EXTREMAL AND CONJUGATE EXTREMAL DISTANCE ON OPEN RIEMANN SURFACES WITH APPLICATIONS TO CIRCULAR-RADIAL SLIT MAPPINGS

BY

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Partition the boundary of a compact bordered Riemann surface $\overline{W}$ into four disjoint sets $\alpha_0, \alpha, \beta, \gamma$ with α_0 and α non-empty. Let $\hat{W}$ denote the compactification of W obtained by adding to W a point for each boundary component. Define

$$F = \{c : c \text{ is an arc in } \hat{W} - \gamma \text{ from } \alpha_0 \text{ to } \alpha\}$$

and $F^* = \{c : c \text{ is a sum of closed curves in } \hat{W} - \beta \text{ such that } c \text{ separates } \alpha_0 \text{ from } \alpha\}$.

Determine the harmonic function u in W by the boundary conditions $u=0$ on α_0, $u=1$ on α, $\partial u/\partial n = 0$ along γ and u is constant on each component β_i in β such that $\int_{\beta_i} du^* = 0$. Then $\lambda(F) = \|du\|^{-2}$, $\lambda(F^*) = \|du\|^2$ (see Lemma III.1.1) where $\lambda(\cdot)$ denotes the extremal length and $\|du\|^2$ the Dirichlet integral. This result was essentially known to Ahlfors and Beurling by the time of their fundamental paper on conformal invariants [1]. We observe that if W is planar and α_0, α are each single boundary components, $\exp 2\pi(u+iu^*)/\|du\|^2$ is a conformal mapping of W into $1 < |z| < \exp 2\pi/\|du\|^2$ and the images of the components in β are circular slits and the images of the components in γ radial slits.

The purpose of this paper is to give a complete generalization of the above result to arbitrary open Riemann surfaces. As a consequence of our work we obtain a new class of conformal mappings of plane regions onto "extremal" slit annuli analogous to the situation described above.

We begin with an open Riemann surface W and partition its ideal boundary into four disjoint sets $\alpha_0, \alpha, \beta, \gamma$ with α_0 and α non-empty and α_0, α and $\alpha_0 \cup \alpha \cup \beta$ closed in the Kerékjárto-Stoilöw compactification $\hat{W}$ of W. Classes of curves $\mathfrak{F}$, $\mathfrak{F}^*$ analogous to F and F^*

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are defined in I.3. In Chapter II we construct by means of an exhaustion the harmonic function u corresponding to the partition $(\alpha_0, \alpha, \beta, \gamma)$ and in a certain sense determine the values of u on the ideal boundary (roughly speaking, in the sense of limits along curves tending to the ideal boundary). We also show (II.5) the existence of a boundary component of maximal "capacity". Corresponding results are derived for a harmonic function v with a logarithmic singularity at a prescribed point.

In Chapter III we prove our main result: a) $\lambda(\mathfrak{F}) = \|du\|^{-2}$ and b) $\lambda(\mathfrak{F}^*) = \|du\|^2$. The proof of a) depends on a highly topological continuity method for extremal length in which arcs in an exhaustion are pieced together to form an arc in $\hat{W}$. This method is ascribed to Beurling and was developed by Wolontis [13] and Strebel [12]. Using it, Strebel proved a) in the case $\beta = \emptyset$. Part b) is proven by establishing the formula $\int_c du^* = \|du\|^2$ for all curves $c \in \mathfrak{F}^*$ except for a subclass of infinite extremal length. Our main theorem also yields some uniqueness theorems for u (III.4).

The information we have previously obtained is specialized in Chapter IV to the case of plane regions W with α_0 and α each consisting of a single boundary contour. We show that $\exp 2\pi(u + iu^*)/\|du\|^2$ is a conformal mapping of W onto an "extremal" slit annulus contained in $1 < |z| < R = \exp 2\pi/\|du\|^2$ such that a) the area of the slits is zero, b) the image of a boundary component in γ is a radial slit (or point), c) the image of a component in β which is isolated from γ is a circular slit (or point), and d) in many other cases the image is circular with radial incisions. An extremal slit annulus is uniquely characterized (to a rotation) by the following property: set $\varrho = (R \log|z|)^{-1}$, then $\int_c \varrho |dz| \geqslant 1$ and $\int_d \varrho |dz| \geqslant 2\pi/\log R$ for all $c \in \mathfrak{F}$ and $d \in \mathfrak{F}^*$, except for subclasses of infinite extremal length. Our results imply the classical properties of extremal circular slit annuli ($\gamma = \emptyset$) obtained by Reich and Warschawski [8, 9] and of extremal radial slit annuli ($\beta = \emptyset$) obtained by Strebel [12] and Reich [7]. Even in these classical cases however, the uniqueness property above is new.

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I. Preliminaries

I.1. Arcs and 1-chains. Given an open Riemann surface W, its Kerékjártó–Stoilöw compactification, in which each ideal boundary component becomes a point, will be denoted by $\hat{W}$ (see Ahlfors–Sario [2]). The following topological model of $\hat{W}$ given by I. Richards [10] is less well known but conceptually quite useful for what follows. Take the extended complex plane and remove a closed subset S of the Cantor set from the real axis (a topological model of the ideal boundary). Then remove a countable or finite number of disks from the open upper half plane so that they accumulate only to points of S (to ideal boundary components of nonplanar character). Next remove symmetrically placed disks from the lower half plane and identify symmetric circumferences by the correspondence $z \to \bar{z}$. When this construction has been suitably carried out the resulting surface will be a topological model of W and the union of W and S will be a topological model of $\hat{W}$.

The definition of *arc* and *open arc* in a topological space is standard. We will use the same terminology and notation for an arc and the point set determined by it.

A *relative* 1-*chain* τ on W is a countable formal sum

$$\tau = \sum c_i \tau_i,$$

where each c_i is a positive or negative integer, each τ_i is an arc or open arc in W, and given any compact set K, $\tau_i \cap K$ is non-empty for only a finite number of i.

The restriction of an arc or closed curve τ in $\hat{W}$ to W will be denoted by $\tau \cap W$. We see that $\tau \cap W$ is then a relative 1-chain on W.

Let $\omega = a\,dx + b\,dy$ be a differential on W. We make the following definitions.

(i). Suppose $\tau : (0,1) \to W$ is an open arc on W and $\{[t_n, t_n']\}$ is a nested sequence of intervals in (0,1) which approach (0,1). Denote the restriction of τ to $[t_n, t_n']$ by τ_n. Then define

$$\omega = \lim_{n \to \infty} \int_{\tau_n} \omega$$

when each term on the right exists and the limit exists independent of the particular exhaustion of (0,1) used.

