)} |\varphi(X) - \varphi(X_0)| \leq c(\varrho/R)^{\alpha/2}, \quad \varrho \in (0, R/4),$$

where $c > 0$ depends only on $\Lambda_0, \Lambda_1, \Lambda_2 R^2, \Lambda_3$ and Λ_4 and where $\alpha \in (0, 1)$ is as in Theorem (2.1); $S_\varrho^*(X_0)$ is as in Lemma (2.2).

Proof. Let X_1 be an arbitrary point of $S_{R/4}(X_0)$. By the Hölder inequality, (1.12) and Theorem (2.1) we have

$$\int_{S_\varrho(X_1)} |\delta\varphi_i| dA \leq c'(c)^{1/2} \varrho(\varrho/R)^{\alpha/2}, \quad \varrho \in (0, R/4), \quad i = 1, \dots, m$$

where c, α are as in Theorem (2.1) and c' depends on Λ_3, Λ_4 . Hence the hypotheses of Lemma (2.2) are satisfied, with $\beta = \alpha/2$ and $K = c'c^{1/2}$.

§ 3. Graphs with (Λ_1, Λ_2) -quasiconformal Gauss map

In this section M will denote the graph $\{X = (x, z): x \in \Omega, z = u(x)\}$ of a $C^2(\Omega)$ function u , where $\Omega \subset \mathbf{R}^2$ is an arbitrary open set. x_0 will denote a fixed point of Ω , and it will be assumed that Ω contains the disc $D_R(x_0) = \{x \in \mathbf{R}^2: |x - x_0| < R\}$. X_0 will denote the point $(x_0, u(x_0))$ of M and ν will denote the Gauss map of M into S_+^2 defined (as in (1.9) (ii)) by setting $\nu(X)$ equal to the upward unit normal at X ; that is,

$$(3.1) \quad \nu(X) \equiv \nu(x) = (1 + |Du(x)|^2)^{-1/2} (-Du(x), 1), \quad X = (x, u(x)), \quad x \in \Omega.$$

We already mentioned in (1.9) (ii) that $J\nu = K = \kappa_1 \kappa_2$ and $|\delta\nu|^2 = \kappa_1^2 + \kappa_2^2$, where κ_1, κ_2 are the principal curvatures of M . Hence the Gauss map ν is (Λ_1, Λ_2) -quasiconformal if and only if

$$(3.2) \quad \kappa_1^2 + \kappa_2^2 \leq \Lambda_1 K + \Lambda_2;$$

this inequality will be assumed throughout this section. The remaining notation and terminology will be as in § 1 and § 2.

In order to effectively apply Theorem (2.2) to the Gauss map, we first need to discuss appropriate choices for the constants Λ_0, Λ_3 and Λ_4 .

To begin with, we have already seen in (1.5) (ii) that in case $N = S_+^2$ we can take $\Lambda_0 = 4$. Next we notice that, since $|\delta\nu|^2 = \kappa_1^2 + \kappa_2^2$, Lemma (2.1) with $\varphi = \nu$ gives $\int_{S_{R/2}(X_0)} (\kappa_1^2 + \kappa_2^2) dA \leq c$, where c depends only on $\Lambda_1, \Lambda_2 R^2$ and Λ_3 . Thus since $\kappa_1^2 + \kappa_2^2 \geq \frac{1}{2}(\kappa_1 + \kappa_2)^2 = \frac{1}{2}H^2$ we can in this case make the choice $\Lambda_4 = 2c$. The next lemma shows that we can choose Λ_3 to depend only on $\Lambda_1, \Lambda_2 R^2$.

LEMMA (3.1). *If $X_1 \in S_R(X_0)$ and $\varrho \in (0, \frac{1}{2}(R - |X_1 - X_0|))$, then*

$$|S_\varrho(X_1)| \leq c\varrho^2,$$

where c is a constant depending only on Λ_1 and $\Lambda_2 R^2$.

Proof. We will use the well-known identities

$$(3.3) \quad \Delta v_l + v_l(\kappa_1^2 + \kappa_2^2) = \delta_l H, \quad l = 1, 2, 3,$$

where $H = \kappa_1 + \kappa_2$ is the mean curvature of M and $\Delta = \sum_{i=1}^3 \delta_i \delta_i$ is the Laplace-Beltrami operator on M . We will also need the first variation formula:

$$(3.4) \quad \int_M \delta_i h dA = \int_M v_i H h dA, \quad i = 1, 2, 3,$$

which is valid whenever h is a C^1 function with compact support on M . Finally, we will need to use the fact that if $\zeta \in C^2(\Omega \times \mathbf{R})$, then

$$(3.5) \quad \Delta(\zeta|M) = \sum_{i,j=1}^3 (\delta_{ij} - v_i v_j) D_{ij} \zeta + H \sum_{i=1}^3 v_i D_i \zeta$$

on M ; one easily checks this by direct computation together with (1.2).

We now let $h \geq 0$ be a $C^2(M)$ function with compact support in M . Multiplying by h in (3.3), with $i=3$, and integrating by parts with the aid of (3.4), we obtain

$$\int_M \{(\kappa_1^2 + \kappa_2^2)h + \Delta h\} v_3 dA = \int_M \{v_3(\kappa_1 + \kappa_2)^2 h - (\kappa_1 + \kappa_2) \delta_3 h\} dA.$$

that is, since $\kappa_1^2 + \kappa_2^2 - (\kappa_1 + \kappa_2)^2 = -2\kappa_1 \kappa_2 = -2K$,

$$-2 \int_M K h v_3 dA = \int_M (-v_3 \Delta h - (\kappa_1 + \kappa_2) \delta_3 h) dA.$$

Choosing h of the form $h(X) \equiv \zeta(x)$, $X = (x, u(x))$, $x \in \Omega$, where $\zeta \in C^2(\Omega)$ has compact support we then deduce, with the aid of (3.5) and (1.1)–(1.2),

$$(3.6) \quad 2 \int_M K \zeta(x) v_3 dA = \int_M v_3 \left\{ \sum_{i,j=1}^2 (\delta_{ij} - v_i v_j) D_{ij} \zeta(x) + 2(\kappa_1 + \kappa_2) \sum_{i=1}^2 v_i D_i \zeta(x) \right\} dA.$$

Replacing ζ by ζ^2 and using (3.2), it is easily seen that this implies

$$\begin{aligned} & \int_M (\kappa_1^2 + \kappa_2^2) \zeta^2(x) v_3 dA \\ & \leq \int_M 2|\Lambda_1| \left\{ |D\zeta(x)|^2 + \zeta(x) \sum_{i,j=1}^2 |D_i \zeta(x)| + |\kappa_1 + \kappa_2| \zeta(x) |D\zeta(x)| \right\} v_3 dA + \Lambda_2 \int_M \zeta^2(x) v_3 dA. \end{aligned}$$

Since we have

$$2|\Lambda_1| |\kappa_1 + \kappa_2| \zeta |D\zeta| \leq \frac{1}{2}(\kappa_1^2 + \kappa_2^2) \zeta^2 + \frac{1}{2}(2\Lambda_1 |D\zeta|)^2,$$

this gives

$$(3.7) \quad \frac{1}{2} \int_M (\kappa_1^2 + \kappa_2^2) \zeta^2(x) \nu_3 dA \leq \int_M \left\{ c_1 (|D\zeta(x)|^2 + \zeta(x) \sum_{i,j=1}^2 |D_{ij}\zeta(x)|) + \Lambda_2 \zeta^2(x) \right\} \nu_3 dA,$$

where c_1 depends only on Λ_1 .

Now let $x^{(1)} \in \Omega$ be such that $X_1 = (x^{(1)}, u(x^{(1)}))$, note that $D_{2\varrho}(x^{(1)}) \subset \Omega$ and choose ζ such that

$$0 \leq \zeta \leq 1 \text{ on } \Omega, \quad \zeta \equiv 1 \text{ on } D_\varrho(x^{(1)}), \quad \zeta \equiv 0 \text{ on } \mathbf{R}^3 - D_{2\varrho}(x^{(1)}),$$

$$\sup_\Omega |D\zeta| \leq c_2/\varrho, \quad \sup_\Omega \sum_{i,j=1}^2 |D_{ij}\zeta| \leq c_2/\varrho^2,$$

where c_2 is an absolute constant. Then, since $\Lambda_2 \leq (\Lambda_2 R^2)/\varrho^2$, (3.7) implies

$$(3.8) \quad \int_M (\kappa_1^2 + \kappa_2^2) \zeta^2(x) \nu_3 dA \leq c_3 \varrho^{-2} \int_{M \cap (D_{2\varrho}(x^{(1)}) \times \mathbf{R})} \nu_3 dA,$$

where c_3 depends only on Λ_1 and $\Lambda_2 R^2$.

