DEFORMATIONS OF LIE SUBGROUPS AND THE VARIATION OF ISOTROPY SUBGROUPS

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Introduction

Let the (real or complex) Lie group G act analytically on the connected (real or complex) analytic manifold M. In this paper we shall study the behaviour of the isotropy subgroups G_x as a function of $x \in M$. If $m = \min_{x \in M} \dim G_x$, it is trivial to show that $M_0 =$

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$\{x \in M \mid \dim G_x = m\}$ is a dense open subset of M, the complement of an analytic subset of M. We shall restrict attention to the case in which $M = M_0$, i.e. we assume that all orbits of G on M have the same dimension. This case is well understood if G is a compact connected Lie group (even for the case of continuous G-actions). There exists a dense open subset U of M such that G_x is conjugate to G_y for $x, y \in U$. Moreover, $M - U$ is of codimension $\geqslant 2$ in M and, if $x \in U$ and $y \in (M-U)$, then G_x is conjugate to a subgroup of G_y. Thus, modulo conjugacy, all isotropy subgroups are the same on U and on $(M-U)$ all that happens is that the isotropy subgroups pick up extra components. In particular, the number of components of the isotropy subgroup G_x is an upper semi-continuous function of $x \in M$.

The situation is quite different in the case of non-compact G. For example, there exists an analytic action of $G = \mathrm{SL}_4(\mathbf{R})$ on an analytic manifold M such that all orbits of G on M have codimension 1 and such that, for $x, y \in M$, G_x is non-isomorphic to G_y unless x and y lie on the same orbit under G. Similar examples exist for semi-simple algebraic groups acting algebraically on quasi-affine algebraic varieties. To give another example, consider the irreducible representation of $G = \mathrm{SL}_2(\mathbf{C})$ on $\mathbf{C}^4$ and let $U = \{x \in \mathbf{C}^4 \mid G_x \text{ is finite}\}$; U is a non-empty Zariski-open subset of $\mathbf{C}^4$. There exists a Zariski closed subset A of U such that $G_x = \{e\}$ for $x \in A$ and G_x is of order three if $x \in (U - C)$. Thus, for the action of G on U, the number of components of G_x is a lower semi-continuous function of x but is not upper semi-continuous. For all of these examples, see § 12.

In order to study the behaviour of the isotropy subgroups, we shall study the more general problem of "deformations" of subgroups of a (real or complex) Lie group G. Roughly, an *analytic family of Lie subgroups of G*, parametrized by an analytic manifold M, is an analytic submanifold $\mathbf{H}$ of $G \times M$ such that the projection $\pi_M \colon \mathbf{H} \to M$ is a submersion and, for every $t \in M$, the fibre $\pi_M^{-1}(t)$ is of the form $H_t \times \{t\}$, where H_t is a Lie subgroup of G. Our basic result concerning such analytic families is the following:

THEOREM 3.1. Let $\mathbf{H} = (H_t)_{t \in M}$ *be an analytic family of Lie subgroups of G, let $t_0 \in M$ and let $H = H_{t_0}$. Let K be a Lie subgroup of H such that the component group K/K^0 is finitely generated and such that the Lie group cohomology space $H^1(K, \mathfrak{g}/\mathfrak{h})$ vanishes. Then there exists an open neighborhood U of t_0 in M and an analytic map $\eta \colon U \to G$ with $\eta(t_0) = e$ such that $H_t \supset \eta(t) K \eta(t)^{-1}$ for every $t \in U$.*

Here $\mathfrak{g}$ (resp. $\mathfrak{h}$) denotes the Lie algebra of G (resp. H) and the K-module structure of $\mathfrak{g}/\mathfrak{h}$ is determined by the adjoint representation of K on $\mathfrak{g}$.

Theorem 3.1 generalizes the result of A. Weil [24, p. 152] which states that if Γ is a discrete, finitely generated subgroup of G such that $H^1(\Gamma, \mathfrak{g}) = 0$, then Γ is "rigid". It also generalizes results of the author [17], [19] on deformations of subalgebras of Lie algebras to the case of Lie subgroups. Theorem 3.1 makes sense in the framework of C^∞–manifolds and we conjecture that it holds there. However, our proof relies heavily on the assumption of analyticity. In particular we require a theorem of M. Artin [1] on replacing formal power series solutions of analytic equations by convergent power series solutions.

If, in particular, $K \subset H$ is a reductive Lie group, then $H^1(K, \mathfrak{g}/\mathfrak{h}) = 0$ and Theorem 3.1 applies.

Let G act analytically on M such that all orbits have the same dimension and let $\mathbf{G} = \{(g, t) \in G \times M \mid g \in G_t\}$. In general $\mathbf{G} = (G_t)_{t \in M}$ is not an analytic family of Lie subgroups of G. We say that $(g, t) \in \mathbf{G}$ is a *regular point* of G if there exists a neighborhood U of t in M and an analytic map $s\colon U \to G$ with $s(t) = g$ such that $s(u) \in G_u$ for every $u \in U$. Let $\mathbf{G}^*$ be the set of regular points of $\mathbf{G}$ and let $G_t^* = \{g \in G_t \mid (g, t) \in \mathbf{G}^*\}$. Then G_t^* is a Lie subgroup of G containing the identity component of G_t and $\mathbf{G}^* = (G_t^*)_{t \in M}$ is an analytic family of Lie subgroups of G. Thus Theorem 3.1 can be applied to the analytic family $\mathbf{G}^*$. If all orbits of G on M are locally closed, it follows from a result of J. Glimm [7] that there exists a dense open subset U of M such that $G_t = G_t^*$ for every $t \in U$.

We obtain our sharpest results in the case of algebraic transformation groups (over $\mathbf{C}$). In this case, our main result is

THEOREM 9.3.1. *Let (G, X) be an algebraic transformation space with G an affine algebraic group. For each $t \in X$, let U_t denote the unipotent radical of G_t and let L_t be a Levi subgroup of G_t. Then there exists a finite family $X_1, \dots, X_n$ of non-singular Zariski-locally closed subsets of X such that the following conditions hold:*

(a) $X = \bigcup_{j=1}^{n} X_j$.

(b) *For each j, X_j is a Zariski open subset of $X - (\bigcup_{i=1}^{j-1} X_i)$.*

(c) *If $x, y \in X_j$, then L_x and L_y are conjugate.*

(d) *For each j, the family $(U_t)_{t \in X_j}$ is an algebraic family of algebraic subgroups of G.*

Many of the results of this paper were announced in [20].

§ 0. Preliminaries

0.1. As usual, $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ denote respectively the set of natural numbers, the ring of integers, and the fields of real and complex numbers.

0.2. Throughout this paper, F will denote either $\mathbf{R}$ or $\mathbf{C}$ and, unless stated otherwise, all Lie groups and analytic manifolds will be taken over F. All Lie groups and analytic manifolds are assumed paracompact and Hausdorff. Our basic reference for Lie groups and analytic manifolds is [22] and we shall follow the terminology therein, except that we denote by df_x: $T_x(M)\to T_{f(x)}(N)$ the differential (tangent linear mapping in the terminology of [22]) at $x\in M$ of an analytic map f: $M\to N$ of analytic manifolds. In particular, we shall consistently use the terminology and elementary results mentioned in [22] on immersions, submersions, subimmersions and fibre products.

0.3. The Lie algebra of a Lie group G, H, etc., will be denoted by the corresponding lower case German letter $\mathfrak{g}$, $\mathfrak{h}$, etc. If V is a finite dimensional vector space over F, then $\mathrm{GL}(V)$ denotes the Lie group of automorphisms of V and $\mathrm{End}(V)$ denotes the F-algebra of endomorphisms of V. If G is a Lie group, then e denotes the identity element of G, G^0 the connected component of e in G, and $\exp_G$: $\mathfrak{g}\to G$ and Ad_G: $G\to\mathrm{GL}(\mathfrak{g})$ denote respectively the exponential map of G and the adjoint representation of G. By a linear representation of G, we shall always mean an *analytic* homomorphism $G\to\mathrm{GL}(V)$, where V is *finite-dimensional* over F. Lie subgroups are as defined in [9] (for the real case; we take the analogous definition in the complex case); in particular, a Lie subgroup of G is not necessarily an analytic submanifold of G and is not required to have the topology induced as a subset of G. If the Lie group G acts as a transformation group on a topological space M and if $t\in M$, then $G_t=\{g\in G\mid g\cdot t=t\}$ is the *isotropy subgroup* of G at t and the Lie algebra $\mathfrak{g}_t$ of G_t is the *isotropy subalgebra* of $\mathfrak{g}$ at t.

0.4. An *open box* in F^r is a set of the form $\{(x_1, \ldots, x_r)\in F^r \mid |x_j|<a \text{ for } j=1, \ldots, r\}$, where a is a positive real number. If V is an r-dimensional vector space over F, an open box in V is the image in V of an open box in F^r under a linear isomorphism φ: $F^r\to V$.

Chapter I. Deformations of Lie subgroups

§ 1. Analytic families of Lie subgroups

Many of our definitions in this section are special cases of a more general situation considered by Douady and Lazard [6], that of an analytic family of Lie groups (or, in the terminology of Douady–Lazard, Lie groups over M). In contrast to [6], we require all manifolds to be Hausdorff.

1.1. *Definition.* Let G be a Lie group and M an analytic manifold. An *analytic family of Lie subgroups* of G, parametrized by M, is an analytic manifold $\mathbf{H}$ satisfying the following conditions:

(a) $\mathbf{H}\subset G\times M$ and the inclusion map i: $\mathbf{H}\to G\times M$ is an immersion.

(b) If π_M: $\mathbf{H}\to M$ denotes the composite map $\mathrm{pr}_M\circ i$, where pr_M denotes the projection $G\times M\to M$, then π_M is surjective and is a submersion.

(c) For every $t \in M$, the fibre $\pi_M^{-1}(t)$ is of the form $H_t \times \{t\}$, where H_t is a subgroup of G.

(d) Let $\mathbf{H} \times_M \mathbf{H}$ denote the fibre product of $\mathbf{H}$ with itself over M with respect to π_M and let μ: $\mathbf{H} \times_M \mathbf{H} \to \mathbf{H}$ and j: $\mathbf{H} \to \mathbf{H}$ be defined by $\mu((x, t), (y, t)) = (xy, t)$ and $j(x, t) = (x^{-1}, t)$. Then μ and j are analytic maps.

1.2. *Remarks.* (a) We do not require that $\mathbf{H}$ be an analytic submanifold of $G \times M$. In particular, the topology on $\mathbf{H}$ is not necessarily the induced topology as a subset of $G \times M$.

(b) If $\mathbf{H}$ is an analytic submanifold of $G \times M$, then condition 1.1. (d) is a consequence of 1.1. (a)–(c).

(c) It follows from the definition that each $H_t (t \in M)$ is a Lie subgroup of G. Consequently $\mathbf{H}$ can be considered as a family of Lie subgroups of G, which depend analytically on the parameter $t \in M$. We shall frequently denote $\mathbf{H}$ by $(H_t)_{t \in M}$.

(d) It follows from 1.1. (b) that $t \mapsto \dim H_t$ is constant on each component of M. We shall assume that $t \mapsto \dim H_t$ is constant on all of M.

(e) For $t \in M$, let $c(t)$ be the number of components of H_t. The function $t \mapsto c(t)$ is not necessarily either upper semi-continuous or lower semi-continuous. For examples, see § 12.

(f) A number of non-trivial examples of analytic families of Lie subgroups are given in § 12.

1.3. *Sections.* Let $\mathbf{H} = (H_t)_{t \in M}$ be an analytic family of Lie subgroups of G and let U be an open subset of M. An *analytic section* of $\mathbf{H}$ over U is an analytic map s: $U \to G$ such that (i) $s(t) \in H_t$ for every $t \in U$ and (ii) the map $t \mapsto (s(t), t)$ is an analytic map of U into $\mathbf{H}$. If $\mathbf{H}$ is an analytic submanifold of $G \times N$, then condition (ii) follows from condition (i), but this is not the case if $\mathbf{H}$ is not an analytic submanifold. It follows easily from 1.1. (b) that for every $(x, t) \in \mathbf{H}$, there exists an open neighborhood U of t in M and an analytic section s: $U \to G$ of $\mathbf{H}$ over U such that $s(t) = x$. Moreover, if U is open in M, then the constant map $t \mapsto e$ is an analytic section over U and, if s_1, s_2 are analytic sections over U, then $t \mapsto s_1(t) s_2(t)^{-1}$ is an analytic section over U.

1.4. *The Lie algebra of* $\mathbf{H}$. For each $t \in M$, let $\mathfrak{h}_t$ denote the Lie algebra of H_t and let $\mathfrak{H} = \{(X, t) \in \mathfrak{g} \times M \mid X \in \mathfrak{h}_t\}$. Then $\mathfrak{H}$ is a sub-bundle of the trivial vector bundle $\mathfrak{g} \times M \to M$; $\mathfrak{H}$ is an analytic vector bundle of Lie algebras over M. We shall call $\mathfrak{H}$ the Lie algebra of the analytic family $\mathbf{H}$. Define $\exp_{\mathbf{H}}$: $\mathfrak{H} \to \mathbf{H}$ by $\exp_{\mathbf{H}}(X, t) = (\exp_G X, t)$. It is not difficult to show that $\exp_{\mathbf{H}}$ is an analytic map. If $t_0 \in M$, the differential of $\exp_{\mathbf{H}}$ at $(0, t_0)$ is a linear isomorphism. It follows from the inverse function theorem that $\exp_{\mathbf{H}}$ defines an analytic manifold isomorphism of an open neighborhood of $(0, t_0)$ in $\mathfrak{H}$ onto an open neighborhood of $(e, 0)$ in $\mathbf{H}$.

1.5. *Coordinates on the Grassmann manifold.* Let $\mathbf{H}=(H_t)_{t\in M}$ be as above, let $t_0 \in M$, let $\mathfrak{h}=\mathfrak{h}_{t_0}$ and let $q=\dim \mathfrak{h}$. In order to make computations involving the family $\mathfrak{H}=(\mathfrak{h}_t)_{t\in M}$ of subalgebras of $\mathfrak{g}$, we shall need to introduce "coordinates" on the Grassmann manifold $\Gamma_q(\mathfrak{g})$ of q-dimensional vector subspaces of $\mathfrak{g}$. Let W be a vector subspace of $\mathfrak{g}$ such that $\mathfrak{g}$ is the vector space direct sum of $\mathfrak{h}$ and W and let Γ_W be the open submanifold of $\Gamma_q(\mathfrak{g})$ consisting of all q-dimensional subspaces of $\mathfrak{g}$ whose intersection with W is $\{0\}$. For each $T \in \mathrm{Hom}_F(\mathfrak{h}, W)$, let $\Phi(T)=\{X+T(X) \mid X \in \mathfrak{h}\}$. Then $\Phi(T) \in \Gamma_W$ and Φ: $\mathrm{Hom}_F(\mathfrak{h}, W) \to \Gamma_W$ is an isomorphism of analytic manifolds. If $F=\mathbf{C}$, then Φ is an isomorphism of algebraic varieties. We denote by $\Theta: \Gamma_W \to \mathrm{Hom}_F(\mathfrak{h}, W)$ the inverse isomorphism. Classically, Γ_W is the big Schubert cell on $\Gamma_q(\mathfrak{g})$ and Θ defines Schubert coordinates on Γ_W.

For each $t \in M$, the Lie algebra $\mathfrak{h}_t$ may be considered as a point of $\Gamma_q(\mathfrak{g})$. It follows easily from the definitions that $t \mapsto \mathfrak{h}_t$ is an analytic map of M into $\Gamma_q(\mathfrak{g})$. Choose an open neighborhood U of t_0 in M such that $\mathfrak{h}_t \in \Gamma_W$ for every $t \in U$ and let φ: $U \to \mathrm{Hom}_F(\mathfrak{h}, W)$ be defined by $\varphi(t)=\Theta(\mathfrak{h}_t)$. Then φ is an analytic map.