(ii). If $\tau=\sum c_i\tau_i$ is a relative 1-chain define

$$\omega=\sum c_i\int_{\tau_i}\omega$$

when all terms on the right exist and the convergence is absolute.

If $\varrho|dz|$ is a linear density and τ an open arc $\int_\tau \varrho|dz|$ is defined as in (i) and always exists $\leqslant\infty$. If $\tau=\sum c_i\tau_i$ is a relative 1-chain we define

$$\int_\tau \varrho|dz|=\sum|c_i|\int_{\tau_i}\varrho|dz|.$$

In particular $|\omega|=(|a|^2+|b|^2)^{\frac{1}{2}}|dz|$ is a linear density. If τ is a relative 1-chain and $\int_\tau|\omega|<\infty$ then $\int_\tau\omega$ exists and $|\int_\tau\omega|\leqslant\int_\tau|\omega|$.

Suppose a triangulation of W is given and $\hat{\tau}\subset\hat{W}$ is a closed curve or arc with both end points ideal boundary points. By the method of simplicial approximation the relative 1-chain $\tau=\hat{\tau}\cap W$ is homologous (singularly) to a simplicial 1-chain τ_s. If σ is a closed differential with compact support,

$$\int_\tau\sigma=\int_{\tau_s}\sigma.$$

We will also make use of the fact that if $\{\Omega_n\}$ is an exhaustion of W (i.e. $\Omega_n<\Omega_{n+1}$ and $\partial\Omega_n$ is smooth) a triangulation of W may be chosen so that $\partial\Omega_n$ for each n appears as a simplicial cycle [2]. We shall use these remarks later to simplify the evaluation of certain integrals.

I.2. Extremal length. If C is a family of rectifiable relative 1-chains and $\varrho|dz|$ is a Borel measurable linear density define

$$L(\varrho,C)=\inf_{c\in C}\int_c\varrho|dz|,$$

$$A(\varrho)=\iint_W\varrho^2dx\,dy,$$

$$\lambda(C)=\sup_\varrho\frac{L^2(\varrho,C)}{A(\varrho)};$$

$\lambda(C)$ is called the extremal length of C.

Traditionally, the elements of C are called "curves". If C is a family of arcs on $\hat{W}$, the same definitions apply except c is to be replaced by the 1-chain $c\cap W$.

Let C be a family of curves. Following M. Ohtsuka [6] (see also [3]) we say that a statement is true for *almost all curves* in C if the subfamily C' of C for which the statement is false has $\lambda(C')=\infty$.

The following lemma will find frequent use in this paper. In a somewhat different form it is due to Fuglede [3]. From the point of view of Riemann surfaces it also has applications to the theory of square integrable differentials [5].

LEMMA I. 2.1. *Let* $\{\varrho_n|dz|\}$ *be a family of linear densities on* W *which satisfy* $\lim A(\varrho_n)=0$ *and suppose* C *is a family of curves. Then there is a subsequence* $\{n\}$ *such that for almost all* $c\in C$,

$$\lim_{n\to\infty}\int_c \varrho_n|dz|=0.$$

Proof. Pick the subsequence $\{n\}$ so that (without changing notation) $A(\varrho_n)<2^{-3n}$. Set

$$A_n=\left\{c\in C:\int_c \varrho_n|dz|>2^{-n}\right\},$$

$$B_n=A_n\cup A_{n+1}\cup A_{n+2}\cup\ldots,$$

$$E=\bigcap_{n=1}^{\infty} B_n.$$

Then for any n, using $\varrho_n|dz|$ as a competing density for $\lambda(A_n)$,

$$\lambda(E)^{-1}\leqslant\lambda(B_n)^{-1}\leqslant\sum_{i=n}^{\infty}\lambda(A_i)^{-1}\leqslant\sum_{i=n}^{\infty}2^{-i}=2^{1-n}\to 0$$

(here we are making use of the well-known result that if a class of curves Γ is contained in a countable union of classes $\bigcup\Gamma_n$ then $\lambda(\Gamma)^{-1}\leqslant\sum\lambda(\Gamma_n)^{-1}$; see [6]). Hence $\lambda(E)=\infty$. If for some $c\in C$, $\limsup\int_c\varrho_n dz>0$ then c belongs to all B_i and therefore to E.

I.3. The classes $\mathcal{F}$ and $\mathcal{F}^*$: statement of the problem. Let W be an arbitrary open Riemann surface and partition the ideal boundary into four disjoint sets α_0, α, β, γ such that

(i) α_0 and α are non-empty

(ii) α_0, α, and $\alpha_0\cup\alpha\cup\beta$ are closed sets in the compactification $\hat{W}$ of W.

Define the classes $\mathcal{F}$, $\mathcal{F}^*$ (or $\mathcal{F}(\alpha_0,\alpha,\beta,\gamma)$, $\mathcal{F}^*(\alpha_0,\alpha,\beta,\gamma)$) as follows.

$\mathcal{F}=\{\tau:\tau$ is an arc in $\hat{W}-\gamma$ with initial point in α_0, end point in $\alpha\}$

$\mathcal{F}^*=\{\tau:\tau$ is a countable union of closed curves in $\hat{W}-\alpha_0-\alpha-\beta$ such that

(a) all limit points of τ are contained in γ, and

(b) no component of $(\hat{W}-\gamma)-\tau$ contains points in *both* α_0 and $\alpha\}$.

Given $\tau\in\mathcal{F}^*$ let $\hat{W}_1$ be the union of the regions in $(\hat{W}-\gamma)-\tau$ which contain points in

α_0 and let τ_1 be the relative boundary of $\hat{W}_1$ so that τ_1 is contained in τ. Then τ_1 satisfies the following property

(c) If τ_i is a closed curve in τ_1, then $\tau_1 - \tau_i$ does *not* satisfy (b).

The main problem of this paper is to find $\lambda(\mathcal{F})$, $\lambda(\mathcal{F}^*)$. Therefore it is no restriction to assume the curves in $\mathcal{F}^*$ also satisfy (c). Then the curves in $\mathcal{F}^*$ may be oriented so that α_0 lies to the left.

The method of solution of these extremal length problems is really dictated by the assumption (ii) on $(\alpha_0, \alpha, \beta, \gamma)$. In Chapter V we will briefly consider the problem of finding $\lambda(\mathcal{F})$ and $\lambda(\mathcal{F}^*)$ when (ii) is replaced by

(ii)′ α_0, α, and $\alpha_0 \cup \alpha \cup \gamma$ are closed in $\hat{W}$.

The method of finding the solution is different, and even when the second partition is obtained from the first by adding to γ and subtracting from β the minimum number of points to make (ii)′ true, the solution may be different.

