# THE HOMOLOGY GROUPS OF SOME ORDERED SYSTEMS

BY

H. B. GRIFFITHS

*The University of Southampton, England*

The considerations of this paper were suggested by the 'abelian' version of a problem concerning a Fuchsian group G. Greenberg [4] has shown that if $\mathcal{J}(G)$ denotes the system of all finitely generated subgroups of $G$, ordered by the relation of being included of finite index, then $\mathcal{J}(G)$ has maximal elements. To comprehend the ordered system $\mathcal{J}(G)$, a first approximation is to look at its 'homology groups', these being definable for any partially ordered set with zero (see below). The resulting problem is still intractable, and it seemed of interest to try the analogous problem when $G$ is replaced by a finitely generated abelian group $M$: the analogue of the maximal elements of $\mathcal{J}(G)$ is then the family $S(M)$ of all direct factors of M. Here, $S(M)$ happens to be a lattice, ordered by inclusion, and we form from it a complex $\Psi M$ whose vertices are the elements of $S(M)$, and whose simplices $(v_0, \dots, v_q)$ are ordered sets of vertices such that $v_0 \cap \dots \cap v_q \neq 0$. The homology groups of $\Psi M$ are then the ones we consider (with related matters) in this paper. A principal result (see section 15) states:

*If $M$ has $n \geqslant 3$ generators, then $\Psi M$ contains a wedge of $(n-2)$-spheres, and the inclusion induces isomorphisms of homology and homotopy groups. The set of spheres is bijective with the group of all $n \times n$ non-singular rational upper triangular matrices, modulo the diagonal matrices.*

Just as $\Psi M$ was formed from the partially ordered set $S(M)$, we can form a complex $\Psi P^n$ from the lattice $\mathrm{Flat}(P^n)$ of flats of a projective $n$-space $P^n(F)$ over a field $F$. It happens that $\Psi M \approx \Psi P^{n-1}(\mathbf{Q})$. When $F$ is finite, the Möbius function $\mu$ of $\mathrm{Flat}(P^n)$ was studied by Rota [11] who related it to the Euler Characteristic $\chi$ of $\Psi P^n$, and calculated $\mu$ and $\chi$. Our treatment, however, is geometrical rather than arithmetical, and thus gives more information (for example, here $\Psi P^n$ has the homotopy type of a wedge of spheres.) We take a more general point of view than Rota, working at first with partially ordered sets rather than lattices, and abstracting the role of the 'support' $v_0 \cap \dots \cap v_q$ of a simplex $(v_0, \dots, v_q)$. It turns out (see section 16) that Rota's equation $\chi = 1 + \mu$ for a general finite lattice expresses the complementary nature of two subcomplexes of an acyclic complex, and several known relations about $\mu$ are easily inferred from this point of view.

The plan of the paper is as follows. It is divided into seventeen sections, and in the first we introduce some notation and three Examples, to which we return several times in later sections, for motivation. In section 2 we look at a special complex $\Psi_E X$ associated with an ordered set $X$ and prove that its homology and homotopy groups depend only on the minimal elements in $X$. A notion of 'dimension' is introduced in section 3, and we show that the fundamental group and 'early' homology groups of $\Psi_E X$ vanish for rather primitive reasons. For an example in groups see 3.8: and for one concerning sections of a vector bundle see 5.9. The later groups are calculated by means of the Mayer-Vietoris sequence in sections 7 and 10; and the associated inductive arguments force us to consider a situation where the simplices have 'supports' in an ordered system $L$, to which we gradually add extra structure (in the form of existence of atoms, descending chains etc.). These considerations occupy the sections from 4 to 10, and confirm the intuitive expectation that the first non-vanishing homology group is free abelian while the higher ones vanish. In section 11 we show how to work with $L$ entirely, and in section 12 we relate the homology groups to the 'order-homology' of $L$ discussed in Pretzel [10] and Rota [12]; we use a 'uniqueness' theorem for homology, established in section 7 by the Mayer-Vietoris technique. This uniqueness theorem enables us also in section 13 to find a significant wedge of spheres in the associated complex $\Psi L$; the theory is shown to work for example, with the geometric lattices considered by Crapo-Rota [1] (see 13.6 below). When $L$ is Flat$(P^n)$, the projective group is used in section 14 to describe the set of spheres of the wedge. In section 15 we relate $\Psi M$ (when $M$ is a free abelian finitely generated group) to $P^n$, but in a more general situation when the embedding, of the ring of integers in the field of rationals, is replaced by a similar kind of embedding of a ring in a (possibly non-commutative) field. Next, in section 16, we show how our geometric theory can be a basis for Rota's theory of the Moebius function. Finally, we look at 'intrinsic' conditions on $L$ which yield the conditions imposed earlier: these indicate the role of geometric lattices in the theory.

## 1. Notation

For partially ordered sets and lattices we use the general terminology of MacLane and Birkhoff [9]; in particular, 'partially ordered set' is abbreviated to 'poset'. Also we use the abbreviation 'izoposet' for a poset with intersection or glb (written $a \wedge b$), zero (**0**) and 'one' or universal upper bound (**1**). Three examples will be kept in mind for illustrative purposes, and for future reference we enumerate them here.

*Example 1.* Let $m>0$ be an integer. Then Fact $(m)$ will denote the isoposet of all factors of $m$, ordered by divisibility; here $\mathbf{0}$ and $\mathbf{1}$ are 1 and $m$ respectively, and $a \wedge b = \mathrm{HCF}(a,b)$. The *atoms* in Fact $(m)$ are the prime divisors of $m$, and the *coatoms* (elements covered by 1) are all factors of the form $m/p$, where $p$ is an atom.

*Example 2.* Let $U$ be a set. Then $\mathcal{P}U$ will denote the family of subsets of $U$, ordered by inclusion. It is an izoposet of course, with extra structure. Its atoms are the singletons, and its coatoms are subsets of the form $U-x$, $x \in U$.

*Example 3.* Let $P^m$ be a projective space of dimension $m$. Then Flat $(P^m)$ will denote the family of all linear subspaces of $P^m$, ordered by inclusion. It is an izoposet, with $\mathbf{0}$ the empty set and $\mathbf{1}$ the whole space; the atoms are the single points, and the coatoms are the hyperplanes. We denote by $\mathrm{Flat}^\dagger P^m$ the dual izoposet, obtained by reversing the order relation and taking $\mathbf{0}$, $\mathbf{1}$ to be $P^m$ and $\varnothing$ respectively. Here atoms and coatoms are hyperplanes and points respectively.

By an *abstract complex,* we mean a family $K$ of subsets $\sigma$ of a set $K_0$ of *vertices* of $K$, for which every non-empty subset of any $\sigma$ in $K$ lies also in $K$. Thus $K$ is a poset, ordered by inclusion. It has a geometrical realisation, in the sense that there is an isomorphism between $K$ and a geometric simplical complex $J$ which preserves the simplicial structure; but $J$ is also a topological space (see Hilton-Wylie [7] p. 46). The homology groups $H_qK$ will be those of the total complex $K^\Omega$ (see Hilton-Wylie, p. 100), whose chain-groups are freely generated by *ordered* simplices $\sigma=(v_0, v_1, \ldots, v_q)$, i.e. points of the weak Cartesian product $K_0^\omega$ for which the unordered set $\{v_0, \ldots, v_q\}$ (deleting repetitions) is a simplex of $K_0$. It is well-known that $H_qK$ is isomorphic to the $q$th singular homology group of $J$. For such matters we follow the approach of Hilton-Wylie.

## 2. The complex of minimal elements

We begin by showing in this section how certain complexes can be associated with a non-empty poset $(X, \leqslant)$. First, we observe that the family $\Phi X$ of all finite non-empty subsets of $X$ is an abstract complex in the sense of section 1. Its vertices are the points of $X$. Thus its associated total complex $(\Phi X)^\Omega$ is simply the weak Cartesian product $X^\omega$, i.e. the set of all finite ordered subsets $\sigma^q=(x_0, \ldots, x_q)$ of $X$, repetitions being allowed in $\sigma^q$. By identifying $x \in X$ with $(x) \in X^\omega$ we allow ourselves to write $X \subseteq X^\omega$, and we call the elements of $X$ the 'vertices' of $X^\omega$. If $q>0$ and $0 \leqslant i \leqslant q$, let $\partial_i\sigma^q$ denote the ordered set obtained from $\sigma^q$ by suppressing the vertex $x_i$; if $q=0$ define $\partial_i\sigma=\sigma$.

The ordering on $X$ induces orderings $(X^\omega, \leqslant)$, $(\Phi X, \leqslant)$ on $X^\omega$, $\Phi X$, if we define $\sigma^q \leqslant \tau^p$ to mean:

*for each vertex $x_i$ of $\sigma^q$, there exists a vertex $y_i$ of $\tau^p$ such that $x_i \leqslant y_i$ in $X$.*

Then $\leqslant$ is transitive on $X^\omega$, $\Phi X$, and agrees with the given ordering on $X$. Note, however that we may have $\sigma^q \leqslant \tau^p \leqslant \sigma^q$ yet $\sigma^q \neq \tau^p$. Also

2.1. *If $\sigma^q \in X^\omega$, then $\partial_i \sigma^q \leqslant \sigma^q \quad 0 \leqslant i \leqslant q$.*

Now $X^\omega$ defines a chain-complex with simplices $\sigma^q$ and face-operators $\partial_i$. If we fix one vertex $x \in X$, then for each $\sigma^q \in X^\omega$, the cone $x\sigma^q = (x, \sigma^q) \in X^\omega$, so $X^\omega$ (and hence $\Phi X$) is acyclic in the sense of homology theory using finite chains. Observe that we have a 'support' function

2.2. $$t: X^\omega \to \Phi X$$

given by $t(x_0, ..., x_q) = \{x_0, ..., x_q\}$, the unordered set with deletion of repetitions. Clearly, $t: (X^\omega, \leqslant) \to (\Phi X, \leqslant)$ is order-preserving.

Since acyclic complexes are not very interesting, we shall suppose that we are given a set $E \subseteq \Phi X$ of 'excluded' simplexes, so that the complex $\Phi X - E$ is the one of interest. $E$ must satisfy:

2.3. (*The Exclusion Condition*). *If $\sigma \leqslant \tau$ in $\Phi X$ and $\tau \notin E$ then $\sigma \notin E$.*

Thus $\Phi X - E$ is a closed subcomplex of $\Phi X$. Further,

2.4. $$\Phi X - E = \Phi(X - E) - E$$

by the Exclusion Condition, so that we may replace $X$ and $E$ by $X' = X - E$, $E' = \Phi(X - E) \cap E$ respectively.

To help illustrate, and motivate the later work, let us now extend the three examples of Section 1. In Example 1, with $X = \text{Fact}\,(m)$, we take $E = E(m)$ to consist of those $\sigma^q = \{x_0, ..., x_q\}$ for which the LCM of the vertices $x_i$ is $m$. In Example 2 with $X = \mathfrak{P}U$, let $E = E(U)$ consist of those $\sigma^q = \{x_0, ..., x_q\}$ for which $x_0 \cup x_q \cup ... \cup x_q = U$; here $E = \emptyset$ if $U$ is infinite. In Example 3, with $X = \text{Flat}\,(P^n)$, let $E = E(P^n)$ consist of those $\sigma^q = \{x_0, ..., x_q\}$ for which $x_0 \cup ... \cup x_q$ spans $P^n$ and if $X = \text{Flat}^\dagger P^n$ we take the dual version of $E$: all $\sigma^q$ for which $x_0 \cap ... \cap x_q = \emptyset$ are to form $E^\dagger = E^\dagger\,(P^n)$. In each example, the Exclusion Condition 2.3 is obviously satisfied. It so happens that in each example (except when $U$ is infinite), $X \cap E \neq \emptyset$; in fact, $X \cap E$ is the 1 of the izoposet $X$. Thus, using the notation following 2.4, we have

2.5. $$\text{Fact}'(m) = \text{Fact}\,(m) - \{m\},\; \mathfrak{P}'U = \mathfrak{P}U - \{U\}, \text{etc}$$

[1] By our definition of a complex in Section 1, a subcomplex is automatically closed, since it is a subset that is a complex.

the latter when $U$ is finite; and then $E'(m)$ consists of those $\sigma \in E(m)$ for which $m$ is not a vertex of $\sigma$.

Here and in the general case, our concern is to give a simplified description of $\Phi X - E$, either by describing its homotopy type (as in 13.4 below) or by computing its homology groups. The latter are the groups of the total complex of $\Phi X - E$; that is to say of $X^\omega - t^{-1}E$, a closed subcomplex of $X^\omega$. By the Exclusion Condition, $t^{-1}E$ satisfies two 'exclusion conditions', one a special case of the other:

2.6. *If $\sigma^q \leqslant \tau^p$ in $X^\omega$ and $\tau^p \notin t^{-1}E$, then $\sigma^q \notin t^{-1}E$.*

2.7. *If $\sigma^q = (x_0, \ldots, x_q) \notin t^{-1}E$, then also $\sigma^{q+1} \notin t^{-1}E$, where $\sigma^{q+1} = (x_0, \ldots, x_i, x_i, \ldots, x_q)$ with $x_i$ repeated.*

For simplicity we write $T$ for $t^{-1}E$. Observe that 2.6 implies for example, that if $x \leqslant y$ in $X$ then $(x, y) \notin T$ if $y \notin T$; for, $(x, y) \leqslant y$ in $X^\omega$. Moreover, $(X - T)^\omega - T = X^\omega - T$, also by 2.6. Hence we shall now suppose that $T \cap X = \varnothing$ (see 2.4). Further, as a first step in our programme for simplifying $\Phi X - E$, we shall suppose also that $X$ *has minimal elements*, i.e. given $x \in X$, there exists $y \in X$ such that $y \leqslant x$ and if $z \leqslant y$ in $X$ then $z = y$. Throughout

$X_*$ *denotes the set of all minimal elements of* $X$;

since $X \neq \varnothing$, then $X_* \neq \varnothing$. To simplify the notation we write $TX$ for the subcomplexes

$$TX = X^\omega - T, \text{ and } TX_* = X_*^\omega - T;$$

note that the ordering on $X_*$ induced by that on $X$ is the relation of equality.

The principal result of this section now follows: compare Folkman [3].

2.8. THEOREM. *The inclusion $TX_* \subseteq TX$ induces isomorphisms*

$$\theta_q\colon H_q(TX_*) \approx H_q(TX), \quad 0 \leqslant q.$$

*Proof.* Recall that to calculate $H_q(TX)$ we use *finite* chains $\gamma^q = \sum n_\sigma \sigma$ where $\sigma$ runs through all $q$-simplices $\sigma^q$ in $TX$, and all but a finite number of the integers $n_\sigma$ vanish. If $n_\sigma \neq 0$ say that $\sigma$ is 'in' $\gamma^q$. If $\sigma$ also lies in $TX_*$, we call $\sigma$ 'minimal'.

For any minimal element $\mu \in X_*$, there is a function $f_\mu\colon X \to X$ defined by setting $f_\mu(x) = \mu$ if $\mu \leqslant x$, and $f_\mu(x) = x$ otherwise. By condition 2.6, $f_\mu$ induces a simplicial mapping $f_\mu\colon TX \to TX$, and on $TX_*$, $f_\mu$ is the identity.

Clearly, if $C$ is a $q$-chain of $K$, the no more $q$-simplexes lie in $f_\mu C$ than in $C$, while for the number $h(C)$ of non-minimal vertices of $C$ we may assert

(a) $$h(f_\mu C) < h(C).$$

The standard formula of Čech theory:

$$D(\sigma^q) = \sum (-1)^i (x_0 \dots x_i x_i' \dots x_q') \quad (x_i' = f_\mu x_i)$$

where $\sigma^q = (x_0, ..., x_q)$, shows as in Wilder [14] p. 128, that $\gamma \sim f_\mu \gamma$ on $TX$, for *every* cycle $\gamma$ on $TX$; for, each term in the above sum lies in $TX$ by 2.6 and 2.7.

Hence, using induction and (a) on $h(\gamma)$ we may assert

(b) *If $\gamma$ is a q-cycle on $TX$, then $\gamma \sim \gamma_1$ on $TX$, where $\gamma_1$ is a q-cycle on $TX_*$.*

Also, since $f_\mu$ is a chain-mapping and $f_\mu | TX_* =$ identity, then by (a) and induction on $h(C)$ we may assert

(c) *If $\gamma = \partial C$ in $TX$ and $\gamma$ is in $TX_*$, then there exists $C'$ in $TX_*$ such that $\gamma = \partial C'$.*

But then (b) and (c) respectively assert that $\theta_q$ in 2.7 is surjective and injective. This completes the proof.

By definition of the homology groups as those of the total complex, we obtain at once the

2.9. COROLLARY. *The inclusion $X'_* = X_* - E \subseteq X - E$ induces isomorphisms*

$$H_q(\Phi X_* - E) \approx H_q(\Phi X - E), \quad 0 \leqslant q.$$

For brevity, we shall write:

$$\Psi_E X = \Phi X - E, \quad \Psi_E X_* = \Phi X_* - E$$

respectively for (geometrical realisations of) the abstract complexes $\Phi X - E$, $\Phi X_* - E$ $(= \Phi X'_* - E)$.

From the proof of Theorem 2.6 we may extract a little more, viz:

2.10. THEOREM. *The inclusion $X'_* \subseteq X - E$ induces an epimorphism $\theta$: $\pi \Psi_E X_* \to \pi \Psi_E X$ of Fundamental groups* (assuming $\Phi X - E$ is connected).

*Proof.* Choose a base point $x \in X'_*$. We may regard the fundamental groups as edge-path groups (see Hilton-Wylie [7] p. 237), so that representative loops may be taken to be simplicial curves $\gamma$ on a finite subcomplex $K = K(\gamma)$. Also by Hilton-Wylie [7] p. 46, we may here replace $TX$ by an isomorphic geometrical realisation.

In the last proof we saw that if $v$ were any vertex of $TX$, then $(v, f_\mu v)$ is a 1-simplex of $TX$. Hence in $\Psi_E X$ there is induced a homotopy $\phi_t(v) = (1-t)v + t f_\mu(v)$ and $\phi_t(x_0)$ remains fixed. Thus by using $h(\gamma)$ as before, we find an edge-path loop $\gamma_1$ in $\Psi_E X_*$ such that $\gamma \simeq \gamma_1$ rel $x_0$. This proves that $\theta$ in 2.10 is onto, as required.

(In applications, $\pi \Psi_E X_*$ will turn out to be zero, so $\theta$ will then be an isomorphism).