Next we notice that, since M is the graph of u , if f is any given continuous function on M then

$$\int_M f dA = \int_\Omega \tilde{f}(x) \sqrt{1 + |Du(x)|^2} dx,$$

where \tilde{f} is defined on Ω by $\tilde{f}(x) = f(x, u(x))$. In particular since $\sqrt{1 + |Du(x)|^2} = (\nu_3(x))^{-1}$, we have

$$(3.9) \quad \int_M f \nu_3 dA = \int_\Omega \tilde{f}(x) dx.$$

Hence (3.8) can be written

$$(3.8)' \quad \int_\Omega (\tilde{\kappa}_1^2 + \tilde{\kappa}_2^2) \zeta^2 dx \leq c_3 \varrho^{-2} \int_{D_{2\varrho}(x^{(1)})} dx = 4c_3 \varrho^{-2} \pi \varrho^2 = 4c_3 \pi$$

where

$$\tilde{\kappa}_i(x) = \kappa_i(x, u(x)), \quad x \in \Omega, \quad i = 1, 2.$$

Writing $\tilde{H} = \tilde{\kappa}_1 + \tilde{\kappa}_2$, noting that $\tilde{H}^2 \leq 2(\tilde{\kappa}_1^2 + \tilde{\kappa}_2^2)$ and using Hölder's inequality, we then have

$$(3.10) \quad \int_\Omega |\tilde{H}| \zeta dx \leq \left\{ \int_\Omega \tilde{H}^2 \zeta^2 dx \right\}^{1/2} |D_{2\varrho}(x^{(1)})|^{1/2} \leq (8c_3 \pi)^{1/2} (4\pi \varrho^2)^{1/2} < 8\sqrt{c_3} \pi \varrho.$$

We now let M_- denote the region below the graph of u ; that is,

$$M_- = \{X = (x, z): x \in \Omega, z < u(x)\}.$$

Also, letting $B_\sigma = \{X \in \mathbf{R}^3: |X - X_1| < \sigma\}$, we take γ to be a C^1 function on \mathbf{R}^3 such that

$$0 \leq \gamma \leq 1 \text{ on } \mathbf{R}^3, \gamma \equiv 1 \text{ on } B_\rho, \gamma \equiv 0 \text{ on } \mathbf{R}^3 - B_{2\rho}, \sup_\Omega |D\gamma| \leq c_2/\rho.$$

Applying the divergence theorem on M_- we have

$$\int_M \gamma \zeta(x) \nu \cdot \nu dA = \int_{M_-} \operatorname{div} (\gamma \zeta(x) \nu) dx dz.$$

Here we take ν to be a $C^1(\Omega \times \mathbf{R})$ function defined by

$$\nu(x, z) \equiv \nu(x) = (1 + |Du(x)|^2)^{-1/2} (-Du(x), 1), \quad x \in \Omega, z \in \mathbf{R}.$$

Hence we obtain

$$(3.11) \quad |S_\rho(X_1)| \leq \left| \int_{M_-} \{\gamma \zeta(x) \operatorname{div} \nu + \nu \cdot D(\gamma \zeta(x))\} dx dz \right|.$$

Finally, noting that

$$(3.12) \quad \operatorname{div} \nu(X) = \sum_{i=1}^2 D_i \nu_i(X) = \sum_{i=1}^2 D_i \nu_i(x) = \tilde{H}(x), \quad X = (x, z) \in \Omega \times \mathbf{R},$$

and using (3.11) together with the fact that $|D(\gamma \zeta(x))| \leq 2c_2 \rho^{-1}$, we easily deduce the required area bound from (3.10).

Thus we have shown that Λ_3, Λ_4 can both be chosen to depend only on $\Lambda_1, \Lambda_2 R^2$. Hence Theorem (2.2) gives the Hölder estimate

$$(3.13) \quad \sup_{X \in S_\rho^*(X_0)} |\nu(X) - \nu(X_0)| \leq c(\rho/R)^\alpha, \quad \rho \in (0, R),$$

where $c > 0$ and $\alpha \in (0, 1)$ depend only on $\Lambda_1, \Lambda_2 R^2$. Notice that we assert (3.13) for all $\rho \in (0, R)$ rather than $\rho \in (0, R/4)$ as in Theorem (2.2). We can do this because $|\nu| = 1$ (which means an inequality of the form (3.13) trivially holds for $\rho \in (R/4, R)$).

We now wish to show that an inequality of the form (3.13) holds with $S_\rho(X_0)$ in place of $S_\rho^*(X_0)$; we will in fact prove that there is a constant $\theta \in (0, 1)$, depending only on $\Lambda_1, \Lambda_2 R^2$ such that $S_\rho^*(X_0) = S_\rho(X_0)$ for all $\rho \leq \theta R$.

We first use (3.13) to deduce some facts about local non-parametric representations for M . Let

$$S = S_{\theta R}^*(X_0)$$

$$\tilde{S} = \{(\xi, \zeta): (\xi, \zeta) = (x - x_0, z - z_0)Q, \quad (x, z) \in S\}$$

where $\theta \in (0, 1)$, $z_0 = u(x_0)$ and Q is the 3×3 orthogonal matrix with rows $e_1, e_2, \nu(X_0)$, where e_1, e_2 are principal directions of M at X_0 . Since M is a C^2 surface we of course know that for *small enough* θ there is a neighbourhood U of $0 \in \mathbb{R}^2$ and a $C^2(U)$ function \tilde{u} with $D\tilde{u}(0) = 0$ and

$$(3.14) \quad \tilde{S} = \text{graph } \tilde{u} = \{(\xi, \zeta) : \xi \in U, \zeta = \tilde{u}(\xi)\}.$$

Furthermore, letting

$$(3.15) \quad \tilde{v}(\xi) = (1 + |D\tilde{u}(\xi)|^2)^{-1}(-D\tilde{u}(\xi), 1), \quad \xi \in U,$$

we have by (3.13) that

$$|\tilde{v}(\xi) - \tilde{v}(0)| \leq c\theta^\alpha, \quad \xi \in U,$$

where c, α are as in (3.13). That is, by (3.15),

$$(1 + |D\tilde{u}(\xi)|^2)^{-1}|D\tilde{u}(\xi)|^2 + ((1 + |D\tilde{u}(\xi)|^2)^{-1} - 1)^2 \leq (c\theta^\alpha)^2, \quad \xi \in U,$$

which implies

$$(3.16) \quad |D\tilde{u}(\xi)| \leq (1 - (c\theta^\alpha)^2)^{-1}c\theta^\alpha < \frac{1}{2}, \quad \xi \in U,$$

provided θ is such that

$$(3.17) \quad c\theta^\alpha \leq 1/4.$$

Because of (3.16), we can infer that a representation of the form (3.14) holds for any θ satisfying (3.17).

For later reference we also note that (3.16) implies

$$(3.18) \quad D_{\theta R/2}(0) \subset U.$$

The next lemma contains the connectivity result referred to above.

LEMMA (3.2). *There is a constant $\theta \in (0, 1)$, depending only on $\Lambda_1, \Lambda_2 R^2$, such that $S_\varrho(X_0)$ is connected for each $\varrho \leq \theta R$.*

Proof. In the proof we will let $c_1, c_2 \dots$ denote constants depending only on $\Lambda_1, \Lambda_2 R^2$. B_σ , for $\sigma > 0$, will denote the open ball $\{X \in \mathbb{R}^3 : |X - X_0| < \sigma\}$.