II. Canonical Harmonic Functions

II.1. Construction of $u(z; \alpha_0, \alpha, \beta, \gamma)$. We begin with an arbitrary Riemann surface W and a partition $(\alpha_0, \alpha, \beta, \gamma)$ of the ideal boundary of W as described in I.3 above. Assumption (ii) implies that there is a Jordan curve in W separating α_0 and α. In this section we will construct the harmonic function $u(z) = u(z; \alpha_0, \alpha, \beta, \gamma)$ determined by the "boundary" conditions $u=0$ on α_0, $u=1$ on α, $u=$ constant on each component β_i in β with $\int_{\beta_i} du^* = 0$ and $\partial u/\partial n = 0$ along γ. These conditions are to be understood as limits in a sense to be made precise below. Finally we wish to emphasize that convergence (of functions and differentials) is considered only in the Dirichlet norm.

Let $\{\Omega_n\}$ be an exhaustion of W such that for each n, Ω_n separates α_0 from α and each component of $\partial\Omega_n$ is a dividing cycle not homologous to zero. Suppose σ is a component of some $\partial\Omega_n$; then σ is the relative boundary of a subregion S of $W - \Omega_n$. If π is an ideal boundary component of S we shall call π a *derivation* of σ.

Using the following rules divide $\partial\Omega_n$ into disjoint collections of components α_{0n}, α_n, β_n and γ_n.

A–1) α_{0n} consists of those components of $\partial\Omega_n$ which have a point of α_0 as derivation.

A–2) α_n consists of those components which have a point of α as derivation.

A–3) A contour of $\partial\Omega_n$ belongs to γ_n if and only if the only ideal boundary points which are derivations of it lie in γ.

A–4) β_n consists of the remaining contours of $\partial\Omega_n$.

Let $\mathbf{\Omega}_n$ denote the possibly noncompact region obtained by adjoining to Ω_n the noncompact components of $W-\Omega_n$ which are bounded by curves in γ_n. We orient the boundary contours so that $\partial\mathbf{\Omega}_n=\alpha_n+\beta_n-\alpha_{0n}$. Choose an exhaustion $\{\Omega_{ni}\}_{i=1}^{\infty}$ of $\mathbf{\Omega}_n$ so that $\partial\Omega_{ni}=\alpha_n+\beta_n+\gamma_{ni}-\alpha_{0n}$. Thus for all n, γ_{ni} is homologous to γ_n.

Define the harmonic function u_{ni} in Ω_{ni} by the boundary conditions

B–1) $u_{ni}=0$ on α_{0n}, $u_{ni}=1$ on α_n

B–2) $\partial u_{ni}/\partial n=0$ along γ_{ni}

B–3) u_{ni} is constant on each component β_{nj} in β_n with the constant chosen so that $\int_{\beta nj}du_{ni}^{*}=0$.

We will first show that $\lim_{i\to\infty}u_{ni}=u_n$ *exists in* $\mathbf{\Omega}_n$ *and is independent of the exhaustion* $\{\Omega_{ni}\}$ *of* $\mathbf{\Omega}_n$.

For $j>i$ the equation

$$(du_{ni},du_{nj})_{\Omega ni}=\int_{\partial\Omega_{ni}}u_{nj}du_{ni}^{*}=\int_{\alpha_n}du_{ni}^{*}=\|du_{ni}\|_{\Omega_{ni}}^{2}$$

implies that

$$\|du_{ni}-du_{nj}\|_{\Omega_{ni}}^{2}<\|du_{nj}\|_{\Omega_{nj}}^{2}-\|du_{ni}\|_{\Omega_{ni}}^{2}. \tag{1}$$

In particular, $\|du_{ni}\|_{\Omega_{ni}}^{2}$ is increasing with i.

Let v be the harmonic function in $\Omega_{n0}=\Omega_n$ defined by the boundary conditions $v=0$ on α_{0n}, $v=1$ on $\partial\Omega_n-\alpha_{0n}$. Then since

$$(du_{ni},dv)=\int_{\partial\Omega_n}vdu_{ni}^{*}=\|du_{ni}\|_{\Omega_{ni}}^{2}$$

we find that

$$\|du_{ni}-dv\|_{\Omega_n}^{2}\leqslant\|dv\|_{\Omega_n}^{2}-\|du_{ni}\|_{\Omega_{ni}}^{2},$$

and therefore $\|du_{ni}\|$ is uniformly bounded. It follows from equation (1) that $\lim_{i\to\infty}u_{ni}=u_n$ exists. Furthermore $\lim_{i\to\infty}\|du_{ni}\|_{\Omega ni}=\|du_n\|_{\Omega_n}$. To prove this last assertion set $A=\lim_{i\to\infty}\|du_{ni}\|_{\Omega_{ni}}^{2}$. Then using a simplified notation, let $i\to\infty$, $i>j$, in the inequality

$$\|du_{ni}-du_{nj}\|_{j}^{2}\leqslant\|du_{ni}\|_{i}^{2}-\|du_{nj}\|_{j}^{2}$$

to obtain

$$\|du_n-du_{nj}\|_{j}^{2}\leqslant A-\|du_{ni}\|_{j}^{2}.$$

Our assertion now follows upon letting $j\to\infty$ in the inequality:

$$\big|\,\|du_n-du_{nj}\|_{j}-\|du_n\|_{j}\,\big|\leqslant\|du_{nj}\|_{j}\leqslant\|du_n-du_{nj}\|_{j}+\|du_n\|_{j}.$$

Clearly u_n is independent of the particular exhaustion $\{\Omega_{ni}\}$ of $\mathbf{\Omega}_n$ and u_n is 0 on α_{0n}, 1 on α_n and constant on each component β_{nj} of β_n with $\int_{\beta nj}du_n^{*}=0$.

Next we show that $\lim du_n = \omega$ *exists as an exact harmonic differential independently of the exhaustion* $\{\Omega_n\}$. If $n > m$ then $\mathbf{\Omega}_m < \mathbf{\Omega}_n$,

$$\partial(\Omega_{ni} \cap \mathbf{\Omega}_m) = \alpha_m + \beta_m - \alpha_{0m} + \gamma_{ni} \cap \mathbf{\Omega}_m,$$

and β_m, α_{0m}, α_m are homologous to components in β_n, γ_{ni}; $\alpha_{0n}, \beta_n, \gamma_{ni}$; $\alpha_n, \beta_n, \gamma_{ni}$, respectively. Hence since $\lim_{i\to\infty} \|du_n - du_{ni}\|_{\Omega_{ni}} = 0$,

$$(du_m, du_n)_{\Omega_m} = \lim_{i\to\infty} (du_m, du_{ni})_{\Omega_{ni} \cap \Omega_m} = \lim_{i\to\infty} \int_{\alpha_m} du_{ni}^* = \|du_n\|^2_{\Omega_m}.$$

Therefore
$$\|du_m - du_n\|^2_{\Omega_m} \leqslant \|du_m\|^2_{\Omega_m} - \|du_n\|^2_{\Omega_n} \tag{2}$$

and we see that $\|du_m\|^2_{\Omega_m}$ is decreasing as m increases and $\lim_{m\to\infty} du_m$ exists. The harmonic limit differential is exact and we denote it by du, u being unique only up to an additive constant. We also see that $\lim \|du_m\|_{\Omega_m} = \|du\|$. The question arises, does $\lim u_n = u$ exist for suitable u? This is false if $\|du\| = 0$ but turns out to be correct if $\|du\| > 0$. The proof follows easily from later results (Theorems II.3.2 and III.3.1). For the present we do not need this fact.