## 3. Dimension

We now concentrate on the groups $H_q \Psi_E X_*$ and write $t_*: X_*^\omega \to \Phi X_*$ for the restriction of $t$; then $E(X_*) = E \cap \Phi X_*$ consistently with the requirement prior to 2.4 that $t_*^{-1} E(X_*) = T \cap X_*^\omega$. Contemplation of the above Examples 1–3 leads us to impose a 'dimensional' restriction in the general case.

3.1. *Definition. The system $(X, \leqslant, E)$ is of dimension $\geqslant n$, provided that for each $\sigma^q \in \Phi X_*$, $\sigma^q \notin E$ if $q < n$. We write*

$$\dim (X, E) \geqslant n.$$

Consider the examples of section 1 using the notation $X_*$, $E(X_*) = E_*$. In Example 1, suppose $m$ has $k$ distinct prime factors. If $m$ is not square free, then $E_*(m) = \emptyset$, so (omitting the subscript $E$ for brevity in standard cases):

3.2 $$\Psi(m) = \mathrm{Fact}_*(m) - E_*(m)$$

is acyclic and (with an obvious notation) $\dim (\mathrm{Fact}(m), E(m)) = \infty$. If $m$ is square-free, then $\Psi(m) = \emptyset$ if $k = 1$, and otherwise

$$\Psi(m) = S^{k-2}, \quad \dim (\mathrm{Fact}(m), E(m)) \geqslant k-2.$$

Observe that the Euler characteristic, $\chi(TX)$, is $\chi(\Psi(m))$ by Corollary 2.9, and this is $1 + \mu(m)$ where $\mu$ denotes the Moebius function. Compare Rota [11] p. 356.

In Example 2, if the set $U$ is empty, then $\dim (\mathfrak{D} U, E(U)) = \infty$ but if card $U = k < \infty$ then

$$\Psi(U) = \Phi \mathfrak{D}_* U - E_*(U) = \Phi Y - \{U\}$$

is empty if $k = 1$, while otherwise

3.3 $$\Psi(U) = S^{k-2}, \quad \dim (\mathfrak{D} U, E(U)) \geqslant k-2.$$

In Example 3, we have

3.4 $$\dim (\mathrm{Flat}(P^m), E(P^m)) = \dim (\mathrm{Flat}^\dagger(P^m), E^\dagger(P^m)) \geqslant m.$$

Here,

3.5 $$\Psi(P^m) = \Phi\, \mathrm{Flat}_*(P^m) - E_*(P^m)$$

has no immediate simple description; but if $P^m$ is the geometry with $q+1$ points on a line $(q < \infty)$ then $\Psi P^m$ is a complex of dimension $\leqslant q^{m-1} + q^{m-2} + \ldots + q + 1$ since any larger number of points spans $P^m$ and therefore forms a simplex in $E$.

Turning to the general case, observe first that for any system $(X, \leqslant, E)$, $\dim (X, E) \geqslant 0$. For calculating homology groups we now prove:

3.6 THEOREM. *If* $\dim(X, E) \geqslant n > 1$ *then* $\Psi_E X$ *is connected and*

$$H_q \Psi_E X = 0, \quad 0 \leqslant q \leqslant n-2$$

(we use reduced homology when $q=0$).

*Proof*. By Theorem 2.8, it suffices, for showing that $\Psi_E X$ is connected, to show that $x \sim y$ in $TX_*$ for any two minimal elements of $X$. But by 2.6 and 2.7 the 1-simplex $(x, y)$ does not lie in $T$ since $1 < n$ by hypothesis. Thus $TX$ is connected, and hence so is $\Psi_E X$.

To compute $H_q TX$ when $q > 0$, it suffices by Theorem 2.8 to show that every $q$-cycle $\gamma^q$ in $TX_*$ bounds if $q \leqslant n-2$. Since $X \neq \emptyset$ we can choose a minimal element $\mu \in X_*$. Then for every simplex $\sigma = (x_0, ..., x_q)$ in $\gamma^q$, $\mu\sigma \notin T$, where $\mu\sigma$ denotes $(\mu, x_0, ..., x_q)$; for, $\mu\sigma$ is a $(q+1)$-simplex in $X_*^\omega$, and $q+1 < n$ if $q \leqslant n-2$. Therefore all the cells of the cone $K = \mu|\gamma^q|$, with vertex $\mu$, exist in $TX_*$; and $\gamma^q \sim 0$ on $K \subseteq YX_0$ so the proof is complete.

3.7 COROLLARY (of proof). *The edge-path group* $\pi\Psi_E X_*$ (and hence $\pi\Psi_E X$) *is trivial if* $\dim(X, E) \geqslant 3$. For, if $\gamma$ in the proof represented an edge-path, it is homotopic to zero on the cone $K$; here $q = 1 \leqslant n-2$, so $n \geqslant 3$. Therefore by 2.10, $\pi\Psi_E X$ is trivial also, when $n \geqslant 3$.

Thus, by 3.4, the complex $\Psi(P^m)$ of 3.5 is simply connected if $m \geqslant 3$ (and it is not connected if $m=1$). Also, by 3.6, if $m > 1$ we have

$$H_q \Psi(P^m) = 0, \quad 0 \leqslant q \leqslant m-2.$$

We shall prove below that

$$H_q \Psi(P^m) = 0, \quad q > m-1$$

while $H_{m-1}\Psi(P^m)$ is free abelian on a (determinable) set of generators. The considerations leading to the proof of these extra facts generate the remaining sections of this paper.

3.8 To conclude the present section, consider the following example. Let $G$ be a non quasi-abelian Fuchsian group (as in Greenberg [4]), and let $X$ denote the set of subnormal subgroups $(\neq 1)$ of $G$. Any two such intersect in a third, by Lemma 3 of Greenberg's paper. Hence if we take $E$ to be empty, the system $(X, E)$ has dimension $\geqslant n$ for all $n$. Thus $\Psi_E X$ here is acyclic.

## 4. Supports

The 'projective' Example 3, of the system $\Psi(P^m)$, suggests that we investigate the following situation. There, each $\sigma^q \in \mathrm{Flat}\,(P^m)$ has a 'support', the smallest subspace $V$ of $P^m$ containing the vertices of $\sigma^q$; and $V$ lies in the poset Flat $(P^m)$. Similarly each $\sigma^q = (x_0, ..., x_q) \in \mathrm{Flat}^\dagger(P^m)$ has 'support' $x_0 \cap ... \cap x_q \in \mathrm{Flat}\,(P^m)$; and in Examples 1 and 2, supports may be readily defined. We therefore now consider the general case where we are given a function

4.1 $$g\colon \Phi X \to L$$

where $(L, \leqslant)$ is a poset with $\mathbf{1}$, while $g\colon (\Phi X, \leqslant) \to (L, \leqslant)$ is order-preserving, and $g$ is *convex* in the sense of

4.2 *Definition. If $\lambda \in L$, $\tau^q = \{x_0, ..., x_q\} \in \Phi X$ and $gx_i \leqslant \lambda$, $(0 \leqslant i \leqslant q)$ then $g\tau^q \leqslant \lambda$.*

Let $E_g = g^{-1}(1)$: *then $E_g$ satisfies the Exclusion Condition* 2.3 *above* since $g$ is order-preserving and in $L$ the order-relation is transitive. If $E_g = \varnothing$ then as observed after 2.1, $\Psi_g X = \Phi X - E_g$ is acyclic. Thus we now assume that $E_g \neq \varnothing$ *and hence that $g$ is onto* (by using Image $(g)$ to replace $L$). Using the 'support' function $t\colon X^\omega \to \Phi X$ of 2.2, we obtain a commutative diagram

4.3 $$\begin{tikzcd} \Psi_g X \subseteq \Phi X \arrow[r, "g"] & L \\ X^\omega \arrow[u, "t"] \arrow[ur, "f"'] & \end{tikzcd}$$

so that $f$ is also order-preserving and convex (i.e. $f(x_i) \leqslant \lambda$ implies $f(x_0, ..., x_q) \leqslant \lambda$). Note that if we regard $(\Phi X, \leqslant)$ as a poset, then $t$ is convex in the sense of 4.2. Since $t$ and $g$ are onto, so is $f$.

Conversely, given a system $(X, \leqslant, E)$ as in 3.1, then there is a convex order-preserving function $q\colon \Phi X \to M$ such that $E = E_q$ and $M$ is a poset with $\mathbf{1}$; for, let $M = (\Phi X - E) \cup \{E\}$, ordered in the obvious way by the ordering induced from $X$, let $E$ be the universal bound $\mathbf{1}$, and let $q$ denote the quotient map. Certainly, $q$ is order-preserving, and the Exclusion Condition 2.3 ensures that $q$ is convex. Thus 4.1 is a natural generalisation of the case considered in section 2.

We introduce a little notation. If $\lambda \in L$ we write:

4.4 $$\lambda.L = \{\mu \in L \mid \mu \leqslant \lambda\}$$

and if $A \subseteq X^\omega$ we write (with $f$ as in 4.3):

$$\lambda.A = f^{-1}(\lambda.L) \cap A \subseteq A.$$

It is easily checked that if $A$ is a sub-complex of $X^\omega$ then so is $\lambda.A$. Also, to avoid brackets, we write $\omega A$ for $A^\omega$. Now, we remarked, following 4.3 that the convexity of $g$ implies that of $f$; and this in turn implies (as is easily checked) that

4.5 $$\lambda.\omega X = \omega(\lambda.X)$$

Further progress requires that we impose extra conditions on $f$ and $L$. Thus, we suppose that in $L$, any two elements $\lambda$, $\mu$ have a glb in $L$ denoted by $x \wedge y$, so that $x \wedge y$ is the maximal $z \in L$ such that $z \leqslant x$ and $z \leqslant y$. It is easily checked that

$$4.6 \qquad \lambda.A \cap \mu.A = \lambda \wedge \mu.A = \lambda.(\mu.A).$$

In 4.3, $t$ is onto, and therefore $f$ is onto if and (here) only if $g$ is onto. The following result will be useful, although it follows at once from the hypotheses.

4.7. Proposition. *Let $f: X^{\omega} \to L$ be an order-preserving surjection. Then the function $\lambda \to \lambda.\omega X$ is an embedding $\eta: L \to \mathfrak{P}X^{\omega}$ of $L$ into the family of subsets of $X^{\omega}$.*

Observe that Propositions 4.6 and 4.7 have analogues for any order-preserving surjection $h: M \to L$ of posets. In particular, if $h$ is the identity mapping on $L$, the corresponding embedding $\eta: L \to \mathfrak{P}L$ has the property that for any *minimal* element $c \in L$, $\eta(c)$ is the singleton $\{c\}$. Thus we shall in future use $\lambda \cap \mu$ for $\lambda \wedge \mu$ and, for minimal elements $c$, write $c \in \lambda$ rather than $c \leqslant \lambda$. We let $L_*$ denote the set of minimal elements of $L$, and suppose that $L \neq \{\varnothing\}$ so that $L_* \neq \varnothing$. Finally we add a zero to $L$, as universal lower bound, and the resulting system becomes an izoposet still denoted by $L$, in which the minimal elements cover $\mathbf{0}$ and hence are *atoms*. This addition of $\mathbf{0}$ leaves the sets $f^{-1}(\lambda.L)$ unchanged since $f^{-1}(\mathbf{0}) = \varnothing$.

4.8 Lemma. *If $f: X^{\omega} \to L$ is convex, and if $c \in L_*$ then $f^{-1}(c) = c.X^{\omega}$ is a cone, and a subcomplex of $X^{\omega}$.*

*Proof.* By our earlier conventions, $f$ is onto $L - \{0\}$, so $f^{-1}(c) \neq \varnothing$. Fix $v \in f^{-1}(c)$, so $f(v) \leqslant c$. If $\sigma^q \in f^{-1}(c)$ then since $f$ is convex, $f(v\sigma^q) \leqslant c$, so $f^{-1}(c)$ is a cone with vertex $v$. Hence $f^{-1}(c)$ is acyclic. If $\sigma^q \in f^{-1}(c)$ then $\partial_i \sigma^q \leqslant \sigma^q$ in $X^{\omega}$ so $\partial_i \sigma^q \in f^{-1}(c)$ since $f$ is order-preserving and $c \in L_*$; hence $f^{-1}(c)$ is a subcomplex of $X^{\omega}$. Finally, $f^{-1}(c) = c.X^{\omega}$ because $c \in L_*$. This completes the proof.

## 5. $X|f$ and its subcomplexes

When computing homology, we emphasise that we are working in $X^{\omega}$ by writing

$$5.1 \qquad X|f = X^{\omega} - t^{-1}E_g = t^{-1}\Psi_g X$$

(the complex $TX$ in the notation 2.8): it is the total complex of $\Psi_g X = \Phi X - E_g$.

The following result is suggested by analogy with Theorem 2.8.

5.2 Proposition. *In the diagram 4.3, we may assume, for calculating homology, that $f$ maps vertices to atoms. More precisely, let*

$$Y = X \cap f^{-1}(L_*).$$

*Then* $Y^\omega \subseteq X^\omega$, *and if* $f'\colon Y^\omega \to L$ *denotes the restriction of* $f$, *the inclusion induces isomorphisms*

$$H_q(Y|f') \approx H_q(X|f),\ q \geqslant 0.$$

*Proof.* First observe that $Y \neq \varnothing$. For, $L_* \neq \varnothing$; and $f$ is onto $L - \{0\}$, so if $c = f(\sigma) \in L_*$, then for each vertex $v$ of $\sigma$, $f(v) \leqslant f(\sigma)$ since $f$ preserves order. But $0 < f(v)$ and $f(\sigma)$ is an atom, so $v \in Y$, and $Y \neq \varnothing$.

Next, let $\leqslant'$ denote a new order-relation on $X$, defined by:

$a \leqslant' b$ *if and only if* $a = b$ *or* $f(a) < f(b)$ *in* $L$.

With $E_g = g^{-1}(1)$ as in section 4, let $F = t^{-1}E_g = f^{-1}(1) \subseteq X^\omega$. Then $F$ satisfies the Exclusion Condition 2.3 relative to the order relation induced by $\leqslant'$ in $X^\omega$; this follows from the convexity of $f$. But the atoms of $X$, relative to $\leqslant'$ are exactly the elements of $Y$. The Proposition now follows directly from Theorem 2.8.

Next we supplement the notation of 4.4 by defining, for each $\lambda \in L$ and subset $B$ of $\Phi X$

$$\lambda.B = g^{-1}(\lambda.L) \cap B \subseteq B.$$

In particular $\lambda.\Phi X \subseteq \Phi(\lambda.X)$ since $g$ preserves order and $X$ was agreed in Section 2 to lie in $X^\omega$. The reverse inclusion holds because $g$ is convex, so we have

$$\lambda.\Phi X = \Phi(\lambda.X).$$

Also, since $f(\lambda.X^\omega) \subseteq \lambda.L$ we shall denote by

$$\lambda.f\colon \lambda.X^\omega \to \lambda.L$$

the restriction of $f$ regarded as having codomain $\lambda.L$. We make similar conventions for $\lambda.g$ and $\lambda.t$, and frequently omit the dots if no ambiguity arises. Then we obtain from 4.3 the commutative diagram

$$\begin{tikzcd} \Psi'_{\lambda g} X & \subseteq & \lambda\cdot\Phi X \arrow[r, "\lambda g"] & \lambda L \\ \lambda X|\lambda f \arrow[u] & \subseteq & \lambda X^\omega \arrow[u, "\lambda t"] \arrow[ur, "\lambda f"'] & \end{tikzcd}$$

where $\lambda.X|\lambda.f = (\lambda t)^{-1}\Psi'_{\lambda g}X$ consistently with 5.1. Now, in 5.1, $t^{-1}E_g = f^{-1}(1_L)$, so

$$(\lambda t)^{-1}E_{\lambda g} = (\lambda f)^{-1}(1_{\lambda L}) = f^{-1}(\lambda) \subseteq \lambda.X^\omega;$$

Therefore, by 5.1

5.2 $$\lambda.X|\lambda.f = \lambda.X^\omega - f^{-1}(\lambda) \neq \lambda.(X|f).$$

We now develop this notation, eventually to express Definition 2.3 in a form (see Section 9) such that if dim $(X, E_g) \geqslant n$ and $\lambda$ is a co-atom of $L$ then dim $(\lambda X, E_{\lambda g}) \geqslant n-1$. This will allow inductive arguments to be made. First, consider the co-atoms of $L$.

So far, $L$ has been an izoposet with atoms. We now assume

5.3 *$L$ has co-atoms; i.e. if $\lambda \in L$ then there exists $\mu \prec 1$ in $L$ with $\lambda \leqslant \mu$, unless $\lambda = \mathbf{1}$.*

[$a \prec b$ in L means that $b$ covers a, i.e. if $a \leqslant c \leqslant b$ then either $a=c$ or $b=c$]. Let $L^*$ denote the set of all co-atoms of $L$.

Now for any atom $c \in L_*$ and co-atom $\mu \in L^*$ then either $c \in \mu$, or the lub of $x$ and $\mu$ is $\mathbf{1}$. There may, however, be other elements $\alpha \in L$ with this property: thus we define

$$L^{\perp}(c) = \{\alpha \in L \,|\, c \vee \alpha = \mathbf{1} \neq \alpha\} \cup L^*$$

where $c \vee \alpha$ denotes lub of $c$ and $\alpha$. The set

$$L^{\perp} = \bigcup \{L^{\perp}(c) \,|\, c \in L_*\}$$

will be called the set of *atomic complements* in $L$. Also let

$$L^*(c) = \{\lambda \in L^* \,|\, c \in \lambda\},$$

so that

$$L^{\perp}(c) \supseteq L^* - L^*(c).$$

The definition of the complex $X|f$ in 5.1 then yields at once (with $\lambda.X|f$) as in 5.2):

5.4 Proposition. $X|f = \bigcup \{\lambda.(X|f) \,|\, \lambda \in L^*\}$.

This proposition suggests that we compute the homology of $X|f$ using the Mayer-Vietoris sequence. As a step towards the computation, we choose an atom $c \in L_*$, with $L^*(c)$ as defined above. A direct argument concerning the convexity of $f$ proves:

5.5 Lemma. *Suppose $f\colon X^{\omega} \to L$ is convex. If $v \in f^{-1}(c)$ and $\lambda \in L^*(c)$, then each $\lambda.(X|f)$ is a cone with vertex $v$. Also*

$$K(c) = \bigcup \{\lambda.(X|f) \,|\, \lambda \in L^*(c)\}$$

*is a cone with vertex $v$, and hence an acyclic subcomplex of $X|f$.*

Now, from 5.4 we may write:

$$X|f = K(c) \cup \bigcup \{\lambda.(X|f) \,|\, \lambda \in L^{\perp}(c) - L^*(c)\}$$

where for a technical reason (see 7.2 below) we replace $L^*$ by $L^{\perp}(c)$.