1.6. *The pull-back of an analytic family.* Let $\mathbf{H}=(H_t)_{t\in M}$ be as above and let f: $N \to M$ be an analytic map of analytic manifolds. We wish to define an analytic family $f^*(\mathbf{H})$ of Lie subgroups of G parametrized by N, the pull-back of $\mathbf{H}$ by f. We set

$$f^*(\mathbf{H})=\{(x, s) \in G \times N \mid x \in H_{f(s)}\}.$$

Let π_N denote the restriction to $f^*(\mathbf{H})$ of the projection $G \times N \to N$. For each $s \in N$, $\pi_N^{-1}(s)= H_{f(s)} \times \{s\}$. In order to define the analytic manifold structure on $f^*(\mathbf{H})$ we consider the analytic maps $1_G \times f$: $G \times N \to G \times M$ and i: $\mathbf{H} \to G \times M$. Since π_M: $\mathbf{H} \to M$ is a submersion, it follows easily that $1_G \times f$ and i are transversal at each point of the fibre product $E= (G \times N) \times_{(G\times M)} \mathbf{H}$. Thus E is a closed analytic submanifold of $(G \times N) \times \mathbf{H}$. One checks easily that the projection $(G \times N) \times \mathbf{H} \to G \times N$ maps E bijectively onto $f^*(\mathbf{H})$. We define an analytic manifold structure on $f^*(\mathbf{H})$ by requiring that this bijection be an isomorphism of analytic manifolds. It is easy to check that $f^*(\mathbf{H})$, with this analytic manifold structure, is an analytic family of Lie subgroups of G.

1.7. *Expansion of an analytic family.* Let $\mathbf{H}=(H_t)_{t\in M}$ be an analytic family of Lie subgroups of G. It is convenient for technical reasons to "expand" the family $\mathbf{H}$ so as to obtain a new family $\tilde{\mathbf{H}}$ which contains all conjugates of all $H_t (t \in M)$. We define $\tilde{\mathbf{H}}= \{(x, t, y) \in G \times M \times G \mid x \in y H_t y^{-1}\}$. Let $\pi_{M\times G}$: $\tilde{\mathbf{H}} \to M \times G$ denote the restriction to $\tilde{\mathbf{H}}$ of the projection $G \times (M \times G) \to M \times G$. For each $(t, y) \in M \times G$, $\pi_{M\times G}{}^{-1}(t, y)=(yH_t y^{-1}) \times \{(t, y)\}$. In order to define the analytic manifold structure on $\tilde{\mathbf{H}}$ we proceed as in the preceding paragraph. We define α: $G \times M \times G \to G \times M$ by $\alpha(x, t, y)=(y^{-1}xy, t)$. One checks easily that α is a submersion. We now form the fibre product L of $G \times M \times G$ and $\mathbf{H}$ over $G \times M$, corresponding to $\alpha: G \times M \times G \to G \times M$ and $i: \mathbf{H} \to G \times M$. Since α is a submersion, it follows immediately that α and i are transversal at each point of the fibre product L. Thus L is a closed analytic submanifold of $(G \times M \times G) \times \mathbf{H}$. The projection $(G \times M \times G) \times \mathbf{H} \to G \times M \times G$ maps L bijectively onto $\tilde{\mathbf{H}}$; we define an analytic manifold structure on $\tilde{\mathbf{H}}$ by requiring that this bijection be an isomorphism of analytic manifolds. One checks easily that $\tilde{\mathbf{H}}$, with the above analytic manifold structure, is an analytic family of Lie subgroups of G. We write $\tilde{\mathbf{H}} = (H_{(t,y)})_{(t,y) \in M \times G}$, where $H_{(t,y)} = yH_t y^{-1}$.

1.8. *The analytic family* $\mathbf{H}^0$. Let $\mathbf{H}^0 = \{(x, t) \in \mathbf{H} \mid x \in H_t^0\}$.

1.8.1. LEMMA. $\mathbf{H}^0$ *is an open subset of* $\mathbf{H}$.

Proof. Let $(x_0, t_0) \in \mathbf{H}^0$. We may write $x_0 = (\exp_G X_1) \dots (\exp_G X_n)$, where $X_1, \dots, X_n \in \mathfrak{h} = \mathfrak{h}_{t_0}$. Let W, U and $\varphi: U \to \mathrm{Hom}_F(\mathfrak{h}, W)$ be as in 1.5. Let $\mathfrak{H}_U = \{(X, t) \in \mathfrak{H} \mid t \in U\}$ and define an analytic map $\alpha: \mathfrak{H}_U \to \mathbf{H}$ by

$$\alpha(X, t) = ((\exp_G X)\ (\exp_G(X_1 + \varphi(t) \cdot X_1)) \dots (\exp_G(X_n + \varphi(t) \cdot X_n)), t);$$

the analyticity of α follows from the analyticity of $\exp_{\mathbf{H}}$ and the fact that the group operations on $\mathbf{H}$ are analytic. We have $\alpha(0, t_0) = (x_0, t_0)$ and $\alpha(\mathfrak{H}_U) \subset \mathbf{H}^0$; moreover, it is easy to check that the differential of α at $(0, t_0)$ is a linear isomorphism. It follows that $\alpha(\mathfrak{H}_U)$ is a neighborhood of (x_0, t_0) in $\mathbf{H}$. This proves 1.8.1.

1.8.2. COROLLARY. $\mathbf{H}^0$ *is an analytic family of Lie subgroups of* G.

1.8.3. *Remark.* In general, it is not true that $\mathbf{H}^0$ is a connected component of $\mathbf{H}$. See § 12.2.

§ 2. The normal displacement of an analytic family

Throughout this section $\mathbf{H} = (H_t)_{t \in M}$ denotes an analytic family of Lie subgroups of the Lie group G. Since we are only interested in the local deformation theory, we assume throughout that M is an open neighborhood of 0 in F^r and we set $H = H_0$. Let W be a vector subspace of $\mathfrak{g}$ such that $\mathfrak{g}$ is the vector space direct sum of $\mathfrak{h}$ and W. The *normal displacement* of the family $\mathbf{H}$ at 0 is an analytic mapping ψ of a neighborhood $\mathcal{U}$ of $H \times \{0\}$ in $H \times M$ into W. The normal displacement determines the family $\mathbf{H}$ locally. Let K be a Lie subgroup of H. The main result of this section, Proposition 2.7.1, says that if we take the Taylor series expansion of the restriction of ψ to $\mathcal{U} \cap (K \times M)$, then the lowest order non-vanishing terms in this expansion can be interpreted as a one-cocycle of K with coefficients in an appropiate K-module.

2.1. *Normal displacement functions.*

2.1.1. *Definition. A normal displacement function* for the analytic family $\mathbf{H}=(H_t)_{t\in M}$ at 0 is an analytic map ψ of an open neighborhood $\mathcal{U}$ of $H\times\{0\}$ in $H\times M$ into W such that the following conditions hold:

(a) $\psi(x, 0)=0$ for every $x\in H$;

(b) For every $(x, t)\in\mathcal{U}$, $(\exp_G\psi(x, t))x\in H_t$ and the map γ: $\mathcal{U}\to\mathbf{H}$ defined by $\gamma(x, t)=((\exp_G\psi(x, t))x, t)$ is an analytic map;

(c) For every $x\in H$, there exists an open neighborhood B_x of x in H and an open neighborhood C_x of 0 in M such that $B_x\times C_x\subset\mathcal{U}$ and such that γ defines an analytic manifold isomorphism of $B_x\times C_x$ onto an open neighborhood of $(x, 0)$ in $\mathbf{H}$.

Roughly speaking, a normal displacement function measures the normal variation of the family $\mathbf{H}$ at 0. The existence of normal displacement functions is demonstrated in 2.3. It is not difficult to show that two normal displacement functions agree in a neighborhood of $H\times\{0\}$.

Let ψ: $\mathcal{U}\to W$ be a normal displacement function for the family $\mathbf{H}$ at 0, let N be an open neighborhood of 0 in F^d and let f: $N\to M$ be an analytic map such that $f(0)=0$. Let $\mathcal{U}_1=\{(x, t)\in H\times N\,|\,(x, f(t))\in\mathcal{U}\}$ and let ψ_1: $\mathcal{U}_1\to W$ be defined by $\psi_1(x, t)=\psi(x, f(t))$. Then it follows immediately from the definitions that ψ_1 is a normal displacement function for the pull-back analytic family $f^*(\mathbf{H})=(H_{f(t)})_{t\in N}$.

2.2. *Local parametrization of a family of submanifolds.* Let A, B and C be open boxes in (resp.) F^p, F^q and F^r and let X be a closed $(q+r)$-dimensional analytic submanifold of $A\times B\times C$. Let π_C denote the restriction to X of the projection $A\times B\times C\to C$. Assume that π_C is surjective and is a submersion. It follows that each fibre $\pi_C^{-1}(z)$ $(z\in C)$ is a closed q-dimensional submanifold of $A\times B$ of the form $X_z\times\{z\}$, where X_z is a closed q-dimensional analytic submanifold of $A\times B$. Thus X can be considered as a parametrized family $(X_z)_{z\in C}$ of analytic submanifolds of $A\times B$. We assume further that $X_0=\{0\}\times B$.

2.2.1. LEMMA. *There exist open boxes $A_1\subset A$, $B_1\subset B$ and $C_1\subset C$ and an analytic map γ: $B_1\times C_1\to A_1$, with $\gamma(y, 0)=0$ for $y\in B_1$, such that the map $(y, z)\mapsto(\gamma(y, z), y, z)$ is an analytic manifold isomorphism of $B_1\times C_1$ onto an open neighborhood of* $(0, 0, 0)$ *in* X.

Proof. Let $\pi_{B\times C}$: $X\to B\times C$ denote the restriction to X of the projection $A\times B\times C\to B\times C$. Since π_C is a submersion and $X_0=\{0\}\times B$, the differential of $\pi_{B\times C}$ at $(0, 0, 0)$ is a linear isomorphism. The proof of 2.2.1. now follows easily from the inverse function theorem.

2.3. *Existence of normal displacement functions.* We define an analytic map $\tau\colon W\times H\times M\to G\times M$ by $\tau(w, x, t)=((\exp_G w)x, t)$. If $x\in H$, then the differential of τ at $(0, x, 0)$ is a linear isomorphism. Hence τ maps a neighborhood of $(0, x, 0)$ isomorphically onto a neighborhood of $(x, 0)$ in $G\times M$.

2.3.1. LEMMA. *Let $x\in H$. There exists an open box A_x in W, an open neighborhood B_x of x in H, an open box $C_x\subset M$ in F^r and an analytic map ψ_x: $B_x\times C_x\to A_x$, with $\psi_x(y, 0)=0$ for every $y\in B_x$, such that the following conditions hold:*

(a) *τ maps $A_x\times B_x\times C_x$ isomorphically onto an open neighborhood of $(0, x, 0)$ in $G\times M$;*

(b) *Let $E_x=\{(w, y, t)\in A_x\times B_x\times C_x \mid w=\psi_x(y, t)\}$. Then τ maps the closed analytic submanifold E_x isomorphically onto an open neighborhood L_x of $(x, 0)$ in* $\mathbf{H}$.

(c) *The map $t\mapsto(\exp_G\psi_x(x, t))x$ of C_x into G is an analytic section of* $\mathbf{H}$ *over C_x.*

The proof of 2.3.1. follows readily from 2.2.1. The function ψ_x measures the normal displacement of the family $\mathbf{H}$ in a neighborhood of $(x, 0)$.

2.3.2. LEMMA. *Let $x, y\in H$. Then $\psi_x(b, t)=\psi_y(b, t)$ for $(b, t)\in(B_x\times C_x)\cap(B_y\times C_y)$.*

Proof. Let $z\in(B_x\cap B_y)$ and let V denote the connected component of z in $B_x\cap B_y$. We may choose open neighborhoods $N(z)$ of z in V and $N(0)$ of 0 in $C_x\cap C_y$ such that, if $(b, t)\in N(z)\times N(0)$, *then* $\tau(\psi_x(b, t), b, t)\in L_x\cap L_y$ and $\tau(\psi_y(b, t), b, t)\in L_x\cap L_y$. It follows easily that $\psi_x(b, t)=\psi_y(b, t)$ for $(b, t)\in N(z)\times N(0)$. Since $V\times(C_x\cap C_y)$ is connected and ψ_x and ψ_y are analytic functions, ψ_x and ψ_y must agree on $V\times(C_x\cap C_y)$. This proves 2.3.2.

Let $\mathcal{U}=\bigcup_{x\in H}(B_x\times C_x)$; $\mathcal{U}$ is an open neighborhood of $H\times\{0\}$ in $H\times M$. We define an analytic function ψ: $\mathcal{U}\to W$ as follows: Let $(b, t)\in\mathcal{U}$. Then $(b, t)\in B_x\times C_x$ for some $x\in H$. We set $\psi(b, t)=\psi_x(b, t)$. It follows from 2.3.2 that $\psi(b, t)$ is independent of the choice of $x\in H$. It is an easy consequence of 2.3.1 that ψ is a normal displacement function of the family $\mathbf{H}$ at 0.

2.4. *The infinitesimal displacement.* Now we want to look at the power series expansion of the normal displacement function ψ. First some notation. If $\alpha=(\alpha_1, \ldots, \alpha_r)\in\mathbf{N}^r$ and $t=(t_1, \ldots, t_r)\in F^r$, we write $t^\alpha=t_1^{\alpha_1}\ldots t_r^{\alpha_r}$ and $|\alpha|=\alpha_1+\ldots+\alpha_r$. For each $x\in H$, the function $t\mapsto\psi(x, t)$ can be expanded in a power series $\psi(x, t)=\sum_{\alpha\in\mathbf{N}^r}u_\alpha(x)t^\alpha$ $(u_\alpha(x)\in W)$ about 0, convergent in some open box about 0 in F^r. Since $\psi(x, t)$ is analytic in both x and t, it follows that the functions $x\mapsto u_\alpha(x)$ are analytic functions defined on all of H with values in W.

For each $n\in\mathbf{N}$, we denote by $P_n(F^r, W)$ the vector space of homogeneous polynomial maps of degree n of F^r into W. Now let K be a Lie subgroup of H. For $m\in\mathbf{N}$ and $x\in K$ we define $S_m(x)\in P_m(F^r, W)$ to be the polynomial map $t\mapsto\sum_{|\alpha|=m}u_\alpha(x)t^\alpha$. Then S_m is an analytic map of K into $P_m(F^r, W)$. If $S_j=0$ for all $j<m$, then S_m is called the *m-th infinitesimal*

displacement along K of the analytic family $\mathbf{H}$. The important term, of course, is the first non-vanishing infinitesimal displacement along K.

In the case $K=H$ the first infinitesimal displacement coincides with the infnitesimal displacement defined by Kodaira [14] for an analytic family of compact complex submanifolds of a complex manifold. To pass from Kodaira's definition to ours, one needs to note that the normal bundle of H in G is trivial, so that an analytic cross section of the normal bundle corresponds to an analytic map of H into W.

2.5. *On the cohomology of Lie groups.* Let L be a Lie group and let $\eta: L \to \mathrm{GL}(V)$ be a linear representation of L. We define $Z^1(L, V)$, the space of one-cocycles of L with coefficients in the L-module V, to be the vector space of all analytic maps $f: L \to V$ such that $f(xy)=f(x)+\eta(x)\cdot f(y)$ for all $x, y \in L$. $B^1(L, V)$, the space of one-coboundaries, is defined to be the vector space of all analytic maps of L into V of the form $x \mapsto \eta(x)\cdot v - v$ for some $v \in V$. We have $B^1(L, V) \subset Z^1(L, V)$ and the quotient vector space $H^1(L, V)=Z^1(L, V)/B^1(L, V)$ is the first cohomology space of the Lie group L with coefficients in the L-module V.

In this paper we shall be interested in the cohomology groups $H^1(K, W)$ and $H^1(K, P_m(F^r, W))$, where K and W are as above. First we need to define the K-module structure on W. Let $p: W \to \mathfrak{g}/\mathfrak{h}$ denote the restriction to W of the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$; p is a vector space isomorphism. The restriction to K of the adjoint representation of G gives a representation of K on $\mathfrak{g}$, $\mathfrak{h}$ is a K-submodule of $\mathfrak{g}$ and thus we have a quotient K-module structure on $\mathfrak{g}/\mathfrak{h}$. We define the K-module structure of W by transporting to W by means of p the K-module structure of $\mathfrak{g}/\mathfrak{h}$, i.e. we require that $p: W \to \mathfrak{g}/\mathfrak{h}$ be a K-module isomorphism. Let $\varrho: K \to \mathrm{GL}(W)$ be the representation defining the K-module structure of W and let $\mathrm{pr}_W: \mathfrak{g} \to W$ and $\mathrm{pr}_{\mathfrak{h}}: \mathfrak{g} \to \mathfrak{h}$ be the projections corresponding to the direct sum decomposition $\mathfrak{g}=\mathfrak{h}+W$. If $x \in K$ and $w \in W$, then $\varrho(x)\cdot w = \mathrm{pr}_W(\mathrm{Ad}_G(x)\cdot w)$.