If $\{\Omega'_n\}$ is another exhaustion of W, then given k there exist m and n such that $\Omega'_k < \Omega_m < \Omega'_n$. It follows that $\mathbf{\Omega}'_k < \mathbf{\Omega}_m < \mathbf{\Omega}'_n$ and we see that indeed du is independent of our choice of exhaustion $\{\Omega_n\}$.

We will have occasion to use the following simple observation. For each n we can find an integer i_n such that

$$\|du_n\|_{\Omega_n} - \|du_{ni_n}\|_{\Omega_{ni_n}} < \frac{1}{n}.$$

Thus on setting $v_n = u_{ni_n}$ and replacing Ω_{ni_n} by Ω_n we see that $\lim \|dv_n\|_{\Omega_n} = \|du\|$. By passage to a subsequence we may assume the Ω_n are nested and thus form an exhaustion.

II.2. Dependence of $u(z; \alpha_0, \alpha, \beta, \gamma)$ **on** α_0. For many applications α_0 is a finite collection of mutually disjoint, piecewise anlytic Jordan curves imbedded in a Riemann surface W in such a way that α_0 does not separate W into two non-compact components. If this is the case the exhaustion $\{\Omega_n\}$ may be taken to be an exhaustion of $W - \alpha_0$ less compact regions bounded by α_0 (if any) so that $\partial\Omega_n = \alpha_n + \beta_n + \gamma_n - \alpha_0$, for all n (α_0 used in this connection may actually consist of both sides of the curves α_0). Then $u = u(z; \alpha_0, \alpha, \beta, \gamma)$ may be constructed as above, and is harmonic in W − compact regions bounded by α_0.

Now suppose that we are given another such collection of curves α'_0 disjoint from α_0. Choose a corresponding exhaustion $\{\Omega'_n\}$ so that $\partial\Omega'_n = \alpha_n + \beta_n + \gamma_n - \alpha'_0$ and construct the harmonic function $u' = u(z; \alpha'_0, \alpha, \beta, \gamma)$. As a consequence of the observation made at the

end of II.1, we may assume that $\{\Omega_n\}$ and $\{\Omega_n'\}$ are chosen so that $\lim\|dv_n\| = \|du\|$, $\lim\|dv_n'\| = \|du'\|$ (see the notation used there) and $\alpha_0 \subset \Omega_n'$, $\alpha_0' \subset \Omega_n$ for all n.

Setting $K_n = \Omega_n \cap \Omega_n'$ we see that $\partial K_n = \alpha_n + \beta_n + \gamma_n - \alpha_0 - \alpha_0'$ and

$$(dv_n, dv_n')_{K_n} = \int_{\partial K_n} v_n dv_n'^* = \int_{\alpha_n} dv_n'^* - \int_{\alpha_0} v_n dv_n'^* = \|dv_n'\|^2_{\Omega_n'} - \int_{\alpha_0} v_n dv_n'^*.$$

Hence
$$\|dv_n - dv_n'\|^2_{K_n} \leqslant \|dv_n\|^2_{\Omega_n} - \|dv_n'\|^2_{\Omega_n} + 2\int_{\alpha_0} v_n dv_n'^*.$$

We conclude that if $\lim\|dv_n\|_{\Omega_n} = \|du\| = 0$, then $\lim v_n = 0$ and since $dv_n'^* \geqslant 0$, $\int_{\alpha_0} dv_n'^* = \|dv_n'\|^2$,

$$\lim_{n\to\infty} \|dv_n - dv_n'\|^2_{K_n} = -\lim\|dv_n'\|^2 = -\|du'\|^2.$$

Therefore $\|du'\| = 0$, and we have proved most of the following theorem.

THEOREM II.2.1. *Given α_0 and α as above, but not necessarily disjoint, then $u(z; \alpha_0, \alpha, \beta, \gamma) = 0$ if and only if $u(z; \alpha_0', \alpha, \beta, \gamma) = 0$.*

Proof. We have just proved that if α_0 and α_0' are disjoint the conclusion holds. If α_0 and α_0' are not disjoint, choose α_0'' disjoint from both α_0 and α_0' and compare u and u' with $u(z; \alpha_0'', \alpha, \beta, \gamma)$.

II.3. The boundary behavior of $u(z; \alpha_0, \alpha, \beta, \gamma)$. In this section we show that in a certain sense $u(z; \alpha_0, \alpha, \beta, \gamma)$ takes the expected values on the ideal boundary.

LEMMA II.3.1. *Notation as in* II.1. *Suppose c is a union of analytic Jordan curves dividing Ω_{ni} into two regions A_1 and A_2 in such a way that ∂A_1 consists only of c and elements of β_n and/or γ_n. Then for all $z \in A_1$,*

$$0 < \min_{p\in c} u_{ni}(p) \leqslant u_{ni}(z) \leqslant \max_{p\in c} u_{ni}(p) < 1.$$

Proof. To simplify notation we will omit the subscripts and superscripts on u and Ω. We prove the right inequality; the left one can be treated similarly. If our assertion is false then $\max_{z\in A_1} u(z)$ is assumed on a component τ of $\partial\Omega$ which belongs to β_n or γ_n. Pick a point $p \in \operatorname{Int} A_1$ such that $u(p) = k > \max_{z\in c} u(z)$ and let l be that portion of the level curve $u = k$ (not necessarily connected) that lies in A_1. Note that l does not meet c. We may assume that l does not pass through a critical point of u and hence does not meet any component of ∂A_1 in β_n. However l may meet components of ∂A_1 which are in γ_n. Let A_1' be the component of $A_1 - l$ which contains τ in its boundary. Since $u(z) > k$ for $z \in A_1'$, $du^* < 0$ along $l' = l \cap \partial A_1'$ (when l' is oriented so that A_1' lies to the left) and we must have $\int_{l'} du^* < 0$. On the other hand, l' possibly together with pieces of components of $\partial A_1'$ in γ_n is homologous

to a combination of components of $\partial A_1'$ which are in β_n and γ_n. Because $du^*=0$ along components in γ_n, we must have then $\int_{l'} du^*=0$. This contradiction proves the lemma.