In case the set $L^{\perp}(c) - L^*(c)$ is infinite, we compute $H_*(X|f)$ as the direct limit of certain groups $H_*K_M$ and the appropriate induced homomorphisms, when $M$ runs through the family $\Phi(L^{\perp}(c)-L^{*}(c))$ of all non-empty finite subsets of $L^{\perp}(c)-L^{*}(c)$, directed by inclusion; here:

5.6 $$K_M = K(c) \cup \bigcup_{\lambda \in M} \lambda T, \quad T = X\,|\,f.$$

Let $M = \{\mu_1, ..., \mu_m\}$, and

5.7 $$K_0 = K(c), \quad K_{i+1} = K_i \cup \mu_{i+1} T, \quad (0 \leqslant i < m).$$

Since we are dealing with complexes, the triads

$$(K_i, J_i, K_{i+1}), \quad J_i = \mu_{i+i} T,$$

are proper (see [2] p. 34) so the Mayer-Vietoris sequence

5.8 $$... \to H_{q+1} K_{i+1} \to H_q(K_i \cap J_i) \to H_q K_i + H_q J_i \to H_q K_{i+1} \to H_{q-1}(K_i \cap J_i) \to ...$$

is exact. We therefore need to know more about $K_i \cap J_i$, and for this we study $L$ further, in the next section.

5.9. However, we pause to consider the following cautionary example. Let $B$ denote a vector bundle over a space $Y$, whose fibre is an $n$-dimensional vector space ($n > 2$). Let $X$ denote the set of all factors $B'$ of $B$, i.e. $B = B' + B''$ (Whitney sum). Thus $B'$ is a continuously varying distribution $s(y)$ of $k$-flats, $s(y)$ in the fibre $B_y$ at $y \in Y$. We order $X$ by: $s \leqslant s'$ if $s(y) \subseteq s'(y)$ in $B_y$ for all $y \in Y$. Unfortunately $X$ is not closed under intersection, nor under the operation $(s \vee s')\,(y) = s(y) \vee s'(y)$, since $s \vee s'$ may not have constant dimension on each fibre. Nevertheless if $\Gamma(B) \subseteq X$ denotes the family of sections of $B$, we can form a complex $\Psi_E \Gamma$ where $E$ consists of those $\sigma = (s_0, ..., s_q) \in \Gamma^{\omega}$ such that for each $x \in Y$, the points $s_0(x), ..., s_q(x)$ span $B_x$. Just as for $\Psi P^{n-1}$ (to which the example reduces if $Y$ is a single point) we find that $H_q \Psi_E \Gamma = 0$ if $0 \leqslant q < n-2$, but perhaps $H_q \Psi_E \Gamma \not\approx H_q \Psi_E X$. It would be interesting to known how to modify the considerations of the later sections to compute $H_q \Psi_E \Gamma$ in general.

## 6. A Uniqueness theorem for homology

It is convenient here to derive from the sequence 5.8 a uniqueness theorem for homology groups, as follows. Suppose that, for each $\lambda \in L$, we can associate a (closed) subcomplex $C_\lambda$ of $X^{\omega}$ such that $C_0 = \emptyset$ but if $0 < \lambda$ then

6.1 (i) *$C_\lambda$ is acyclic, $C_\lambda \subseteq \lambda.(X\,|\,f)$, and $C_\lambda \cap C_\mu \subseteq C_{\lambda \cap \mu}$.*

Let $c$ be an atom, and let $Q(c) = \bigcup \{C_\lambda \,|\, \lambda \in L^{*}(c)\}$. Suppose

6.1 (ii) *$Q(c)$ is acyclic.*

With $M$ as in 5.6, we can form complexes $Q_M$, $Q_i$, $P_i$ analogously to the complexes $K_M$, $K_i$, $J_i$ in 5.7, but using $C_\lambda$ instead of $\lambda.T$. We thus derive an exact Mayer-Vietoris sequence $S_Q^i$ analogous to the sequence $S_K^i$ of 5.8, and the inclusions $C_\lambda \subseteq \lambda.T$ induce a homomorphism $s_i$: $S_Q^i \to S_K^i$ of exact sequences.

6.2 Proposition. *If, for each $i$ in 5.8, the inclusion induces isomorphisms $H_*(P_i \cap Q_i) \approx H_*(K_i \cap J_i)$ then $s_i$: $S_Q^i \to S_K^i$ is an isomorphism of exact sequences. Therefore*

$$H_* Q_M \approx H_* K_M$$

*Proof.* The conditions of the "Five Lemma" (see [2] p. 19) are satisfied when $i=0$, since $K_0$, $J_0$, $Q_0$, $P_0$ are all acyclic and $H_*(P_0 \cap Q_0) \approx H_*(K_0 \cap P_0)$. Thus $s_0$ is an isomorphism, so by induction on $i$, $s_i$ is always an isomorphism and the proposition follows.

This proposition will be worked into a more useful form in 7.8 below.

## 7. The Mayer–Vietoris Sequence

Going back to the situation in 5.7, we shall now compute $K_i \cap J_i = K_i \cap \mu_{i+1} I$, in the notation there, assuming that $L$ satisfies condition $\mathbf{L}_1$ below. Recall from 5.6 that $M = \{\mu_1, \ldots, \mu_m\}$, and we may assume that $M$ is labelled so that $\mu_i < \mu_j$ implies $i < j$.

7.1 Lemma. *For each $i = 0, \ldots, m-1$,*

$$K_i \cap J_i \subseteq \mu X | \mu f \quad (\mu = \mu_{i+1}).$$

*Proof.* If $\sigma \in K_i \cap J_i$ then we may suppose $\sigma \in \lambda.T$ by definition of $K_i$, where either $\lambda \in L^*(c)$ or $\lambda = \mu_j$ for some $j \leqslant i$. Hence $\sigma \in (\lambda \cap \mu).T$, using 4.6. It remains to prove that $\lambda \cap \mu < \mu$. But, if $\lambda \in L^*(c)$, then $c \vee \lambda = \lambda$; so if $\lambda \cap \mu = \mu$ we would have

$$\mathbf{1} = c \vee \mu = c \vee (\lambda \cap \mu) \leqslant c \vee \lambda = \lambda,$$

a contradiction. And, if $\lambda = \mu_j$, then again $\lambda \cap \mu \neq \mu$, otherwise $\mu < \lambda = \mu_i$ contrary to our agreed manner of labelling $M$. This completes the proof.

To obtain equality in 7.1 we need to suppose that $L$ satisfies:

$\mathbf{L}_1$. *For each atom $c$, and $\lambda \in L^\perp(c)$, the set $\lambda.L^\perp(c)$ is finite.* (Recall: $\mathbf{1} \notin L^\perp(c)$).

This enables us to suppose that the set $M$ in 5.6 is *complete*, in the sense that if $\alpha \in M$ and $\beta \in L^\perp(c)$ then $\beta \in M$ if $\beta < \alpha$. Every such $M$ lies in a complete member of $\Phi(L^\perp(c) - L^*(c))$, since we need only take the union $N$ of all sets $\alpha.L^\perp(c)$ as $\alpha$ runs through $M$; by $\mathbf{L}_1$, $N$ is finite and complete. Thus the family of complete subsets is cofinal in $\Phi(L^\perp(c) - L^*(c))$, so we may confine ourselves to this family when taking direct limits of the homology groups $H_q K_M$. With this agreement we have

7.2 LEMMA. *If $M$ is complete, then* $K_i \cap J_i = \mu X|\mu f$, $(0 \leqslant i < m)$.

*Proof.* By 7.1 we must prove that $\mu X|\mu f \subseteq K_i \cap J_i$. If $\sigma \in \mu X|\mu f$, then $\alpha = f(\sigma) < \mu$, and then either (i) $c$ and $\alpha$ are $\leqslant \lambda$ for some $\lambda \in L^*$, or (ii) $c \vee \alpha = \mathbf{1}$, so $\alpha \in L^{\perp}(c)$. If (i) holds, then $\sigma \in \lambda T \subseteq K_0 \subseteq K_i$, while $\sigma \in \mu T = J_i$, so $\sigma \in K_i \cap J_i$ as required. If (ii) holds, then by the labelling of $M$, $\alpha = \mu_j \in M$ for some $j < i+1$ since $\alpha < \mu = \mu_{i+1}$ and $M$ is complete. Therefore $\sigma \in \mu_j T \subseteq K_j \subseteq K_i$. Since $\alpha < \mu$ then $\sigma \in \mu T = J_i$, so $\sigma_i \in K \cap J_i$ as required. This establishes the lemma.

The last lemma suggests an inductive approach to the problem of computing $H_*(X|f)$, since the Mayer-Vietoris sequence 5.8 now reduces to:

$$(7.3) \quad \ldots \to H_{q+1}(K_{i+1}) \to H_q(\lambda X|\lambda f) \to H_q(K_i) \to H_q(K_{i+1}) \to H_{q-1}(\lambda X|\lambda f) \to \ldots$$
$$\ldots \to H_0'(\lambda X|\lambda f) \to H_0' K_i + H_0' J_i \to H_0' K_{i+1} \to 0$$

when $\lambda = \mu_{i+1}$, using Lemmas 7.1 and 7.2. (By 5.5, $J_i$ is acyclic; we use *unreduced* homology groups $H_0'$ in this sequence.)

One useful conclusion can be drawn at once from the sequence 7.3, and the hypotheses will be shown below to arise 'naturally'. To indicate reduced homology groups we use $H$.

7.4 THEOREM. *Suppose there exists $n > 0$ such that, for all $\lambda \in L^{\perp}(c)$, $H_q(\lambda X|\lambda f) = 0$ if $q \neq n-1$. Then $H_q(X|f) = 0$ if $q \neq n$.*

*Proof.* The exact sequence 7.3 yields

$$\text{(a)} \qquad 0 = H_q K_0 \approx H_q K_1 \approx \ldots \approx H_q K_i \approx H_q K_{i+1},$$

if $q \neq n-1$ or $n$; and for these two dimensions

$$0 \to H_n K_i \to H_n K_{i+1} \to H_{n-1}(\lambda X|\lambda f) \to H_{n-1} K_i \to H_{n-1} K_{i+1} \to 0,$$

is exact. Since $K_0$ is acyclic, the last sequence yields $H_{n-1} K_i = 0$ by induction on $i$, and leaves us with the exact sequence

$$\text{(b)} \qquad 0 \to H_n K_i \to H_n K_{i+1} \to H_{n-1}(\lambda X|\lambda f) \to 0.$$

Thus (a) holds if $q \neq n$, whence as $K_m = K_M$ in 5.6, $0 = H_q K_m = H_q(X|f)$ if $q \neq n$.

7.5 COROLLARY. *If also, each group $H_{n-1}(\lambda X|\lambda f)$ is free abelian, so is $H_n(X|f)$.*

For, since $H_n K_0 = 0$ in (b), induction on $i$ shows that $H_n K_{i+1} \approx H_n K_i + H_{n-1}(\lambda X|\lambda f)$ is free abelian. Hence $H_n(X|f)$, as a direct limit of the union of free abelian groups $H_m K_M$ in 5.6, is free abelian because if $M \subseteq N$ then $H_n K_M$ is a direct summand of $H_n K_N$. This establishes the corollary.

Clearly, we now need $L$ to be such that assumptions about $X|f$ will be inherited (with appropriate dimension shift) by the subcomplexes $\lambda X|\lambda f$. This explains the style of the digression in the next section, to study $L$ further. Meanwhile, the case when $L$ is finite is of interest. For then, $X|f$ is $K_M$ for a suitable $M$, and the proof of Corollary 7.5 shows immediately:

7.6 $$\operatorname{rank} H_n(X|f)=\sum_{i=1}^{m}\operatorname{rank} H_{n-1}(\lambda_i X|\lambda_i f)$$

where $\lambda_i$ runs through $L^{\perp}(c)-L^*(c)$, of cardinal $m$.

Instead of using the complexes $K_i$, $J_i$, $T$ in the proofs of Lemmas 7.1, 7.2 we could have used the complexes $Q_i$, $P_i$, $C_\lambda$ of 5.9, with $\mu X|\mu f$ replaced by the complex

7.7(i) $$D_\mu=\{\sigma|\sigma\in D \text{ and } f(\sigma)<\mu\}\subseteq \mu X|\mu f$$

and, guided by 5.4, we define the complex

7.7(ii) $$D=\bigcup\{C_\lambda|\lambda\in L^*\}\subseteq X|f.$$

For, the arguments in the lemmas used only inclusion relations between complexes, because the rest of the argument concerned $L$ and $M$. Just as with $X|f$, we define the homology of $D$ to be the direct limit of $H_*Q_M$, with $Q_M$ as in 6.2, and where $M$ runs through the complete subsets of $\Phi(L^{\perp}(c)-L^*(c))$ described prior to 7.2. Hence we obtain at once from Proposition 6.2 the result:

7.8 THEOREM. *Let the izoposet $L$ satisfy condition $\mathbf{L}_1$. For each $\lambda\in L^{\perp}(c)-L^*(c)$, suppose that in* 7.7 (i), *the natural homomorphism $H_qD_\lambda\to H_q(\lambda X|\lambda f)$ is an isomorphism*, for all $q\geqslant 0$. *Then in* 7.7 (ii) *also, each $H_qD\to H_q(X|f)$ is an isomorphism.*

## 8. A filtration on $L$

In order to allow inductive arguments to be made, we shall now suppose that the izoposet $L$ is *filtered*; that is to say, there is a monotonic, strictly increasing function $\phi$ from $L$ to the non-negative integers, with $\phi(0)=0$. (For example, if $L$ is finite then it is filtered: take $\phi(x)$ to be the number of $y\leqslant x$). Since $L$ has atoms and co-atoms, we have immediately.

8.1. *If $0<x<1$ in $L$ then $0<\phi(x)<\phi(1)$.*

*Therefore $L$ contains no infinite chains, so lub's exist in $L$, i.e. $L$ is a lattice.* Hence, also, if $\phi(1)=1$ then $L$ consists solely of $0$ and $1$.

8.2 LEMMA. *If $\phi(1)=2$, then $L_*=L^*$.*

*Proof.* If $\lambda \in L^*$, then there exists an atom $c \in \lambda$; so by strict monotonicity, $0 < \phi(c) \leqslant \phi(\lambda) < 2$, whence $\phi(c) = \phi(\lambda) = 1$, and $c = \lambda$ again by strict monotonicity. Thus $L^* \subseteq L_*$. Similarly $L_* \subseteq L^*$, and the lemma follows.

As a first application, consider condition 6.1 (ii) on the system $C_\lambda$. We use the notation of 6.1.

8.3 LEMMA. *If $L$ is filtered, then $Q(c)$ is acyclic.*

*Proof.* It suffices to prove that if $c \in \lambda(i) \in L$, $1 \leqslant i \leqslant k$, then $C_{\lambda(1)} \cup \ldots \cup C_{\lambda(k)}$ is acyclic. We prove this by induction on $n = \max \phi(\lambda(i))$, starting with $n = 1$ since no $\lambda(i)$ is zero. But if $n = 1$, then each $\lambda(i)$ is an atom by strict monotonicity, so each $\lambda(i)$ is $c$, and $C_c$ is acyclic by condition 6.1 ($i$). If $n > 1$, we first observe that the inclusion $C_\lambda \cap C_\mu \subseteq C_{\lambda \cap \mu}$ in 6.1 (i) implies $C_{\lambda \cap \mu} = C_\lambda \cap C_\mu$. Hence, for each $r = 2, \ldots, k$,

$$E_r = C_{\lambda(r)} \cap \bigcup_{i<r} C_{\lambda(i)} = \bigcup C_{\lambda(r) \cap \lambda(i)}.$$

Since $C_\lambda$ increases with $\lambda$ we may assume that for no $i < r$ can $\lambda(i) \leqslant \lambda(r)$. Hence $\lambda(r) \cap \lambda(i) < \lambda(i)$, so by strict monotonicity of $\phi$, $\phi(\lambda(r) \cap \lambda(i)) < n$. Therefore, by an inductive hypothesis on $n$, $E_r$ is acyclic since $c \in \lambda(r) \cap \lambda(i)$. But then, by the Mayer-Vietoris theorem and induction on $r$, $C_{\lambda(r)} \cup \bigcup_{i<r} C_{\lambda(i)}$ is acyclic; therefore the lemma follows by induction on $n$.

## 9. Filtration and dimension

For the system $f: \Phi X \to L$ in 4.1, an appropriate version of the notion of dimension (see 3.1), when $L$ is filtered by a function $\phi$, is given by:

9.1 *Definition. We write* $\dim(X, g) \geqslant n$ *provided that for all $\sigma^q \in \Phi X$ and $0 \leqslant q < n$, then*

$$\phi(g\sigma^q) \leqslant \varphi(1) - (n - q).$$

Thus since $E_g = g^{-1}(1)$ as in 4.1, then $\dim(X, E_g) \geqslant n$ in the sense of 3.1, by strict monotonicity of $\phi$. Hence by Theorem 3.6, we have

9.2 THEOREM. *If* $\dim(X, g) \geqslant n > 0$, *then* $H_q(X \mid f) = 0$, $0 \leqslant q \leqslant n - 2$.

Always, $\dim(X, g) \geqslant 0$. Further, if $n > 0$ and $\dim(X, g) \geqslant n$, then clearly $\dim(X, g) \geqslant n - 1$.

A further consequence of strict monotonicity is

9.3 LEMMA. *If* $\dim(X, g) \geqslant n > 0$, *and* $a \in L$, *then*

$$\dim(a.X, a.g) \geqslant n - (\phi(1) - \phi(a)).$$

Recall from 5.2 the set $Y \subseteq X$ of vertices mapped by $f$ to atoms of $L$. Since $\Phi Y \subseteq \Phi X$ there, let $g' = g | \Phi Y$; then $\dim (Y, g') \geq n$ if $\dim (X, g) \geq n$. Now if $n \geq 3$, both $\Psi_{g'} Y$ and $\Psi_g X$ are simply connected, by 3.7. Hence, a well-known result of homotopy theory (see Hu [5] p. 167) allows us to add to Proposition 5.2 the

9.4 COROLLARY. *Suppose* $\dim (X, g) \geq 3$, *and* $Y = X \cap f^{-1} L_*$. *Then the inclusion* $Y \subseteq X$ *induces isomorphisms of the homotopy groups of* $\Psi_{g'} Y$ *on those of* $\Psi_g X$, *in all dimensions.*

One might now investigate, along the lines of dimension theory, the consequences of defining $\dim (X, g)$ to be $n$ if $\dim (X, g) \geq n$ and $\dim (X, g) \ngeq n+1$; in particular to aim for an analogue of Lemma 9.3. It seems more direct, however, to investigate the filtration further because Proposition 10.2 below is more precise than Theorem 9.2.