We define a representation ϱ_m of K on $P_m(F^r, W)$ as follows: if $f \in P_m(F^r, W)$ and $x \in W$, then $\varrho_m(x)\cdot f = \varrho(x)\circ f$. Equivalently, note that $P_m(F^r, W)$ is canonically isomorhic to $P_m(F^r, F)\otimes W$, where $P_m(F^r, F)$ is the vector space of all homogeneous polynomial functions of degree m on F^r. Then ϱ_m is the tensor product of the trivial representation of K on $P_m(F^r, F)$ and the representation ϱ of K on W. In particular, the K-module $P_m(F^r, W)$ is the direct sum of K-modules isomorphic to W. Thus the vanishing of $H^1(K, W)$ implies that $H^1(K, P_m(F^r, W))$ vanishes.

2.6. *Two technical lemmas.*

2.6.1. LEMMA. *Let $C \subset M$ be an open box in F^r and let $f: C \to \mathfrak{g}$ be an analytic map such that $f(t) \in \mathfrak{h}_t$ for every $t \in C$. Expand f in a power series about 0: $f(t)=\sum_\alpha c_\alpha t^\alpha$ $(c_\alpha \in \mathfrak{g})$, convergent* *in some open box, and assume that the power series is of order* $\geqslant m$, *i.e.* $c_\alpha = 0$ *whenever* $|\alpha| < m$. *Then* $c_\alpha \in \mathfrak{h}$ *for every* α *such that* $|\alpha| = m$.

Proof. Choose C small enough so that $\mathfrak{h}_t \in \Gamma_W$ for every $t \in C$, let φ: $C \to \mathrm{Hom}_F(\mathfrak{h}, W)$ be as in 1.5 and assume that the power series expansion of φ converges in C. Since $f(t) \in \mathfrak{h}_t$ for $t \in C$, we have $f(t) = u(t) + \varphi(t) \cdot u(t)$, where $u(t) = pr_{\mathfrak{h}} f(t)$. Take power series expansions for $u(t)$ and $\varphi(t)$:

$$u(t) = \sum_{|\alpha| \geqslant m} b_\alpha t^\alpha \quad (b_\alpha \in \mathfrak{h})$$

$$\varphi(t) = \sum_{|\alpha| \geqslant 1} a_\alpha t^\alpha \quad (a_\alpha \in \mathrm{Hom}_F(\mathfrak{h}, W)).$$

Thus
$$f(t) = u(t) + \varphi(t) \cdot u(t) = \sum_{|\alpha| = m} b_\alpha t^\alpha + \text{(higher order terms)}.$$

Consequently $c_\alpha = b_\alpha \in \mathfrak{h}$, for $|\alpha| = m$. This proves 2.6.1.

Let D be an open box in g such that $\exp_G$ defines an analytic manifold isomorphism of D onto an open neighborhood D_1 of e in G. Similarly, let E be an open neighborhood of $(0, 0)$ in $\mathfrak{H}$ such that $\exp_{\mathbf{H}}$ maps E isomorphically onto an open neighborhood E_1 of $(e, 0)$ in $\mathbf{H}$.

2.6.2. Lemma. *Let* $C \subset M$ *be an open box and let* ζ: $C \to G$ *be an analytic section of* $\mathbf{H}$ *over* C *such that* $\zeta(0) = e$ *and* $\zeta(C) \subset D_1$. *Let* $\log_G$: $D_1 \to D$ *denote the inverse of* $\exp_G$ *and expand* $\log_G \zeta(t)$ *in a power series about* 0

$$\log_G \zeta(t) = \sum_\alpha a_\alpha t^\alpha \quad (a_\alpha \in \mathfrak{g}),$$

convergent in some open box, and assume that this power series is of order $\geqslant m$. *Then* $a_\alpha \in \mathfrak{h}$ *whenever* $|\alpha| = m$.

Proof. We may assume C chosen small enough so that $(\zeta(t), t) \in E_1$ whenever $t \in C$. If $t \in C$, it follows that $\log_G \zeta(t) \in \mathfrak{h}_t$. The conclusion now follows from 2.6.1.

2.7. *The first non-vanishing infinitesimal displacement is a one-cocycle.* Let ψ and $S_j (j > 0)$ be as in 2.4. and let m be the smallest positive integer such that $S_m \neq 0$; the analytic function S_m: $K \to P_m(F^r, W)$ is the first non-vanishing infinitesimal displacement of $\mathbf{H}$ along K.

2.7.1. Proposition. S_m *is a one-cocycle.*

Proof. Let $x, y \in K$. We must show that

$$S_m(xy) = S_m(x) + \varrho(x) S_m(y).$$

For $z \in K$ and $t \in F^r$, let $s_m(z, t) = S_m(z)\ (t)$. It suffices to show that

$$s_m(xy, t) = s_m(x, t) + \varrho(x)\, s_m(y, t)$$

for all t in some neighborhood of 0 in F^r. Choose an open box C in F^r such that $C \subset C_x \cap C_y \cap C_{xy}$, where C_x, C_y and C_{xy} are as in Proposition 2.3.1. Define analytic maps σ_x, σ_y and σ_{xy} of C into G by: $\sigma_x(t) = (\exp_G \psi(x, t))x$; $\sigma_y(t) = (\exp_G \psi(y, t))y$; and $\sigma_{xy}(t) = (\exp_G \psi(xy, t))xy$. It is a consequence of 2.3.1. (c) that σ_x, σ_y and σ_{xy} are analytic sections of $\mathbf{H}$ over C. Define an analytic section ζ: $C \to G$ by

$$\begin{aligned}\zeta(t) = \sigma_x(t)\sigma_y(t)\sigma_{xy}(t)^{-1} &= \exp_G \psi(x, t)\ (x(\exp_G \psi(y, t))x^{-1}) \exp_G - \psi(xy, t)\\ &= \exp_G \psi(x, t) \exp_G(\mathrm{Ad}_G x \cdot \psi(y, t)) \exp_G - \psi(xy, t).\end{aligned}$$

We have $\zeta(0) = e$. From the first order terms of the Campbell-Hausdorff formula, we see that the power series expansion about 0 of $\log_G \zeta(t)$ has the form

$$\log_G \zeta(t) = s_m(x, t) + \mathrm{Ad}_G x \cdot s_m(y, t) - s_m(xy, t) + (\text{terms of order } > m \text{ in } t).$$

This series converges in some open box $C_1 \subset C$. It follows from Lemma 2.6.2. that

$$0 = \mathrm{pr}_W(s_m(x, t) + \mathrm{Ad}_G x \cdot s_m(y, t) - s_m(xy, t)) = s_m(x, t) + \varrho(x)\, s_m(y, t) - s_m(xy, t), \quad \text{for} \quad t \in C_1.$$

This proves 2.7.1.

§ 3. Proof of Theorem 3.1

This section will be devoted to the proof of the following theorem.

3.1. THEOREM. *Let $\mathbf{H} = (H_t)_{t \in M}$ be an analytic family of Lie subgroups of G, let $t_0 \in M$ and let $H = H_{t_0}$. Let K be a Lie subgroup of H such that the component group K/K^0 is finitely generated and such that $H^1(K, \mathfrak{g}/\mathfrak{h}) = 0$. Then there exists an open neighborhood U of t_0 in M and an analytic map η: $U \to G$ such that $\eta(t)K\eta(t)^{-1} \subset H_t$ for every $t \in U$.*

The proof of Theorem 3.1 falls into two parts. By using Proposition 2.7.1 and the vanishing of $H^1(K, \mathfrak{g}/\mathfrak{h})$, we show that there exists a formal power series solution of the problem. Then we use a recent theorem of M. Artin [1] to show that the formal power series solution can be replaced by a convergent power series solution.

We assume throughout § 3 that M is an open neighborhood of 0 in F^r and that $t_0 = 0$.

3.2. *Enlarging the family H.* Let $\tilde{\mathbf{H}} = (H_{(t,x)})_{(t,x) \in M \times G}$ be the "expanded" family defined in 1.7. Define an analytic map σ: $M \times W \to M \times G$ by $\sigma(t, w) = (t, \exp_G w)$ and let $\mathcal{H}$ denote the pull-back family $\sigma^*(\tilde{\mathbf{H}})$. Thus we may write $\mathcal{H} = (H_{(t,w)})_{(t,w) \in M \times W}$, where $H_{(t,w)} = (\exp_G w)H_t(\exp_G - w)$. We let $\mathcal{U}_0$ be a neighborhood of $H \times \{(0, 0)\}$ in $H \times M \times W$ and Ψ: $\mathcal{U}_0 \to W$ be a normal displacement function for the analytic family $\mathcal{H}$.

In order to be able to use standard notation for the power series expansion of Ψ we assume that we are given a basis $(X_1, ..., X_p)$ of W and we identify F^p with W by the linear isomorphism

$$w=(w_1, \ldots, w_p) \mapsto w_1 X_1 + \ldots + w_p X_p.$$

With this identification, we have the power series expansion

$$\Psi(x, t, w) = \sum_{(\alpha, \beta)} a_{(\alpha,\beta)}(x)\, t^\alpha w^\beta.$$

Here the sum is taken over all $(\alpha, \beta) \in \mathbf{N}^r \times \mathbf{N}^p$ and, for each (α, β), $x \mapsto a_{(\alpha,\beta)}(x)$ is an analytic map from H into W. For each $x \in H$, the corresponding power series in (t, w) is convergent in some open box in $F^r \times F^p$.

3.2.1. LEMMA. $\sum_{|\beta|=1} a_{(0,\beta)}(x)\, w^\beta = w - \varrho(x) \cdot w$.

Proof. Define γ: $W \to M \times W$ by $\gamma(w)=(0, w)$. Then $\gamma^*(\mathcal{H}) = (H_w)_{w \in W}$, where $H_w = (\exp_G w) H (\exp_G - w)$. Let $\mathcal{U} = \{(y, w) \in H \times W \mid (y, 0, w) \in \mathcal{U}_0\}$ and define φ: $\mathcal{U} \to W$ by $\varphi(y, w) = \Psi(y, 0, w)$; then φ is a normal displacement function for the family $\gamma^*(\mathcal{H})$. Let C be a (sufficiently small) open box in W, let $x \in H$ and define an analytic section ζ: $C \to G$ of $\gamma^*(\mathcal{H})$ over C by

$$\zeta(w) = (\exp_G w) x (\exp_G - w) x^{-1} \exp_G \varphi(x, w).$$

We have $\zeta(0)=e$. The power series expansion of $\log_G \zeta(w)$ about 0 has the form

$$\log_G \zeta(w) = w - \mathrm{Ad}_G(x) \cdot w - \sum_{|\beta|=1} a_{(0,\beta)}(x)\, w^\beta + (E),$$

where (E) consists of terms of degree higher than one in w. It follows from 2.6.2 that

$$0 = pr_W(w - \mathrm{Ad}_G(x) \cdot w - \sum_{|\beta|=1} a_{(0,\beta)}(x)\, w^\beta) = w - \varrho(x) \cdot w - \sum_{|\beta|=1} a_{(0,\beta)}(x)\, w^\beta.$$

This proves 3.2.1.

3.2.2. *Remark.* Equivalently, 3.2.1 says that $D_3\Psi$ (0, 0, 0), the partial derivative of Ψ with respect to w at (0, 0, 0) is $1_W - \varrho(x)$. (1_W denotes the identity map of W.)

3.3. *The formal power series solution.*

3.3.1. PROPOSITION. *There exists a sequence $(g_j)_{j \in \mathbf{N}}$ of homogeneous polynomial maps $g_j \in P_j(F^r, W)$ such that, if $m \in \mathbf{N}$ and $f_m = g_0 + \ldots + g_m$, then the power series expansion about* 0 *of the analytic function $t \mapsto \Psi(x, t, f_m(t))$ is of order $\geqslant m+1$ for every $x \in K$.*

Proof. By induction on m. If we define $g_0 = f_0 = 0$, then it is clear that $t \to \Psi(x, t, f_0(t))$ is of order >1 for every $x \in K$. Assume that we have have defined $g_j \in P_j(F^r, W)$ for $j=1, \ldots, m$ satisfying the conditions of the proposition. Let $f_m = g_0 + \ldots + g_m$ and let φ_m: $M \to M \times W$ denote the analytic map $t \mapsto (t, f_m(t))$. Let $\mathcal{U}_m = \{(x, t) \in H \times M \mid (x, t, f_m(t) \in \mathcal{U}_0\}$ and let ψ_m: $U_m \to W$ be defined by $\psi_m(x, t) = \Psi(x, t, f_m(t))$; ψ_m is is a normal displacement function for the pull-back family $\varphi_m^*(\mathcal{H})$. Thus we have a power series expansion about 0 (in t) $\psi_m(x, t) = \sum_{\alpha\in \mathbf{N}^r} c_\alpha(x)t^\alpha$; for each α, c_α is an analytic map of H into W and, for each $x \in H$, the series converges for t in some open box in F^r. It follows from the induction hypothesis that, for each α such that $|\alpha| < m+1$, the restriction of c_α to K vanishes. We define q_{m+1}: $H \times F^r \to W$ by $q_{m+1}(x, t) = \sum_{|\alpha|=m+1} c_\alpha(x)t^\alpha$ and we let Q_{m+1}: $K \to P_{m+1}(F^r, W)$ be defined by $Q_{m+1}(x)(t) = q_{m+1}(x, t)$. By Proposition 2.7.1, applied to the analytic family $\varphi_m^*(\mathcal{H})$, we see that $Q_{m+1} \in Z^1(K, P_{m+1}(F^r, W))$. Since $H^1(K, P_{m+1}(F^r, W)) = 0$, there exists $g_{m+1} \in P_{m+1}(F^r, W)$ such that $Q_{m+1}(x) = \varrho(x) \circ g_{m+1} - g_{m+1}$ for every $x \in K$.

Set $\psi_{m+1}(x, t) = \Psi(x, t, f_m(t) + g_{m+1}(t))$, for (x, t) in a sufficiently small neighborhood of $H \times \{0\}$ in $H \times M$. Let $y \in K$. We want to show that the power series expansion about 0 of $t \mapsto \psi_{m+1}(y, t)$ is of order $> m+1$.

If $\beta \in \mathbf{N}^p$, we define $\beta! = \prod_{j=1}^p \beta_j!$. We need the multivariable form of the binomial expansion:

3.3.2. If $a, b \in F^p$ and $\beta \in \mathbf{N}^p$ then

$$(a+b)^\beta = \sum_{\delta \in \mathbf{N}^p, \delta \leqslant \beta} \binom{\beta}{\delta} a^{\beta-\delta} b^\delta.$$

Here $\binom{\beta}{\delta} = (\beta!)(\delta!)^{-1}((\beta-\delta)!)^{-1}$, and $\delta \leqslant \beta$ means $\beta - \delta \in \mathbf{N}^p$.

If we use 3.2.2 and expand $t \to \psi_{m+1}(y, t)$ in a power series about 0, we obtain

$$\psi_{m+1}(y,t) = \sum_{(\alpha,\beta,\delta)} a_{(\alpha,\beta)}(y) \binom{\beta}{\delta} t^\alpha f_m(t)^{\beta-\delta} g_{m+1}(t)^\delta;$$

here the sum is taken over all triples $(\alpha, \beta, \delta) \in \mathbf{N}^r \times \mathbf{N}^p \times \mathbf{N}^p$ such that $\delta \leqslant \beta$. If we sum over all terms with $\delta = 0$, we obtain the power series expansion of $\psi_m(y, t)$. Thus we have

$$\psi_{m+1}(y,t) = \psi_m(y,t) + \sum_{(\alpha,\beta,\delta), \delta \neq 0} a_{(\alpha,\beta)}(y) \binom{\beta}{\delta} t^\alpha f_m(t)^{\beta-\delta} g_{m+1}(t)^\delta.$$

Let $\varphi_{(\alpha,\beta,\delta)}(t) = \binom{\beta}{\delta} t^\alpha f_m(t)^{\beta-\delta} g_{m+1}(t)^\delta$; $\varphi_{(\alpha,\beta,\delta)}$ is a polynomial in t. The order of $\varphi_{(\alpha,\beta,\delta)}$ (as a formal power series in t) is $\geqslant |\alpha| + |\beta-\delta| + (m+1)|\delta|$. Thus, if $\delta \neq 0$, $\varphi_{(\alpha,\beta,\delta)}$ is of order $> m+1$ unless $|\alpha| = |\beta-\delta| = 0$ and $|\delta| = 1$. The homogeneous component of degree $m+1$ in the power series expansion of $\psi_m(y, t)$ is

$$q_{m+1}(y, t) = \varrho(y) \cdot g_{m+1}(t) - g_{m+1}(t).$$

Thus the homogeneous component of degree $m+1$ in the power series expansion of $\psi_{m+1}(y, t)$ is

$$\varrho(y)\cdot g_{m+1}(t)-g_{m+1}(t)+\sum_{|\delta|=1} a_{(0,\delta)}(y)g_{m+1}(t)^{\delta};$$

it follows from Lemma 3.2.1 that this homogeneous component is 0. Thus the power series expansion of $t\to\psi_{m+1}(y,t)$ about 0 is of order $>m+1$. This completes the induction and concludes the proof of 3.3.1.