THEOREM II. 3.2. *If $\lambda(\mathfrak{F})<\infty$ there exists a function $u=u(z;\alpha_0,\alpha,\beta,\gamma)$ such that*

a) $\lim u_n=u$ *(see II.1) and for almost all curves $\tau\in\mathfrak{F}$*

$$\int_\tau du=1.$$

b) *For each component $\beta_i\in\beta$, there exists a number b, $0<b<1$, such that for almost all arcs τ in $\hat{W}-\gamma$ with initial point $\tau(0)$ in W and end point on β_i,*

$$\int_\tau du=b-u(\tau(0)).$$

(This result holds for α_0 and α with b replaced by 0 or 1 respectively.)

Proof. The class of curves to be considered in b) may be written as the union of classes Γ_n such that for $\tau\in\Gamma_n$, τ is an arc in $\hat{W}-\gamma$ with $\tau(0)\in\Omega_n$. Hence it is sufficient to prove b) for curves τ with $\tau(0)\in\Omega_1$. Let c_n be the component of $\partial\mathbf{\Omega}_n$ determined by β_i (or α_0 or α) —see the notation in II.1. By eliminating a class of curves of infite extremal length we may assume that there are only a finite number of components of $\tau\cap\mathbf{\Omega}_n$ which meet more than one component of $\partial\mathbf{\Omega}_n$, for all n. If τ leaves $\mathbf{\Omega}_n$ by crossing a component $c\neq c_n$ of $\partial\mathbf{\Omega}_n$ then the next return of τ to $\mathbf{\Omega}_n$ is by crossing c again; τ leaves $\mathbf{\Omega}_n$ for the last time by crossing c_n. Let u_n have the value b_n on c_n. If c_n is determined by α_0 or α, then b_n is 0 or 1 respectively.

The differential du is uniquely determined as $\lim du_n$. Set $u_n=0$ in $W-\mathbf{\Omega}_n$. The corresponding linear densities $\varrho_n|dz|=|du-du_n|$ $(=|\text{grad }(u-u_n)|\,|dz|)$ satisfy

$$A(\varrho_n)=\|du-du_n\|^2\to 0 \text{ as } n\to\infty.$$

Lemma I.2.1 asserts that for almost all τ, there exists a sequence $\{m\}$, not depending on τ, such that

$$\lim_{m\to\infty}\left|\int_\tau du-(b_m-u_m(\tau(0))\right|=\lim\left|\int_\tau du-du_m\right|\leqslant\lim\int_\tau|du-du_m|=0.$$

(This formula contains the assertion that $\int_\tau du$ exists. A simplicial approximation to τ may be used to evaluate $\int_\tau du_m$.)

Now suppose that $\lambda(\mathfrak{F})<\infty$. By assumption there is a Jordan curve J separating α_0 from α. Since $\lambda(\mathfrak{F})<\infty$ the extremal length of the class of curves in $\hat{W}-\gamma$ which travel from J to α_0 is finite. Given a point $p\in W$ we can connect each such curve (i.e. arc) to p by an arc from p to J. Since $\lim\int_c|du-du_n|=0$ on every compact curve c we can reapply the reasoning above to obtain

$$\lim_{m\to\infty}\left|\int_\tau du + u_m(p)\right| = 0.$$

This proves that $\lim_{m\to\infty} u_m(p)$ exists and more generally that, for the original sequence, $\lim u_n(p)$ exists. Hence the function u is uniquely determined as $\lim u_n$.

Next we obtain from above

$$\lim_{m\to\infty}\left|\int_\tau du - (b_m - u(\tau(0)))\right| = 0.$$

Hence $\lim b_m$ and more generally $b = \lim b_n$ exists. Since α_0 and α are closed sets in $\hat{W}$, every $\beta_i \in \beta$ is isolated from α_0 and α. Lemma II.3.1 and the maximum principle imply therefore that $0 < b < 1$.

The remainder of b) and a) now follow easily using the above methods.

Remark. In the case that τ is in $\hat{W} - \beta$,

$$\int_\tau du = \lim_{z\to\tau(1)} u(z) - \lim_{z\to\tau(0)} u(z),$$

where $\tau(1)$ and $\tau(0)$ are the end point and initial point respectively of τ in $\hat{W}$.

II.2.4. The function $v(z; p, \alpha, \beta, \gamma)$. Suppose now that we have a partition of the ideal boundary into three disjoint sets α, β, γ such that α and $\alpha \cup \beta$ are closed in the compactification $\hat{W}$ of W and α is not empty. We will construct a harmonic function $v(z) = v(z; p, \alpha, \beta, \gamma)$ with a singularity $\log|z|$ at a prescribed point p $(z(p) = 0)$ and behavior on α, β, γ the same as that of u.

As in II.1 we start with an exhaustion $\{\Omega_n\}$ so that $p \in \Omega_n$ for all n and divide $\partial\Omega_n$ into three sets $\alpha_n, \beta_n, \gamma_n$ by the rules A–2), A–3), A–4). Then we construct $\mathbf{\Omega}_n$ with $\partial\mathbf{\Omega}_n = \alpha_n + \beta_n$ and take an exhaustion $\{\Omega_{ni}\}$ of $\mathbf{\Omega}_n$ so that $\partial\Omega_{ni} = \alpha_n + \beta_n + \gamma_{ni}$ where γ_{ni} is homologous to γ_n. Let v_{ni} be harmonic in Ω_{ni} except for the singularity $\log|z|$ at p and satisfying the boundary conditions $v_{ni} = 1$ on α_n, $v_{ni} =$ constant on each component β_i in β_n so that $\int_{\beta_i} dv_{ni}^* = 0$, and $\partial v_{ni}/\partial n = 0$ along γ_{ni}.

First we show that $\lim_{i\to\infty} v_{ni} = v_n$ exists. Extend v_{n1} to $\mathbf{\Omega}_n$ by setting $v_{n1} = 0$ in $\mathbf{\Omega}_n - \Omega_{n1}$. Dropping the subscript n, we find for $j > i$,

$$(d(v_j - v_1), d(v_1 - v_i))_{\Omega_i} = \int_{\gamma_1} (v_j - v_1) d(v_1 - v_i)^* + \int_{\gamma_1} v_j dv_i^* = \int_{\gamma_1} v_1 dv_i^* = -\| d(v_1 - v_i) \|_{\Omega_i}^2.$$

Hence $\| d(v_j - v_i) \|_{\Omega_i}^2 = \| d(v_j - v_1) + d(v_1 - v_i) \|_{\Omega_i}^2 \leq \| d(v_j - v_1) \|_{\Omega_j}^2 - \| d(v_i - v_1) \|_{\Omega_i}^2$.