## 10. The non-vanishing homology group of $X|f$

It is now possible to complete the calculations of homology, from Section 7. We use the notation of 4.2, and assume $L$ is filtered by $\phi$.

10.1 LEMMA. *Suppose $L$ is such that* $\phi(1) = 2$. *Then* $H_q(X|f) = 0$ *if* $q \neq 0$, *and* $H_0(X|f)$ *is free abelian on $N$ generators, where* $N + 1 = \text{card}\,(L_* \cap f X^{\omega})$.

*Proof.* By Lemma 8.2, $L^* = L_*$. Now by Proposition 5.4, $X|f$ is the union of the subcomplexes $\lambda.(X|f)$, $\lambda \in L^*$: but since $L^* = L_*$, then $\lambda.X|f = f^{-1}(\lambda)$, so these subcomplexes are mutually disjoint, while each is acyclic (if non-empty) by Lemma 5.5. The lemma follows.

Lemma 10.1 starts an inductive calculation of $H_*(X|f)$ if we impose two further conditions on $L$:

$\mathbf{L}_2$. *The filtration is minimal*, i.e. *if* $a \succ b$ *in $L$ then* $\phi(a) = \phi(b) + 1$.

$\mathbf{L}_3$. *For all* $b \in L$ *and atoms* $c \notin b$, $c \vee b \succ b$ (*if* $c \vee b$ *exists*).

Thus, $\mathbf{L}_2$ and $\mathbf{L}_3$ together imply that *atomic complements are co-atoms.* For, by $\mathbf{L}_2$, and the strict monotonicity of $\phi$, $b$ in $\mathbf{L}_3$ is a co-atom in $(c \vee b).L$. Therefore for each $a \in L$, the set $aL$ is an izoposet satisfying $\mathbf{L}_1 - \mathbf{L}_3$, with $a$ as $\mathbf{1}$, and filtration $\phi_a = \phi | aL$. Thus $aL$ has no infinite chains, so atoms and co-atoms exist in $aL$, and $aL$ is a lattice, as remarked after 8.1

10.2 PROPOSITION. *If $L$ satisfies* $\mathbf{L}_1 - \mathbf{L}_3$ *and* $\phi(\mathbf{1}) = n + 2$, *then* $H_q(X|f) = 0$ *if* $q \neq n$, *while* $H_n(X|f)$ *is free abelian.*

*Proof.* Atomic complements are co-atoms, so each $\lambda$ in 7.4 lies in $L^*$. Thus $1 \succ \lambda$ so $\phi(\lambda) = n + 1$ by $\mathbf{L}_2$. Hence as observed above, $\lambda L$ is a lattice satisfying $\mathbf{L}_1 - \mathbf{L}_3$. Therefore induction on $n$ can be used, with Lemma 10.1 for $n=0$. The proposition now follows at once from Theorem 7.4 and its corollary. The relationship of $L$ with geometric lattices (see 13.6) is discussed in 17.20.

The next problem of interest is to describe the set of generators of $H_n(X|f)$. Returning first to the case when $L$ is finite (see 7.6), let us suppose that $w \in L$, $\varphi(\mathbf{1})=n+2$ and $\varphi(w)=n+2-j$; so by 10.2,

$$\varrho(w)=\operatorname{rank} H_{n-j}(wX \mid wf)$$

may be non-zero, and we compute it in certain cases as follows.

Suppose that $L$ has the following 'homogeneity' property, which is true of the izoposet Flat $(P^n)$ filtered by dimension (see also 17.5 below):

$\mathbf{L}_4$. *For each element $w \in L$, atom $c \in W=w.L$, and co-atom $\lambda$ of $W$, then $(\lambda W)^* \approx W^*(c)$; and* card $W^*=$ card $(vL)^*$ *if $\varphi(w)=\varphi(v)$.*

Then we have the lemma:

10.3. LEMMA. *Suppose $f$ is onto $L-\{\mathbf{0}\}$, while $L$ is finite and $L$ satisfies $\mathbf{L}_1-\mathbf{L}_4$. If $v, w \in L$ and $\phi(v)=\phi(w)$, then $\varrho(v)=\varrho(w)$. In fact, if $j=\phi(\mathbf{1})-\phi(w)$, then*

$$\varrho(w)=(c_j-c_{j+1})(c_{j+1}-c_{j+2}) \dots (c_{n-1}-c_n)(c_n-1)$$

*where* card $(wL)^*=c_j$ *and* $n=\phi(\mathbf{1})-2 \geqslant 0$.

*Proof.* By induction on $\phi(w)$; for when $\phi(w)=2$, $\varrho(w)=\operatorname{rank} H_0(w.L \mid w.f)$ by definition; since $f$ is onto, then $L_* \subseteq fX^\omega$, and by 8.2, $(w.L)_*=(w.L)^*$, whence by Lemma 10.1, $\varrho(w)=c_n-1$, independently of $w$ (by $\mathbf{L}_4$). To complete the proof, we observe from 6.6 that, for general $j$,

$$\varrho(w)=\varrho(j)=(c_j-c_{j+1})\,\varrho(j+1),$$

if we make an obvious inductive hypothesis and use $\mathbf{L}_4$ to write card $(L^*-L^*(c))=c_j-c_{j+1}$. (Since atomic complements are here co-atoms, $L^*=L^\perp(c)$).

10.4 COROLLARY. *In Example 3, where $L=$ Flat $(P^n)$ and $P^n$ is the finite geometry with $q+1$ points on each line, then*

$$\operatorname{rank} H_{n-1}(\Psi P^n)=q^{\binom{n+1}{2}} \quad (n \geqslant 1).$$

[For here, $c_j=q^{n-j}+q^{n-j-1}+\dots+q+1$; $\phi(\mathbf{1})=n+1$ since $\phi(x)=1+\dim x$ to make $\phi(\varnothing)=0$].

This result augments that of Rota [11], where he calculates rank $H_{n-1}(\Psi P^n)$ by working out the Moebius function of Flat $(P^n)$.

## 11. The replacement of $X$ by $L_*$

When $L$ is not finite, we can simplify the study of the complex $\Psi^{\cdot}_g X$ by constructing a new complex from $L$,—at least when $L$ satisfies a new condition:

$\mathbf{L}_5$. *Atoms add in $L$, i.e. if $a_0, \ldots, a_r$ are atoms in $L$, then they have a lub in $L$, denoted by $a_0 \vee \ldots \vee a_r$.*

If $L$ satisfies $\mathbf{L}_5$ we show that $L$ itself contains all the information about the homology and homotopy of $\Psi^{\cdot}_g X$. Thus $X$ and $g$ can in fact be discarded for the computation. Throughout this section, then, $L$ is an izoposet with atoms, satisfying $\mathbf{L}_5$. *Conditions $\mathbf{L}_1 - \mathbf{L}_4$ are not needed.*

To compute the homology of $\Psi^{\cdot}_g X$ we may assume, by 5.2, that $f(X) \subseteq L_*$. Hence if, as in section 1, we form the ordered sets

$$\Phi A,\ A^{\omega},\ \ (A = L_*)$$

there is a commutative diagram, extending 4.3:

(11.2)
$$\begin{tikzcd}
 & \Phi X \arrow[r, "g"] & L & \Phi A \arrow[l, "m"'] & \\
\Psi^{\cdot}_g X \arrow[ur] \arrow[r] & X^{\omega} \arrow[u, "t"] \arrow[ur, "f"] \arrow[rr, "h"'] & & A^{\omega} \arrow[u, "s"'] \arrow[ul, "l"'] & \Psi^{\cdot}_m A \arrow[l] \arrow[ul]
\end{tikzcd}$$

which we explain as follows. First, since $f(X) = L_*$ and $E_g = g^{-1}(\mathbf{1})$, then $X \cap E_g = \emptyset$ if $d(\mathbf{0}) > 1$ in $L$ (as we now assume). The simplicial maps $s$, $m$ are defined by

$$s(a_0, \ldots, a_q) = \{a_0, \ldots, a_q\}, \text{ and } m\{a_0, \ldots, a_q\} = a_0 \vee \ldots \vee a_q$$

and $l$ by commutativity: $l = m \circ s$; since $\vee$ is a lub operation, it is associative and commutative, so $m$ is well-defined. Further, the order properties of $\vee$ ensure that $m$ (and hence $l$) is order-preserving and convex.

To define the function $h$, we use the fact that $f(X) = L_*$ to set

$$h(x_0, \ldots, x_q) = (a_0, \ldots, a_q),\ a_i = f(x_i) = g(x_i).$$

To establish the commutative relation $l \circ h = f$, we observe that $x_i \leqslant (x_0, \ldots, x_q)$ whence $a_i = f(x_i) \leqslant f(x_0, \ldots, x_q)$ since $f$ is order-preserving, so

$$l \circ h(x_0, \ldots, x_q) = l(a_0, \ldots, a_q) = a_0 \vee \ldots \vee a_q \leqslant w \leqslant f(x_0, \ldots, x_q):$$

but each $a_i \leqslant w$, so $f(x_0, \ldots, x_q) \leqslant w$ by convexity of $f$, whence $l \circ h = f$ as required. [N.B. In $L$, the order relation is antisymmetric].

The remaining arrows in diagram 11.2 indicate inclusions, in the following sense. If we well-order $A$, we turn $\Phi A$ into an oriented complex, so there is an embedding of $\Phi A$

in $A^\omega$, whose image $I$ we identify with $\Phi A$. Thus $\Psi_m A$ is identified with the oriented sub-complex $I - m^{-1}(1) \subseteq I \subseteq A^\omega$.

The well-ordering in $A$ induces one among the sets $\{h^{-1}(a)\}_{a \in A}$, and we well-order each one, to obtain a well-ordering in $X$, which is preserved by $h$. We then identify $\Phi X$ and $\Psi_g X$ with oriented sub-complexes of $X^\omega$, just as we did in $A^\omega$. It can be verified that (with these identifications) the restrictions of $s$, $t$ to $\Phi A$, $\Phi X$ respectively are the identity functions, and $h(\Phi X) \subseteq \Phi A$ whence $h(\Psi_g X) \subseteq \Psi_m A$. Thus we have a function

11.3 $$k: \Psi_g X \to \Psi_m A, \quad k = h \mid \Psi_g X$$

which is onto since $g(X) = g(X_*) = f(X_*) = A$; also $k$ is simplicial and commutes with the face operators $\partial_i$ since the same is true of $h$. Similarly since $X \mid f = (\Psi_g X)^\Omega$, we have a map of the total complexes, induced by $k$: –

11.4 $$k^\Omega: X \mid f \to A \mid l.$$

For the purposes of the following theorem, we now identify $X^\omega$ and $A^\omega$ with geometrical relisations, so that all the function in 11.2 may be regarded as continuous (as well as simplicial) mappings. We still write '$K$' for both a complex and its underlying topological space. Then we have

11.5 THEOREM. *The mappings $k$, $k^\Omega$* (in 11.3, 11.4) *are homotopy equivalences.*

*Proof.* Given the simplex $\gamma = (a_0, \ldots, a_q) \in A^\omega$, then $h^{-1}(\gamma)$ consists of all $\sigma^q = (x_0, \ldots, x_q) \in X^\omega$ for which $g(x_i) = a_i$ $(0 \leqslant i \leqslant q)$. Since $g$ is onto, this means that

(a) $$F = h^{-1}(\gamma) = g^{-1}(a_0) \times g^{-1}(a_1) \times \ldots \times g^{-1}(a_q);$$

but each factor $g^{-1}(a_i)$ is a cone, by 5.5, whence $F$ is contractible. If $\gamma \in A \mid l$, then $m \circ s(\gamma) < 1$, whence $m \circ s \circ h(\sigma) = g \circ t(\sigma) < 1$ so $\sigma \in X \mid f$. If $\gamma \in \Psi_m A$ then $a_0 < \ldots < a_q$ in the well-ordering of $A$ mentioned above, so for any $\sigma \in F$, $x_0 < \ldots < x_q$ by construction of the well ordering of $X$, whence $\sigma = (x_0, \ldots, x_q)$ lies in the subcomplex of $X^\omega$ identified earlier with $\Psi_g X$.

Both for $k$ and $k^\Omega$ then, the inverse image of a simplex is a product of cones, like $F$ in (a) above.

Next, we define a simplicial mapping $p: A^\omega \to X^\omega$ by assigning to each vertex $c \in L^\omega$ some vertex $x \in h^{-1}(c)$ and extending by linearity on each simplex; clearly $h \circ p$ is the identity on $A^\omega$. By convexity of $f$, $p$ maps $A \mid l$ into $X \mid l$; and for any simplex $\gamma \in \Psi_m A$, $p(\gamma) \in h^{-1}(\gamma) \subseteq \Psi_g X$, since $h \circ p = 1_A$, so $p$ maps $\Psi_m A$ into $\Psi_g X$. Thus the restrictions $u$, $v$ of $p$ to $\Psi_m A$, $A \mid l$ respectively are right inverses for $k$, $k^\Omega$. It remains to prove that $u \circ k \simeq 1$, $v \circ k^\Omega \simeq 1$,—identity maps on $\Psi_g X$, $X \mid f$ respectively. But for each vertex $x$ of $X^\omega$, we know that $h^{-1}(hx)$ is a cone in which $x$ and $y=(p\circ h)x$ may be joined by the segment $\{(1-\xi)x+\xi y\}_{0\leqslant\xi\leqslant 1}$. This yields a homotopy $\Psi_\xi$ from $p\circ h$ to the identity on $X$, and its restrictions to $\Psi_g X$, $X|f$ respectively are the required homotopies of $u\circ k$, $v\circ k^\Omega$. Thus $k$, $k^\Omega$ have the homotopy inverses $u$, $v$, and the proof is complete.

We can now concentrate our attention on $L$. In particular, if $L$ satisfies $\mathbf{L}_2$ and $\mathbf{L}_3$ (and hence $\mathbf{L}_5$ by 10.2) then for each simplex $\sigma^q\in\Phi A$, we have $\phi(l(\sigma^q))\leqslant q+1$. Hence, in the sense of definition 9.1, we have

$$\dim(A, m)\geqslant\phi(\mathbf{1})-1$$

so that, in addition to 10.2 we have (by 3.1):

11.6 LEMMA. *If $\phi(\mathbf{1})\geqslant 4$, then $\Psi A$ and $\Psi_g X$ are simply connected.*

## 12. The order-homology of an izoposet

In [11] and [12], Rota relates the Moebius function of a lattice to its 'order-homology'; and we shall throw light on the relationship by using the uniqueness theorem 7.8. Thus, let $L$ denote an izoposet, and let $OL\subseteq L^\omega$ denote the (closed) subcomplex consisting of all simplices $(u_0, \ldots, u_q)$ such that $u_0<u_1<\ldots<u_q$ in the ordering of $L$. The homology groups of $OL$ constitute the 'order-homology' of $L$ (see also Pretzell [10]). Note that simplices of the form $\mathbf{0}<u_1<\ldots<\mathbf{1}$ are allowed, the 'chains of $L$ stretched from $\mathbf{0}$ to $\mathbf{1}$' used in [11] p. 346. If on the other hand we do not allow such simplices, but only those with either $u_0\neq\mathbf{0}$ or $u_q\neq\mathbf{1}$, we obtain a closed subcomplex $M\subseteq OL$; and if we insist that both $u_0\neq\mathbf{0}$, and $u_q\neq\mathbf{1}$, we obtain a subcomplex $oL\subseteq M$ of which $M$ is the suspension, $SoL$. Now $OL$ is a cone with vertex $\mathbf{1}$ (or vertex $\mathbf{0}$) so it is acyclic. Thus it is the homology of $oL$ or of $SoL$ which is of interest; and Pretzell works with $oL$, which is valid for posets without $\mathbf{0}$ or $\mathbf{1}$. But it is well-known that $SoL$ is connected, while the inclusions induce isomorphisms

12.1 $$H_q(oL)\approx H_{q+1}(SoL)\quad(q\geqslant 0).$$

Suppose further that $L$ has atoms that add (i.e. $\mathbf{L}_5$ holds). Let $L_0=L-\{\mathbf{0}\}$. Then by 11.2 we have convex functions $l\colon L_0^\omega\to L_0$, $m\colon \Phi L_0\to L_0$ given by

$$l(u_0, \ldots, u_q)=u_0\vee u_1\vee\ldots\vee u_q=m\{u_0, \ldots, u_q\},$$

whence $L_0|l$ consists of all simplices $\sigma$ of $L_0^\omega$ such that $l(\sigma)<\mathbf{1}$, and similarly for $\Psi L_0=\Phi L_0-m^{-1}(\mathbf{1})$. As usual let $A=L_*$. We already know by 5.1 and 5.2 that (using $l$ also to denote its restrictions)

12.2 $$H_*(\Psi_m A)=H_*(A|l)\approx H_*(L_0|l)=H_*(\Psi_m L_0)$$

and now we prove the following result, *which needs none of the conditions* $\mathbf{L}_1-\mathbf{L}_4$.

12.3 THEOREM. *Suppose the izoposet $L$ is filtered, and atoms add (i.e. $\mathbf{L}_5$ holds). Then there exist natural isomorphisms $H_q(oL) \approx H_q(L_0|l)$, $q \geq 0$.*

*Proof.* If $\varphi(\mathbf{1}) = 2$, then the proof of 10.1 showed that $L_0|l$ is a disjoint union of acyclic subcomplexes $K_a$, $a \in L_*$; in this case, too, $oL$ consists solely of the vertices $a$, $a \in L_*$. Hence the inclusion $oL \subseteq L_0|l$ certainly induces isomorphisms of homology in each dimension.

We now use induction on $\phi(\mathbf{1})$, because for each $w \in L$, $w.L$ is an izoposet, filtered by $\phi|w.L$.