Let $\theta \in (0, 1)$ satisfy (3.17), let $\varrho = \theta R/2$, let $\beta \in (0, \frac{1}{4})$ and define \mathcal{S}_β to be the collection of those components of $S_{\varrho/2}(X_0)$ which intersect the ball $B_{\beta\varrho}$. For each $S \in \mathcal{S}_\beta$ we can find $X_1 \in S \cap B_{\varrho/4}$ such that

$$(3.19) \quad S \subset S_\beta^*(X_1),$$

and hence, replacing X_0 by X_1 and R by $R/2$ in the discussion preceding the lemma, we see that S can be represented in the form (3.14), (3.16). Using such a non-parametric representation for each $S \in \mathcal{S}_\beta$ and also using the fact that no two elements of \mathcal{S}_β can intersect, it follows that the *union* of all the components $S \in \mathcal{S}_\beta$ is contained in a region bounded between two parallel planes π_1, π_2 with

$$(3.20) \quad d(\pi_1, \pi_2) \leq c_1(\beta + \theta^\alpha)\varrho.$$

Here $d(\pi_1, \pi_2)$ denotes the distance between π_1 and π_2 and α is as in (3.17).

Our aim now is to show that, for suitable choices of β and θ depending only on Λ_1 and $\Lambda_2 R^2$, there is only one element (viz. $S_{\varrho/2}^*(X_0)$) in \mathcal{S}_β . Suppose that in fact there are two distinct elements $S_1, S_2 \in \mathcal{S}_\beta$. We can clearly choose S_1, S_2 to be adjacent in the sense that the volume V enclosed by S_1, S_2 and $\partial B_{\varrho/2}$ intersects no other elements $S \in \mathcal{S}_\beta$. Thus $V \cap B_{\rho_0}$ consists either entirely of points above the graph M or entirely of points below M ; it is then evident that if the unit normal ν points out of (into) V on S_1 , then it also points out of (into) V on S_2 . Furthermore by (3.20) we have

$$(3.21) \quad \text{volume}(V) \leq c_2(\beta + \theta^\alpha)\varrho^2,$$

$$(3.22) \quad \text{area}(V \cap \partial B_{\varrho/2}) \leq c_3(\beta + \theta^\alpha)\varrho^2.$$

An application of the divergence theorem over V then gives

$$\int_{S_1} \nu \cdot \nu dA + \int_{S_2} \nu \cdot \nu dA = \pm \left\{ \int_V \text{div } \nu dx dz - \int_{\partial B_{\varrho/2} \cap V} \eta \cdot \nu dA \right\},$$

where η is the outward unit normal of $\partial B_{\varrho/2}$. By (3.22) and (3.12) this gives

$$(3.23) \quad \text{area}(S_1) + \text{area}(S_2) \leq \int_V |\hat{H}(x)| dx dz + c_3(\beta + \theta^\alpha)\varrho^2.$$

Also, by (3.8)' and (3.21),

$$\begin{aligned} \int_V |\hat{H}(x)| dx dz &\leq \left(\int_V \hat{H}^2(x) dx dz \right)^{1/2} \{\text{volume}(V)\}^{1/2} \\ &\leq \left(\int_{B_{\varrho/2}} \hat{H}^2(x) dx dz \right)^{1/2} \{c_2(\beta + \theta^\alpha)\varrho^2\}^{1/2} \\ &\leq (c_4\varrho)^{1/2} \{c_2(\beta + \theta^\alpha)\varrho^2\}^{1/2} = \sqrt{c_4 c_2} (\beta + \theta^\alpha) \varrho^2. \end{aligned}$$

Hence (3.23) gives

$$(3.24) \quad \text{area}(S_1) + \text{area}(S_2) \leq c_5 \sqrt{\beta + \theta^\alpha} \varrho^2.$$

On the other hand by using a non-parametric representation as in (3.14), (3.16) we infer that

$$(3.25) \quad \text{area}(S) \geq c_6 \varrho^2$$

for each $S \in \mathcal{S}_\beta$, where $c_6 > 0$ is an absolute constant.

(3.24) and (3.25) are clearly contradictory if we choose β, θ small enough (but depending only on Λ_1 and $\Lambda_2 R^2$). For such a choice of β, θ we thus have

$$S_{\beta\varrho}(X_0) = M \cap B_{\beta\varrho} = S_{\varrho/2}(X_0) \cap B_{\beta\varrho} = S_{\varrho/2}^*(X_0) \cap B_{\beta\varrho}.$$

But by using a representation of the form (3.14), (3.16) for $S_{\varrho/2}^*(X_0)$, we clearly have $S_{\varrho/2}^*(X_0) \cap B_{\beta\varrho}$ connected. Thus $S_{\beta\varrho}(X_0) = S_{\beta\theta R/2}(X_0)$ is connected. The lemma follows because the choice of β, θ depended only on $\Lambda_1, \Lambda_2 R^2$.

Because of the above connectivity result we can replace $S_\varrho^*(X_0)$ in (3.13) by $S_\varrho(X_0)$ for $\varrho \leq \theta R$. However since $|\nu| = 1$, an inequality of the form (3.13) is trivial for $\varrho > \theta R$. Hence we have the result of the following theorem.

THEOREM (3.1). *For each $\varrho \in (0, R)$ we have*

$$\sup_{X \in S_\varrho(X_0)} |\nu(X) - \nu(X_0)| \leq c(\varrho/R)^\alpha,$$

where $c > 0$ and $\alpha \in (0, 1)$ depend only on $\Lambda_1, \Lambda_2 R^2$.

Remark. The above inequality implies

$$(3.26) \quad |\nu(X) - \nu(\bar{X})| \leq c'(|X - \bar{X}|/R)^\alpha, \quad X, \bar{X} \in S_{R/2}(X_0)$$

($c' = 4^\alpha c$). This is seen by using \bar{X} in place of X_0 and $R/4$ in place of R .

§ 4. Graphs with $(\Lambda_1, 0)$ -quasiconformal Gauss map

Here the notation will be as in § 3, except that we take $\Lambda_2 = 0$ always; that is, we assume that the graph M of u has $(\Lambda_1, 0)$ -quasiconformal Gauss map. It will be shown that there are a number of special results which can be established in this case.

We note that, in particular, the graph of a solution of any *homogeneous* equation of mean curvature type (i.e. an equation as in (1.9) (ii) with $b \equiv 0$) is $(\Lambda_1, 0)$ -quasiconformal.

Hence the results of this section apply in particular to these equations. (See [7] for further discussion.)

Our first observation is that if $\Omega = \mathbf{R}^2$, then we can let $R \rightarrow \infty$ in (3.26) to obtain $v \equiv \text{const.}$; that is, u is linear. Thus we have

THEOREM (4.1). *Suppose $\Omega = \mathbf{R}^2$ and v is $(\Lambda_1, 0)$ -quasiconformal. Then u is a linear function.*

Remark. Actually this theorem can be deduced directly from Theorem (2.1) (by letting $R \rightarrow \infty$) without first proving (3.26) (or even (3.13)). However note that Lemma (3.1) is still needed to show that Λ_3 can be chosen to depend only on Λ_1 .

Before proceeding further, we want to establish an interesting integral identity (equation (4.5) below) involving the Gauss curvature K of the graph M .