Thus $\| d(v_i - v_1) \|_{\Omega_i}$ is monotonically increasing in i; it is also uniformly bounded. For let w have the singularity $\log|z|$ at p and boundary value 1 on $\partial\Omega_1$. We find that

$$\|d(v_i-w)\|^2_{\Omega_i}=\|d(w-v_1)\|^2_{\Omega_1}-\|d(v_i-v_1)\|^2_{\Omega_i}.$$

Hence $\|d(v_i-v_1)\|^2_{\Omega_i}$ is uniformly bounded and $\lim_{i\to\infty} v_{ni}=v_n$ exists in $\mathbf{\Omega}_n$ with

$$\lim_{i\to\infty}\|d(v_n-v_{ni})\|_{\Omega_i}=0 \ (u_{ni}=1 \text{ on } \alpha_n \text{ for all } i).$$

Next we show that $\lim v_n=v$ *exists in* W *as a harmonic function with singularity* $\log|z|$ *at* p. Let Δ be the parametric disk $|z|\leqslant 1$ center at p and define v_0 as

$$v_0\equiv 1+\log|z|,\ z\in\Delta,\quad v_0=0,\ z\notin\Delta.$$

On breaking the Dirichlet integral into two parts we find for $n>m$,

$$\begin{aligned}(d(v_n-v_0),d(v_m-v_0))_{\Omega_m}&=\lim_{i\to\infty}(d(v_{ni}-v_0),d(v_m-v_0))_{\Omega_{ni}\cap\Omega_m}\\&=\lim_{i\to\infty}\left[\int_{\alpha_m}dv^*_{ni}-\int_{\partial\Delta}v_m dv^*_{ni}+\int_{\partial\Delta}(v_m-1)d(v_{ni}-v_0)^*\right]\\&=-\int_{\partial\Delta}(v_m-1)dv^*_0=\|d(v_m-v_0)\|^2_{\Omega_m}.\end{aligned}$$

Hence $$\|d(v_n-v_m)\|^2_{\Omega_m}\leqslant\|d(v_n-v_0)\|^2_{\Omega_n}-\|d(v_m-v_0)\|^2_{\Omega_m}.$$

Since $\|d(v_n-v_0)\|^2_{\Omega_n}=-\int_{\partial\Delta}(v_n-1)dv^*_0$, we see that either $\lim v_n=-\infty$ or $\lim v_n=v$ exists as a harmonic function in $W-\{p\}$ with singularity $\log|z|$ at p.

As in II.1, v is independent of the exhaustion $\{\Omega_n\}$.

Suppose $v\equiv v(z;\,p,\,\alpha,\,\beta,\,\gamma)$ and v_k have the expansions at p

$$v=\log|z|+c+o(1),\qquad v_k=\log|z|+c_k+o(1).$$

Then $c_k>c_{k+1}$ *for all* k *and* $\lim c_k=c$.

For set $v_0=1+\log(|z|/r)$ in the disk $\Delta:|z|\leqslant r$. We find

$$\begin{aligned}\|d(v_k-v_0)\|^2_{\Omega_k}&=-\int_{\partial\Delta}(v_k-1)dv^*_0=-\int_0^{2\pi}(\log r+c_k-1+o(1))d\theta\\&=-2\pi\log r-2\pi c_k-2\pi+o(1).\end{aligned}$$

Since $\|d(v_k-v_0)\|_{\Omega_k}$ is increasing, c_k is decreasing. Since $\lim v_k=v$, $\lim c_k=c$.

Choose k so small that the level curve α_0 on which $v=k$ bounds a relatively compact subregion K of W containing p. *Then* $u(z;\,\alpha_0,\,\alpha,\,\beta,\,\gamma)=(1-k)^{-1}\,[v(z;\,p,\,\alpha,\,\beta,\,\gamma)-k]$ *in* $W'=W-K$.

For we may take the exhaustion $\{\Omega_n\}$ of W so that $K\subset\Omega_n$ for all n. Use the notation u_{ni} for the approximations to $u(z;\,\alpha_0,\,\alpha,\,\beta,\,\gamma)$ with respect to the exhustion $\{\Omega'_n=\Omega_n-K\}$ of W' and v_{ni} for the approximations to $v(z;\,p,\,\alpha,\,\beta,\,\gamma)$ with respect to $\{\Omega_n\}$ as constructed in the paragraphs above. We find

$$\left\| d\left[u_{ni} - \frac{1}{1-k}(v_{ni}-k)\right]\right\|^2_{\Omega_{ni}} = \frac{1}{1-k}\int_{\alpha_0}(v_{ni}-k)d\left(u_{ni}-\frac{1}{1-k}v_{ni}\right)^*.$$

Letting $i\to\infty$ and then $n\to\infty$ we obtain

$$\left\| d\left[u(z;\alpha_0,\alpha,\beta,\gamma) - \frac{1}{1-k}(v(z;p,\alpha,\beta,\gamma)-k)\right]\right\|^2_{W'} = 0$$

since convergence is uniform on α_0.

Using the results of II.2 we see that $v(z;p,\alpha,\beta,\gamma) = -\infty$ *if and only if* $v(z;q,\alpha,\beta,\gamma) = -\infty$ *for any pair of points* p *and* q.

The boundary behavior of $v(z;p,\alpha,\beta,\gamma)$ is determined exactly as in II.3.

II. 5. An extremal problem. Partition the ideal boundary into three disjoint sets α_0, β' and γ' where β' is closed in $\hat{W}$ and $\beta'\cup\gamma'$ is isolated from α_0. Suppose α consists of a single boundary component and allow α to range over the set $S=\beta'\cup\gamma'$. When α is chosen, set $\beta=\beta'-\alpha$, $\gamma=\gamma'$ or $\beta=\beta'$, $\gamma=\gamma'-\alpha$, depending on whether $\alpha\in\beta'$ or $\alpha\in\gamma'$. Indicate the dependence of $u(z;\alpha_0,\alpha,\beta,\gamma)$ on α by the notation u_α. Thus $\alpha\to\|du_\alpha\|$ is a real valued function on the compact subset S of $\hat{W}$.

THEOREM II.5.1. *The function* $\alpha\to\|du_\alpha\|$ *is u.s.c. Hence there is a component* $\alpha\in S$ *which maximizes* $\|du_\alpha\|$.