We apply Theorem 7.8, and begin by defining the complex $C_\lambda$ to be $\lambda.oL$, so that $D_\lambda$ in 7.7(i) becomes

$$D_\lambda = o(\lambda.L), \quad (\lambda \in L).$$

Then conditions 6.1 (i) and 6.1 (ii) need to be verified. This is done by noting first that since $\lambda$ is a vertex of $C_\lambda$, then $C_\lambda$ is always a cone with vertex $\lambda$. Obviously $C_\lambda \cap C_\mu = C_{\lambda \cap \mu}$, and $C_\lambda \subseteq \lambda.(X|f)$. Thus condition 6.1 (i) holds, while 6.1 (ii) holds by Lemma 8.3. Note that in the notation of 7.7 (ii), $D = oL$. Also $\lambda.L_0 = (\lambda.L)_0$, and in $\lambda.L$, $\phi(\mathbf{1}_{\lambda.L}) = \phi(\lambda) < \phi(\mathbf{1})$. Thus the statement of 12.3 with $\phi(\mathbf{1}) = n$ yields the hypothesis of 7.8, whose conclusion then yields the case $\phi(\mathbf{1}) = n+1$ of 12.3. This completes the proof of 12.3, by induction.

For brevity we write $\Psi L$ for $\Psi_m L_0$. Thus by 12.2 we obtain

12.4 COROLLARY. $H_q(oL) \approx H_q(\Psi L)$, $q \geq 0$.

## 13. The homotopy type of $\Psi L$

In this section, let $L$ be an izoposet satisfying $\mathbf{L}_2$ and $\mathbf{L}_3$, (so $L$ is a lattice), with $A$ as its set $L_*$ of atoms. Thus, in the notation of 12.4, $\Psi_m A \subseteq \Psi L$; we now write $\Psi A$, dropping the subscript $m$. The main result of the section is Theorem 13.2, which shows that $\Psi A$ is rather lika wedge of spheres contained within it. We first describe the sort of wedge that concerns us.

Suppose that we have in $A^\omega$ a set $\{\sigma_\alpha\}_{\alpha \in Q}$ of $q$-simplexes, for which there is an $r$-simplex $\tau (r < q)$ such that $\sigma_\alpha \cap \sigma_\beta \supseteq \tau$ for all $\alpha$, $\beta$ in $Q$ (perhaps $Q$ is empty). Then we call $\tau \cup \bigcup_{\alpha \in Q} \sigma_\alpha$ a '$\tau$-wedge' $W$ in $A^\omega$. We set $\partial W = \bigcup_{\alpha \in Q} \partial \sigma_\alpha$, where $\partial \sigma$ denotes the subcomplex of $A^\omega$ consisting of $\tau$ and all boundary simplexes of $\sigma$. We call $W$ and $\partial W$ 'degenerate' if and only if $Q = \emptyset$; and then $W = \partial W = \tau$. Thus $\partial W$ is a union of $\tau$ with some $(q-1)$-spheres, all having $\tau$ in common; and any two intersecting in a simplex. We call $\partial W$ a $\tau$-wedge (or 'wedge') of $(q-1)$-spheres.

If $v$ is a vertex of $A^\omega$, not in $W$, we may form the cone $vW$ on $W$ with vertex $v$; thus

$vW = \bigcup_{\alpha \in Q} v\sigma_\alpha \subseteq A^\omega$, and $vW$ is a $v\tau$-wedge. Also $\partial vW$ is a $w\tau$-wedge of $q$-spheres $S_\alpha = \partial v\sigma_\alpha \cup \sigma_\alpha$.

Now each group $H_{q-1}(\partial\sigma_\alpha)$ is infinite cyclic (even when $q=1$ since we use reduced homology), generated by $\xi_\alpha$ say; whence

13.1 *$H_{q-1}(\partial W)$ is free abelian, freely generated by the*(1) *family* $\{\xi_\alpha\}_{\alpha \in Q} (q \geqslant 1)$.

Therefore $H_q(\partial vW)$ is free abelian, freely generated by the images $v\xi_\alpha$ of $H_q S_\alpha$ under the inclusion homomorphisms $H_q S_\alpha \to H_q(\partial vW)$.

As a first example, consider Lemma 10.1 again, when $X = A$ (so $L_* \cap fX^\omega$ there, is now $A$ also.) Recalling our conventions concerning diagram 11.2, so that $A$ is well-ordered, let $c$ denote the first element of $A$, and for each $a \in A - c$, let $\sigma_a$ denote the 1-simplex $\sigma_a = \{c, a\}$ of $\Phi A$. Then $W = c \cup \bigcup_{a \in A-c} \sigma_a$ is a $c$-wedge, (degenerate if and only if $A = \{c\}$) and $\partial W = A \subseteq \Psi_m A \subseteq \Psi L$. Since $\phi(1) = 2$ in $L$ then $c \vee a = 1$, so $\sigma_a \in E_m$; therefore $\Psi_m A = A = \partial W$. Moreover, the proof of Lemma 10.1 showed that $A|l$ consisted of mutually disjoint cones $K_a$, $a \in A$. Therefore the inclusion $\partial W \subseteq A|l$ is a homotopy equivalence, and $H_0(\partial W)$ is freely generated by the generators $\xi_a (a \in A - c)$ of 13.1. Note that $\partial W$ is a wedge of zero-spheres, degenerate if and only if $A = \{c\}$. This example leads us to formulate:

13.2 THEOREM. *If $\phi(1) = n+2$ $(n \geqslant 0)$ then $\Psi L$ contains a $c$-wedge $V \subseteq \Psi A$, of $n$-spheres such that the inclusion induces isomorphisms of homology groups in all dimensions. If $n > 0$, the wedge is non-degenerate if an only if for some co-atom $\alpha$ of $\lambda.L$, $c \notin \alpha$ and $H_{n-1}\Psi(\alpha.L)_* \neq 0$.*

*Proof.* With $A$ well-ordered as before, we pick from each non-zero $\alpha \in L$ its first atom $c(\alpha) = c_\alpha \in \alpha$. Thus $c(1) = c$ above; and if $\alpha \in A$ then $c_\alpha = \alpha$.

If $\sigma = (v_0, \ldots, v_q) \in A^\omega$ we call $\sigma$ *full* if each $v_i$ precedes $v_{i+1}$ in the well-ordering of $A$, while $c_{l(\sigma)} \in \sigma$ and $\phi(l(\sigma)) = 1 + \dim \sigma$. Then $\sigma$ is *very full* if each (non-empty) face $\tau \subseteq \sigma$ is full (so $\tau$ is also very full). Thus every 0-simplex is very full, and no full simplex $\sigma$ has a repeated vertex. In order to apply Theorem 7.8, we now define subcomplexes $C_\lambda \subseteq A^\omega$ by

$$C_\lambda = \{\sigma \mid \sigma \text{ is very full and } l(\sigma) \leqslant \lambda\}.$$

Since every face of $\sigma \in C_\lambda$ is very full, then $C_\lambda$ is a (closed) subcomplex of $A^\omega$; $C_0$ is empty, and if $\lambda < 1$, then $C_\lambda \subseteq \lambda.(A|l)$. Moreover, $C_\lambda \cap C_\mu = C_{\lambda \cap \mu}$.

Further each $C_\lambda$ is acyclic $(\lambda \neq 0)$. For, if $\sigma \in C_\lambda$ then $l(\sigma) \leqslant \lambda$ so $c_\lambda$, being the first atom in $\lambda$, must precede or equal $c_{l(\sigma)}$. Thus either $c_\lambda \in \sigma$, or $c_\lambda \notin \sigma \subseteq c_\lambda \sigma = \tau \in A^\omega$. But then $l(\tau) \leqslant \lambda$

(1) This terminology is to imply that if the family is empty, then the group is zero.

and, $\tau$ being ordered correctly, it suffices to prove that $\tau$ is very full, to show that $C_\lambda$ is in fact a cone with vertex $c_\lambda$. Now, by $\mathbf{L}_2$ and $\mathbf{L}_3$,

$$\phi(l(\tau)) = \phi(c_\lambda \vee l(\sigma)) = 1 + \phi(l(\sigma)) = 2 + \dim \sigma = 1 + \dim \tau;$$

while, if $l(\tau) = \gamma$, then $c_\lambda \in \gamma \leqslant \lambda$ so $c_\lambda$, being the first atom in $\lambda$, must also be the first in $\gamma$. Hence $c_\lambda = c_{l(\tau)} \in \tau$. Therefore, $\tau$ is full. Of its faces, those in $\sigma$ are very full, like $\sigma$, while those of the form $c_\lambda \varrho$ with $\varrho \subseteq \sigma$, are full by the same argument as for $\tau$. Hence $\tau$ is very full and therefore $\tau \in C_\lambda$ as required, so $C_\lambda$ is acyclic.

These considerations verify condition 6.1(i), while 6.1(ii) now holds by Lemma 8.3. Before we can apply 7.8, however, we need to know the homology of the sets $D$, $D_\mu$ of 7.7. In the case at hand, $D$ consists of all very full $\sigma$ with $l(\sigma) \leqslant \lambda$ for some $\lambda \in L^*$; so $\phi(l(\sigma)) \leqslant n + 1 = \phi(\mathbf{1}) - 1$ and $\dim \sigma \leqslant n$. Similarly if $\mu \in L$, then $D_\mu \subseteq C_\mu$ and consists of all very full $\sigma$ with $l(\sigma) < \mu$, so $\phi(l(\sigma)) \leqslant \phi(\mu) - 1$, and $\dim \sigma \leqslant \phi(\mu) - 2$. Therefore if $\phi(\mu) = 2$, then $D_\mu$ consists of 0-simplices, the atoms of $\mu.L$; so $H_* D_\mu \approx H_*(\mu A \,|\, \mu l)$ in this case. Hence by an inductive hypothesis and 7.8,

$$H_* D_\lambda \approx H_*(\lambda A \,|\, \lambda l), \quad 0 < \lambda \in L;$$

(and when $\lambda = \mathbf{1}$, $D_\lambda = D$).

To get at the wedge $S_\lambda$ required in the statement of the theorem, define

$$V_\lambda = \{\sigma \in C_\lambda \,|\, l(\sigma) = \lambda\} \subseteq C_\lambda \subseteq \lambda.A^\omega;$$

then $V_\lambda$ is a (possibly degenerate) $c_\lambda$-wedge since $c_\lambda = c_{l(\sigma)} \in \sigma$. The vertices of each $\sigma$ in $V_\lambda$ lie in $(\lambda.L)_*$, so $V_\lambda$ is bijective with the wedge $T_\lambda = s V_\lambda \subseteq \Psi(\lambda.L)_*$, where $s\colon A^\omega \to \Phi A$ is the function defined in 11.2 which ignores the ordering of $\sigma \in A^\omega$. If $\sigma(\lambda) = 2$, this definition agrees with that of $W$, in the example prior to 13.2.

Now let $W_\lambda$ denote the subcomplex of $C_\lambda$, that consists of the simplices of $V_\lambda$ together with all their faces; then if $S_\lambda = s(\partial W_\lambda)$, we have $S_\lambda = \partial T_\lambda$. The case when $\phi(\lambda) = 2$ having already been considered, we assume $\phi(\lambda) > 2$. But then, by the last isomorphism and 10.2, each (reduced) group $H_q D_\lambda$ is zero unless $q = \phi(\lambda) - 2$, the dimension of $D_\lambda$. Therefore the homology of $D_\lambda$ (as an ordered complex )depends only on its $(\phi(\lambda) - 2)$-simplices, and these consist of those in $\partial W_\lambda$ together with a set $R_\lambda$ of others. Now, when we earlier proved $C_\lambda$ to be acyclic, we saw that if $\sigma \in C_\lambda$ then either $c_\lambda \in \sigma$ or $\sigma \subseteq c_\lambda \sigma \in C_\lambda$. Therefore, if also $\sigma \in R_\lambda$, then $c_\lambda \in \sigma$ since $\sigma$ is not a face of any $\tau \in V_\lambda$. Hence the simplices of $R_\lambda$, together with their faces, form an acyclic subcomplex of $D_\lambda$, so the inclusion $\partial W_\lambda \subseteq D_\lambda$ (of ordered complexes) induces isomorphisms $H_q(\partial W_\lambda) \approx H_q D_\lambda$ in all dimensions.

Since $\partial W_\lambda$ is $\partial S_\lambda$ together with a particular ordering, we have shown

$$H_*(\partial S_\lambda) = H_q(\partial W_\lambda) \approx H_q(\lambda A\,|\,\lambda l) = H_q\Psi(\lambda.L)_*$$

and it remains to establish the final sentence of 13.2. But if $\sigma(\lambda) > 2$ and $S_\lambda$ is non-degenerate, then $V_\lambda \neq \varnothing$, so there exists $\sigma \in V_\lambda$ and $c_\lambda = c_{l(\sigma)} \in \sigma$; and the order in $\sigma$ is induced from $A$, so $\sigma = c_\lambda \tau$ where $\tau$ is very full. Since $c_\lambda \notin \tau$ and $l(\tau) \leqslant \lambda$, then $c_\lambda \notin l(\tau)$ by the 'first' property of $c_\lambda$; so $l(\sigma) = \lambda = c_\lambda \vee l(\tau) \succ l(\tau)$ by $\mathbf{L}_3$. Thus $\tau \in V_{l(\tau)}$ where $\alpha = l(\tau)$ is a co-atom of $\lambda$ not containing $c_\lambda$. Hence $T_\alpha = sV_\alpha \neq \varnothing$, so $H_q\Psi(\alpha.L)_* \neq 0$ when $q = \phi(\alpha) - 2 = \phi(\lambda) - 3 \geqslant 0$. The argument reverses. Note that it allows us to write, *provided* $W_\lambda$ *is non-degenerate:*

$$W_\lambda = \bigcup\{c_\lambda W_\alpha\,|\,c_\lambda \notin \alpha \prec \lambda\}.$$

This completes the proof of Theorem 13.2.

We write $W_\lambda = W$, $T_\lambda = T$ when $\lambda = \mathbf{1}$. Then:

13.3 COROLLARY. *If $\phi(\mathbf{1}) > 2$, the inclusions $\partial W \subseteq A\,|\,l$, $\partial T \subseteq \Psi L$ induce isomorphisms of homotopy groups.*

If $\phi(\mathbf{1}) \geqslant 4$, $\partial W$ and $A\,|\,l$ are connected and simply connected by 11.6, so the result follows from homotopy theory (see Hu [5] p. 167) and Theorem 13.2. But if $\phi(\mathbf{1}) = 3$, $A\,|\,l$ is connected and its fundamental group is the edgepath group $G$ of its 2-skeleton $B$. Now $(a, b, c)$ is a 2-simplex of $B$ if and only if $l(a, b, c) = a \vee b \vee c \neq \mathbf{1}$; therefore *either* at least two vertices coincide, *or* $a \vee b = b \vee c = c \vee a$. In either case, then, if any two simplicial loops on the 1-skeleton $B^1$ of $B$ are equivalent (in the sense of $G$), then they are homotopic on $B^1$. Hence $G \approx \pi_1(B^1)$, so $\pi_1(A\,|\,l)$ is free on the same set of generators as $H_1(A\,|\,l)$. Therefore since $\partial W \subseteq A\,|\,l$ induces $H_1(\partial W) \approx H_1(A\,|\,l)$, it induces $\pi_1(\partial W) \approx \pi_1(A\,|\,l)$ as required. A similar argument establishes that $\pi_1(\partial T) \approx \pi_1(\Psi L)$ completing the proof.

13.4 COROLLARY. *If $L$ is finite, then the inclusions $\partial T \subseteq \Psi A \subseteq \Psi L$ are homotopy equivalences.*

(For then $\partial T$ and $\Psi L$ are finite, and a theorem of J. H. C. Whitehead (see Hilton [6] p. 107) may be used with Corollary 13.3). This shows the advantage of $\Psi A$ over $A\,|\,l$, since $A\,|\,l$ is rarely even locally finite.

Further corollaries yield information about the atoms of $L$.

13.5 COROLLARY. *If $\phi(\mathbf{1}) = n + 2 \geqslant 2$, then $H_n\Psi A \neq 0$ if and only if $\mathbf{1}$ is a join of atoms.*

For, by 13.2, if $H_n\Psi A \neq 0$ then $W$ contains at least one $(n+1)$-simplex $\sigma$ and $l(\sigma) = \mathbf{1}$, since $\sigma$ is full; thus $\mathbf{1}$ is the join of the atoms in $\sigma$. Conversely suppose $\mathbf{1} = v_0 \vee v_1 \vee \dots \vee v_{n+1}$, where $n \geqslant 0$ and the $v$'s are atoms. Let $\alpha(i) = v_i \vee \dots \vee v_{n+1}$. We can always well-order $A$ so that $v_0 = c_1$, and $v_i = c_{\alpha(i)}$. Then the wedge $W_{\alpha(n)}$ is non-degenerate by the example following 13.1; so by 13.2 the wedge $W_{\alpha(n-j)}$ is non-degenerate for each $j$, by induction on $j$. In particular $W_{\alpha(0)}$ is non-degenerate, so $H_n\Psi A \approx H_n(\partial W_{\alpha(0)}) \neq 0$, and the corollary is proved.

Let us say that $\alpha \in L - \{0\}$ is *without complementary atoms* provided that there exists $\beta \prec \alpha$ such that for no atom $x \in L$ is $x \vee \beta = \alpha$ (so $(\alpha.L)_* = (\beta.L)_*$).

13.6 COROLLARY. *If $H_n\Psi A = 0$ then some $\alpha$ in $L$ is without complementary atoms.*

For, co-atoms exist in each izoposet $\alpha.L$ by the strict monotonicity of $\phi$. Hence if no element of $L - \{0\}$ is without complementary atoms, $\mathbf{1}$ is a join $x \vee a$, $a \in L^*$. Hence by induction, $\mathbf{1}$ is a join of atoms. The corollary follows therefore from 13.5.

In particular when $L = \text{Flat}\ (P^n)$ and $P^n$ is the geometry over a finite field, then (using 13.4) $\Psi P^n$ has the homotopy type of a non-degenerate wedge of $(n-1)$-spheres; we analyse this further in the next section.

13.7 As another example, let $\Gamma$ be a finite *geometric lattice* of rank $n+2$. Thus (see Rota [11]) every element is a join of atoms and $\Gamma$ is filtered by the rank function which is strictly monotonic and satisfies $r(x \vee y) + r(x \cap y) \leqslant r(x) + r(y)$, $r(\text{atom}) = 1$. Hence $\mathbf{L}_2$ and $\mathbf{L}_3$ both hold in $\Gamma$, while atoms add (Condition $\mathbf{L}_5$). Therefore 13.5 applies to $\Gamma$, so $H_n(\Psi\Gamma) \neq 0$ and $\Psi\Gamma$ has the homotopy type of a non-degenerate wedge of $n$-spheres. The number in the wedge is the $n$th Betti-number $\beta_n$ of $\Gamma$; so the Euler characteristic $\chi = \chi(\Psi\Gamma)$ is $1 + (-)^n\beta_n$ (if $n > 0$). But $\chi - 1$ is the Moebius function of $\Gamma$ (see 16.1) so $\beta_n = (-1)^n\mu$. Compare Folkman [3] Theorem 4.1, and see 17.20 below.