3.4. *M. Artin's theorem.* If the formal power series $f(t)=\sum_{j=0}^{\infty} g_j(t)$ converges (where the $g_j(t)$ are chosen as in Proposition 3.3.1), then setting $\eta(t)=\exp_G -f(t)$, the conclusion of Theorem 3.1 would be satisfied. We shall use a recent theorem of M. Artin [1] to show that the $g_j(t)$ can be chosen so that $\sum_{j=1}^{\infty} g_j(t)$ converges in a neighborhood of 0. In order to apply this theorem directly, however, we need to reformulate things slightly, so there is still more work to be done.

3.4.1. PROPOSITION (Artin). *Let f_j $(j=1,\dots,m)$ be analytic functions defined in a neighborhood of $(0,0)$ in $F^r\times F^p$ such that $f_j(0,0)=0$ for $j=1,\dots,m$. Assume there exist formal power series γ_i $(i=1,\dots,p)$ in r variables with 0 constant term such that the formal power series $f_j(x,\gamma(x))$ $(j=1,\dots,m)$ vanish identically (here $x=(x_1,\dots,x_r)$ and $\gamma(x)=(\gamma_1(x),\dots,\gamma_p(x))$. Then there exist analytic functions τ_j $(j=1,\dots,p)$, defined in a neighborhood A of 0 in F^r, such that $f_i(x,\tau(x))=0$ for $x\in A$ and $i=1,\dots,m$. Moreover, one may require that the power series expansion of $\gamma_j-\tau_j$ $(j=1,\dots,p)$ about 0 is of order $>n$, where n may be chosen arbitrarily.*

3.5. *A technical lemma.* Asumme that $M\subset F^r$ is chosen small enough that $\mathfrak{h}_t\in\Gamma_W$ for every $t\in M$, let $\Theta\colon\Gamma_W\to\mathrm{Hom}_F(\mathfrak{h},W)$ be defined as in 1.5 and define the analytic map φ: $M\to\mathrm{Hom}_F(\mathfrak{h},W)$ by $\varphi(t)=\Theta(\mathfrak{h}_t)$. Expand φ in a power series about 0.

$$\varphi(t)=\sum_{\alpha\in\mathbf{N}^r}\varphi_\alpha t^\alpha\quad(\varphi_\alpha\in\mathrm{Hom}_F(\mathfrak{h},W)).$$

Let $r\colon\mathrm{Hom}_F(\mathfrak{h},W)\to\mathrm{Hom}_F(\mathfrak{k},W)$ denote the restriction map. Then the power series expansion of $r\circ\varphi$ about 0 is given by $r\circ\varphi(t)=\sum_\alpha(r\circ\varphi_\alpha)t^\alpha$. If the terms of order less than m in the power series expansion of $r\circ\varphi$ vanish, then the terms of order m in this expansion give the *mth order infinitesimal displacement along* $\mathfrak{k}$ of the analytic family $\mathfrak{H}=(\mathfrak{h}_t)_{t\in M}$ of subalgebras of $\mathfrak{g}$. It is intuitively clear that if the mth order infinitesimal displacement along K of the analytic family $\mathbf{H}=(H_t)_{t\in M}$ vanishes, then so does the mth order infinitesimal displacement along $\mathfrak{k}$ of the analytic family $\mathfrak{H}=(\mathfrak{h}_t)_{t\in M}$. This is proved in the lemma below.

3.5.1. LEMMA. *Let ψ be a normal displacement function for the analytic family $\mathbf{H}=(H_t)_{t\in M}$ and, for $j\in N$, let $S_j\colon K\to P_j(F^r,W)$ be defined as in 2.4. Assume that S_j vanishes for $j\leqslant m$. Then the power series expansion of $r\circ\varphi$ about 0 is of order $>m$.*

Proof. For every $X \in \mathfrak{g}$, let $\nu(X) \in \operatorname{End}_F(\mathfrak{g})$ be defined by the convergent power series

$$\nu(X) = \sum_{j=0}^{\infty} \frac{(\operatorname{ad} X)^j}{(j+1)!};$$

symbolically, one can write $\nu(X) = (e^{\operatorname{ad} X} - 1)/\operatorname{ad} X$. For $x \in G$, let $R(x)\colon G \to G$ denote right translation by x, $y \mapsto yx$. It is well known that

$$(3.5.2) \qquad d(R(\exp_G - X) \circ \exp_G)_X = \nu(X)$$

(see, e.g. [9, p. 95]).

Let s be the order of the power series expansion of $r \circ \varphi$ about 0 and assume that $s \leqslant m$. We shall show that this leads to a contradiction. We may choose $X \in \mathfrak{k}$ such that the following conditions hold: (i) (X) is invertible; (ii $)\nu(X)(W) \cap \mathfrak{h} = \{0\}$; and (iii) there exists $\alpha \in \mathbf{N}^r$ such that $|\alpha| = s$ and $\varphi_\alpha(X) \neq 0$. Let $Q \in P_s(F^r, W)$ be defined by

$$Q(t) = pr_W(\nu(X)\,(\sum_{|\alpha|=s} \varphi_\alpha(X)\, t^\alpha).$$

It follows from (i), (ii), (iii) above that $Q \neq 0$.

Let $C \subset M$ be a (sufficiently small) open box in F^r and define $\tau\colon C \to \mathfrak{g}$ by

$$\tau(t) = \log_G(\exp_G(X + \varphi(t)\, X) \exp_G - X).$$

Then $\tau(0) = 0$ and it is a consequence of (3.5.2) that the power series expansion of τ about 0 takes the form

$$(3.5.3) \qquad \tau(t) = \nu(X)\,(\sum_{|\alpha|=s} \varphi_\alpha(X) t^\alpha) + (\text{terms of order } > s).$$

Define an analytic section $\zeta\colon C \to G$ by

$$\zeta(t) = (\exp_G(X + \varphi(t)\, X)\ (\exp_G - X)\ (\exp_G - \psi\ (\exp_G X, t)).$$

We have $\zeta(0) = e$. Since the power series expansion of $t \to \psi(\exp_G X, t)$ is of order $> s$, we see from (3.5.3) that the power series expansion of $\log_G \circ \zeta$ takes the form

$$\log_G \zeta(t) = \nu(X)\,(\sum_{|\alpha|=s} \varphi_\alpha(X) t^\alpha) + (\text{terms of order } > s).$$

It follows from Lemma 2.6.2 that $\operatorname{pr}_W(\nu(X)\, \varphi_\alpha(X)) = 0$ for every α such that $|\alpha| = s$. This implies that

$$Q(t) = \operatorname{pr}_W(\nu(X)\,(\sum_{|\alpha|=s} \varphi_\alpha(X) t^\alpha))$$

vanishes identically in t, which gives a contradiction.

3.6. *The convergent power series solution.* Let Ψ be as in 3.1, let $g_j \in P_j(F^r, W)$ $(j \in \mathbf{N})$ be as in Proposition 3.3.1 and let $f(t)$ denote the formal power series $\sum_{j=0}^{\infty} g_j(t)$. Then it follows from Proposition 3.3.1 that, for every $x \in K$, the formal power series $t \mapsto \Psi(x, t, f(t))$ vanishes identically. Choose elements $x_1, \ldots, x_n \in K$ such that the cosets $x_1 K^0, \ldots, x_n K^0$ generate the component group K/K^0. Let $C \subset M$ and A be (sufficiently small) open boxes in (resp.) F^r and W and define analytic function λ_j: $C \times A \to W$ $(j = 1, \ldots, n)$ by $\lambda_j(t, w) = \Psi(x_j, t, w)$. Define an analytic map λ_0: $C \times A \to \mathrm{Hom}_F(\mathfrak{k}, W)$ by $\lambda_0(t, w) = r \circ \Theta(\mathrm{Ad}_G(\exp_G w)(\mathfrak{h}_t))$; note that $\mathfrak{k} \subset \mathrm{Ad}_G(\exp_G w)(\mathfrak{h}_t)$ if and only if $\lambda_0(t, w) = 0$. It follows immediately that the formal power series $\lambda_j(t, f(t))$ $(j = 1, \ldots, n)$ vanish identically. Let $m \in \mathbf{N}$, let $f_m(t) = g_0(t) + \ldots + g_m(t)$, let φ_m: $M \to M \times W$ be as in the proof of Proposition 3.3.1 and let $\psi_m(x, t) = \Psi(x, t, f_m(t))$ (ψ_m is defined in a neighborhood of $H \times \{0\}$ in $H \times M$). It follows from Proposition 3.3.1 and Lemma 3.5.1, applied to the analytic family $\varphi_m^*(\mathcal{H})$ with normal displacement function ψ_m, that the power series expansion of $t \to \lambda_0(t, f_m(t))$ about 0 is of order $\geqslant m+1$. Thus the formal power series $\lambda_0(t, f(t))$ vanishes identically. Hence we can apply Artin's theorem to the family of analytic functions $\lambda_0, \ldots, \lambda_n$. Thus there exist an open box $U \subset C$ in F^r and an analytic map β: $U \to W$ such that $\beta(U) \subset A$ and such that $\lambda_j(t, \beta(t)) = 0$ for $t \in U$ and $j = 0, \ldots, n$. Define η: $U \to G$ by $\eta(t) = \exp_G - \beta(t)$. Since $\lambda_0(t, \beta(t)) = 0$, it follows that $\mathfrak{k} \subset \mathrm{Ad}_G(\eta(t)^{-1})(\mathfrak{h}_t)$, hence that $K^0 \subset \eta(t)^{-1} H_t \eta(t)$ or, equivalently, that $\eta(t) K^0 \eta(t)^{-1} \subset H_t$. Similarly, for $j > 0$, $\lambda_j(t, \beta(t)) = 0$ implies that $x_j \in \eta(t)^{-1} H_t \eta(t)$, or that $\eta(t) x_j \eta(t)^{-1} \in H_t$. Since $K^0 \cup \{x_1, \ldots, x_n\}$ generates K, it follows that $\eta(t) K \eta(t)^{-1} \subset H_t$ for $t \in U$. This completes the proof of Theorem 3.1.

§ 4. Relation of Theorem 3.1 to results on deformations on subalgebras of Lie algebras

Let G be a Lie group with Lie algebra $\mathfrak{g}$, let $q < \dim \mathfrak{g}$ and let A_q be the Zariski closed subset of $\Gamma_q(\mathfrak{g})$ consisting of all q-dimensional subalgebras of $\mathfrak{g}$. The following result was proved in [19, § 9]:

4.1. Let $\mathfrak{h}$ be a q-dimensional subalgebra of $\mathfrak{g}$ and let $\mathfrak{k}$ be a subalgebra of $\mathfrak{h}$ such that the Lie algebra cohomology space $H^1(\mathfrak{k}, \mathfrak{g}/\mathfrak{h})$ vanishes. Then there exists an open neighborhood V of $\mathfrak{h}$ in A_q and an analytic map β: $V \to G$ such that $\mathrm{Ad}_G(\beta(\mathfrak{a}))(\mathfrak{k}) \subset \mathfrak{a}$ for every $\mathfrak{a} \in V$.

Now let $\mathbf{H} = (H_t)_{t \in M}$ be an analytic family of Lie subgroups of G, let $t_0 \in M$ and let $H = H_{t_0}$. Let K be a *connected* Lie subgroup of H such that $H^1(\mathfrak{k}, \mathfrak{g}/\mathfrak{h}) = 0$. Then it follows from 4.1 that there exists an open neighborhood U of t_0 in M and an analytic map η: $U \to G$ such that $\mathrm{Ad}_G(\eta(t))(\mathfrak{k}) \subset \mathfrak{h}_t$ for every $t \in U$. This implies that $\eta(t) K \eta(t)^{-1} \subset H_t$ for $t \in U$. Thus, if K is a connected Lie subgroup and the Lie algrebra cohomology space $H^1(\mathfrak{k}, \mathfrak{g}/\mathfrak{h})$ vanishes, the conclusion of Theorem 3.1 follows from 4.1.

If K is connected and ϱ: $K \to \mathrm{GL}(V)$ is a representation, then the Lie group cohomology space $H^1(K, V)$ can be canonically identified with a subspace of $H^1(\mathfrak{k}, V)$. Thus, $H^1(\mathfrak{k}, V) = 0$

implies that $H^1(K, V)=0$. The converse statement does not hold, e.g. for K a torus and ϱ the trivial representation. There are many examples of analytic families $\mathbf{H}=(H_t)_{t\in M}$ and connected $K\subset H_{t_0}$ as above for which $H^1(K, \mathfrak{g}/\mathfrak{h})=0$, but $H^1(\mathfrak{k}, \mathfrak{g}/\mathfrak{h})\neq 0$. Consequently, even if K in Theorem 3.1 is assumed connected, the conclusion of Theorem 3.1 is stronger than any corollaries one can derive from 4.1.

§ 5. Variations on Theorem 3.1

5.1. *An extension of Theorem 3.1.* Let the Lie group G be a closed normal Lie subgroup of the Lie group L. If $x\in L$, then $\mathrm{Int}_L x$: $L\to L$ denotes the inner automorphism $y\mapsto xyx^{-1}$. If $x\in G$, then $\mathrm{Int}_L x$ maps G into G and induces the identity isomorpism of L/G. Since $\mathrm{Ad}_L x$ is the differential of $\mathrm{Int}_L x$, it follows that $\mathrm{Ad}_L x$ induces the identity map of $\mathfrak{l}/\mathfrak{g}$; equivalently, if $X\in\mathfrak{l}$, then $(\mathrm{Ad}_L x\cdot X-X)\in\mathfrak{g}$.

5.1. Theorem. *Let G be a closed normal Lie subgroup of the Lie group L, let $\mathbf{H}=(H_t)_{t\in M}$ be an analytic family of Lie subgroups of G, let $t_0\in M$ and let $H=H_{t_0}$. Let K be a Lie subgroup of H such that the component group K/K^0 is finitely generated. Assume that there exists a vector subspace V of $\mathfrak{l}$ such that the following condition holds: For every $\varphi\in Z^1(K, \mathfrak{g}/\mathfrak{h})$, there exists $v\in V$ such that $\varphi(x)=\pi_{\mathfrak{g}/\mathfrak{h}}(\mathrm{Ad}_L x\cdot v-v)$ for every $x\in K$ (here $\pi_{\mathfrak{g}/\mathfrak{h}}$: $\mathfrak{g}\to\mathfrak{g}/\mathfrak{h}$ denotes the canonical projection). Then there exists an open neighborhood U of t_0 in M and an analytic map η: $U\to V$ such that $(\exp_L \eta(t)K(\exp_L-\eta(t))\subset H_t$ for every $t\in U$.*

We shall only sketch the proof of Theorem 5.1, which is essentially the same as that of Theorem 3.1. Let M be an open neighborhood of 0 in F^r and let $t_0=0$. First define an expanded analytic family $\tilde{\mathbf{H}}=(H_{(t,z)})_{(t,z)\in M\times L}$ as in 1.7 (with G replaced be L; $\tilde{\mathbf{H}}$ is an analytic family of Lie subgroups of G since G is normal in L). Let α: $M\times V\to M\times L$ be the map $(t, v)\to(t, \exp_L v)$ and let Ψ be a normal displacement function for the pull-back analytic family $\alpha^*(\tilde{\mathbf{H}})$. As in § 3, one defines a formal power series $f(t)=\sum_{j=0}^{\infty} g_j(t)$ $(g_j\in P_j(F^r, V))$ such that, for every $x\in K$, the formal power series $t\to\Psi(x, t, f(t))$ vanishes identically. Using Artin's theorem, one can replace $f(t)$ by a convergent power series $-\eta(t)$, and $\eta(t)$ satisfies the conclusion of Theorem 5.1.