Proof. Let $a=\lim\sup_{\alpha_n\to\bar{\alpha}}\|du_{\alpha_n}\|$. We take a sequence α_n such that $\lim_n\|du_{\alpha_n}\| = a$, $\alpha_n\to\bar{\alpha}$.

Consider $u_{\bar{\alpha}}$ and let $\mathbf{\Omega}_k$ be one of the regions in the definition of $u_{\bar{\alpha}}$ with u_k the approximation to $u_{\bar{\alpha}}$ in $\mathbf{\Omega}_k$. For large n, u_k is also an approximation to u_{α_n}; from II.1 we have, for sufficiently large n,

$$\|du_k\|_{\mathbf{\Omega}_k} \geqslant \|du_{\alpha_n}\|,$$

and thus,

$$\|du_k\|_{\mathbf{\Omega}_k} \geqslant a.$$

This implies that

$$\|du_{\bar{\alpha}}\| \geqslant a.$$

There is a corresponding theorem for the functions $v(z;p,\alpha,\beta,\gamma)$ as follows. Let β' and γ' be a partition of the ideal boundary into two disjoint sets so that β' is closed in $\hat{W}$. Let α be a single component ranging over the ideal boundary and once α is chosen, set $\beta=\beta'-\alpha$, $\gamma=\gamma'$ or $\beta=\beta'$, $\gamma=\gamma'-\alpha$, whichever is relevant. Writing v_α for $v(z;p,\alpha,\beta,\gamma)$, about p, v_α has the expansion

$$v_\alpha = \log|z| + c_\alpha + o(1).$$

THEOREM II.5.2. *The function $\alpha \to c_\alpha$ is u.s.c. Hence there is an ideal boundary point α which maximizes c_α.*

Proof. Set $b = \limsup_{\alpha_n \to \underline{\alpha}} c_{\alpha_n}$ and choose a sequence α_n such that $\alpha_n \to \alpha$ and $\lim_n c_{\alpha_n} = b$.

Let $\{\Omega_k\}$ be an exhaustion used to define $v_{\underline{\alpha}}$. Let v_k be the approximation to $v_{\underline{\alpha}}$ in Ω_k. But for large n, v_k is also an approximation to v_{α_n}. Hence $c_k \geqslant c_{\alpha_n}$ for sufficiently large n. It follows that $c_k \geqslant b$ and then $c_{\underline{\alpha}} \geqslant b$.

Note that c_α depends on the local parameter z. If $w = w(z)$ is another local parameter with $w = 0$ at p and c'_α corresponds to c_α for w, then

$$c'_\alpha = c_\alpha + \log \left| \frac{dz}{dw} \right|_p .$$

Thus the component which maximizes c_α does not depend on z.

III. The Extremal Length Problems

III.1. Compact bordered surfaces. Suppose the boundary components of a compact bordered Riemann surface Ω are divided into four disjoint sets α_0, α, β, γ. Let $u = u(z; \alpha_0, \alpha, \beta, \gamma)$ be the harmonic function in Ω determined by the boundary conditions $u = 0$ on α_0, $u = 1$ on α, $u =$ constant on each component β_i in β in such a way that $\int_{\beta_i} du^* = 0$, and $\partial u / \partial n = 0$ along γ.

Denote the compactification of Ω by $\hat{\Omega}$ (then α_0, α, β, γ are each interpreted as a finite point set in $\hat{\Omega}$) and define the classes F, F^* of curves as in I.3. In the present situation however, when $c \in F$ or F^*, we may assume $c \cap \Omega$ has a finite number of components.

Denoting the extremal length of F and F^* by $\lambda(F)$, $\lambda(F^*)$, respectively, the object of this section is to prove the following Lemma.

LEMMA III.1.1. a) $\lambda(F) = \|du\|^{-2}$, b) $\lambda(F^*) = \|du\|^2$.

Proof of a). The proof is accomplished by finding a parametrization l_s of level curves of du^* such that $l_s \in F$ for almost all s.

The boundary $\partial\Omega$ is oriented so that Ω lies on the left. Fix a point p on α_0 as the origin and fix other points on α_0 so that there are two points specified on each component of α_0. Then using these points determine a route which traverses α_0 exactly once in the positive direction, beginning and ending at p. Since du^* is strictly negative along this route, setting $u^*(p) = 0$, define $u^*(z)$, $z \in \alpha_0$, so that as z passes along this circuit $u^*(z)$ takes on each value s, $0 \geqslant s \geqslant -\|du\|^2$, exactly once.

We will always orient the level curves of du^* so that u is increasing in the positive direction. For each s, $-\|du\|^2 < s \leqslant 0$, a single level curve $\boldsymbol{l}_s$ of u^* leaves α_0. In the succeeding paragraphs we shall formulate rules for dividing the interval $-\|du\|^2 < s \leqslant 0$ into two complementary sets, S and C.

If the connected level curve $\boldsymbol{l}_s$ passes through a critical point of u in $\overline{\Omega}$ (=closure of Ω), put s in C.

If $s \notin C$, then $\boldsymbol{l}_s$ ends at either α or β. If $\boldsymbol{l}_s$ ends at α, put s in S and write $\boldsymbol{l}_s$ simply as l_s.

Suppose $\boldsymbol{l}_s$ ends at some component $\beta_i \in \beta$. If p_s denotes the end point of $\boldsymbol{l}_s$, follow β_i from p_s in the direction of increasing du^*. When β_i is traced in this manner the arc $\boldsymbol{l}_s$ and hence Ω lies to the left. Let q_s be the first point for which $\int du^* = 0$, the integral being taken over the arc on β_i from p_s to q_s.

If q_s is a critical point of u, put s in C. Otherwise we see that a level curve $\boldsymbol{l}'_s$ of du^* at q_s *begins* at q_s since if $u = c_i$ on β_i, the fact that u is $< c_i$ to the left of p_s implies that u is $> c_i$ to the left of q_s. We will refer to $\boldsymbol{l}'_s$ as the continuation of $\boldsymbol{l}_s$.

If $\boldsymbol{l}'_s$ goes through a critical point of u on $\overline{\Omega}$, we put s in C. Otherwise we repeat the above procedure. Note that $\boldsymbol{l}'_s$ cannot return to β_i without passing through a critical point of u.

Finally we end with the following situation. Each s, $-\|du\|^2 < s \leqslant 0$, is either in C or there is an l_s, which is a finite union of connected level curves of du^*, one the continuation of the preceding, which runs from α_0 to α. For $s \in S$, each l_s can be regarded as an arc in $\hat{\Omega}$ running from α_0 to α, that is $l_s \in F$.

We make the following two observations.