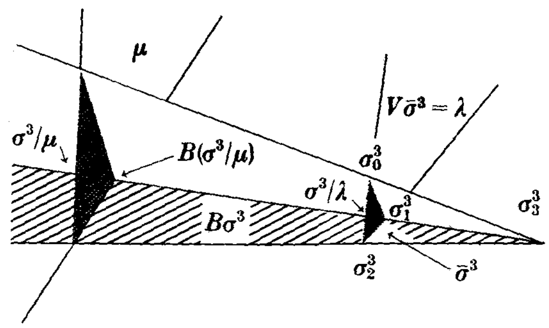
## 14. The case when $L$ is Flat $(P^n)$

When $L$ is Flat $(P^n)$ and $P^n$ is defined over a field $F$, we can specify more precisely the wedge $V = \partial W$ in 13.2.

14.1 THEOREM. *Within the group $GL_{n+1}(F)$ let $T_{n+1}$, $D_{n+1}$ denote the subgroups of triangular and diagonal matrices, respectively. Let $\Delta^n$ denote the $n$-simplex of reference in $P^n$. Then for the wedge $W$ in $\Psi P^n$ we may take the orbit of $\Delta^n$ under the projective transformations corresponding to $T_{n+1}$. Hence the spheres in $\partial W$ correspond biuniquely with the elements of $T_{n+1}/D_{n+1}$.*

Before giving the proof, we observe that, since $F$ is a (possibly non-commutative) field, then each coset in $T_{n+1}/D_{n+1}$ has a unique representative matrix with 1's down the main diagonal. The set of such representative matrices is bijective with $F^{\binom{n+1}{2}}$, and hence it has $q^{\binom{n+1}{2}}$ elements when $F$ is finite with $q$ elements. This result augments that of Corollary 10.4.

The proof of Theorem 14.1, while basically simple, involves a lot of notation, so we explain this in the following paragraphs before giving the proof proper. The reader may find the Figure helpful.




As in section 4, we use $\omega X$ for $X^\omega$. Given a $q$-simplex $\sigma^q \in \omega P^n$, we call $\sigma^q$ *full* if its vertices are distinct and in general position; thus there is a $q$-dimensional subspace $V\sigma^q = \sigma_q^q \vee \ldots \vee \sigma_0^q$ containing the vertices $(\sigma_q^q, \ldots, \sigma_0^q)$ of $\sigma^q$ such that no $r$-dimensional subspace has this property if $r < q$. The set of $(q-1)$-hyperplanes of $V\sigma^q$ will be denoted by $V^*\sigma^q$. We write

14.1 $$\mathfrak{P}\sigma^q = (\sigma_q^q, \sigma_q^q \vee \sigma_{q-1}^q, \ldots, \sigma_q^q \vee \ldots \vee \sigma_1^q)$$

for the ordered family of $q$ subspaces of $V\sigma^q$, and the last term is of special importance, being independent of the last vertex $\sigma_0^q$; we write

$$B\sigma^q = \sigma_q^q \vee \ldots \vee \sigma_1^q \in V^* \sigma^q.$$

Let $\bar{\sigma}^q$ denote the $(q-1)$-simplex $\bar{\sigma}^q = (\sigma_{q-1}^q, \ldots, \sigma_0^q)$, (when $q > 0$). Then $\bar{\sigma}^q$ is also full, and $V\bar{\sigma}^q \in V^*\sigma^q$. Moreover, if $\sigma_q^q \notin \lambda \vee \mu$, where $\lambda, \mu \in V^*\sigma^q$, there is a projective transformation

$$\pi_{\lambda\mu}: V\sigma^q \to V\sigma^q,$$

such that $$\pi_{\lambda\mu}\, \sigma_q^q = \sigma_q^q, \quad \pi_{\lambda\mu}(\lambda) = \mu.$$

Taking $\lambda$ to be $V\bar{\sigma}^q$, let $\sigma^q/\mu$ denote the $(q-1)$-simplex $(v_{q-1}, \ldots, v_0)$ such that $v_i = \pi_{\lambda\mu}(\sigma_i^q)$, $0 \leqslant i < q$. Again $\sigma^q/\mu$ is full, while $V(\sigma^q/\mu) = \mu$; also $\pi_{\lambda\mu}\bar{\sigma}^q = \sigma^q/\mu$, and

$$B\sigma^q \supseteq B(\sigma^q/\mu) = \pi_{\lambda\mu}(B\bar{\sigma}^q), \quad (\lambda = V\bar{\sigma}^q).$$

Note that, if $\nu \in V^*\sigma^q$ and $\sigma_q^q \notin \nu$.

14.2 $$\pi_{\mu\nu}\, \mathfrak{P}(\sigma^q/\mu) = \mathfrak{P}(\sigma^q/\nu),$$

while $\pi_{\mu\nu}$ *leaves* $\mathfrak{P}\sigma^q$ *invariant,* i.e.

14.3 $$\pi_{\mu\nu}(\sigma_q^q \vee \sigma_{q-1}^q \vee \ldots \vee \sigma_i^q) \subseteq \sigma_q^q \vee \ldots \vee \sigma_i^q, \quad 1 \leqslant i \leqslant q.$$

We shall need the following result which follows easily from the definitions, concerning the set of points of the affine space $V\sigma^q - B\sigma^q$:

14.4 Proposition. $V\sigma^q - B\sigma^q = \bigcup \{V(\sigma^q/\mu) - B(\sigma^q/\mu) \mid \mu \in V^*\sigma^q, \sigma_q^q \notin \mu\}$.

The last equation in the proof of Theorem 13.2, suggests an inductive process for associating with $\sigma^q$ a wedge $W(\sigma^q)$ of $q$-simplexes in $\omega(V\sigma^q)$, together with a subsidiary set $Sx(\sigma^q)$ of $(q-1)$-simplexes. Thus when $q=1$, $Sx(\sigma^1)$ is to consist of the single 0-simplex $(\sigma_1^1)$ while we express $W(\sigma^1)$ in the unnecessarily complicated but usefully suggestive manner as:

$$W(\sigma^1) = (\tau y \mid \tau \in Sx(\sigma^1) \quad \text{and} \quad y \in V\sigma^1 - B\sigma^1\}.$$

Observe that $\sigma^1 \in W(\sigma^1)$. If $q > 1$, then we set inductively

$$Sx(\sigma^q) = \{\sigma_q^q \tau \mid \tau \in Sx(\sigma^q/\mu) \quad \text{and} \quad \mu \in V^*\sigma^q, \ \sigma_q^q \notin \mu\},$$

and take $W(\sigma^q)$ to be the cone

$$W(\sigma^q) = \sigma_q^q \bigcup \{W(\sigma^q/\mu) \mid \mu \in V^*\sigma^q, \sigma_q^q \notin \mu\}$$

(which contains $\sigma^q = \sigma_q^q(\sigma^q/\lambda)$ when $\lambda = V^{-q}\sigma$, by an inductive hypothesis and the last observation.) Since $W(\sigma^q/\mu)$ consists of simplices in $\mu$, it follows that $W(\sigma^q/\mu) \cap W(\sigma^q/\nu) = \emptyset$ if $\mu \neq \nu$, so $W(\sigma^q)$ is indeed a non-degenerate wedge. Two results now follow easily by induction:

14.5 Proposition. $W(\sigma^q) = \{\tau y \mid \tau \in Sx(\sigma^q)$ *and* $y \in V\sigma^q - B\sigma^q\}$.

14.6 Proposition. *If* $\tau^q$ *is full and* $\tau_i^q = \sigma_i^q$ *if* $1 \leqslant i \leqslant q$, *then*

$$Sx(\tau^q) = Sx(\sigma^q).$$

Next, let $G(\sigma^q)$ denote the group of projective transformations of $V\sigma^q$ *which leave* $\mathfrak{P}\sigma^q$ (see 14.1) *invariant,*

i.e. $$g(\sigma_q^q \vee \sigma_{q-1}^q \vee \ldots \vee \sigma_i^q) \subseteq \sigma_q^q \vee \sigma_{q-1}^q \vee \ldots \vee \sigma_i^q, \quad 1 \leqslant i \leqslant q.$$

We denote by Stab $(\sigma^q)$ the subgroup of $G(\sigma^q)$ consisting of transformations which leave every vertex of $\sigma^q$ fixed. In terms of $G(\sigma^q)$, Proposition 14.5 will now be shown to have the more precise form:

14.7. Lemma. $W(\sigma^q)$ *is the orbit of* $\sigma^q$ *under* $G(\sigma^q)$. *Hence its simplices correspond biuniquely with* $G(\sigma^q)/\text{Stab}\,(\sigma^q)$.

*Proof.* The invariance of $\mathfrak{P}\sigma^q$ under $g \in G(\sigma^q)$ shows easily by induction on $q$ that

(A) *$g$ fixes all vertices of $\sigma^q$ except $\sigma_0^q$. Hence $g(B\sigma^q) \subseteq B\sigma^q$.*

The next step is to prove

(B) *$g$ preserves the wedge $W(\sigma^q)$, i.e. $gW(\sigma^q) \subseteq W(\sigma^q)$.*

Here, the result is obvious when $q=1$ since $g(\sigma_1^1)=\sigma_1^1$. When $q>1$, we first observe that by (A) above, if $\mu \in V^*\sigma^q$, then both $\sigma^q/g\mu$ and $g\sigma^q/g\mu$ have identical vertices except their last, so by Proposition 14.6,

$$Sx(\sigma^q/g\mu) = Sx(g\sigma^q/g\mu)$$

By Proposition 14.5, $W(\sigma^q)$ consists of $q$-simplices of the form

$$(*) \qquad \varrho = \sigma_q^q \tau y, \quad \{\tau \in Sx(\sigma^q/\mu),\ \mu \in V^*\sigma^q, \text{ and } y \in V\sigma - B\sigma\}, \sigma = \sigma^q.$$

Hence $g\varrho = \sigma_q^q\tau'y'$ where $\tau'=g\tau$ and $y'=gy \in V(g\sigma)-B(g\sigma)$. But since $\tau \in Sx(\sigma^q/\mu)$, then $\tau' \in Sx(g\sigma^q/g\mu)=Sx(\sigma^q/g\mu)$ as seen above; while $g$ leaves $V^*\sigma$, $V\sigma$ and $B\sigma$ invariant. Therefore $g\varrho \in W(\sigma^q)$, and (B) follows.

To complete the proof, it suffices to show that given $\varrho$ as in (*) above, then there exists $g \in G(\sigma^q)$ such that $\varrho = g(\sigma^q)$ (since $\sigma^q \in W(\sigma^q)$ as we saw earlier). When $q=1$, this follows at once by the Fundamental Theorem of Projective Geometry, and we now suppose $q>1$. Consider then a $q$-simplex $\varrho$ of $W(\sigma^q)$ as above, and let $\lambda = V\bar{\sigma}^q$. Then using the notation of 14.2,

$$\varrho' = \pi_{\mu\lambda}\varrho = \sigma_q^q\tau'y', \quad \text{where} \quad \tau' = \pi_{\mu\lambda}\tau,\ y' = \pi_{\mu\lambda}y:$$

and $\tau' \in Sx(\pi_{\lambda\mu}(\sigma^q/\mu)) = Sx\bar{\sigma}^q$ while $y' \in V\bar{\sigma}^q - B\bar{\sigma}^q$. Moreover, by 14.3, $\pi_{\mu\lambda} \in G(\sigma^q)$. But now, by an inductive hypothesis, there exists $h \in G(\bar{\sigma}^q)$ which maps $\bar{\sigma}^q$ onto $\tau'y' \in W(\bar{\sigma}^q)=W(\sigma^q/\lambda)$. Since $h$ is a projective transformation of $\lambda$ which leaves $\mathfrak{P}(\bar{\sigma}^q)$ invariant, $h$ has an extension $k$: $V\sigma^q \to V\sigma^q$ such that $k(\sigma_q^q)=\sigma_q^q$ and hence $k$ leaves $\mathfrak{P}\sigma^q$ invariant. Therefore $k \in G(\sigma^q)$ and $k^{-1}\pi_{\mu\lambda}(\varrho)=k^{-1}(\sigma_q^q\tau'y')=\sigma_q^q g^{-1}(\tau'y')=\sigma_q^q\bar{\sigma}^q=\sigma^q$. Hence $\pi_{\lambda\mu}\circ k$ is the required element $g \in G(\sigma^q)$ which maps $\sigma^q$ to $\varrho$, and the lemma is proved.

The proof of Theorem 14.1 now follows by linear algebra. For, we may regard the projective group as the factor group of $GL_{n+1}(F)$ by its centre $C$, when the points of $P^n$ are the lines through $0 \in F^{n+1}$. We choose a basis $e_0, \ldots, e_n$ of vectors in $F^{n+1}$, where $e_i$ lies on the line corresponding to the vertex $\sigma_i^n$ of the $n$-simplex of reference, $\Delta_n$.

Relative to this basis, the elements of $G(\Delta^n)$ have triangular matrices (mod $C$) because they leave $\mathfrak{P}\Delta^n$ invariant; while the elements of Stab $(\Delta^n)$ are represented by diagonal matrices (mod $C$). Hence in the notation of 14.1 and 14.7

$$G(\Delta^n)/\text{Stab}\ (\Delta^n) \approx T_{n+1}/D_{n+1}.$$

and the proof of Theorem 14.1 is complete.

## 15. Extension to lattices of sub-modules

In the introduction to this paper, we mentioned the problem of describing the complex $\Psi'(M)$ associated with the lattice of free factors of a finitely generated free abelian group $M$. Now $M$ is a $\mathbf{Z}$-module, so the problem can be generalised to replace $\mathbf{Z}$ by a (unitary) ring $R$, and $M$ by a free $R$-module. Just how general $R$ can be, we do not know, and merely choose a situation in this section, where the theory works, and which applies to the original case when $R=\mathbf{Z}$. It would be interesting to see a theory for a regular ring $R$, in the light of the work of von Neumann and others on continuous projective geometries (see Skornyakov [13]).

For our purposes here, let $R$ denote a ring with unit, let $K$ be a (left) $R$-module, and let $M$ be a sub-module of $K$. We call $M$ 'saturated' provided

$$rk \in M \Rightarrow k \in M$$

whenever $0 \neq r \in R$, and $k \in K$. Then the family $\mathcal{S}(K)$ of saturated submodules of $K$ forms an izoposet under intersection, with $\mathbf{0}=0$, and $\mathbf{1}=K$. Observe that if $K$ is a free, finitely generated abelian group and $R=\mathbf{Z}$, then $M$ is saturated if and only if $K/M$ has no divisors of zero, and then $M$ is a free factor of $K$.

Next let $F$ be a ring with left inverses (i.e. a field), such that $R \subseteq F$. For each $n \geqslant 1$, we regard $R^n$ as a (left) $R$-submodule of $F^n$ (*qua* $R$-module). Let $\mathcal{L}(F^n)$ denote the lattice of all (left) subspaces of $F^n$; since $F$ has left inverses then each $A \in \mathcal{L}(F^n)$ is saturated. Also, $A \cap R^n$ is an $R$-saturated $R$-submodule of $R^n$. Moreover we may saturate each submodule $B \subseteq R^n$, by forming $\sigma B \in \mathcal{L}(F^n)$, the subspace of $F^n$ spanned by $B$; thus $B \subseteq \sigma B$. It here suffices to consider the case when the following condition holds:

*Condition* **A**. *For each* $A \in \mathcal{L}(F^n)$, $\sigma(A \cap R^n)=A$ $(n>1)$.

This condition implies that $A$ intersects $R^n$ non-trivially, unless $A=0$, so $R^n$ is 'dense' in $F^n$. More precisely, the relation between $R$ and $F$ is given by

*Condition* **B**. *Given* $s \in F$, *there exists* $r \in R$ *with* $0 \neq r$ *and* $rs \in R$.

15.1 PROPOSITION. *Conditions* **A** *and* **B** *are equivalent.*

*Proof.* If Condition **A** holds, and $s \in F$ is given then we take $A$ to be the submodule of $F^n$ generated by the vector $v=(s, 1, \ldots, 1)$; thus $A \cap R^n \neq 0$ so there exists $t \in F$ such that $tv \in R^n$. Checking coordinates we have $ts \in R$ and (since $n>1$), $t \in R$. Thus Condition **B** holds.

If Condition **B** holds, we first observe that for any $v=(v_1, \ldots, v_n) \in F^n$, there exists $r \in R$ such that $r \neq 0$ and $rv \in R^n$; for this is Condition **B** when $n=1$, and if $(rv_1, \ldots, rv_{n-1}) \in R^n$, then there exists $t \in R$ with $t \neq 0$ and $t(rv_n) \in R$, so $(tr)v \in R^n$ as required since $tr \neq 0$ by the existence of left inverses in $F$.

Now let $A \in \mathcal{L}(F^n)$ be given. Obviously $\sigma(A \cap R^n) \subseteq A$, so we suppose there exists $v \in A - \sigma(A \cap R^n)$. By the observation above, there exists $r \in R$ such that $r \neq 0$ and $rv \in R^n$. But $rv \in A$ also, whence $rv \in A \cap R^n$. Therefore since $r \neq 0$, then $v = r^{-1}(rv) \in \sigma(A \cap R^n)$, a contradiction. This completes the proof.

The convenience of Condition **A** for our 'structural' point of view is shown by the following

15.2 PROPOSITION. *The assignment $A \mapsto A \cap R^n$ is an isomorphism $\theta$: $\mathcal{L}(F^n) \to \mathcal{S}(R^n)$ of izoposets.*

*Proof.* Clearly, $\theta$ preserves $\cap$, $\mathbf{0}$, and $\mathbf{1}$; and by Condition **A**, $\sigma$ is a left-inverse of $\theta$, so $\theta$ is one-one. If $B \in \mathcal{S}(R^n)$ then we assert that $B = (\sigma B) \cap R^n$; for $B \subseteq (\sigma B) \cap R^n = J$, and if $v \in J$, then $v = su$ for some $s \in F$, $u \in B$. By Condition **B**, there exists $r \neq 0$ in $R$ such that $rs \in R$. Then $rv \in J$ so $v$ is an element of $R^n$ with $rv \in B$, whence $v \in B$ since $B$ is saturated. Hence $B = J = \theta(\sigma(B))$ and $\theta$ is onto. This completes the proof.