5.2. *Remark.* Let the Lie group A act analytically on G by automorphisms and let $L=G\times A$ be the corresponding semi-direct product. The Lie algebra $\mathfrak{l}$ can be identified with the vector space direct sum $\mathfrak{g}\oplus\mathfrak{a}$. Let $\mathbf{H}$, H and K be as in Theorem 5.1 and assume that $\mathfrak{a}$ satisfies the hypothesis on V in the statement of Theorem 5.1. In this case, Theorem 5.1 can be regarded as the extension of Theorem 3.1 from the case of the group of inner automorphisms to the case of an arbitrary Lie group of automorphisms of G.

5.3. *More on the cohomolgy of Lie groups.* Let H be a Lie group, let L be a closed normal subgroup of H and let V be an H-module. Set $V^L = \{v \in V \mid gv = v$ for every $g \in L]$. Let $Z^1(H, L, V) = \{\varphi \in Z^1(H, V) \mid \varphi(L) = \{0\}\}$ and let $H^1(H, L, V)$ denote the image of $Z^1(H, L, V)$ in $H^1(H, V)$.

5.3.1. LEMMA. *Let the notation be as above. Then $H^1(H, L, V)$ is canonically isomorphic to $H^1(H/L, V^L)$.*

The proof of 5.3.1 is elementary and will be omitted.

5.4. *Analytic families containing a fixed normal subgroup.*

5.4.1. PROPOSITION. *Let $\mathbf{H} = (H_t)_{t \in M}$ be an analytic family of Lie subgroups of G, let $t_0 \in M$ and set $H = H_{t_0}$. Assume that the component group H/H^0 is finitely generated and that there exists a normal Lie subgroup L of G such that L is a closed Lie subgroup of H_t for every $t \in M$ and such that $H^1(H/L, (\mathfrak{g}/\mathfrak{h})^L) = 0$. Then there exists an open neighborhood U of t_0 in M and an analytic map $\eta\colon U \to G$ with $\eta(0) = e$ such that $\eta(t) H \eta(t)^{-1} \subset H_t$ for every $t \in U$.*

We shall only sketch the proof, which is similar to the proof of Theorem 3.1. Let $\tilde{\mathbf{H}} = (H_{(t,x)})_{(t,x) \in M \times G}$ be the expanded analytic family, defined as in 1.7, let $\mathfrak{g} = \mathfrak{h} \oplus W$ as in § 2 and let $\alpha\colon M \times W^L \to M \times G$ be defined by $\alpha(t, w) = (t, \exp_G w)$. Let Ψ be a normal displacement function for the pull-back family $\alpha^*(\tilde{\mathbf{H}})$. It follows from the normality of L and the definition of $\tilde{\mathbf{H}}$ that $L \subset H_{(t,w)}$ for every $(t, w) \in M \times W^L$. Therefore $\Psi(x, t, w) = 0$ for every $(x, t, w) \in L \times M \times W^L$. Consequently, if $\mathbf{H}'$ is any analytic family obtained by pull-back from the family $\alpha^*(\tilde{\mathbf{H}})$, it follows that the first non-vanishing infinitesimal displacement along H of the family $\mathbf{H}'$ is an element of $Z^1(H, L, \mathfrak{g}/\mathfrak{h})$. Since $H^1(H, L, \mathfrak{g}/\mathfrak{h}) = H^1(H/L, (\mathfrak{g}/\mathfrak{h})^L) = 0$ one can, exactly as in 3.6, find a formal power series $f(t) = \sum_{j=0}^{\infty} g_j(t)$ ($g_j \in P_j(F^r, W^L)$ such that, for every $x \in H$, the formal power series $t \to \Psi(x, t, f(t))$ vanishes identically. The rest of the proof is as in § 3.

5.4.2. COROLLARY. *Let $\mathbf{H}$, G and H be as in 5.4.1. Assume that M is connected, that $H_t^0 = H^0$ for every $t \in M$ and that H_t/H_t^0 is of (finite) order m for every $t \in M$. Then H_t is conjugate to H for every $t \in M$.*

Proof. Let N be the normalizer of H^0 in G. Then N is a closed Lie subgroup of G and $H_t \subset N$ for every $t \in M$; thus $\mathbf{H}$ is an analytic family of Lie subgroups of N. Therefore we may assume that $G = N$, i.e. that H^0 is normal in G. Let $s \in M$. Since H_s/H_s^0 is finite, the cohomology group $H^1(H_s/H_s^0, (\mathfrak{g}/\mathfrak{h}_s)^{H^0})$ vanishes, and we may apply Proposition 5.4.1. This, plus the fact that the order of H_t/H_t^0 is independent of $t \in M$, implies that H_t is conjugate to H_s for every t in a neighborhood of s. Since M is connected, the conclusion of 5.4.2 now follows.

§ 6. Algebraic families of algebraic subgroups

In this section we shall deal with (complex) algebraic families of algebraic subgroups of a (complex) algebraic group. Our methods involve a somewhat unsatisfactory mixture of transcendental and algebraic techniques.

6.1. *Conventions.* Our basic reference for algebraic geometry and algebraic groups is [2] and we shall follow the notation and terminology therein. All algebraic varieties and algebraic groups are taken over $\mathbf{C}$. If X is an algebraic variety, then X has an underlying structure of (reduced) complex space, which we denote by the same letter X. Thus there are two topologies given on X, the Zariski topology and the usual topology of X as a complex space; we shall refer to the latter topology as the Hausdorff topology of X. If $f: X \to Y$ is a morphism of algebraic varieties, then f is also a holomorphic map of the underlying complex spaces. An *algebraic subgroup* of an algebraic group G is a subgroup of G which is a Zariski closed subset of G.

6.2. *Algebraic families of algebraic subgroups.*

6.2.1. *Definition.* Let G be an algebraic group and M a non-singular algebraic variety. An algebraic family of algebraic subgroups of G, parametrized by M, is a non-singular Zariski locally closed subvariety $\mathbf{H}$ of $G \times M$ satisfying the following conditions:

(a) Let $\pi_M: \mathbf{H} \to M$ denote the restriction of the projection $\mathrm{pr}_M: G \times M \to M$. Then π_M is surjective and is a submersion (i.e., the differential of π_M is surjective at each point of $\mathbf{H}$).

(b) For every $t \in M$, the fibre $\pi_M^{-1}(t)$ is of the form $H_t \times \{t\}$, where H_t is a subgroup of G.

It follows easily from the definitions that an algebraic family of algebraic subgroups of G has an underlying structure of a complex analytic family of Lie subgroups of the complex Lie group G.

Let $\mathbf{H} = (H_t)_{t \in M}$ be an algebraic family of algebraic subgroups of the algebraic group G. It follows from the definition that $\dim H_t$ is constant on each component of M.

6.2.2. Lemma. *Let $\mathbf{H} = (H_t)_{t \in M}$ be as above, assume that M is connected and let q be the common dimension of the $H_t (t \in M)$. Then the map $t \to \mathfrak{h}_t$ of M into $\Gamma_q(\mathfrak{g})$ is a morphism of algebraic varieties.*

The proof is elementary and will be omitted.

6.3. *Some lemmas on algebraic families.* Let $\mathbf{H} = (H_t)_{t \in M}$ be an algebraic family of algebraic subgroups of the algebraic group G and let K be an algebraic subgroup of G.

6.3.1. Lemma. *Let $A = \{t \in M \mid H_t \supset K\}$. Then A is a Zariski locally closed subset of M.*

Proof. Since $H_t \supset K^0$ if and only if $\mathfrak{h}_t \supset \mathfrak{k}$, it follows immediately from Lemma 6.2.2 that $A_0 = \{t \in M \mid H_t \supset K^0\}$ is a Zariski closed subset of M. Let $x_1, ..., x_n$ be a complete set of representatives for the cosets of K modulo K^0 and let $A_j = \{t \in M \mid x_j \in H_t\}$ $(j = i, ..., n)$. Then $H \cap (\{x_j\} \times M) = \{x_j\} \times A_j$ is Zariski locally closed and hence A_j is a Zariski locally closed subset of M. Thus $A = \bigcap_{j=0}^{n} A_j$ is Zariski locally closed.

We recall that a subset C of an algebraic variety X is *constructible* if C is the union of a finite number of Zariski locally closed subsets of X. A constructible subset of X contains a Zariski open, Zariski dense subset of its Zariski closure. If C is a constructible subset of X and $\varphi: X \to Y$ is a morphism of algebraic varieties, then $\varphi(C)$ is a constructible subset of Y.

6.3.2. LEMMA. *Let $B = \{t \in M \mid H_t$ contains a conjugate of $K\}$. Then B is a constructible subset of M.*

Proof. Define $\alpha: G \times M \times G \to G \times M$ by $\alpha(x, t, y) = (y^{-1}xy, t)$, let $\tilde{\mathbf{H}} = \alpha^{-1}(\mathbf{H})$ and let $\pi_{M \times G}: \tilde{\mathbf{H}} \to M \times G$ denote the restriction to $\tilde{\mathbf{H}}$ of the projection $\mathrm{pr}_{M \times G}: G \times (M \times G) \to M \times G$. Then the fibre $\pi_{M \times G}{}^{-1}(y, t)$ has the form $(yH_t y^{-1}) \times \{(t, y)\}$. Let $H_{(t,y)} = yH_t y^{-1}$. One checks easily that $\tilde{\mathbf{H}} = (H_{(t,y)})_{(t,y) \in M \times G}$ is an algebraic family of subgroups of G. By Lemma 6.3.1, $A = \{(t, y) \in M \times G \mid yH_t y^{-1} \supset K\}$ is a Zariski locally closed subset of $M \times G$. Let $\mathrm{pr}_M: G \times M \to M$ denote the projection. Then $B = \mathrm{pr}_M(A)$ is a constructible subset of M.

6.3.3. LEMMA. *Let X be an algebraic variety, let U be a subset of X which is open with respect to the Hausdorff topology of X and let B be a constructible subset of X which contains U. Then there exists a Zariski open subset V of X such that $U \subset V \subset B$.*

Proof. By induction on dim X (dimension as an algebraic variety). The lemma is trivial for dim $X = 0$. Let $n = \dim X$ and assume that the result holds for varieties of dimension less than n. Let $X_1, ..., X_r$ be the irreducible components of X which do not meet U and let $X' = X - \bigcup_{j=1}^{r} X_j$. Set $B' = X' \cap B$. Then X' is Zariski open in X and B' is a constructible subset of X'. Replacing X and B by (resp.) X' and B', we may assume that X meets every irreducible component of X. It is clear that U, and hence B, is Zariski dense in X. Thus B contains a Zariski open, Zariski dense subset V_0 of X. Let $C = X - V_0$. Then $U \cap C \subset B \cap C$ and dim $C < n$. Thus, by the inductive hypothesis, there exists a Zariski open subset V_1 of C such that $U \cap C \subset V_1 \subset B \cap C$. Let $V = V_0 \cup V_1$. Then V is Zariski open in X and $U \subset V \subset B$.

6.4. *Application of Theorem 3.1 to algebraic families of algebraic subgroups.*

6.4.1. THEOREM. *Let $\mathbf{H} = (H_t)_{t \in M}$ be an algebraic family of algebraic subgroups of the algebraic group G, let $t_0 \in M$ and let $H = H_{t_0}$. Let K be an algebraic subgroup of H such that $H^1(K, \mathfrak{g}/\mathfrak{h}) = 0$. Then there exists a Zariski open subset V of M containing t_0 such that H_t contains a conjugate of K for every $t \in V$.*

Proof. Let $B=\{t\in M \mid H_t$ contains a conjugate of $K\}$. Then B is constructible and by Theorem 3.1 there exists a Hausdorff open neighborhood U of t_0 such that $U\subset B$. By Lemma 6.3.3, there exists a Zariski open subset V of M such that $U\subset V\subset B$. This completes the proof.

6.4.2. *Remark.* The cohomology group which occurs in the statement of Theorem 6.4.1 is defined using holomorphic cochains. If L is an algebraic group and $\varrho: L\to \mathrm{GL}(V)$ a rational representation, one can define $Z^1_{\mathrm{alg}}(L, V)$ to be the set of all morphisms $\varphi: L\to V$ which satisfy the cocycle condition. We set $H^1_{\mathrm{alg}}(L, V)=Z^1_{\mathrm{alg}}(L, V)/B^1(L, V)$. We have $H^1_{\mathrm{alg}}(L, V)\subset H^1(L, V)$. If L is an affine algebraic group, then it is known [12] that $H^1(L, V)=H^1_{\mathrm{alg}}(L, V)$. However, this equality does not necessarily hold for the case of an arbitrary algebraic group (e.g. an extension of an abelian variety by the "additive group" $\mathbf{C}$). Thus our statement of Theorem 6.4.1 is somewhat unsatisfactory, except for the case of affine algebraic groups. It seems virtually certain that the statement of Theorem 6.4.1 remains valid with "$H^1(K, \mathfrak{g}/\mathfrak{h})=0$" replaced by "$H^1_{\mathrm{alg}}(K, \mathfrak{g}/\mathfrak{h})=0$". However, the proof will probably require more sophisticated algebraic techniques.

6.5. *Variation of the unipotent radical.* Let V be a finite dimensional vector space over $\mathbf{C}$. For every subalgebra $\mathfrak{a}$ of the Lie algebra $\mathfrak{gl}(V)$, let $\mathfrak{u}(\mathfrak{a})$ be the maximal ideal of nilpotence of the $\mathfrak{a}$-module V (see [3, p. 60] for the definition of the maximal ideal of nilpotence). If $\mathfrak{a}$ is the Lie algebra of an algebraic subgroup A of $\mathrm{GL}(V)$, then $\mathfrak{u}(\mathfrak{a})$ is the Lie algebra of the unipotent radical of A [5]. It is known [3, p. 127, Ex. 11] that $\mathfrak{u}(\mathfrak{a})$ can be characterized as follows:

6.5.1. Let E be the associative subalgebra of End(V) generated by 1_V and $\mathfrak{a}$. Then $\mathfrak{u}(\mathfrak{a})=\{x\in\mathfrak{a} \mid Tr(xu)=0$ for every $u\in E\}$.

Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$, let $n<\dim\mathfrak{g}$ and let $L_n(\mathfrak{g})$ be the algebraic variety of all n-dimensional subalgebras of $\mathfrak{g}$; $L_n(\mathfrak{g})$ is considered in the obvious way as a closed subvariety of the Grassmann variety $\Gamma_n(\mathfrak{g})$. For each $m\leqslant n$, let $L_{m,n}(\mathfrak{g})=\{\mathfrak{a}\in L_n(\mathfrak{g}) \mid \dim \mathfrak{u}(\mathfrak{a})=m\}$. Then, using 6.5.1, it is not difficult to prove:

6.5.2. (a) For each $r\leqslant n$, $\bigcup_{m\leqslant r} L_{m,n}(\mathfrak{g})$ is a Zariski open subset of $L_n(\mathfrak{g})$; in particular, each $L_{m,n}(\mathfrak{g})$ is a Zariski locally closed subset of $L_n(\mathfrak{g})$.

(b) For each $m\leqslant n$, the map $\mathfrak{a}\mapsto\mathfrak{u}(\mathfrak{a})$ of $L_{m,n}(\mathfrak{g})$ into $L_m(\mathfrak{g})$ is a morphism of algebraic varieties.

Now let G be an algebraic subgroup of $\mathrm{GL}(V)$ and let $\mathbf{H}=(H_t)_{t\in M}$ be an algebraic family of algebraic subgroups of G. For each $t\in M$, let $\mathfrak{u}_t=\mathfrak{u}(\mathfrak{h}_t)$; if U_t is the unipotent radical of H_t, then $\mathfrak{u}_t$ is the Lie algebra of U_t. Let $m=\min_{t\in M}\dim\mathfrak{u}_t$ and set $S=\{t\in M \mid \dim\mathfrak{u}_t=m\}$.

Then S is a Zariski open subset of M and the map β: $S \to L_m(\mathfrak{g})$ defined by $\beta(t) = \mathfrak{u}_t$ is a morphism of algebraic varieties.

Let $\mathfrak{N}$ (resp. $\mathfrak{U}$) be the Zariski closed subset of $\mathfrak{gl}(V)$ (resp. GL(V)) consisting of all nilpotent (resp. unipotent) endomorphisms of V and let η: $\mathfrak{N} \to \mathfrak{U}$ denote the restriction of the exponential map $\exp_{GL(V)}$: $\mathfrak{gl}(V) \to GL(V)$. Since $\mathfrak{N}$ consists of nilpotent matrices, η is a polynomial mapping. In fact, η is an isomorphism of algebraic varieties; the inverse of η is given by the log series.