Recall first that K is the area magnification factor for the Gauss map; hence since the area form for S_+^2 is $d\omega$, where $\omega(X) = (1 + x_3)^{-1}(-x_2 dx_1 + x_1 dx_2)$ (see (1.5) (ii)), we have the identity

$$(4.1) \quad KdA = d\omega^*, \quad \omega^* = v^\# \omega = (1 + v_3)^{-1}(-v_2 dv_1 + v_1 dv_2),$$

where dA is the area form for M . Since $\sum_{i=1}^3 v_i^2 = |v|^2 = 1$, we have

$$(4.2) \quad dv_3 = -v_3^{-1}(v_1 dv_1 + v_2 dv_2),$$

and using this in (4.1) yields the identity

$$(4.3) \quad KdA = v_3^{-1} dv_1 \wedge dv_2.$$

Now, by using (4.1) together with Stoke's theorem, we deduce

$$(4.4) \quad \int_M \zeta K dA = - \int_M d\zeta \wedge \omega^*$$

for any $\zeta \in C^1(M)$ with compact support in M . In particular, choosing ζ of the form $\zeta = \gamma(v_3)\zeta_1$, where γ is a $C^1(\mathbf{R})$ function and $\zeta_1 \in C^1(M)$ has compact support in M , it can be checked, by using (4.2) and (4.3), that (4.4) implies

$$\int_M \zeta_1(\gamma(v_3) - (1 - v_3)\gamma'(v_3)) K dA = - \int_M \gamma(v_3) d\zeta_1 \wedge \omega^*,$$

which can be written

$$(4.5) \quad \int_M \zeta_1((1 - v_3)\gamma(v_3))' K dA = \int_M \gamma(v_3) d\zeta_1 \wedge \omega^*.$$

We will subsequently need the following inequalities for the principal curvatures κ_1, κ_2 of M :

$$(4.6) \quad (1 - \nu_3^2) \min \{\kappa_1^2, \kappa_2^2\} \leq |\delta \nu_3|^2 \leq (1 - \nu_3^2) \max \{\kappa_1^2, \kappa_2^2\}.$$

To prove this, first recall that the 3×3 matrix $(\delta_i \nu_j)$ is the second fundamental form for M in the sense that there are orthogonal tangent vectors (principal directions) $e^{(i)} = (e_1^{(i)}, e_2^{(i)}, e_3^{(i)})$, $i=1, 2$, such that

$$\sum_{j=1}^3 (\delta_j \nu_k) e_j^{(i)} = \kappa_i e_k^{(i)}, \quad i=1, 2, \quad k=1, 2, 3.$$

Since $\sum_{j=1}^3 (\delta_j \nu_k) \nu_j = 0$, $k=1, 2, 3$, we can set $k=3$ in these identities to give

$$(\delta \nu_3) = (\kappa_1 e_3^{(1)}, \kappa_2 e_3^{(2)}, 0) Q,$$

where Q is the orthogonal matrix with rows $e^{(1)}, e^{(2)}, \nu$. Thus

$$|\delta \nu_3|^2 = \kappa_1^2 (e_3^{(1)})^2 + \kappa_2^2 (e_3^{(2)})^2.$$

(4.6) now easily follows by noting that $(e_3^{(1)})^2 + (e_3^{(2)})^2 = 1 - \nu_3^2$, because $(e_3^{(1)}, e_3^{(2)}, \nu_3)$ is the third column of the orthogonal matrix Q .

Now we are assuming the Gauss map of M is $(\Lambda_1, 0)$ quasiconformal; that is

$$(4.7) \quad |\delta \nu|^2 = \kappa_1^2 + \kappa_2^2 \leq \Lambda_1 \kappa_1 \kappa_2 = \Lambda_1 K.$$

This implies

$$\max \{\kappa_1^2, \kappa_2^2\} \leq \Lambda_1^2 \min \{\kappa_1^2, \kappa_2^2\},$$

and hence, since $|\delta \nu|^2 = \kappa_1^2 + \kappa_2^2$, (4.6) implies

$$(4.8) \quad \frac{1}{2}(1 - \nu_3^2) \Lambda_1^{-2} |\delta \nu|^2 \leq |\delta \nu_3|^2 \leq (1 - \nu_3^2) |\delta \nu|^2.$$

This inequality will be needed in the proof of the following theorem, which gives an interesting Harnack inequality for the quantity $v(X)$, defined by

$$v(X) = \sqrt{1 + |Du(x)|^2}, \quad X = (x, u(x)), \quad x \in \Omega.$$

(Note that $v = \nu_3^{-1}$ on M .)

THEOREM (4.2). *If ν is $(\Lambda_1, 0)$ -quasiconformal, then*

$$\sup_{S_{\frac{1}{2}R}(X_0)} v \leq c \inf_{S_{\frac{1}{2}R}(X_0)} v,$$

where v is as defined above and c is a constant depending only on Λ_1 .

Before giving the proof of this theorem we note the following corollary.

COROLLARY. *If $u \geq 0$ on the disc $D_R(x_0)$, then*

$$|Du(x_0)| \leq c_1 \exp \{c_2 u(x_0)/R\},$$

where c_1 and c_2 depend only on Λ_1 .

Proof of Corollary. Let

$$G = \{x \in \bar{D}_{R/2}(x_0) = u(x) \leq u(x_0)\}$$

and let $y \in G$ be such that

$$|Du(y)| = \inf_G |Du|.$$

Now take a sequence X_0, X_1, \dots, X_N of points in $M \cap (G \times \mathbf{R})$ with $|X_i - X_{i-1}| \leq \frac{1}{4}R$, $i = 1, \dots, N$, and with $X_N = (y, u(y))$. Clearly, repeated applications of Theorem (4.2) imply

$$(4.9) \quad \sqrt{1 + |Du(x_0)|^2} \leq c^N \sqrt{1 + |Du(y)|^2}.$$

Also, it is not difficult to see that it is possible to choose N such that

$$(4.10) \quad N \leq c_1(1 + u(x_0)/R),$$

where c_1 is an absolute constant. The required result now follows from (4.9) and (4.10), because $|Du(y)| \leq 2u(x_0)/R$. (To see this, we note that either $Du(y) = 0$, or else one can apply the mean value theorem to the function $\varphi(s) = u(x(s))$, $s \in [0, R/2]$, where $x(s)$ is the solution of the ordinary differential equation $dx(s)/ds = -Du(x(s))/|Du(x(s))|$, $s \in [0, R/2]$, with $x(0) = x_0$.)

Proof of Theorem (4.2). Since we can vary X_0 , it suffices to prove the lemma with θR in place of R , where $\theta \in (0, 1)$, provided the eventual choice of θ depends only on Λ_1 .

We first consider the case when $\nu_3(X) > \frac{1}{2}$ at some point of $S_{\theta R}(X_0)$. Then provided θ is small enough to ensure $c\theta^\alpha < \frac{1}{2}$, where c and α are as in Theorem (3.1), we can use Theorem (3.1) to deduce $\nu_3(X) \geq c_1 > 0$ at each point X of $S_{\theta R}(X_0)$, where c_1 depends only on Λ_1 . Then, since $v = \nu_3^{-1}$, the required result is established in this case. Hence we can assume $\nu_3(X) < \frac{1}{2}$ at each point of $S_{\theta R}(X_0)$. In this case we can replace $\gamma(\nu_3)$ in (4.3) by $\gamma(\nu_3)/(1 - \nu_3)$, provided $\gamma(\nu_3)\zeta_1$ has support contained in $S_{\theta R}(X_0)$. This gives

$$(4.11) \quad \int_M \zeta_1 \gamma'(\nu_3) K dA = \int_M \frac{\gamma(\nu_3)}{1 - \nu_3} d\zeta_1 \wedge \omega^*.$$

Now one easily checks that

$$(4.12) \quad |d\zeta_1 \wedge \omega^*| \leq |\delta\zeta_1| |\delta\nu|,$$

and, by the quasiconformal condition (4.7) we can use (4.8) to deduce

$$(4.13) \quad \int_M \zeta_1 \gamma'(\nu_3) |\delta\nu_3|^2 dA \leq c \int_M \gamma(\nu_3) |\delta\zeta_1| |\delta\nu_3| dA$$

whenever $\zeta_1 \gamma(\nu_3)$ has support contained in $S_{\theta R}(X_0)$, where c depends only on Λ_1 .

Now if we also take θ small enough (depending on Λ_1) to ensure that (3.17) and the conclusion of Lemma (3.2) both hold, then $S_{\theta R}(X_0)$ is topologically a disc, and one can easily check that (4.13) implies that ν_3 satisfies a maximum and a minimum principle on each open subset of $S_{\theta R}(X_0)$. (If, for example, $\nu_3(X_1) > \sup_{\partial U} \nu_3$ for some $X_1 \in U \subset S_{\theta R}(X_0)$, U open, then we choose γ such that $\gamma(t) \equiv 0$ for $t < \frac{1}{2}\{\nu_3(X_1) + \sup_{\partial U} \nu_3\}$, $\gamma'(t) > 0$ for $t > \frac{1}{2}\{\nu_3(X_1) + \sup_{\partial U} \nu_3\}$ (so that $\gamma(\nu_3(X_1)) > 0$) and choose $\zeta_1 \equiv 0$ on $S_{\theta R}(X_0) \sim U$ and $\zeta_1 \equiv 1$ on $\{X \in U: \nu_3(X) > \frac{1}{2}(\nu_3(X_1) + \sup_{\partial U} \nu_3)\}$. Then $\delta\zeta_1 \equiv 0$ when $\gamma(\nu_3) \neq 0$, and hence (4.13) gives

$$\int_U \gamma'(\nu_3) |\delta\nu_3|^2 dA = 0;$$

that is, $\nu_3 \equiv \text{const.}$ on each component of $\{X: \nu_3(X) > \frac{1}{2}(\nu_3(X_1) + \sup_{\partial U} \nu_3)\}$, which is clearly absurd. Similarly one proves that ν_3 satisfies a minimum principle on U .)