Now $\mathcal{L}(F^n)$ is isomorphic to Flat $(P^{n-1}(F))$, when $n > 1$. Therefore by 15.2 the entire theory of Flat $(P^{n-1})$ applies to $\mathcal{S}(R^n)$. In particular, when $R = \mathbf{Z}$ and $F$ is the field of rationals, we obtain at once from 13.2 and 14.1 the theorem about the abelian group $M$, stated in the introduction; for Condition **B** is obviously satisfied in this case.

## 16. Some geometrical aspects of the Moebius function

The conclusion of section 11 was that we could replace the study of the function $g$: $\Phi X \to L$ of 4.3 by that of the function $m$: $\Phi A \to L$ when $A = L_*$ and $L$ is an izoposet in which atoms add (Condition $\mathbf{L}_5$).

We shall suppose in this section that $L$ is a finite non-empty lattice, so that the complex $\Psi L$ (see 12.2) is finite. Thus the Euler characteristic $\chi(L)$ of $\Psi L$ is defined, and equals $\chi(\Psi A)$. In the terminology of [11], $A$ is a 'cross-cut' of $L$, whence we obtain from Rota's Theorem 3 the equation:

16.1 $$\chi(L) = 1 + \mu(L)$$

where $\mu$ is the Moebius function $L \times L \to \mathbf{Z}$ defined inductively by

16.2 $$\mu(x, x) = 1, \quad \mu(x, y) = -\sum_{x \leqslant z < y} \mu(x, z),$$

whenever $x \leqslant y$ in $L$; otherwise $\mu(x, y) = 0$. Rota's point of view was essentially 'arithmetic' because of the applications he had in mind, but some of his results have a 'geometric' content which we now discuss.

Recall from 12.2, that $\Psi L = \Phi L - E_m$, a closed subcomplex of the acyclic complex $\Phi L$ (we assume $L \neq \emptyset$). From the exact sequence

$$\ldots \to H_q(\Psi L) \to H_q(\Phi L) \to H_q(\Phi L, \Psi L) \to H_{q-1}(\Psi L) \to \ldots$$

we obtain for the Euler characteristic (with *non-reduced* homology)

$$\chi(\Phi L) = \chi(\Psi L) + \chi(\Phi L, \Psi L) \tag{16.3}$$

using an obvious notation. Since $\Phi L$ is acyclic, then $\chi(\Phi L) = 1$ and $\chi(\Psi L) = \chi(L)$ in the notation of 16.1. Thus 16.3 is Rota's equation 16.1 with

$$-\mu(L) = \chi(\Phi L, \Psi L) = \chi(\Phi L - \Psi L), \tag{16.4}$$

since the RHS is $\chi(\Phi L - \Psi L)$ and the chain groups $C_q(\Phi L, \Psi L)$ have ranks

$$\gamma_q = \phi_q - \psi_q$$

where $\phi_q = \text{rank } C_q(\Phi L)$, $\psi_q = \text{rank } C_q(\Psi L)$. Thus

$$\mu(L) = -\gamma_0 + \gamma_1 - \gamma_2 + \ldots$$

an equation obtained in a different way by Rota (with $\gamma_i = q_{i+1}$ in his notation), to begin his proof of 16.1. Our interpretation of $-\mu$ as the $\chi$ of the 'excluded complex' $E$ explains at once the multiplicative property of $\mu$ (Proposition 5 in Rota [11]). This property is *not* explained so easily if we use 16.1 to regard $-\mu$ as the 'reduced $\chi$' of $\Psi L$ (i.e. computing $\chi$ with reduced homology groups). Further we see by 13.7 why, if $x \leqslant y$ in a finite geometric lattice then $\mu(x, y) \neq 0$; and that if $x \prec y$ then $\mu(z, x)$ and $\mu(z, y)$ are of opposite signs if $z \leqslant x$ because the wedges of spheres in the two cases have dimensions differing by 1 (in each case, $\mu = (-1)^{\dim} \times \text{Betti number}$).

It is interesting to see that $\mu(L)$ can be expressed as in 16.4, but using either the pair $(\Phi A, \Psi A)$,—to work with 'small' complexes—or the order homology explained in section 12. Thus, *we now establish 16.5 below*. Let $F(L)$ denote the exact sequence of the pair $(\Phi L, \Psi L)$ used for 16.3, and let $F(A)$ denote that of $(\Phi A, \Psi A)$. The inclusion $(\Phi A, \Psi A) \subseteq (\Phi L, \Psi L)$ induces a homomorphism $f\colon F(A) \to F(L)$ of which the components corresponding to the inclusions $\Phi A \subseteq \Phi L$, $\Psi A \subseteq \Psi L$ are isomorphisms by acyclicity and 2.9 respectively. Hence by the 'Five' Lemma,

$$H_*(\Phi A, \Psi A) \approx H_*(\Phi L, \Psi L);$$

thus in 16.4 we may write $-\mu(L) = \chi(\Phi A, \Psi A)$.

For the order-homology, we proceed as follows, at least when $L$ is a lattice (so it satisfies $\mathbf{L}_5$). As for 11.2, we may allow the inclusion

$$(\Phi L, \Psi L) \subseteq (L_0^\omega, L_0|l), \quad L_0 = L - \{0\},$$

where $l$ was defined for 11.2. Then $L_0^\omega$ is acyclic, while $L_0|l$ is the total complex of $\Psi L$; hence we have the isomorphisms required by the 'Five' Lemma, when applied to the induced homomorphism $F(L) \to F(L_0)$ to deduce

$$H_*(\Phi L, \Psi L) \approx H_*(L^\omega, L_0|l).$$

Now let $OL$, $oL$ denote the complexes for order-homology, as explained in section 12. Thus $oL \subseteq SoL \subseteq OL$ where $SoL$ denotes the suspension of $oL$. Let $K_0$, $K_1$ denote the closed subcomplexes of $OL$ formed by joining $oL$ to zero and $\mathbf{1}$ respectively: each is a cone and

$$SoL = K_0 \cup K, \quad K_1 \cap K_2 = oL.$$

Then $(K_0, oL) \subseteq (SoL, K_1) \subseteq (L_0^\omega, L_0|l)$, and by the exision axiom of homology theory

$$H_*(K_0, oL) \approx H_*(SoL, K_1).$$

But $K_0$ is a cone and therefore acyclic, while $H_*(oL) \approx H_*(L_0|l)$ by 12.3. Hence by the 'Five' Lemma applied to the homomorphism

$$F(K_0, oL) \to F(L_0^\omega, L_0|l)$$

of homology sequences, we get

$$H_*(K_0, oL) \approx H_*(L_0^\omega, L_0|l).$$

From all these isomorphisms then, we see that $\mu(L)$ can be defined in various ways, by

16.5 $$-\mu(L) = \chi(\Phi L, \Psi L) = \chi(\Phi A, \Psi A) = \chi(SoL, K_1) = \chi(K_0, oL).$$

These arguments tell us something of the geometrical nature of $\mu$, and have implications when $L$ is not finite.

The question arises, whether 16.2 has 'geometrical content': or rather, whether 16.2 is *deducible* if we define $\mu$ by 16.4. (Observe that the set $[x, y] = \{z | x \leqslant z \leqslant y\}$ is a lattice with $\mathbf{0} = x$, and $\mathbf{1} = y$: while $[x, x]$ is an ordered set with $\mathbf{0} = x$ and without atoms, so $\Phi A$ here is $\varnothing$ (not acyclic) and we need a convention in place of 16.4. Thus we set $\mu(x, x) = 1$). Let us now show how to deduce 16.2 from this geometrical definition of $\mu$.

First, choosing an atom $c \in L$, we obtain the 'geometrical' relation:

16.6 $$\mu(L) = -\sum \mu(0, \gamma_i)$$

*the sum over all atomic complements* $\gamma \in L^\perp(c) - L^*(c)$ *in the notation of* 6.5.

*Proof.* Taking Euler characteristics of the sequence 7.3 we get (in that notation but taking $\lambda = \gamma_1$, $X = A$, $f = l$):

$$\chi(K_{i+1}) = \chi(K_i) + \chi(J_i) - \mu(\gamma_i A \mid \gamma_i l).$$

But $\chi(J_i) = 1$, so we subtract 1 from each term and use induction on $i$ to get

$$\mu(L) = -\sum \mu(\gamma_i A \mid \gamma_i l) = -\sum \mu(0, \gamma_i)$$

as required, since $K_0$ is acyclic with $\mu(K_0) = 0$. [Note: 16.6 is the especially simple case of Weisner's Theorem, discussed in Wilson [15] § 4.]

It appears that 16.6 is the 'geometrical' part of 16.2, the remaining terms summing to zero by a 'book-keeping' argument. Thus, let us suppose that we have established 16.2 for all filtered lattices $L$ whose '$\mathbf{1}$' has $\phi(\mathbf{1}) < n$. If $\phi(\mathbf{1}) = 1$, then $L$ consists only of $\mathbf{0}$ and $\mathbf{1}$: so $A = \{\mathbf{1}\} = \Phi A$, $\Psi L = \emptyset$ and $\chi L = 0$; therefore by 16.1, $\mu(L) = -1$ and 16.2 holds. For larger $n$, we are trying to prove that, if $\phi(\mathbf{1}) = n$ in $L$, then

16.7 $$\mu(\mathbf{0}, \mathbf{1}) + \Sigma_c \mu(\mathbf{0}, \gamma) + \Sigma' \mu(\mathbf{0}, x) = 0$$

where $\Sigma_c$ is summed over all $\gamma \in L^{\perp}(c) - L^*(c)$ for a fixed atom $c$ and $\Sigma'$ is summed over the set $V$ of all remaining $x \neq \mathbf{1}$ in $L$. By 16.6, it suffices then to prove $\Sigma' = 0$. Now, all intervals $[\mathbf{0}, v]$ in $V$ are lattices; hence $\mu$ on $[\mathbf{0}, v]$ satisfies 16.6. Thus for any atom $a \in [\mathbf{0}, v]$, $\sum_x \mu(\mathbf{0}, x) = 0$, where the sum is taken over all atomic complements $x \in [u, v]$ such that $x \vee a = v$. We therefore take $a = c$, and $v$ to be a coatom in $L^*(c)$ to see that the sum $\Sigma'$ is equal to a sum over $x$ in a smaller set $V_1 \subseteq V$, with $v \notin V_1$. By proceeding in this way (varying the choice of atom $a$) we finally obtain $\Sigma' = 0$ and hence 16.2 follows by induction.

This justifies our earlier remark about the 'geometric' content of 16.2 and 16.6.

Our discussion so far in this section has assumed that $L$ is a finite lattice, whereas the Moebius function $\mu(x, y)$ is defined in Rota [11] when $x$, $y$ lie in any locally finite ordered set. Hence, in the general case, $\mu(x, y) = \mu(L)$ for the *finite* ordered set $[x, y] = L$. To express $-\mu$ as an Euler characteristic in the above manner we have to find a set $E$ of 'excluded simplices' in the complex $\Phi L$, to construct $\Psi L$ as $\Phi L - E$. Proposition 5.4 tells us how to find $E$: we simply include in $\Psi L$ only those simplices $\sigma$ of $\Phi L$ for which there exists a coatom $\lambda \in L$ such that $v \leqslant \lambda$ for each vertex $v$ of $\sigma$. Thus, equation 16.6 is deducible as before, and hence $-\chi(\Psi L)$ is $\mu(L)$ as deduced previously, after 16.7. With the order-homology of course, there is no problem of definition.

## 17. Some intrinsic conditions for $L$

In this final section we consider some intrinsic conditions that an izoposet $L$ may satisfy, which allow us to assume that the image of $g$: $\Phi X \to L$ in section 4 shall satisfy Conditions $\mathbf{L}_2$, $\mathbf{L}_3$, $\mathbf{L}_5$ above. The choice of intrinsic conditions is imposed on us naturally because of our policy of working inductively 'downwards' from $\mathbf{1}$ through the co-atoms of $L$. Always, $L$ is an izoposet with non-empty sets $L_*$, $L^*$ of atoms and co-atoms. The first two conditions are

$\mathbf{A}_1$: *For all co-atoms $\lambda$, $\mu \in L^*$, if $\lambda \neq \mu$ then*

$$\lambda \cap \mu \in (\lambda . L)^* \cap (\mu . L)^*.$$

$\mathbf{A}_2$: *Given $x \in L_*$ and $\lambda \prec \mu \prec \mathbf{1}$, a unique* lub *$x \vee \lambda \in L$ is defined and $x \vee \lambda \neq \mathbf{1}$.*

(Thus $x \vee \lambda \geqslant x$, $x \vee \lambda \geqslant \lambda$, whence $x \vee \lambda = \lambda$ if $x \in \lambda$; and if also $x \vee \lambda'$ is defined and $\lambda \geqslant \lambda'$, then $x \vee \lambda \geqslant x \vee \lambda'$). Because of the possibility of parallels, condition $\mathbf{A}_2$ is not satisfied by the lines and hyperplanes in the lattice of flats of Euclidean space, if hyperplanes are atoms and $\mathbf{1}$ is the empty flat. The condition implies that atomic complements are co-atoms, by the next lemma; and the later ones (17.2, 17.3) are weak forms of the modular law—not surprisingly since $\mathbf{A}_1$ will hold whenever the dual $L^\dagger$ of $L$ is semi-modular (see 17.19).

17.1 LEMMA. *If $\lambda \in L^*$, $\nu \in (\lambda . L)^*$, and $x \in L_*$, $x \notin \lambda$, then $x \vee \nu \in L^*$.*

*Proof.* Since $\nu \prec \lambda \prec \mathbf{1}$, then by $\mathbf{A}_2$, $\alpha = x \vee \nu$ is defined in $L$ and $\alpha \neq \mathbf{1}$. Hence there exists $\mu \in L^*$ such that $\alpha \leqslant \mu$. Then by the isotone properties of $\cap$ and $\vee$,

$$\nu = \nu \cap \lambda \leqslant \alpha \cap \lambda \leqslant \mu \cap \lambda.$$

Now $\mu \cap \lambda \neq \lambda$, otherwise $x \in \mu \subseteq \lambda$, contrary to $x \notin \lambda$; also by $\mathbf{A}_1$, $\mu \cap \lambda \in (\lambda . L)^*$, and since distinct co-atoms of $\lambda . L$ are not comparable, then $\nu = \mu \cap \lambda$. But again by $\mathbf{A}_1$, $\mu \cap \lambda \in (\mu . L)^*$, so $\nu$ is a co-atom in $\mu . L$. However, $\alpha > \nu$, otherwise $x \in \alpha = \nu \leqslant \lambda$, contrary to hypothesis. But $\alpha \in \mu . L$, so $\alpha = \mu$. Therefore $x \vee \nu \in L^*$, as asserted.

17.2 LEMMA. *With $\nu$ and $x \notin \lambda$, as above, $(x \vee \nu) \cap \lambda = \nu$.*

*Proof.* We have just seen that $\alpha = x \vee \nu \in L^*$. Since $x \in \alpha$ and $x \notin \lambda$, then $\alpha \neq \lambda$. Hence $\alpha \cap \lambda \in (\lambda . L)^*$, by $\mathbf{A}_1$. But $\alpha \cap \lambda \geqslant \nu \cap \lambda = \nu$ (since $\nu \in \lambda . L$), so $\alpha \cap \lambda = \nu$ since distinct co-atoms are not comparable. This completes the proof.

17.3 COROLLARY. $\nu \in (x \vee \nu . L)^*$.

For, by Lemma 17.2, $\nu = (x \vee \nu) \cap \lambda \in (x \vee \nu . L)^*$ by $\mathbf{A}_1$.

17.4 LEMMA. *If $x$ is an atom such that $x \notin \lambda$, $x \in \mu$, and $\lambda$, $\mu \in L^*$ then*

$$x \vee (\lambda \cap \mu) = \mu.$$

*Proof.* Since $x \in \mu$ and $\lambda \cap \mu \leqslant \mu$, then the lub property of $\vee$ implies that $\gamma = x \vee (\lambda \cap \mu) \leqslant \mu$. Thus $\gamma \in \mu . L$. Also, by $\mathbf{A}_1$,

$$\mu \geqslant \gamma \geqslant \lambda \cap \mu \in (\mu.L)^*.$$

But $x \in \gamma$, and $x \notin \lambda \cap \mu$, so $\gamma > \lambda \cap \mu$ in $\mu.L$. Therefore $\gamma = \mu$, as required.

For each atom $x \in L_*$, define $L^*(x)$ as in section 5 by

$$L^*(x) = \{\mu \in L^* \mid x \in \mu\}.$$

Suppose $\lambda \in L^* - L^*(x)$; we prove the following lemma, whose interpretation in Flat $(P^n)$ is familiar. (It is part of Condition $\mathbf{L}_4$ in section 10).

17.5 LEMMA. *There is a bijection* $\phi\colon L^*(x) \approx (\lambda.L)^*$, *given by*

$$\phi(\mu) = \mu \cap \lambda. \quad (x \notin \lambda \in L^*).$$

*Proof.* By $\mathbf{A}_1$, $\phi$ is a function. It is onto, because each $\nu \in (\lambda.L)^*$ can be written $\nu = (x \vee \nu) \cap \lambda$ by 17.2, and $x \vee \nu \in L(x)$ by 17.1.

To see that $\phi$ is one-one, suppose $\phi(\mu) = \phi(\mu')$, i.e. $\mu \cap \lambda = \mu' \cap \lambda$. Hence, $x \vee (\mu \cap \lambda) = x \vee (\mu' \cap \lambda)$; so $\mu = \mu'$, by 17.4. This completes the proof.

Next we attempt to define a filtration on $L$, but we work 'down from $\mathbf{1}$' rather than 'up from zero'. Suppose then that $w \in L$ is such that there exists in $L$ a chain:

17.6 $$w = w_k \prec w_{k-1} \prec \ldots \prec w_1 \prec \mathbf{1}.$$

The least such $k$ will be denoted by $d_L(w)$ with the subscript often omitted: clearly if $w \in L^*$ when $d(w)$ is defined and $d(w) = 1$. We extend the definition of $d$ to all $L$ by setting $d(\mathbf{1}) = 0$, with $d(w) = \infty$ if $d(w)$ is not finite. Thus we have a function $d\colon L \to \Delta = \{0\} \cup \mathbf{N} \cup \{\infty\}$; the corresponding function $d_{a.L}\colon a.L \to \Delta$, for any $a \in L$, will be abbreviated to $d_a$. An easy consequence of the definitions is:

17.7 LEMMA. *If* $a \in L^*$ *and* $d_a(w) < \infty$, *then* $d(w) < \infty$ *and* $d_a(w) \geqslant d(w) - 1$.

To prove a converse (and for other purposes) we now impose an extra condition on $L$.