The set $\{(x, t) \in \mathfrak{g} \times S \mid x \in \mathfrak{u}_t\}$ is a Zariski closed subset of $\mathfrak{g} \times S$. Since η: $\mathfrak{N} \to \mathfrak{U}$ is an isomorphism, we readily see that $\mathbf{U} = \{(g, t) \in G \times S \mid g \in U_t\}$ is a Zariski closed subset of $G \times S$ and that $\mathbf{U} = (U_t)_{t \in S}$ is an algebraic family of algebraic subgroups of G. Thus we have proved:

6.5.3. Lemma. *Let G be an affine algebraic group, let $\mathbf{H} = (H_t)_{t \in M}$ be an algebraic family of algebraic subgroups of G and, for each $t \in M$, let U_t be the unipotent radical of H_t. Let $m = \min_{t \in M}$* dim *U_t, let $S = \{t \in M \mid \dim U_t = m\}$ and let $\mathbf{U} = \{(g, t) \in G \times S \mid g \in U_t\}$. Then S is a Zariski open subset of M and $\mathbf{U} = (U_t)_{t \in S}$ is an algebraic family of algebraic subgroups of G.*

6.6. *Algebraic families of subgroups of an affine algebraic group.* If G is an affine algebraic group and U is the unipotent radical of G, then an algebraic subgroup L of G is a *Levi subgroup* of G if the canonical homomorphism $G \to G/U$ defines an isomorphism (of algebraic groups) of L onto G/U. It is known (in the characteristic zero case) that G admits Levi subgroups. A Levi subgroup L of G is reductive and any reductive subgroup of G is conjugate to a subgroup of L.

If R is a reductive affine algebraic group and ϱ: $R \to GL(V)$ is a rational representation of R, then $H^1(R, V) = H^1_{\mathrm{alg}}(R, V) = 0$ [12].

6.6.1. Proposition. *Let $\mathbf{H} = (H_t)_{t \in M}$ be an algebraic family of algebraic subgroups of the affine algebraic group G and let M be irreducible. For each $t \in M$ let L_t be a Levi subgroup of H_t. Let $r = \max_{t \in M} \dim L_t$* and let *$S = \{t \in M \mid \dim L_t = r\}$. Then S is a Zariski open subset of M and L_s^0 is conjugate to L_t^0 for every $s, t \in S$. Moreover, there exists a non-empty Zariski open subset $T \subset S$ such that L_s is conjugate to L_t for every $s, t \in T$.*

Proof. The fact that S is Zariski open follows from 6.5.3. It is a consequence of Theorem 6.4.1. that, for every $t \in S$, there exists a Zariski open subset S_t of S, containing t, such that, if $t_1 \in S_t$, then $L_{t_1}^0$ is conjugate to L_t^0. Let $s, t \in S$. Since M is irreducible, S_s and S_t intersect. This implies that L_s^0 and L_t^0 are conjugate.

Let p be the maximum number of components of L_t as t ranges over S and let $t_0 \in S$ be such that L_{t_0} has p components. It follows from Theorem 6.4.1 that there exists a Zariski open subset T of M containing t_0 such that H_t contains a conjugate of L_{t_0} for every $t \in T$. The conjugacy of Levi subgroups and the maximality of the number of components of L_{t_0} implies that L_t is conjugate to L_{t_0} for every $t \in T$. This proves Proposition 6.6.1.

§ 7. Complex-analytic families of subgroups of complex Lie groups

There exist reasonable analogues of the results of § 6 in the complex-analytic case, provided we restrict attention to *connected* Lie subgroups.

7.1. *An extension of Theorem 3.1 for complex-analytic families.* Let V be a finite dimensional complex vector space and let G be a connected complex Lie subgroup of $\mathrm{GL}(V)$. Let G' denote the Zariski closure of G in $\mathrm{GL}(V)$. If $g \in G'$, then $\mathfrak{g}$ is stable under $\mathrm{Ad}_{\mathrm{GL}(V)} g$ and consequently G is stable under $\mathrm{Int}_{\mathrm{GL}(V)}$: $x \mapsto gxg^{-1}$. By abuse of notation, we denote by $\mathrm{Int}_G g$: $G \to G$ $(g \in G')$ the restriction to G of $\mathrm{Int}_{\mathrm{GL}(V)}\, g$.

7.1.1. THEOREM. *Let $G \subset \mathrm{GL}(V)$ and G' be as above and let $\mathbf{H} = (H_t)_{t \in M}$ be a complex analytic family of complex Lie subgroups of G, parametrized by a connected complex manifold M. Let $t_0 \in M$, let $H = H_{t_0}$ and let K be a connected complex Lie subgroup of H such that $H^1(K, \mathfrak{g}/\mathfrak{h}) = 0$. Then there exists a connected dense open neighborhood U of t_0 in M, such that the following condition holds: for every $t \in U$, there exists $g \in G'$ such that* $(\mathrm{Int}_G\, g)\ (K) \subset H_t$.

We shall need the following lemma for the proof of Theorem 7.1.1.

7.1.2. LEMMA. *Let φ: $M \to X$ be a holomorphic map of a connected complex manifold M into an algebraic variety X, let $t \in M$ and assume that there exists an open neighborhood U of t and a constructible subset A of X such that $\varphi(U) \subset A$. Then there exists a connected dense open subset V of M, with $t \in V$, such that $\varphi(V) \subset A$.*

We omit the proof, which is straightforward.

7.1.3. *Proof of Theorem 7.1.1.* Let $q = \dim H$ and let $E \subset G' \times \Gamma_q(\mathfrak{g})$ be defined by

$$E = \{(g, W) \in G' \times \Gamma_q(\mathfrak{g}) \mid (\mathrm{Ad}_{\mathrm{GL}(V)} g\ (W) \subset \mathfrak{k}\};$$

E is a Zariski closed subset of $G' \times \Gamma_q(g)$. Let pr_2 denote the projection $G' \times \Gamma_q(\mathfrak{g}) \to \Gamma_q(\mathfrak{g})$

and let $A = \mathrm{pr}_2(E)$. Then A is a constructible subset of $\Gamma_q(\mathfrak{g})$; we note that $W \in \Gamma_q(\mathfrak{g})$ is in A if and only if W contains an $\mathrm{Ad}_{\mathrm{GL}(V)}(G')$ conjugate of $\mathfrak{k}$. Define $\varphi: M \to \Gamma_q(\mathfrak{g})$ by $\varphi(t) = \mathfrak{h}_t$. It is an easy consequence of Theorem 3.1 that there exists an open neighborhood U_0 of t_0 in M such that $\varphi(U_0) \subset A$. The proof now follows from Lemma 7.1.2.

7.1.4. *Remark.* If G is complex-analytically isomorphic to a complex linear algebraic group, then one may assume that $G = G'$ in Theorem 7.1.1. It has been shown by Hochschild and Mostow [11] that G is complex-analytically isomorphic to a complex linear algebraic group if and only if the factor group of the radical of G modulo its maximal normal connected nilpotent complex Lie subgroup is reductive.

7.2. *Reductive complex Lie groups.* A complex Lie group G is *reductive* if it satisfies the following conditions: (i) component group G/G^0 is finite; (ii) G admits a faithful (holomorphic) representation; and (iii) every linear representation of G is semi-simple. We shall need the following results concerning reductive complex Lie groups.

7.2.1. Let $\varrho: G \to \mathrm{GL}(V)$ be a linear representation of the reductive complex Lie group G. Then $H^1(G, V) = 0$.

7.2.2. Let G be a connected complex Lie group which admits a faithful representation. Then G has a maximal reductive complex Lie subgroup H, which is closed and connected, and every reductive complex Lie subgroup of G is conjugate to a subgroup of H.

The proof of 7.2.1 is an easy consequence of the semi-simplicity of linear representations of a reductive group. 7.2.2. is proved in [11].

7.2.3. PROPOSITION. *Let G be a connected complex Lie group which admits a faithful linear representation Let $\mathbf{H} = (H_t)_{t \in M}$ be a complex-analytic family of complex Lie subgroups of G, parametrized by a connected complex manifold M, such that each $H_t (t \in M)$ is connected. For each $t \in M$, let L_t be a maximal connected reductive complex Lie subgroup of H_t. Then there exists a connected dense open subset U of M such that if $s, t \in U$, then L_s is conjugate to L_t.*

The proof is an easy consequence of 7.1.1, 7.2.1 and 7.2.2.

Chapter II. The variation of isotropy subgroups

§ 8. The analytic family $\mathfrak{G}^*$

8.1. Let the Lie group G act analytically on the connected analytic manifold M, let $q = \min_{t \in M} \dim G_t$ and let $M_0 = \{t \in M \mid \dim G_t = q\}$. An elementary argument (see e.g. [18, § 2]) shows that M_0 is a dense open subset of M, the complement of an analytic subset. We assume throughout § 8.1 that $M_0 = M$, i.e. that all orbits of G on M have the same dimension. We set $\mathfrak{G} = \{(g, t) \in G \times M \mid g \in G_t\}$. In general, $\mathfrak{G} = (G_t)_{t \in M}$ is *not* an analytic family of Lie subgroups of G.

We say that a point $(g, t) \in \mathfrak{G}$ is *regular* if there exists an open neighborhood U of t in M and an analytic map $s: U \to G$ such that $s(u) \in G_u$ for every $u \in U$ and such that $s(t) = g$. Let $\mathfrak{G}^*$ denote the set of regular points of $\mathfrak{G}$ and, for $t \in M$, let $G_t^* = \{g \in G_t \mid (g, t)$ is a regular point of $\mathfrak{G}\}$.

8.1.1. LEMMA. *G_t^* is a closed Lie subgroup of G_t and $G_t^0 \subset G_t^*$.*

Proof. Let $g_1, g_2 \in G_t^*$ and let $s_j: U \to G$ $(j = 1, 2)$ be an analytic map of an open neighborhood U of t in M into G such that $s_j(u) \in G_u$ for every $u \in U$ and $s_j(t) = g_j$. Define an analytic map $s: U \to G$ by $s(u) = s_1(u) s_2(u)^{-1}$. Then $s(u) \in G_u$ for every $u \in U$ and $s(t) = g_1 g_2^{-1}$. Thus $g_1 g_2^{-1} \in G_t^*$. This shows that G_t^* is a subgroup of G.

Let $\mathfrak{h} = \mathfrak{g}_t$, let W be an r-dimensional subspace of $\mathfrak{g}$ such that $\mathfrak{g}$ is the vector space direct sum of $\mathfrak{h}$ and W and let Γ_W and $\Theta: \Gamma_W \to \mathrm{Hom}_F(\mathfrak{h}, W)$ be as in 1.5. Choose an open neighborhood U of t such that $\mathfrak{g}_u \in \Gamma_W$ for $u \in U$ and define an analytic map $\varphi: U \to \mathrm{Hom}_F(\mathfrak{h}, W)$ by $\varphi(u) = \Theta(\mathfrak{g}_u)$. Let $g \in G_t^0$. We may write $g = \exp_G(X_1) \dots \exp_G(X_n)$, where $X_1, \dots, X_n \in \mathfrak{h}$. Define an analytic map $s: U \to G$ by $s(u) = \exp_G(X_1 + \varphi(u) X_1) \dots \exp_G(X_n + \varphi(u) X_n)$. Then $s(u) \in G_u^0$ for every $u \in U$ and $s(t) = g$. Thus $g \in G_t^*$ and we have shown that $G_t^0 \subset G_t^*$. Since G_t^0 is a closed Lie subgroup of G_t, so is G_t^*.

8.1.2. LEMMA. *Let $r = \dim M$ and let p be the common dimension of the orbits of G on M. Let $\psi: G \times M \to M \times M$ be defined by $\psi(g, t) = (g \cdot t, t)$. Then the differential of ψ at each point of $G \times M$ is of rank $(p + r)$. In particular, ψ is a subimmersion.*

The proof of 8.1.2 follows from an easy computation of differentials and will be omitted.

8.1.3. PROPOSITION. (a) *$\mathfrak{G}^*$ is an open subset of $\mathfrak{G}$.* (b) *$\mathfrak{G}^*$ is an analytic submanifold of $G \times M$.* (c) *$\mathfrak{G}^*$ is an analytic family of Lie subgroups of G.*

Proof. Let $(g, t) \in G^*$ and let $s: U \to G$ be an analytic map of an open neighborhood U of t into G such that $s(u) \in G_u$ for every $u \in U$ and such that $s(t) = g$. Since ψ is a subimmersion, there exists an open neighborhood S of (g, t) in $G \times M$, an open neighborhood T of (t, t) in $M \times M$ and a closed submanifold V of T such that $\psi(S) = V$ and such that ψ: $S \to V$ is a submersion. Choose an open neighborhood A of t in M and an open neighborhood B of g in G such that the following conditions hold: $A \times A \subset T$; $B \times A \subset S$; $A \subset U$; and $s(A) \subset B$. Let $S_1 = S \cap \psi^{-1}(A \times A)$; S_1 is an open neighborhood of (g, t) in $G \times M$. Let ψ_1: $S_1 \to A \times A$ denote the restriction of ψ and let $V_1 = V \cap (A \times A)$. Then V_1 is a closed analytic submanifold of $A \times A$ and ψ_1: $S_1 \to V_1$ is a submersion.

Let Δ_A denote the diagonal of A; Δ_A is a closed analytic submanifold of $A \times A$. If $u \in A$, then $(s(u), u) \in B \times A$ and $\psi(s(u), u) = (u, u)$. Therefore $\Delta_A \subset V_1$. Consequently Δ_A is a closed analytic submanifold of V_1 and, since ψ_1: $S_1 \to V_1$ is a submersion, $\psi_1^{-1}(\Delta_A) = S_1 \cap \mathfrak{G}$ is a closed analytic submanifold of S_1.

Assume that U is sufficiently small so that $\mathfrak{g}_u \in \Gamma_W$ for every $u \in U$ and let φ: $U \to \operatorname{Hom}_F(\mathfrak{h}, W)$ be as above. Define the analytic map $\alpha: \mathfrak{h} \times U \to G \times M$ by $\alpha(X, u) = (s(u) \exp_G(X + \varphi(u) \cdot X), u)$. For $u \in U$ we have $(X + \varphi(u) \cdot X) \in \mathfrak{g}_u$ and thus $(s(u) \exp_G(X + \varphi(u) \cdot X)) \in G_u$. Consequently $\alpha(\mathfrak{h} \times U) \subset \mathfrak{G}$. One checks easily that, in fact, $\alpha(\mathfrak{h} \times U) \subset \mathfrak{G}^*$. Moreover, the differential $d\alpha_{(0, t)}$ is injective. An easy dimension count shows that $\dim(\mathfrak{h} \times U) = \dim(S_1 \cap \mathfrak{G})$. It follows from the inverse function theorem that α maps an open neighborhood of $(0, t)$ in $\mathfrak{h} \times U$ isomorphically onto an open neighborhood of (g, t) in $S_1 \cap \mathfrak{G}$. This proves 8.1.3. (*a*)–(*b*). It follows from the definition of $\mathfrak{G}^*$ that π_M: $\mathfrak{G}^* \to M$ is a submersion, which proves 8.1.3. (c).

Combining 8.1.3, 8.1.1 and 8.1.2, we obtain

8.1.4. Corollary. *Let* $\mathfrak{G}^0 = \{(g, t) \in G \times M \mid g \in G_t^0\}$. *Then* $\mathfrak{G}^0$ *is an analytic family of Lie subgroups of* G.

We note that $\mathfrak{G}^*$ can be equivalently defined as follows. Let $\mathfrak{G}_1$ be the subset of G consisting of all points $(g, t) \in \mathfrak{G}$ such that $\mathfrak{G}$ is locally a submanifold of $G \times M$ at (g, t). (See [22, p. LG 3.22] for the appropriate definition.) Let π_M: $\mathfrak{G}_1 \to M$ denote the restriction to $\mathfrak{G}_1$ of the projection and let $\mathfrak{G}_1^*$ be the set of points $(g, t) \in G_1$ at which the differential $d(\pi_M)_{(g, t)}$ is surjective. Then $\mathfrak{G}_1^* = \mathfrak{G}^*$. This follows easily from 8.1.3 and the implicit function theorem.