We now choose ζ_1 in (4.13) such that $\zeta_1 \equiv 1$ on $S_{3\theta R/4}(X_0)$, $\zeta_1 \equiv 0$ outside $S_{\theta R}(X_0)$ and $\sup_M |\delta\zeta_1| \leq 5/(\theta R)$. Also we choose $\gamma(\nu_3) \equiv \nu_3^{-1}$. Then using the Cauchy inequality, Lemma (3.1) and (4.13) we can prove

$$\int_{S_{3\theta R/4}(X_0)} |\delta w|^2 dA \leq c,$$

where $w = \log \nu_3^{-1}$ (so that $\delta w = -\nu_3^{-1} \delta\nu_3$) and where c depends only on Λ_1 . Thus, again using Cauchy's inequality and Lemma (3.1), we have

$$(4.14) \quad \int_{S_{3\theta R/4}(X_0)} |\delta w| dA \leq c'R,$$

with c' depending only on Λ_1 .

Now let

$$\bar{w} = \sup_{S_{\theta R/2}(X_0)} w, \quad \underline{w} = \inf_{S_{\theta R/2}(X_0)} w,$$

and, for $\lambda \in (\underline{w}, \bar{w})$, define

$$E_\lambda = \{X \in S_{3\theta R/4}(X_0) : w(X) > \lambda\},$$

$$C_\lambda = \{X \in S_{3\theta R/4}(X_0) : w(X) = \lambda\}.$$

By the co-area formula

$$(4.15) \quad \int_w^{\bar{w}} \mathcal{H}^1(C_\lambda) d\lambda = \int_{E_{\bar{w}} \sim E_w} |\delta w| dA \leq \int_{S_{3\theta R/4}(X_0)} |\delta w| dA.$$

However we note that

$$C_\lambda \cap \partial S_\rho(X_0) \neq \emptyset$$

for each $\rho \in (\frac{1}{2}\theta R, \frac{3}{4}\theta R)$, $\lambda \in (w, \bar{w})$. (Otherwise either E_λ or $\sim E_\lambda$ has a component contained in $S_\rho(X_0)$, which contradicts the maximum/minimum principle for v_3 on open subsets of $S_{\theta R}(X_0)$.) Hence

$$(4.16) \quad \mathcal{H}^1(C_\lambda) \geq \frac{\theta R}{4}, \quad \lambda \in (w, \bar{w}).$$

Combining (4.14), (4.15) and (4.16) we then have

$$\bar{w} - w \leq \bar{c},$$

i.e.

$$\sup_{S_{\theta R/2}(X_0)} v_3 \leq e^{\bar{c}} \inf_{S_{\theta R/2}(X_0)} v_3,$$

where \bar{c} depends only on Λ_1 . This is the required result because $v = v_3^{-1}$.

We can use the Harnack inequality of Theorem (4.2) to prove the following strengthened version of (3.26)

THEOREM (4.3). *Suppose v is $(\Lambda_1, 0)$ -quasiconformal. Then*

$$|v(X) - v(\bar{X})| \leq c \left\{ \inf_{S_{R/2}(X_0)} v_3 \right\} \left\{ \frac{|X - \bar{X}|}{R} \right\}^\alpha, \quad X, \bar{X} \in S_{R/2}(X_0),$$

where $c > 0$ and $\alpha \in (0, 1)$ depend only on Λ_1 .

Proof. Supposing that $v_3 > \frac{1}{2}$ at some point of $S_{R/2}(X_0)$, the theorem is a trivial consequence of Theorem (4.2) and (3.26). Hence we assume that $v_3 \leq \frac{1}{2}$ at each point of $S_{R/2}(X_0)$. We can then use (4.5) with $\gamma(v_3) \equiv v_3/(1 - v_3)$, thus giving (by (4.12))

$$\left| \int_M \zeta_1 K dA \right| \leq c \int_M v_3 |\delta \zeta_1| |\delta v| dA,$$

where c depends only on Λ_1 and ζ_1 has support in $S_{R/2}(X_0)$. Then by Theorem (4.2) we obtain

$$(4.17) \quad \left| \int_M \zeta_1 K dA \right| \leq c' \left\{ \inf_{S_{R/2}(X_0)} \nu_3 \right\} \int_M |\delta \zeta_1| |\delta \nu| dA,$$

where c' depends only on Λ_1 . Then by an argument almost identical to that used in the proof of Lemma (2.1), we see that (4.17) implies

$$\int_{S_{R/4}(X_0)} |\delta \nu|^2 dA \leq c'' \left\{ \inf_{S_{R/2}(X_0)} \nu_3 \right\}^2,$$

where c'' depends only on Λ_1 . Thus in the case $\varphi = \nu$, with ν ($\Lambda_1, 0$)-quasiconformal, we see that the inequality (2.12) can be improved by the addition of the factor $\{\inf_{S_{R/2}(X_0)} \nu_3\}^2$ on the right. (Note however that we must now use $R/2$ in place of R in (2.12).) Then Theorem (2.1) gives in this case

$$\int_{S_{R/8}(X_1)} |\delta \nu|^2 dA \leq c \left\{ \inf_{S_{R/2}(X_0)} \nu_3 \right\}^2 (\varrho/R)^\alpha$$

whenever $X_1 \in S_{R/8}(X_0)$ and $\varrho \in (0, R/8)$, where $c > 0$ and $\alpha \in (0, 1)$ depend only on Λ_1 . Then applying Lemma (2.2) as before, we obtain an inequality of the required form.

Next we wish to point out the following global Hölder continuity result for graphs with ($\Lambda_1, 0$)-quasiconformal Gauss map.

THEOREM (4.4). *Suppose u is continuous on $\bar{\Omega}$, graph $(u|_{\Omega})$ has ($\Lambda_1, 0$)-quasiconformal Gauss map ν , and let φ be a Lipschitz function on \mathbf{R}^2 with $|D\varphi(x)| \leq L$, $x \in \mathbf{R}^2$. Then, if $u \equiv \varphi$ on $\partial\Omega$, we have*

$$|u(\bar{x}) - u(x)| \leq c \{M^{1-\alpha} + |x - \bar{x}|^{1-\alpha}\} |x - \bar{x}|^\alpha, \quad x, \bar{x} \in \bar{\Omega},$$

where $M = \sup_{\Omega} |u - \varphi|$ and where $c > 0$ and $\alpha \in (0, 1)$ are constants depending only on L .

Remarks. 1. Note that there is no dependence in this estimate on Ω .

2. Using the above estimate as a starting point, various local estimates for the modulus of continuity of u can be obtained near boundary points at which u is continuous. (See Theorems 3 and 4 of [8].)