$\mathbf{A}_1'$. *If* $d(w) < \infty$ *then* $w.L.$ *satisfies* $\mathbf{A}_1$, *and* $(w.L)^* \neq \varnothing$ *if* $w \neq \mathbf{0}$.

17.8. LEMMA. *Suppose that* $L$ *satisfies* $\mathbf{A}_1'$. *If* $a \in L^*$ *and* $d(w) < \infty$ *then* $d_a(w) = d(w) - 1 < \infty$, $(w \leqslant a)$.

*Proof.* Suppose $d(w) = k < \infty$, so that there is a shortest chain as in 17.1 above. We thus have a chain (in $a.L$):

(i) $$w = a \cap w_k \leqslant a \cap w_{k-1} \leqslant \ldots \leqslant a \cap w_1 \leqslant a.$$

If $a = w_1$, there is nothing to prove. If $a \neq w_1$, then by $\mathbf{A}_1$ for $L$, $a \cap w_1 \prec a$, and $a \cap w_1 \in (w_1.L)^*$.

Suppose inductively that $a \cap w_j \prec a \cap w_{j-1} \prec \ldots \prec a \cap w_1 \prec a$ and $a \cap w_j \in (w_j.L)^*$, or $a \cap w_{j-1} = w_j$. Now $d(w_j) < \infty$ since 17.1 gives a finite chain, in $L$ from $w_j$ to $\mathbf{1}$, so by $\mathbf{A}_1'$, $w_j.L$ satisfies $\mathbf{A}_1$ and has co-atoms. Thus, since $a \cap w_j$ and $w_{j+1}$ both lie in $(w_j.L)^*$, then either $a \cap w_j = w_{j+1}$, or

$$a \cap w_{j+1} = (a \cap w_j) \cap w_{j+1} \in (a \cap w_j.L)^* \cap (w_{j+1}.L)^*$$

whence $a \cap w_{j+1} \prec a \cap w_j$ and $a \cap w_{j+1} \in (w_{j+1}.L)^*$. This justifies the inductive supposition, and proves that there is a chain in $a.L$:

$$w = w_k \prec \ldots \prec w_{j+1} \prec a \cap w_{j-1} \prec a \cap w_{j-2} \prec \ldots \prec a \cap w_1 \prec a$$

with $k-1$ terms; if the equality $a \cap w_{j-1} = w_j$ never occurred then in particular we would have both $a \cap w_{k-1}$ and $w_k$ in $(w_{k-1}.L)^*$ with $w_k \leqslant a \cap w_{k-1}$ so $w_k = a \cap w_{k-1}$. Thus we could start the chain with $w_k = a \cap w_{k-1} \prec \ldots$ which is again of length $k-1$, and the proof is complete.

17.9 Corollary. *If $d(a) < \infty$, then $d_a(w) = d(w) - d(a)$, $(w \leqslant a)$.*
(For, $a \in (b.L)^*$ for some $b$ with $d(b) = d(a) - 1$.)

17.10 Corollary. *If $u \geqslant v$ in $L$ then $d(u) \leqslant d(v)$, provided $d(u) < \infty$.*
(For, $v \in u.L$ and $0 \leqslant d_u(v) = d(z) - d(u)$).

17.11. Corollary. *If $a \in L^*$ and $d(\mathbf{0}) = m < \infty$, then $d_a(\mathbf{0}) = m - 1$.*

17.12 Proposition. *Let $L_F$ denote the subset of $L$, consisting of all $x$ of finite depth. If $d(\mathbf{0}) < \infty$ then $L_F$ is a sub-izoposet of $L$.*

*Proof*. Clearly $\mathbf{0}$ and $\mathbf{1}$ lie in $L_F$, so it remains to prove that if $u$, $v \in L_F$, then $d(u \cap v) < \infty$. This is clear, if $d(\mathbf{0}) = 1$ or if

$$d(u) + d(v) = 1.$$

Suppose now that the proposition holds for all izoposets $M$ with $d_M(\mathbf{0}) < d_L(\mathbf{0})$ and for all $u'$, $v'$ in $L$ with $d(u') + d(v') \leqslant n$. If $u$, $v$ in $L$ satisfy $d(u) + d(v) = n$ we may suppose $n > 1$ (remarked above); if $d(u) = d(v) = 1$, then $d(u \cap v) = 2$ by condition $\mathbf{A}_1$. Suppose then that $d(v) \geqslant 2$, so there exists $w \in L$ such that $v \prec w \neq \mathbf{1}$, and $0 < d(w) \leqslant d(v) - 1 < \infty$. Therefore $d(u) + d(w) < n$ so $d(u \cap w) < \infty$. In $w.L$ we have, by Corollary 17.9, that $d_w(\mathbf{0}) = d_L(\mathbf{0}) - d(w) < d_L(\mathbf{0})$ so by the inductive hypothesis applied to $w.L$,

$$d_w((u \cap w) \cap v) < \infty$$

whence $d(u \cap v) = d_w(u \cap v) + d(w) < \infty$ by 17.9 again. This completes the proof, by induction.

Next we strengthen our conditions further by assuming that $L$ satisfies the following condition, which holds in the dual $\Gamma^{\dagger}$ of any finite geometric lattice $\Gamma$(since $\mathbf{A}_1'$ in $\Gamma^{\dagger}$ is the condition of semi-modularity in $\Gamma$):

$\mathbf{A}_3$. *If* $d(w)<\infty$, *then* $w.L$ *satisfies* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ (*with* $(w.L)^*\neq\emptyset$ *if* $w\neq \mathbf{0}$). (Note: $w.L$ has atoms if $L$ has, and $(w.L)_*=w.L_*$.)

17.13 LEMMA. *If* $d(\mathbf{0})=k+1<\infty$, *then* $d(a)=k$ *for each atom* $a\in L_*$.

*Proof.* There is a chain $\mathbf{0}\prec w_{k+1}\prec w_k\prec\ldots\prec w_1\prec\mathbf{1}$ in $L$, where $w_{k+1}=b$ must be an atom. Certainly then, $d(b)\leqslant k$; and we cannot have $d(a)<k$ for any atom $a$, otherwise (since $\mathbf{0}\prec a$) $d(\mathbf{0})\leqslant d(a)+1<k+1$. Hence $d(b)=k$. Now let $a$ be an atom, $a\neq b$. Then there exists a greatest $j$ such that $a\notin w_{j+1}$, $a\in w_j$ ($w_0=\mathbf{1}$).

Thus $w_{j+2}\in(w\tau_{+1}.L)^*$ in $w_j.L$, and $a\in(w_j.L)_*$, so by $\mathbf{L}_3$ and Lemma 17.1 for $w_j.L$, we have $a\vee w_{j+2}\prec w_j$. Also by Lemma 17.3 for $w_j.L$, $w_{j+2}=(a\vee w_{j+2})\cap w_{j+1}\in(a\vee w_{j+2}.L)^*$ by $\mathbf{A}_1$. Now suppose inductively that $a\vee w_{j+p}\prec a\vee w_{j+p-1}\prec\ldots\prec a\vee w_{j+2}\prec w_j$ and $w_{j+p}\prec a\vee w_{j+p}$. Then by Lemmas 17.1 and 17.2 with $x$, $v$, $\lambda$ taken to be $a$, $w_{j+p+1}$, $w_{j+p}$ in $a\vee w_{j+p}.L$ we get $a\vee w_{j+p+1}\prec a\vee w_{j+p}$ and $w_{j+p+1}=(a\vee w_{j+p+1})\cap w_{j+p}\prec a\vee w_{j+p+1}$, justifying the inductive supposition, $1\leqslant p\leqslant k+1-j$. Hence we have a chain

$$a\prec a\vee w_{k+1}\prec\ldots\prec a\vee w_{j+2}\prec w_j\prec\ldots\prec\mathbf{1},$$

so $d(a)\leqslant k$. We observed that $d(a)\not< k$, so $d(a)=k$ as required.

17.14 LEMMA. *If* $d(w)<\infty$, *and* $a\in L_*$, *then* $a\vee w$ *is defined: if* $a\in w$, *then* $a\vee w=w$; *if* $a\notin w$, *then* $w\in(a\vee w.L)^*$, *and* $d(a\vee w)=d(w)-1$.

*Proof.* If $d(w)\leqslant 2$, then the statement holds by $\mathbf{A}_2$ and Corollary 17.4.

Suppose then that $d(w)=k$ so that there is a chain $w\prec w'\prec w''\prec\ldots\prec\mathbf{1}$. Clearly $d(w')\leqslant k-1$, and if $d(w')<k-1$ then $d(w)=d(w')+1<k$. Thus $d(w')=k-1$.

Therefore by an inductive hypothesis, $v=a\vee w'$ is defined, $d(v)=k-2$, and both $a$ and $w'$ lie in $v.L=M$. Moreover (by the hypothesis) $w'\in M^*$. Thus, $a\vee w$ is defined since $M$ satisfies $\mathbf{A}_2$ (for $L$ satisfies $\mathbf{A}_3$). If $a\notin w$, then by Lemma 17.1, $a\vee w\in M^*$, and by Corollary 17.3, $w\in(a\vee w.L)^*$. Also by Corollary 17.4, $d(a\vee w)=d_v(a\vee w)+d(v)=1+(k-2)=d(w)-1$ and the lemma follows by induction.

17.15. LEMMA. *In the set* $L_F$ *of elements of* $L$ *of finite depth, we have* $(w.L_F)^*\cap L_F\neq\emptyset$, *provided* $w\in L_F$ *and* $w.L_F\neq\{w\}$.

*Proof.* Since $w.L_F\neq\{w\}$, there exists $v\in wL_F$ with $v<w$ such that $a\leqslant w$. If $v$ is not a co-atom of $w.L_F$, there exists $u\in w.L_F$ with $v<u<w$. Now by 17.9, $d_w(u)$ exists, so $0<d_w(u)=d(w)-d(u)$, whence $d(v)>d(u)>d(w)$. Therefore we reach a co-atom $y$ of $w.L_F$ from $v$ after at most $d_w(v)-1$ steps, and $d(y)<\infty$ so $(w.L_F)^*\cap L_F\neq\emptyset$.

17.16 COROLLARY OF PROOF. *If* $u < w \in L_F$, *then* $d(w) < d(u)$; *if* $u \prec w$ *then* $d(w) = d(u) + 1$ (because $d_w(u) = 1$).

We can now extend 17.12 to:

17.17 THEOREM. *If* $d(0) < \infty$ *in* $L$, *and* $L$ *satisfies* $\mathbf{A}_3$, *then* $L_F$ *is a sub-izoposet of* $L$ *which has the same atoms as* $L$ *and which also satisfies* $\mathbf{A}_3$.

*Proof.* We know from 17.12 that $L_F$ is a sub-izoposet, while by 17.13, $L_* = (L_F)_*$. Also, by 17.15, if $\mathbf{0} < w \in L_F$, each set $(w.L_F)^*$, taken relative to $L_F$, is non-empty. If $\lambda \neq \mu$ in $(w.L_F)^*$ then by $\mathbf{A}_1$ for $w.L$, $\lambda \cap \mu \in (\lambda L)^* \cap (\mu L)^*$; therefore $\lambda \cap \mu \prec \lambda$, so $\lambda \cap \mu \in L_F$ by 17.16, whence $\lambda \cap \mu \in (\lambda.L_F)^*$. This verifies $\mathbf{A}_1'$ for $L_F$. To show that $w.L_F$ satisfies $\mathbf{A}_2$ we proceed similarly, using 17.14. Therefore $L_F$ satisfies $\mathbf{A}_3$ as required.

Now recall conditions $\mathbf{L}_2 - \mathbf{L}_5$ used in previous sections. Let us denote by $\mathbf{L}_{3\frac{1}{2}}$ the condition $\mathbf{L}_4$ with the clause about cardinals deleted.

17.18 THEOREM. *If* $L$ *is an izoposet satisfying* $\mathbf{A}_3$ *such that every element is of finite depth, then* $L$ *is a lattice and satisfies* $\mathbf{L}_1 - \mathbf{L}_{3\frac{1}{2}}$.

*Proof.* By 17.16, the function $\phi$ is a filtration on $L$, where $\phi(x) = d(0) - d(x)$, and $\phi$ is minimal (i.e. $\mathbf{L}_2$ holds). By 17.14, condition $\mathbf{L}_3$ holds; so $\mathbf{L}_1$ holds and $L$ is a lattice as we observed prior to 10.2. Since 17.5 holds in each set $w.L$, by $\mathbf{A}_3$ and 17.15, then $\mathbf{L}_{3\frac{1}{2}}$ follows. Thiscompletes the proof.

A consequence of this theorem is that we can forget about those mysterious elements of $L$ which have infinite depth, when computing the homology of $\Psi_g X$ in 4.1. For, we saw in section 11, that starting with the function $g\colon \Phi X \to L$ we could pass to $m\colon \Phi A \to L$ in 11.2; and we can change $L$ to the image of $m$, which is generated by atomic sums and hence lies in $L_F$ if $L$ satisfies $\mathbf{A}_3$. Therefore, *we can change from* $L$ *to* $L_F$, *which then satisfies Conditions* $\mathbf{L}_1 - \mathbf{L}_5$.

As a sort of converse of 17.18, we prove the next result which leads to a curious conclusion about geometric lattices (see 13.6).

17.19. THEOREM. *Let* $L$ *be a submodular lattice with* $\mathbf{0}$ *and* $\mathbf{1}$, *satisfying* $\mathbf{L}_2$ *and* $\mathbf{L}_3$. *Then the dual,* $L^\dagger$, *of* $L$ *satisfies* $\mathbf{A}_3$ *and all elements of* $L^\dagger$ *are of finite depth. Further, all maximal chains in* $L^\dagger$ *between fixed end-points are of constant length.*

*Proof.* The strict monotonicity of $\phi$ excludes infinite chains in $L$, so all elements $x \in L$ have finite depth $d(x)$, and $x$ has depth $\phi(\mathbf{1}) - d(x)$ in $L^\dagger$. By $\mathbf{L}_2$, the length of any maximal chain in $L$ (or $L^\dagger$) between $x$ and $y$ is $|\phi(x) - \phi(y)|$. Since there are no infinite chains, co-atoms always exist, so $(xL^\dagger)^* \neq \varnothing$ unless $x=\mathbf{1}$ in $L$. Now $\mathbf{A}_1$ holds in $xL^\dagger$ since it is *precisely* the condition of submodularity in $L$. Therefore $L^\dagger$ satisfies $\mathbf{A}_1'$, and it satisfies $\mathbf{A}_2'$ by $\mathbf{L}_3$. Hence $L^\dagger$ satisfies $\mathbf{A}_3$ as required. The relationship between our condition $\mathbf{A}_3$ and geometric lattices is given by 13.6 and our final

17.20 THEOREM. *Let $L$ be a finite lattice with $\mathbf{0}$ and $\mathbf{1}$ satisfying $\mathbf{A}_3$. If $d(\mathbf{0})=n+2$, and $H_n\Psi L\neq 0$, then the dual of $L$ is a geometric lattice.*

For, $d$ is always finite, so $L$ satisfies $\mathbf{L}_2$ and $\mathbf{L}_3$ by 17.18. Hence, for each $x \in L$, all maximal chains from $\mathbf{0}$ and $x$ in the dual $L^\dagger$ are of constant length, by 17.19. Also are remarked above, $L^\dagger$ is submodular since $\mathbf{A}_1'$ holds in $L$. It remains to verify that $\mathbf{1}^\dagger$ is a join of atoms in $\mathbf{L}^\dagger$. Now, by definition of orderhomology in section 12, $oL=oL^\dagger$ so

$$0 \neq H_n\Psi L \approx H_n(oL) \approx H_n(oL^\dagger) \approx H_n(\Psi L^\dagger)$$

using the hypothesis of the theorem, with 12.2.

By 17.19, $L^\dagger$ satisfies $\mathbf{A}_3$, so $L^\dagger$ is filtered by the depth function of $L$, by 17.18, and $\phi^\dagger(\mathbf{1}^\dagger)=d(\mathbf{0})=n+2$. Therefore by 13.4, $\mathbf{1}^\dagger$ is a join of atoms in $L^\dagger$ as required.

## References


[1]. Crapo, H. & Rota, G. C., *Combinatorial Geometries* (mimeographed), 1968.
[2]. Eilenberg, S. & Steenrod, N. E., *Foundations of algebraic topology*. Princeton, 1952.
[3]. Folkman, J., The homology groups of a lattide. *J. Math. Mech.* 15 (1966), 631–636.
[4]. Greenberg, L., Discrete groups of motions. *Can. J. Math.* 12 (1960), 414–425.
[5]. Hu, S. T., *Homotopy Theory*. Academic Press, 1962.
[6]. Hilton, P. J., *Introduction to Homotopy Theory*. Cambridge, 1953.
[7]. Hilton, P. J. & Wylie, S., *Homology Theory*. Cambridge, 1960.
[8]. Lefschetz, S., *Algebraic Topology*. AMS Colloqu. Publications, XXVII (1955).
[9]. MacLane, S. & Birkhoff, G., *Algebra*. Macmillan, 1967.
[10]. Pretzel, O., A Meyer-Vietoris type sequence in the homology of partially ordered sets. *J. Lond. Math. Soc.*, 44 (1969), 408–413.
[11]. Rota, G. C., On the foundations of combinatorial theory I. Theory of Moebius functions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 2 (1964), 340–368.
[12]. Rota, G. C., Remarks on Moebius functions and the Euler characteristic. *Studies in Pure Mathematics* (Richard Rado Birthday volume) Academic Press, 1971.
[13]. Skornyakov, L. A., *Complemented modular lattices and regular rings*. Oliver and Boyd (1964).
[14]. Wilder, R. L., *Topology of manifolds*. AMS Colloqu. Publications, XXXII (1949).
[15]. Wilson, R. J., The Möbius function in combinatorial mathematics. *Proceedings of the Oxford Conference on Combinatorial Mathematics and its applications*, Academic Press, 1971.



*Received July 17, 1970*

*Received in reviced form February 13, 1972*