8.2. *An example.* If the Lie group G acts analytically on the analytic manifold M, it is not necessarily the case that there exists a non-empty open subset U of M such that

$G_t = G_t^*$ for every $t \in U$. One counterexample goes as follows (see [23, p. 758 and p. 797] for more details):

Let f be the 2×2 matrix $\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}$. Then f induces a diffeomorphism f_0 of the torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ and the periodic points of f_0 are dense. Let $\mathbf{Z}$ act on $T^2\times\mathbf{R}$ by $n\cdot(t, s) = (f_0^n(t), s+n)$ and let M be the orbit space $(T^2\times\mathbf{R})/\mathbf{Z}$. Then M is a compact three-dimensional manifold and the suspension of f_0 gives a flow, i.e. an action of $\mathbf{R}_1$ on M; the set P of points lying on periodic orbits is dense in M. For this action of $G=\mathbf{R}$ on M we have $G_t^* = \{0\}$ for every $t\in M$, but $G_t \neq G_t^*$ for t belonging to the dense subset P of M.

8.3. *Glimm's results on the continuity of the isotropy subgroups.* Let the Lie group G act continuously on the locally compact space M. Following J. Glimm [7], we say that the isotropy subgroups are *continuous* at $t\in M$ if, for every sequence (t_j) in M converging to t and for every $g\in G_t$, there exists a sequence (g_j) in G converging to g such that $g_j\in G_{t_j}$ for every j. Glimm proves the following result:

8.3.1. Let the Lie group G act continuously on the locally compact space M and assume that every orbit of G in M is locally closed. Then there exists a dense open subset U of M such that the isotropy subgroups are continuous at each point of M.

In order to apply this result to analytic transformation groups we need the following lemma:

8.3.2. LEMMA. *Let the Lie group G act analytically on the analytic manifold M such that all orbits of G on M have the same dimension. Let $t\in M$ and let $g\in G_t$. Then the following conditions are equivalent:*

(i) *For every sequence (t_j) in M converging to t, there exists a sequence (g_j) in G converging to g such that $g_j\in G_{t_j}$ for every j.*

(ii) $g\in G_t^*$.

Proof. It is immediate that (ii) implies (i). Assume that (i) is satisfied, let $\mathfrak{G}$ be as in 8.1 and let π_M: $\mathfrak{G}\to M$ denote the restriction of pr_M: $G\times M\to M$. Let U be a neighborhood of (g, t) in $\mathfrak{G}$. Then I claim that $\pi_M(U)$ is a neighborhood of t in M. For assume that $\pi_M(U)$ is not a neighborhood of t. Then there exists a sequence (t_j) in M converging to t such that $t_j\notin\pi_M(U)$ for every j. By (i), there exists a sequence (g_j) in G converging to g such that $g_j\in G_{t_j}$ for every j. Then $(g_j, t_j)\in\mathfrak{G}$ for every j and (g_j, t_j) converges to (g, t). For j sufficiently large, $(g_j, t_j)\in U$ and hence $t_j\in\pi_M(U)$, which is a contradiction.

Let ψ: $G\times M\to M\times M$ be as in 8.1.2; ψ is a subimmersion. If V is a neighborhood of (g, t) in $G\times M$, then $\psi(\mathbf{G}\cap V)=\{(v, v)\,|\,v\in\pi_M(\mathbf{G}\cap V)\}$, and $\pi_M(\mathbf{G}\cap V)$ is a neighborhood of t in M. Since ψ is a subimmersion, we may choose an open neighborhood V of (g, t) in $G\times M$, an open neighborhood A of t in M and a closed analytic submanifold V_1 of $A\times A$ such that the following conditions hold: $\psi(V)=V_1$; ψ: $V\to V_1$ is a submersion; and $\Delta_A\subset V_1$. It follows that $\mathbf{G}\cap V=\psi^{-1}(\Delta_A)$ is a closed analytic submanifold of V. Moreover, since ψ maps $\mathbf{G}\cap V$ submersively onto Δ_A, it follows easily that there exists an analytic map s: $A\to G$ such that $s(v)\in G_v$ for every $v\in A$ and such that $s(t)=g$. Thus $g\in G_t^*$ and 8.3.2 is proven.

As an easy consequence of Proposition 8.1.3 and Lemma 8.3.2, we have

8.3.3. Proposition. *Let the Lie group G act analytically on the connected analytic manifold M such that each orbit of G on M is locally closed. Then there exists a dense open subset U of M such that all orbits of G on U have the same dimension and such that $G_t=G_t^*$ for every $t\in U$. Thus if $\mathbf{G}'=\{(g, t)\in G\times U\,|\,g\in G_t\}$, then $\mathbf{G}'=(G_t)_{t\in U}$ is an analytic family of Lie subgroups of G.*

8.3.4. *Remark.* If (G, M) is an algebraic transformation space (over $\mathbf{C}$), then each orbit of G on M is locally closed. Using this, one can show, for example, that if ϱ: $G\to \mathrm{GL}(V)$ is a linear representation of a (real or complex) reductive Lie group, then every orbit of G on V is locally closed. Thus 8.3.3 applies to all of these examples. For the case of algebraic transformation spaces, however, we have a much stronger result (see § 9).

§ 9. Algebraic transformation groups

9.1. Let (G, M) be an algebraic transformation space (over $\mathbf{C}$), where M is a non singular algebraic variety. We assume that all orbits of G on M have the same dimension. Let $\mathbf{G}=\{(g, t)\in G\times M\,|\,g\cdot t=t\}$, let $\mathbf{G}_1$ be the set of simple points of $\mathbf{G}$ and let $\mathbf{G}^*$ be the set of points $(g, t)\in \mathbf{G}_1$ at which the differential $d(\pi_M)_{(g,t)}$: $T_{(g,t)}(\mathbf{G}_1)\to T_t(M)$ is surjective. Then $\mathbf{G}^*$ is a Zariski open subset of $\mathbf{G}$, hence $\mathbf{G}^*$ is a Zariski locally closed subset of $G\times M$. For $t\in M$, let $G_t^*=\{g\in G\,|\,(g, t)\in \mathbf{G}^*\}$. Then, as in 8.1, G_t is a subgroup of G and $G_t^0\subset G_t^*\subset G_t$; thus G_t^* is an algebraic subgroup of G. Therefore $\mathbf{G}^*=(G_t^*)_{t\in M}$ is an algebraic family of algebraic subgroups of G.

9.1.1. Proposition. *Let (G, M) be an algebraic transformation space with M a non-singular algebraic variety and assume that all orbits of G on M have the same dimension.*

Let $t_0 \in M$, let $H = H_{t_0}$ and let K be an algebraic subgroup of $H^* = G^*_{t_0}$ such that $H^1(K, \mathfrak{g}/\mathfrak{h}) = 0$. Then there exists a Zariski open subset U of M containing t_0 such that G^*_t contains a conjugate of K for every $t \in U$.

The proof follows immediately from 6.4.1.

9.1.2. COROLLARY. *Let G, M, t_0 and H be as in 9.1.1. Assume that H contains a connected reductive affine algebraic group R. Then there exists a Zariski open subset U of M containing t_0 such that G_t contains a conjugate of R for every $t \in U$. In particular, assume that H is a reductive affine algebraic group. Then there exists a Zariski open neighborhood U of t_0 in M such that G^0_t is conjugate to H^0 for every $t \in U$.*

9.2. *Existence of a Zariski open subset on which the isotropy subgroups form an algebraic family.* Let (G, X) be an algebraic transformation space. Rosenlicht [21] has proved that there exists a non-empty G-stable Zariski open subset Y of X such that there exists a quotient π: $Y \to G\backslash Y$ for the action of G on Y. (See [2, pp. 171–180] for a discussion of such quotients.) We may assume that Y and $G\backslash Y$ are non singular and that π is a submersion. Let ψ: $G \times Y \to Y \times Y$ be defined by $\psi(g, t) = (g \cdot t, t)$ and let $E = \psi(G \times Y)$; E is the graph of the equivalence relation defined by the orbits of G on Y. We note that E is the inverse image of $\Delta_{G\backslash Y}$ under $\pi \times \pi$: $Y \times Y \to (G\backslash Y) \times (G\backslash Y)$. Since $\pi \times \pi$ is a submersion and Δ_Y is a non-singular subvariety of $Y \times Y$, it follows that E is a Zariski closed non-singular subvariety of $Y \times Y$. But, by 8.1.2, ψ is of constant rank (it follows from the existence of the quotient $G\backslash Y$ that all orbits of G on Y have the same dimension). Thus ψ maps $G \times Y$ submersively onto E. Consequently $\mathfrak{G} = \psi^{-1}(\Delta_Y)$ is a closed non-singular subvariety of $G \times M$. But $\mathfrak{G} = \{(g, t) \in G \times Y \mid g \cdot t = t\}$. Thus $\mathfrak{G} = (G_t)_{t \in Y}$ is an algebraic family of algebraic subgroups of G. Thus we have proved:

9.2.1. LEMMA. *Let (G, X) be an algebraic transformation space. Then there exists a non-empty, non-singular, G-stable, Zariski open subset Y of X such that $\mathfrak{G} = (G_t)_{t \in Y}$ is an algebraic family of algebraic subgroups of G. Moreover, we may assume that the number of connected components of G_t is independent of $t \in Y$.*

The last statement in 9.2.1 follows immediately from standard properties of morphisms of algebraic varieties (see e.g. [2, p. 39]).

9.3. *Affine algebraic transformation groups.*

9.3.1. THEOREM. *Let (G, X) be an algebraic transformation space, where G is an affine algebraic group. For each $t \in X$, let U_t be the unipotent radical of G_t an let L_t be a Levi subgroup* *of G_t. Then there exists a finite set $V_1, \ldots, V_m$ of G-stable, non-singular, Zariski locally closed subvarieties of X such that the following conditions hold:*

(i) $X = \bigcup_{j=1}^m V_j$.

(ii) *For each $j = 1, \ldots, m$, V_j is a Zariski open subset of $X - \bigcup_{i=1}^{j-1} V_i$.*

(iii) *If $s, t \in V_j$, then L_s is conjugate to L_t.*

(iv) *$\mathbf{U}_j = (U_t)_{t \in V_j}$ is an algebraic family of algebraic subgroups of G.*

Proof. It follows from 9.1.2, 9.2.1 and 6.5.3 that there exists a non-empty, G-stable, non-singular Zariski open subset V_1 of X satisfying conditions (iii) and (iv). The proof follows by an easy induction, using the fact that X is a Noetherian space with respect to the Zariski topology.

§ 10. Analytic transformation groups

10.1. PROPOSITION. *Let the Lie group G act analytically on the analytic manifold M such that all orbits of G on M have the same dimension. Let $t_0 \in M$, let $H = H_{t_0}$ and let K be a Lie subgroup of $H^* = G_{t_0}^*$ such that K/K^0 is finitely generated and $H^1(K, \mathfrak{g}/\mathfrak{h}) = 0$. Then there exists an open neighborhood U of t_0 in M and an analytic map η: $U \to G$ such that $\eta(t)K\eta(t)^{-1} \subset G_t$ for every $t \in U$.*

The proof of Proposition 10.1 follows immediately from Theorem 3.1 and Proposition 8.1.3.

10.2. *Reductive subroups of isotropy subgroups.* A real Lie group R is *reductive* if: (i) R/R^0 is finite; (ii) R admits a faithful linear representation; and (iii) every linear representation of R is semi-simple. If R is reductive and if ϱ: $R \to \mathrm{GL}(V)$ is a linear representation of R, then it follows immediately from (iii) that $H^1(R, V) = 0$. Let R be a real Lie group which satisfies (i) and (ii) above. Then it is known [10] that R is reductive if and only if the center Z of R is compact and R/Z is a semisimple Lie group.

Let the notation be as in 10.1 and let K be a reductive (real or complex) Lie subgroup of H. Then $H^1(K, \mathfrak{g}/\mathfrak{h}) = 0$ and therefore the conclusion of 10.1 holds. Thus, for example, in the real-analytic case, if G_t contains a compact connected subgroup K, then G_s contains a conjugate of K for every s in a neighborhood of t. In the complex-analytic case, we can obtain somewhat stronger results.

10.3. *Complex-analytic transformation groups.* Let V be a finite-dimensional complex vector space and let G be a connected complex Lie subgroup of $\mathrm{GL}(V)$. Let G' denote the

Zariski closure of G in $\mathrm{GL}(V)$ and, for $g \in G'$, let $\mathrm{Int}_G g$: $G \to G$ be defined as in 7.1. Let G act complex-analytically on the connected complex manifold M and assume that all orbits of G on M have the same dimension.

10.3.1. PROPOSITION. *Let (G, M) be as above, let $t_0 \in M$ and let $H = G_{t_0}$. Let K be a connected Lie subgroup of H such that $H^1(K, \mathfrak{g}/\mathfrak{h}) = 0$. Then there exists a dense open neighborhood U of t_0 in M, such that the following condition holds: for every $t \in U$, there exists $g \in G'$ such that* $(\mathrm{Int}_G g)\ (K) \subset G_t$.

10.3.2. PROPOSITION. *Let (G, M) be as above and, for every $t \in M$, let L_t be a maximal connected reductive complex Lie subgroup of G_t. Let $m = \max_{t \in M}$* dim L_t and *let $U = \{t \in M \mid \dim L_t = m\}$. Then U is the complement of an analytic subset of M and, for every s, $t \in U$, L_s is conjugate to L_t.*

The proof of Proposition 10.3.1 follows from 8.1.4 and 7.1.1. Proposition 10.3.2 is an immediate consequence of 8.1.4 and 7.2.3.

Our methods seem unsatisfactory in dealing with components of the isotropy subgroups, even when all isotropy subgroups have a finite number of components. Along this line, we conjecture that the following complex-analytic analogue of 9.3.1 holds.

10.3.3. CONJECTURE. *Let (G, M) be as above and assume that each isotropy subgroup has only a finite number of components. For each $t \in M$, let L_t be a maximal reductive complex Lie subgroup of G_t. Then there exists a connected dense open subset U of M, such that L_s is conjugate to L_t for every $s, t \in U$.*

(We note that if K is a complex Lie subgroup which admits a faithful linear representation and if the component group K/K^0 is finite, then K admits a maximal reductive complex Lie subgroup and any two such subgroups are conjugate in K. The proof is the same as in the case in which K is connected [10]. Thus the groups L_t in the above conjecture exists.)

§ 11. Relation between analytic families of subgroups and families of isotropy subgroups

11.1. One question which naturally arises is whether every analytic family of Lie subgroups of G can be obtained as the family of isotropy subgroups for an analytic action of G on an analytic manifold. It is clear that the answer to the question as stated is negative. For example, it is easy to find analytic families $\mathbf{H} = (H_t)_{t \in M}$ which do not contain every conjugate of every H_t; such a family cannot be the family of isotropy subgroups for an analytic action of G on M. However, the following result shows that every *closed* analytic family of Lie subgroups of G can be obtained by pull-back from a family of isotropy subgroups.

11.1.1. PROPOSITION. *Let* $\mathbf{H}=(H_t)_{t\in M}$ *be an analytic family of Lie subgroups of the Lie group* G, *and assume that* $\mathbf{H}$ *is a closed analytic submanifold of* $G\times M$. *Then there exist an analytic manifold* N, *an analytic action of* G *on* N *and an analytic map* $\varphi: M\to N$ *such that the following conditions hold:*

(a) $\mathbf{G}=(G_t)_{t\in N}$ *is an analytic family of Lie subgroups of* G.

(b) $\varphi^*(\mathbf{G})=\mathbf{H}$ (*this implies in particular that* $G_{\varphi(t)}=H_t$ *for every* $t\in M$).

Before giving the proof of Proposition 11.1, we need to recall some elementary facts concerning equivalence relations on manifolds.

11.2. Let X be an analytic manifold and let $\mathcal{R}\subset X\times X$ be an equivalence relation on X. Then the following conditions on $\mathcal{R}$ are equivalent:

(a) The quotient set $X/\mathcal{R}$ admits a structure of analytic manifold such that the canonical map $p: X\to X/\mathcal{R}$ is a submersion.

(b) Let $\pi_X: \mathcal{R}\to X$ denote the restriction to $\mathcal{R}$ of the projection pr_2 of $X\times X$ on its second factor. Then $\mathcal{R}$ is a closed analytic submanifold of $X\times X$ and π_X is a submersion.

The proof of 11.2 is given in [22, p. LG-3.27].