Proof of Theorem (4.4). As described in § 1 of [8], it suffices to establish the gradient bound

$$\sup_{\Omega_{x_0, \varrho/2}} |D(u - \varphi)^{\kappa+1}| \leq \{c_1(1 + L)^{1+1/n} M\}^\kappa, \quad \kappa = c_2(1 + L + M/\varrho),$$

where

$$\Omega_{x_0, \sigma} = \{x \in \Omega : |x - x_0| < \sigma\} \cap \bar{\Omega}, \quad M = \sup_{\Omega_{x_0}} |u - \varphi|,$$

and where c_1, c_2 depend only on Λ_1 . This can be proved by a method similar to the method used in the proof of Theorem 1 of [8]. Two main modifications are necessary to adapt the proof to the present setting:

(i) In the proof of Lemmas 1 and 2 of [8] we need an inequality of the form [8], (3.12). Such an inequality can be obtained in the present setting by choosing $\gamma(\nu_3) = \nu_3^{-1}\chi(w)$ (where $w = \log \nu_3^{-1}$ and χ is non-decreasing on $(0, \infty)$) in (4.5). By (4.7) and the right-hand inequality in (4.8) this gives (since $\chi(w)$ is a decreasing function of ν_3)

$$(4.18) \quad \int_M \chi(w) (\nu_3^{-1} |\delta\nu|^2 + (1 - \nu_3) |\delta w|^2) \zeta_1 dA \\ \leq -|\Lambda_1| \int_M \chi(w) \nu_3^{-1} d\zeta_1 \wedge \omega^* \leq |\Lambda_1| \int_M \chi(w) \nu_3^{-1} |\delta\zeta_1| |\delta\nu| dA$$

by (4.12). Now for $\nu_3 > \frac{1}{2}$ we have $|\delta w|^2 = \nu_3^{-2} |\delta\nu_3|^2 \leq 4|\delta\nu|^2$, while for $\nu_3 < \frac{1}{2}$ we have by (4.8) that $|\delta\nu|^2 \leq 3\Lambda_1^2 |\delta\nu_3|^2$. One easily sees that then (4.18) implies

$$(4.19) \quad \int_M \chi(w) (\nu_3^{-1} |\delta\nu|^2 + |dw|^2) \zeta_1 dA \leq c \int_M \chi(w) |\delta\zeta_1| (|\delta\nu| + |\delta w|) dA,$$

where c depends only on Λ_1 . Replacing ζ by ζ_1^2 and using Cauchy's inequality on the right, we then deduce

$$(4.20) \quad \int_M \chi(w) (\nu_3^{-1} |\delta\nu|^2 + |\delta w|^2) \zeta_1^2 dA \leq c' \int_M \chi(w) |\delta\zeta_1|^2 dA,$$

which gives

$$(4.21) \quad \int_M \chi(w) (|\delta\nu|^2 + |\delta w|^2) \zeta_1^2 dA \leq c' \int_M \chi(w) |\delta\zeta_1|^2 dA,$$

where c' depends only on Λ_1 . This is precisely an inequality of the form [8], (3.12).

(ii) The only other essential modification required is in the proof of Lemma 2 of [8]. In this proof equation (0.1) of [8] was used. In place of this equation we can in the present setting use the mean curvature equation (3.12). It is necessary to note however the bound

$$\int_M \nu_3^{-1} |\delta\nu|^2 \zeta_1^2 dA \leq c' \int_M |\delta\zeta_1|^2 dA$$

(which is true by (4.20)). Using this bound we can easily see that

$$\int_\Omega (1 + |Du|^2) \tilde{H}^2 \tilde{\zeta}_1^2 dx \leq c' \int_M |\delta\zeta_1|^2 dA,$$

where \tilde{H} is as in (3.12) and $\tilde{\zeta}_1$ is defined by $\tilde{\zeta}_1(x) = \zeta_1(x, u(x))$, $x \in \Omega$. This is sufficient to

ensure that the argument of Lemma 2 of [8] can be successfully modified (in such a way that (3.12) can be used in place of equation (0.1) of [8]).

It should be pointed out that there is an error in equality (3.3) of [8]; the correct inequality has $\sup_{\Omega} (u - \varphi)$ in place of Δ^* on the right. (This is obtained by making the choice $\rho = \infty$ in (3.2).) This causes no essential change in the proof of Theorem 1 on pp. 270–271 of [8].

We have already pointed out that the above theory applies to any solution u of a homogeneous equation of mean curvature type; we wish to conclude this section with an application to the minimal surface *system* with 2 independent variables.

We suppose that $u = (u^3, \dots, u^n)$ ($n \geq 3$) is a C^2 solution of the minimal surface system

$$(4.22) \quad \sum_{i,j=1}^2 b^{ij} D_i u^\alpha = 0, \quad \alpha = 3, \dots, n,$$

on $\Omega \supset D_R(0) = \{x \in \mathbb{R}^2 : |x| < R\}$, where

$$(4.23) \quad b^{ij} = \delta_{ij} - \frac{D_i u \cdot D_j u}{1 + |Du|^2}, \quad i, j = 1, 2.$$

Suppose also that we have an a-priori bound for the gradient of each component u^α of u , *except possibly for u^3* ; thus

$$(4.24) \quad \sup_{\Omega} |Du^\alpha| \leq \Gamma_1, \quad \alpha = 4, \dots, n,$$

where Γ_1 is some given constant.

We claim that, because of (4.24), setting $\alpha = 3$ in (4.22) gives (after multiplication by a suitable constant) a homogeneous equation of mean curvature type for u^3 , with λ_1 in (1.9) (ii) (a) depending only on Γ_1 (and with (1.9) (ii) (b) holding with $\lambda_2 = 0$). This clearly follows from the fact that

$$(4.25) \quad c_0 \sum_{i,j=1}^2 g^{ij} \xi_i \xi_j \leq \sum_{i,j=1}^2 b^{ij} \xi_i \xi_j \leq c_1 \sum_{i,j=1}^2 g^{ij} \xi_i \xi_j, \quad \xi \in \mathbb{R}^2,$$

where (b^{ij}) is as in (4.23) and (g^{ij}) is given by

$$g^{ij} = \delta_{ij} - \frac{D_i u^3 D_j u^3}{1 + |Du^3|^2}, \quad i, j = 1, 2,$$

and where c_0, c_1 are positive constants determined by Γ_1 . The inequality (4.25) is proved by first noting that

$$|b^{ij} - g^{ij}| \leq c(1 + |Du^3|^2)^{-1}, \quad i, j = 1, 2,$$

with c depending only on Γ_1 , and then using the facts that (b^H) , (g^H) are both positive definite, with (g^H) having eigenvalues $1, (1 + |Du^3|^2)^{-1}$.

We thus have the following theorem.

THEOREM (4.5). *The results of Theorem (4.2), and its corollary, and Theorem (4.4) are applicable to the component u^3 of the vector solution u of (4.22), (4.24), with constants c, α, c_1, c_2 depending only on Γ_1 .*

One can of course also prove that the graph of u^3 satisfies an estimate like that in Theorem (4.3). It then follows that each of the components $u^\alpha, \alpha=3, \dots, n$, of the vector solution u of (4.22), (4.24) satisfies the estimate of the following theorem.

THEOREM (4.6). *Let M_α denote the graph $\{X = (x_1, x_2, x_3): x_3 = u^\alpha(x_1, x_2), (x_1, x_2) \in D_R(0)\}$ and let $\nu^\alpha = 1 + |Du^\alpha|^2)^{-1/2}(-Du^\alpha, 1)$ denote the upward unit normal. Then, writing $S_{R/2} = \{X \in M_\alpha: |X - (0, u^\alpha(0))| < R/2\}$, we have*

$$|\nu^\alpha(X) - \nu^\alpha(\bar{X})| \leq c \left\{ \frac{|X - \bar{X}|}{R} \right\}^\beta, \quad X, \bar{X} \in S_{R/2},$$

where $c > 0, \beta \in (0, 1)$ depend only on Γ_1 .

If (4.22), (4.24) hold over the whole of \mathbf{R}^2 , then we can let $R \rightarrow \infty$ in the above, thus giving the following corollary.

COROLLARY. *Suppose (4.22), (4.24) hold over the whole of \mathbf{R}^2 . Then u is linear.*

It is appropriate here to point out a result of R. Osserman [6] concerning removability of isolated singularities of solutions of (4.22). As we have done above, Osserman also considers the case when all but one component of u satisfies an a -priori restriction (in [6] continuity is the restriction imposed).

§ 5. Concluding Remarks

We wish to conclude this paper with some remarks about the extension of the results of § 3 and § 4 to *parametric surfaces* M . This can be partly achieved provided there is a constant $\gamma > -1$ such that the Gauss map ν of M maps into $S_\gamma^2 = \{X = (x_1, x_2, x_3) \in S^2: x_3 > \gamma\}$; that is, provided $\nu_3(X) > \gamma > -1$ for each $X \in M$. If this is assumed then the proof of the main Hölder estimate carries over in a straightforward manner, giving

$$(5.1) \quad \sup_{X \in S_\gamma^2(X_0)} |\nu(X) - \nu(X_0)| \leq c \{\varrho/R\}^\alpha,$$

where $c > 0$ and $\alpha \in (0, 1)$ depend on $\gamma, \Lambda_1, \Lambda_2 R^2$ and $R^{-2}|S_R(X_0)|$. However no appropriate

analogues of Lemmas (3.1), (3.2) are known, even if M is assumed to be simply connected. Hence $S_\varrho^*(X_0)$ cannot be replaced by $S_\varrho(X_0)$ in (5.1), and the constants c , α depend on $R^{-2}|S_R(X)|$. In case $\Lambda_2=0$ Theorem (4.3) also has an analogue for the parametric surface M . In fact one can prove, by a straightforward modification of the method of § 4, that

$$(5.2) \quad \sup_{X \in S_\varrho^*(X_0)} |\nu(X) - \nu(X_0)| \leq c \inf_{S_{R/2}^*(X_0)} (\nu_3 - \gamma) \{\varrho/R\}^\alpha$$

for $\varrho \in (0, R/2)$. However the constants c , α again depend on $R^{-2}|S_R(X_0)|$.

In the case when the principal curvatures κ_1, κ_2 of the surface M satisfy a relation

$$(5.3) \quad \alpha_1(X, \nu(X))\kappa_1 + \alpha_2(X, \nu(X))\kappa_2 = \beta(X, \nu(X))$$

at each point $X \in M$ (cf. (1.9) (ii)), where $\alpha_1, \alpha_2, \beta$ are Hölder continuous functions on $M \times S^2$ with

$$1 \leq \alpha_i(X, \nu) \leq \lambda_1, \quad i = 1, 2, \quad |\beta(X, \nu)| \leq \lambda_2, \quad (X, \nu) \in M \times S^2,$$

one can easily show (by using a non-parametric representation near X_0 , cf. the argument of [1]) that (5.1) implies

$$(\kappa_1^2 + \kappa_2^2)(X_0) \leq c/R^2,$$

where c depends on $\gamma, R^{-2}|S_R(X_0)|, \lambda_1$ and $\lambda_2 R$. As far as the author is aware, the only other result of this type previously obtained, in case $\lambda_2 \neq 0$, was the result of Spruck [10] for the case $\alpha_1 = \alpha_2 \equiv 1, \beta \equiv \text{constant}$. In the case $\beta \equiv 0$ we can use (5.2) instead of (5.1) to obtain the stronger inequality

$$(\kappa_1^2 + \kappa_2^2)(X_0) \leq c(\nu_3(X_0) - \gamma)^2/R^2.$$

Such an inequality was proved by Osserman [5] in the minimal case ($\alpha_1 = \alpha_2 \equiv 1, \beta \equiv 0$) and by Jenkins [1] for the case when the surface M is stationary with respect to a "constant coefficient" parametric elliptic functional (such surfaces always satisfy an equation of the form (5.3) with $\alpha_i(X, \nu) \equiv \alpha_i(\nu)$ and $\beta \equiv 0$; see [1] and [7] for further details). The results in [5] and [7] are obtained with constant c independent of $R^{-2}|S_R(X_0)|$, unlike the inequality above. (We should mention that of course one can obtain a bound for $R^{-2}|S_R(X_0)|$ if M globally minimizes a suitable elliptic parametric functional.)

Appendix. Area bounds and a proof of the Morrey-type lemma for 2 dimensional surfaces

The first variation formula for M (cf. (3.4)) is

$$\int_M \delta \cdot f dA = \int_M f \cdot H dA,$$

valid for any C^1 vector function $f = (f_1, \dots, f_n)$ with compact support in M , where H is the mean curvature vector (see [4]) of M and $\delta \cdot f = \sum_{i=1}^n \delta_i f_i$ (=divergence of f on M). We begin by replacing f by $\varphi(r)(X - X_1)h$, where φ, h are non-negative functions, where $X_1 = (x'_1, \dots, x'_n) \in S_R(X_0)$, and where $r(X) \equiv r_{X_1}(X) = |X - X_1|$. Since, by (1.1),

$$\delta \cdot X = \text{trace } (\tilde{g}^{ij}(X)) = 2$$

and

$$(X - X_1) \cdot \delta \varphi(r) = r^{-1} \varphi'(r) \sum_{i,j=1}^n (x_i - x'_i) \tilde{g}^{ij}(X) (x_j - x'_j) = r \varphi'(r) |\delta r|^2,$$

this gives

$$(A.1) \quad 2 \int_M \varphi(r) h dA + \int_M r \varphi'(r) h |\delta r|^2 dA = \int_M \varphi(r) (X - X_1) \cdot (-\delta h + Hh) dA.$$

Now one easily checks that this holds if φ is merely continuous and *piecewise* C^1 (rather than C^1) on \mathbf{R} , provided we define $\varphi'(r(X))$ in some arbitrary way (e.g. $\varphi'(r(X)) = 0$) for those X such that φ is not differentiable at $r(X)$. (The proof of this is easily based on the fact that the set $\{X \in M: r(X) = \varrho \text{ and } \delta r(X) \neq 0\}$ has zero \mathcal{H}^2 -measure for each $\varrho \in (0, R - |X_1 - X_0|)$. Hence we can replace φ by the function φ_ε , defined by $\varphi_\varepsilon(t) = 1$ for $t < \varrho - \varepsilon$, $\varphi_\varepsilon(t) = 0$ for $t > \varrho$, and $\varphi_\varepsilon(t) = \varepsilon^{-1}(\varrho - t)$ for $\varrho - \varepsilon \leq t \leq \varrho$. Substituting this in (A.1) and letting $\varepsilon \rightarrow 0_+$, we obtain

$$(A.2) \quad 2 \int_{S_\varrho} h dA - \varrho \frac{d}{d\varrho} \int_{S_\varrho} |\delta r|^2 dA = \int_{S_\varrho} (X - X_1) \cdot \{-\delta h + hH\} dA.$$

Here and subsequently $S_\varrho = S_\varrho(X_1)$ and $\varrho \in (0, R - |X_1 - X_0|)$.

Noting that $H \cdot \delta = 0$ (since H is normal to M), we have from Cauchy's inequality that

$$\begin{aligned} (X - X_1) \cdot H &= r \left(\frac{X - X_1}{r} - \delta r \right) \cdot H \leq 2 \left| \frac{X - X_1}{r} - \delta r \right|^2 + \frac{1}{8} r^2 H^2 \\ &= 2(1 - |\delta r|^2) + \frac{1}{8} r^2 H^2. \end{aligned}$$

(The work of Trudinger [11] suggests handling the term $(X - X_1) \cdot H$ in this manner.)

Hence we deduce from (A.2) that

$$2 \int_{S_\varrho} |\delta r|^2 h dA - \varrho \frac{d}{d\varrho} \int_{S_\varrho} |\delta r|^2 h dA \leq \int_{S_\varrho} \left(\frac{1}{8} r^2 H^2 h + r |\delta h| \right) dA.$$

This last inequality can be written

$$-\frac{d}{d\varrho} \left\{ \varrho^{-2} \int_{S_\varrho} |\delta r|^2 h dA \right\} \leq \varrho^{-3} \int_{S_\varrho} \left(\frac{1}{8} r^2 H^2 h + r |\delta h| \right) dA,$$

and hence, integrating from σ to ϱ , we have

$$(A.3) \quad \sigma^{-2} \int_{S_\sigma} h |\delta r|^2 dA \leq \varrho^{-2} \int_{S_\varrho} h |\delta r|^2 dA + \int_0^\varrho \left\{ \tau^{-3} \int_{S_\tau} \left(\frac{1}{8} r^2 H^2 h + r |\delta h| \right) dA \right\} d\tau.$$

But

$$\int_0^\varrho \tau^{-3} \left(\int_{S_\tau} r^2 H^2 h dA \right) d\tau = \frac{1}{2} \int_{S_\varrho} (1 - r^2/\varrho^2) H^2 h dA \leq \frac{1}{2} \int_{S_\varrho} H^2 h dA,$$

and hence (A.3) implies

$$(A.4) \quad \begin{aligned} \sigma^{-2} \int_{S_\sigma} h |\delta r|^2 dA &\leq \varrho^{-2} \int_{S_\varrho} h |\delta r|^2 dA + 2^{-4} \int_{S_\varrho} H^2 h dA + \int_0^\varrho \tau^{-3} \left(\int_{S_\tau} r |\delta h| dA \right) d\tau \\ &\leq \varrho^{-2} \int_{S_\varrho} h dA + 2^{-4} \int_{S_\varrho} H^2 h dA + \int_0^\varrho \tau^{-2} \left(\int_{S_\tau} |\delta h| dA \right) d\tau. \end{aligned}$$