11.3. *Proof of Proposition 11.1.* Let $\mathcal{R}$ be the equivalence relation on $G\times M$ defined by: $(x, t)\sim_{\mathcal{R}}(y, s)$ if and only if $t=s$ and $x^{-1}y\in H_t$. We claim that:

(i) $\mathcal{R}$ is a closed analytic submanifold of $(G\times M)\times(G\times M)$. For the proof of (i), let

$$\varphi: (G\times M)\times(G\times M)\to G\times M\times M$$

be defined by $\varphi((x, t), (y, s))=(x^{-1}y, t, s)$. Then φ is a submersion. Let

$$\mathbf{H}'=\{(x, t, s)\in G\times M\times M \mid s=t \text{ and } x\in H_t\}.$$

Then $\mathbf{H}'$ is the image of $\mathbf{H}$ under the analytic manifold isomorphism of $G\times M$ onto $G\times\Delta_M$ given by $(x, t)\to(x, (t, t))$. Since $\mathbf{H}$ is a closed analytic submanifold of $G\times M$, and $G\times\Delta_M$ is a closed analytic submanifold of $G\times M\times M$, we see that $\mathbf{H}'$ is a closed analytic submanifold of $G\times M\times M$. Since φ is a submersion, $\mathcal{R}=\varphi^{-1}(\mathbf{H}')$ is a closed analytic submanifold of $(G\times M)\times(G\times M)$, which proves (i).

Let $\pi_2: \mathcal{R}\to G\times M$ denote the restriction to $\mathcal{R}$ of the projection $\mathrm{pr}_2: (G\times M)\times(G\times M)\to G\times M$.

(ii) π_2 is a submersion.

Let $((x, t_0), (y, t_0)) \in \mathcal{R}$ Choose an open neighborhood U of t_0 in M and an analytic map α: $U \to G$ such that $\alpha(t) \in H_t$ for every $t \in U$ and $\alpha(t_0) = x^{-1}y$. Define an analytic map β: $G \times U \to (G \times U) \times (G \times U)$ by $\beta(z, t) = ((z\alpha(t)^{-1}, t), (z, t))$. Then, since $(z\alpha(t)^{-1})^{-1}z = \alpha(t)$, we see that $\beta(G \times U) \subset \mathcal{R}$. Furthermore, $\pi_2 \circ \beta$ is the identity map of $G \times U$. It follows that the differential of π_2 at $((x, t_0), (y, t_0))$ is surjective. This shows that π_2 is a submersion.

It is a consequence of (i) and (ii) that there exists a unique structure of analytic manifold on the quotient $N = (G \times M)/\mathcal{R}$ such that the canonical map p: $G \times M \to N$ is a submersion.

We define an analytic action of G on $G \times M$ by $x(y, t) = (xy, t)$ for $x, y \in G$ and $t \in M$. Using 11.2 and functorial properties of quotient structures, it is not difficult to show that there is an induced analytic action of G on N such that p: $G \times M \to N$ is a G-equivariant map. Furthermore, a similar argument shows that $\mathfrak{G} = (G_t)_{t \in N}$ is analytic family of Lie subgroups of G. Define an analytic map φ: $M \to N$ by $\varphi(t) = p(e, t)$. Then it is easy to check that $\varphi^*(\mathfrak{G}) = \mathbf{H}$. This proves 11.1.1.

11.4. *Remark.* The only place in the above proof in which we used the fact that $\mathbf{H}$ was a *closed* analytic submanifold of $G \times M$ was to insure that N was Hausdorff. Thus if we drop our requirement that manifolds be Hausdorff, Proposition 11.1 remains true if we only assume that $\mathbf{H}$ is an analytic submanifold of $G \times M$.

§ 12. Examples

12.1. Let $G = \mathrm{SL}_2(\mathbf{C})$, let V be the four-dimensional vector space of homogeneous polynomials of degree three in two indeterminates X and Y, and let G act on V in the usual way. Every $P \in V$ can be written as a product of three linear forms. It is a classical fact that $\mathrm{PSL}_2(\mathbf{C}) = \mathrm{SL}_2(\mathbf{C})/\{\pm 1\}$ acts simply transitively on the set of triples of distinct lines in$\mathbf{C}^2$. Using this fact, it is easy to show by direct computation that if $P \in V$ has no linear factor of multiplicity greater than one, then the isotropy subgroup G_P is a cyclic group of order three. However, if P has a linear factor of multiplicity two, e.g. $P = X^2Y$, then $G_P = \{e\}$. Let $V_0 = \{P \in V \mid G_P \text{ is finite}\}$; V_0 consists of all cubic forms with no linear factor of multiplicity three. Let $Q = X^2Y$. The orbit $G(Q)$ is a Zariski closed subset of V_0 and we have: $G_P = \{e\}$ for $P \in G(Q)$; $G(P)$ is cyclic of order three for $P \in (V_0 - G(Q))$. For the action of G on V_0, the number of components of G_P is a lower semi-continuous function of $P \in V_0$, but is not an upper semi-continuous function. It is trivial to check that $G_P = G_P^*$ for every $P \in V_0$, hence $(G_P)_{P \in V_0}$ is an algebraic family of algebraic subgroups of G.

12.2. Let $G=\mathrm{SL}_n(\mathbf{C})$, let $\mathfrak{g}=\mathfrak{sl}_n(\mathbf{C})$ and let G act on $\mathfrak{g}$ via the adjoint representation. We say that an element $x\in\mathfrak{g}$ is *regular* if the isotropy subgroup G_x is of dimension $n-1$ and we let V_0 be the set of regular elements of $\mathfrak{g}$; V_0 is a Zariski open subset of $\mathfrak{g}$. It follows from the results of Kostant [15] on the adjoint representation that $\mathfrak{G}=(G_x)_{x\in V_0}$ is an algebraic family of subgroups of G. If $x\in V_0$ is semi-simple, then G_x is an algebraic torus and hence is connected; the regular semi-simple elements form a Zariski-open subset V_1 of V_0. If $x\in(V_0-V_1)$, it is not necessarily the case that G_x is connected. For example, if x is the regular nilpotent element $E_{12}+E_{23}+\dots+E_{n-1,n}$ (E_{ij} denotes the matrix with 1 in the i–j position and 0 elsewhere), then G_x has n connected components. For the action of G on V_0, we can prove the following properties:

(a) The number of connected components of G_x is an upper semi-continuous function of $x\in V_0$.

(b) $\mathfrak{G}=(G_x)_{x\in V_0}$ is a connected analytic manifold, but $\mathfrak{G}\neq\mathfrak{G}^0$ $\quad(\mathfrak{G}^0=(G_x^0)_{x\in V_0})$.

12.3. The following example was pointed out by Mostow [16]. Let $N\subset\mathrm{GL}_3(\mathbf{R})$ be the two-dimensional vector group consisting of all matrices

$$\tau(a,b)=\begin{pmatrix}1 & a & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\quad(a,b\in\mathbf{R}).$$

N acts on $\mathbf{R}^3$ by matrix multiplication and the hyperplane $A=\{(x,y,z)\in\mathbf{R}^3\,|\,z=1\}$ is N-stable. If $t=(x,y,1)\in A$, then the isotropy subgroup N_t is given by $N_t=\{\tau(a,b)\,|\,ay+b=0\}$. Thus each orbit of N on A is of codimension one (a line) and, since N is abelian, if $s,t\in A$, then N_s and N_t are conjugate if and only if s and t lie on the same N-orbit.

Let $\mathfrak{s}$ be the Lie subalgebra of $\mathfrak{gl}_3(\mathbf{R})$ spanned by E_{11}, E_{33}, E_{12} and E_{13} and let S be the corresponding connected Lie subgroup of $\mathrm{GL}_3(\mathbf{R})$; S is a closed, simply connected subgroup of $\mathrm{GL}_3(\mathbf{R})$. Let $d\varrho\colon\mathfrak{s}\to\mathfrak{n}$ be the Lie algebra homomorphism defined by $d\varrho(E_{11})=E_{12}$, $d\varrho(E_{33})=E_{13}$, $d\varrho(E_{12})=d\varrho(E_{13})=0$ and let $\varrho\colon S\to N$ be the corresponding homomorphism of Lie groups. We let S act on A by means of ϱ. If $t=(x,y,1)\in A$, then the isotropy subalgebra $\mathfrak{s}_t$ is spanned by $(E_{11}-yE_{33})$, E_{12} and E_{13}. If $t'=(x',y',1)$, then it follows from the classification of three-dimensional Lie algebras [13, p. 12] that $\mathfrak{s}_t$ is isomorphic to $\mathfrak{s}_{t'}$ if and only if either $y=y'$ or $(1+y)\,(1+y')=1$. Let $B=\{(x,y,1)\in A\,|\,y>0\}$. Then B is N-stable, and hence S-stable, and, for $t,t'\in B$, $\mathfrak{s}_t$ is isomorphic to $\mathfrak{s}_{t'}$ if and only if t and t' lie on the same S-orbit.

The homomorphism $\sigma\colon S\to\mathrm{SL}_4(\mathbf{R})$ given by

$$\sigma(g)=\begin{pmatrix}(\det g)^{-1} & 0\\ 0 & g\end{pmatrix}$$

maps S isomorphically onto a closed Lie subgroup S' of $\mathrm{SL}_4(\mathbf{R})=G$; to simplify notation we identify S with S' by means of σ. Thus $G\to G/S$ is a principal fibre bundle with structure group S. Since S acts on B we may form the associated fibre bundle $E\to G/S$ with fibre B; E is an analytic manifold and G acts on E in the usual way. For this action of G on E, each orbit is closed and of codimension one and, for $s, t\in E$, we see that G_s^0 is not isomorphic to G_t^0 unless s and t lie on the same orbit. Thus we have shown:

12.3.1. There exists an analytic action of $G=\mathrm{SL}_4(\mathbf{R})$ on an analytic manifold E such that each orbit of G on E is closed and of codimension one and such that, for $s, t\in E$, G_s^0 is not isomorphic to G_t^0 unless s and t lie on the same G-orbit.

12.4. The example given in 12.3. is not an algebraic transformation group. Similar examples exist in the case of algebraic transformation groups, but it seems to be difficult to give an explicit description of them. Let (G, X) be a (complex) algebraic transformation space, with G an affine algebraic group, and, for $x\in X$, let U_x be the unipotent radical of G_x and let L_x be a Levi subgroup of G_x. It follows from Theorem 9.3.1 that the L_x fall into a finite number of conjugacy classes, so one is led to study the conjugacy classes of the unipotent radicals. The problem here is that it is hard to find explicit invariants to distinguish between conjugacy classes of unipotent subgroups.

First we shall show that if there exists an infinite number of isomorphism classes of isotropy subalgebras for the action of G on X, then there exists a G-stable Zariski locally closed subvariety V of X such that (G, V) has properties similar to the example of 12.3.

Let $\mathfrak{L}_n$ be the set of Lie algebra multiplications on $\mathbf{C}^n$. $\mathfrak{L}_n$ is a Zariski closed subset of the vector space of all alternating bilinear maps of $\mathbf{C}^n\times\mathbf{C}^n$ into $\mathbf{C}^n$ and thus has a natural structure of an affine algebraic variety. The general linear group $H=\mathrm{GL}_n(\mathbf{C})$ acts in a canonical manner on $\mathfrak{L}_n$ and the orbits of H on $\mathfrak{L}_n$ are just the isomorphism classes of Lie algebra structures on $\mathbf{C}^n$. Applying the result of Rosenlicht [21] mentioned in 9.2 to the algebraic transformation space $(H, \mathfrak{L}_n)$, we see that there exists a partition of $\mathfrak{L}_n$ into a finite family of disjoint, Zariski locally closed, non-singular H-stable subsets $V_1, \ldots, V_m$ such that the following conditions hold for each $j=1, \ldots, m$: (i) V_j is a Zariski open subset of $\mathfrak{L}_n - \bigcup_{i=1}^{j-1} V_i$; (ii) there exists a quotient $\pi_j\colon V_j\to(H\diagdown V_j)=Z_j$ for the action of H on V_j. Let Z be the disjoint union of the algebraic varieties $Z,\ldots,Z_m$. The points of Z correspond bijectively to the isomorphism classes of n-dimensional complex Lie algebras.

Now let (G, X) be an algebraic transformation space and assume that all isotropy subgroups $G_x (x\in X)$ have dimension n. For each $x\in X$, let $\psi(x)\in Z$ denote the isomorphism class of $\mathfrak{g}_x$. For j$=1, \ldots, m$, let $X_j=\{x\in X\mid\psi(x)\in Z_j\}$ and let $\psi_j\colon X_j\to Z_j$ denote the restriction of ψ. Then a straightforward argument shows that X_j is a Zariski locally closed subset of X and that ψ_j is a morphism of algebraic varieties. We note that ψ_j is constant on G-orbits.

12.4.1. PROPOSITION. *Let (G, X) be an algebraic transformation space and assume that there is an infinite number of distinct isomorphism classes (resp. conjugacy classes) of isotropy subalgebras $\mathfrak{g}_x (x \in X)$. Then there eixsts a Zariski locally closed, non-singular G-stable subvariety V of X such that the following conditions hold:*

(a) *All orbits of G on V have the same dimension and $\dim G(x) < \dim V$ for $x \in V$.*

(b) *For every $x \in V$, there exists an open neighborhood U of x in V (with respect to the Hausdorff topology of V) such that if $y, z \in U$, then $\mathfrak{g}_y$ is not isomorphic (resp. conjugate) to $\mathfrak{g}_z$ unless y and z lie on the same G-orbit.*

Proof. We shall give the proof for the case of isomorphism classes. By passing to a Zariski locally closed subvariety of X, we may assume that all isotropy subgroups $G_x (x \in X)$ have dimension n and that there exists i $(1 \leqslant i \leqslant m)$ such that $\psi(x) \in Z_i$ for every $x \in X$, where ψ is as defined above. By a straightforward argument, involving only standard properties of morphisms of algebraic varieties and the result of Rosenlicht mentioned above, we can show that there exists a G-stable, non-singular Zariski locally closed subvariety V of X and a non-singular Zariski locally closed subvariety A of Z_i such that the following conditions hold: (i) There exists a quotient π: $V \to (G \diagdown V) = B$ for the action of G on V; (ii) B is non-singular and $\dim B > 0$; (iii) $\psi(V) = B$; and (iv) If τ: $B \to A$ is the morphism induced by ψ_i, then for every $b \in B$ the differential $d\tau_b$: $T_b(B) \to T_{\tau(b)}(A)$ is a linear isomorphism.

Conclusion (b) of 12.4.1 follows immediately from (iv) via the implicit function theorem. This proves 12.4.1 for the case of isomorphism classes.

To prove 12.4.1 in the case of conjugacy classes, we let $A_n(\mathfrak{g})$ be the algebraic variety of all n-dimensional subalgebras of g and we apply [21] to the action of G on $A_n(\mathfrak{g})$ determined by the adjoint representation of G. The rest of the proof goes as above.

12.4.2. It was shown in [18, p. 432] that for n sufficiently large, there exists a rational representation of $G = \mathrm{SL}_n(\mathbf{C})$ on a vector space W such that there exists an infinite number of isomorphism classes of isotropy subalgebras $\mathfrak{g}_x (x \in W)$ for the action of G on W. Thus Proposition 12.4.1 applies to (G, W).

We can give explicit examples of representations of $\mathrm{SL}_n(\mathbf{C})$ for which there exists an infinite number of conjugacy classes of isotropy subalgebras. Write $\mathbf{C}^{2n}$ as the direct sum $V_1 \oplus V_2$ of two n-dimensional subspaces and let $\mathfrak{h}$ be the abelian subalgebra of $\mathfrak{sl}_{2n}(\mathbf{C})$ defined by

$$\mathfrak{h} = \{T \in \mathfrak{sl}_{2n}(\mathbf{C}) \mid T(V_1) \subset V_2 \text{ and } T(V_2) = \{0\}\}.$$

Every k-dimensional subspace of $\mathfrak{h}$ is the Lie algebra of a commutative unipotent algebraic subgroup of $G = \mathrm{SL}_{2n}(\mathbf{C})$. For n sufficiently large and $k < n^2$ chosen properly (e.g. $n = 4$, $k = 8$), an easy dimension count shows that there exists an infinite number of conjugacy classes of k-dimensional subalgebras of $\mathfrak{h}$. Let $m = 4n^2 - k$; m is the number of linear equations needed to determine a k-dimensional subspace of $\mathfrak{gl}_{2n}(\mathbf{C})$. Let $(\mathbf{C}^{2n})^*$ denote the dual space of $\mathbf{C}^{2n}$ and let E be the mth exterior power of the vector space $\mathbf{C} + (\mathbf{C}^{2n})^*$; E has natural structure of a G-module. An argument due to Chevalley (see [4, pp. 161–171]) shows that each k-dimensional subalgebra of $\mathfrak{h}$ occurs as an isotropy subalgebra for the action of G on E. Thus, if k and n are properly chosen, there exists an infinite number of conjugacy classes for the action of G on E, and Proposition 12.4.1 applies to (G, E).

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