We can also see from (A.2), by again using Cauchy's inequality,

$$\begin{aligned} 2 \int_{S_\varrho} h dA - \varrho \frac{d}{d\varrho} \int_{S_\varrho} h |\delta r|^2 dA &\leq \int_{S_\varrho} (r |H| h + r |\delta h|) dA \\ &\leq \int_{S_\varrho} \left(1 + \frac{1}{4} r^2 H^2 \right) h + r |\delta h| dA, \end{aligned}$$

so that

$$\int_{S_\varrho} h dA \leq \varrho \frac{d}{d\varrho} \int_{S_\varrho} h |\delta r|^2 dA + \int_{S_\varrho} \left(\frac{1}{4} r^2 H^2 h + r |\delta h| \right) dA.$$

Integrating this over $\varrho \in (\sigma/2, \sigma)$, we deduce that

$$\begin{aligned} \int_{\sigma/2}^\sigma \left(\int_{S_\varrho} h dA \right) d\varrho &\leq \int_{\sigma/2}^\sigma \left(\varrho \frac{d}{d\varrho} \int_{S_\varrho} h |\delta r|^2 dA \right) d\varrho + \int_{\sigma/2}^\sigma \left(\int_{S_\varrho} \left(\frac{1}{4} r^2 H^2 h + r |\delta h| \right) dA \right) d\varrho \\ &\leq \sigma \int_0^\sigma \left(\frac{d}{d\varrho} \int_{S_\varrho} h |\delta r|^2 dA \right) d\varrho + \frac{\sigma}{2} \int_{S_\sigma} \frac{1}{4} r^2 H^2 h dA + \sigma \int_{\sigma/2}^\sigma \left(\int_{S_\varrho} |\delta h| dA \right) d\varrho \\ &\leq \sigma \int_{S_\varrho} h |\delta r|^2 dA + \frac{\sigma^3}{8} \int_{S_\sigma} H^2 dA + 4\sigma^3 \int_0^\sigma \left(\varrho^{-2} \int_{S_\varrho} |\delta h| dA \right) d\varrho. \end{aligned}$$

In obtaining the last term on the right here, we have used the inequality $\sigma^{-2} \leq 4\varrho^{-2}$ for $\varrho \in (\sigma/2, \sigma)$. Multiplication by $8\sigma^{-3}$ now yields

$$(A.5) \quad (\sigma/2)^{-2} \int_{S_{\sigma/2}} h dA \leq 8\sigma^{-2} \int_{S_\sigma} h |\delta r|^2 dA + \int_{S_\sigma} H^2 dA + 32 \int_0^\sigma \left(\varrho^{-2} \int_{S_\varrho} |\delta h| dA \right) d\varrho.$$

Combining this with (A.4) gives

$$(A.6) \quad \sigma^{-2} \int_{s_\sigma} h \, dA \leq 40 \left\{ \int_{s_\sigma} (\varrho^{-2} + H^2) h \, dA + \int_0^\varrho \left(\tau^{-2} \int_{s_\tau} |\delta h| \, dA \right) d\tau \right\}$$

for each σ, ϱ with $0 < \sigma \leq \varrho < R - |X_1 - X_0|$. Notice that (A.4) and (A.5) initially only yield (A.6) for $\sigma \leq \varrho/2$; however (A.6) holds trivially for $\sigma \in (\varrho/2, \varrho)$ because of the term $40\varrho^{-2} \int_{s_\sigma} h \, dA$ on the right.

It clearly follows from this (by setting $h \equiv 1$) that (1.12) holds, as claimed in § 1.

If we let $\sigma \rightarrow 0$ in (A.6), then we have

$$(A.7) \quad h(X_1) \leq \frac{40}{\pi} \left\{ \int_{s_\varrho} (\varrho^{-2} + H^2) h \, dA + \int_0^\varrho \left(\tau^{-2} \int_{s_\tau} |\delta h| \, dA \right) d\tau \right\}.$$

Next we note that if h is of arbitrary sign and if we apply (A.7) with $\psi \circ h$ in place of h (where ψ is a non-negative C^1 function on \mathbf{R}), then we obtain

$$(A.8) \quad \psi(h(X_1)) \leq \frac{40}{\pi} \left\{ \int_{s_\varrho} (\varrho^{-2} + H^2) \psi(h) \, dA + \sup_{\mathbf{R}} |\psi'| \int_0^\varrho \left(\tau^{-2} \int_{s_\tau} |\delta h| \, dA \right) d\tau \right\}.$$

Using this inequality we can prove the Morrey-type lemma, Lemma (2.2), for the surface M . In fact, if h is as in Lemma (2.2), then (A.8) implies

$$(A.9) \quad \psi(h(X_1)) \leq \frac{40}{\pi} \int_{s_\varrho} (\varrho^{-2} + H^2) \psi(h) \, dA + \frac{40}{\pi} \sup_{\mathbf{R}} |\psi'| K \beta^{-1} (\varrho/R)^\beta.$$

We now suppose $\varrho \in (0, R/4)$ and $X_1 \in S_\varrho(X_0)$, and we define

$$\bar{h} = \sup_{s_\varrho^*(X_0)} h, \quad \underline{h} = \inf_{s_\varrho^*(X_0)} h,$$

and

$$\gamma = \frac{1}{2} \{40 K \beta^{-1} (\varrho/R)^\beta\}^{-1}.$$

If $\bar{h} - \underline{h} < 2\gamma^{-1}$, then Lemma (2.2) is established with $c=160$. If on the other hand $\bar{h} - \underline{h} \geq 2\gamma^{-1}$, then we let N be the largest integer less than $(\bar{h} - \underline{h})\gamma$. Thus we have

$$(A.10) \quad N \geq \frac{1}{2} (\bar{h} - \underline{h})\gamma,$$

and, furthermore, we can subdivide the interval $[\underline{h}, \bar{h}]$ into N pairwise disjoint intervals I_1, I_2, \dots, I_N , each of length $\geq \gamma^{-1}$. For each $j=1, \dots, N$ we then let ψ_j be a non-negative $C^1(\mathbf{R})$ function with support contained in I_j , $\max_{\mathbf{R}} \psi_j = 1$ and $\max_{\mathbf{R}} |\psi_j'| \leq 3\gamma$. (It is clear

that such a function ψ_j exists because $\text{length } I_j \geq \gamma^{-1}$.) Since $S_\varrho^*(X_0)$ is connected, we know that for each $j=1, \dots, N$ we can find a point $X^{(j)} \in S_\varrho^*(X_0)$ such that $\psi_j(h(X^{(j)}))=1$. Then, assuming $\varrho < R/4$, we can use (A.9) with $X^{(j)}$ in place of X_1 and with ψ_j in place of ψ , thus giving

$$\begin{aligned} 1 &\leq \frac{40}{\pi} \int_{S_\varrho(X^{(j)})} (\varrho^{-2} + H^2) \psi_j(h) dA + \pi^{-1} \gamma^{-1} 3\gamma/2 \\ &\leq \frac{40}{\pi} \int_{S_{2\varrho}(X_0)} (\varrho^{-2} + H^2) \psi_j(h) dA + \frac{1}{2}; \end{aligned}$$

that is,

$$1 \leq \frac{80}{\pi} \int_{S_{2\varrho}(X_0)} (\varrho^{-2} + H^2) \psi_j(h) dA.$$

Summing over $j=1, \dots, N$, noting that $\sum_{j=1}^N \psi_j(t) \leq 1$ for each $t \in \mathbb{R}$, we then deduce

$$N \leq \frac{80}{\pi} \int_{S_{2\varrho}(X_0)} (\varrho^{-2} + H^2) dA \leq c(\Lambda_3 + \Lambda_4).$$

Lemma (2.2) now follows from (A.10).

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