(1) If $s_1, s_2 \in S$, $s_1 \neq s_2$, then $l_{s_1} \cap l_{s_2} = \emptyset$. To prove this it is enough to observe that if $\boldsymbol{l}_{s_1}$ and $\boldsymbol{l}_{s_2}$ end on the same component $\beta_i \in \beta$, the continuation of $\boldsymbol{l}_{s_1}$ must be different from the continuation of $\boldsymbol{l}_{s_2}$. Indeed, if the direction on β_i determined by $\boldsymbol{l}_{s_2}$ is the same as that determined by $\boldsymbol{l}_{s_2}$, then $q_{s_1} = q_{s_2}$ would require $q_{s_1} = p_{s_2}$, which is impossible. If the direction on β_i determined by $\boldsymbol{l}_{s_1}$ were opposite to that determined by $\boldsymbol{l}_{s_2}$ yet $q_{s_1} = q_{s_2}$, then Ω would lie both to the left of q_{s_1} and to the right of q_{s_1}, an absurdity.

(2) C consists of a finite number of points. For u has only a finite number of critical points in $\overline{\Omega}$ and only a finite number of Jordan arcs along which $du^* = 0$ pass through each critical point. In other words, except for a finite number of s, $\boldsymbol{l}_s$ can be continued until it reaches α.

Except at the finite number of critical points of u, $u + i\,u^*$ can be used as a local parameter and Ω can be paved with little rectangles determined by the level curves of u and u^*. If $\varrho\,|dz|$ is a linear density, from the Schwarz inequality if $s \in S$,

$$\int_{l_s} \varrho\,|dz|^2 = \int_{l_s} \varrho\,du^2 \leqslant \left(\int_{l_s} \varrho^2\,du \int_{l_s} du\right) = \int_{l_s} \varrho^2\,du.$$

Integrating in s from $-\|du\|^2$ to 0, $s \in S$, and we obtain

$$\inf_s \int_{l_s} \varrho\,|dz|^2 \leqslant \|du\|^{-2} A(\varrho)$$

and hence $\lambda(F) \leqslant \|du\|^{-2}$. However, using $\varrho\,|dz| = |\text{grad } u|\,|dz|$ as an admissible density we obtain

$$\left(\int_0^1 du\right)^2 \|du\|^{-2} \leqslant \left(\inf_s \int_{l_s} \varrho\,|dz|\right) \|du\|^{-2} \leqslant \lambda(F)$$

thus obtaining the proof of a).

Proof of b). The level locus (not necessarily connected) $u = c$ for $0 < c < 1$ contains a curve in F^*. The proof is completed by a repetion of the argument using the Schwarz inequality given immediately above.

III.2. Continuity lemma. We will now make use of an extremal length technique due to Beurling and developed by Wolontis [14] and, most closely approximating our present context, by Strebel [12].

Let $\{\Omega_n\}$ be an exhaustion of W of the type considered in II.1, and let F_n be the class of curves in $\bar{\Omega}_n - \gamma_n$ which go from α_{0n} to α_n, via possibly some contours in β_n (see III.1). More precisely, $l \in F_n$ if and only if the domain of l consists of a finite union of closed intervals $[a_0, a_1] \cup [a_2, a_3] \cup \ldots \cup [a_{j-1}, a_j]$ with $a_0 < a_1 < \ldots < a_j$, l is a continuous mapping into $\bar{\Omega}_n - \gamma_n$, $l(a_0) \in \alpha_{0n}$, $l(a_j) \in \alpha_n$, and for odd $i < j$, $l(a_i)$ and $l(a_{i+1})$ belong to the same component of β_n.

LEMMA III.2.1. $\lim_{n\to\infty} \lambda(F_n) \geqslant \lambda(\mathfrak{F})$.

Proof. Restatement. We shall define a family $\mathfrak{F}'$ of relative 1-chains on W such that

(a) $\text{rest}_W(\mathfrak{F}) \subset \mathfrak{F}'$

(b) $\overline{\lim}\, \lambda(F_n) \geqslant \lambda(\mathfrak{F}')$

(c) $\lambda(\mathfrak{F}') = \lambda(\mathfrak{F})$,

where $\text{rest}_W(\mathfrak{F}) = \{l \cap W : l \in \mathfrak{F}\}$.

Once this is done the proof of the lemma will be complete. Recall that $\mathbf{\Omega}_n$ was obtained from Ω_n by attaching to it all components of $W - \Omega_n$ whose boundaries belong to γ_n. Let $\mathfrak{F}_n$ be the family of curves in $\text{Cl}_W \mathbf{\Omega}_n$ which go from α_{0n} to α_n via possibly some β_n's. More precisely, $l \in \mathfrak{F}_n$ if and only if the domain of l consists of a finite union of closed in-

tervals $[a_0, a_1] \cup [a_2, a_3] \cup \ldots \cup [a_{j-1}, a_j]$ with $a_0 < a_1 < \ldots < a_j$, l is a continuous mapping into the closure (with respect to W) of Ω_n, $l(a_0) \in \alpha_{0n}$, $l(a_j) \in \alpha_n$, and for all odd $i < j$, $l(a_i)$ and $l(a_{i+1})$ belong to the same component of β_n. Then $F_n \subset \mathfrak{F}_n$ so $\lambda(F_n) \geqslant \lambda(\mathfrak{F}_n)$. Hence instead of (b) it suffices to prove

(b') $\overline{\lim}\, \lambda(\mathfrak{F}_n) \geqslant \lambda(\mathfrak{F}')$.

We wish to replace (b') by another condition. Choose any $x < \lambda(\mathfrak{F}')$. Choose a linear density $\varrho|dz|$ such that $L^2(\mathfrak{F}', \varrho) > x$ and $A(\varrho) = 1$. To prove (b') it is enough to show $\sup_n L^2(\mathfrak{F}_n, \varrho) \geqslant x$. If that failed to hold there would exist a subsequence along which $L^2(\mathfrak{F}_n, \varrho)$ had a limit $y < x$. Let $L(l, \varrho) = \int_l \varrho|dz|$. If we can prove that to each $\varepsilon > o$ there is an $l(\varepsilon) \in \mathfrak{F}'$ satisfying

$$L^2(l(\varepsilon), \varrho) \leqslant y + 7\varepsilon$$

then we would have the desired contradiction since

$$y < x < L^2(\mathfrak{F}', \varrho) \leqslant L^2(l(\varepsilon), \varrho) \leqslant y + 7\varepsilon.$$

(b'') Given a density $\varrho|dz|$ on W with $L^2(\mathfrak{F}_n, \varrho) \to y$ as $n \to \infty$. Then to each $\varepsilon > o$ there is an $l(\varepsilon) \in \mathfrak{F}'$ to be defined according to (a), (c) such that

$$L^2(l(\varepsilon), \varrho) \leqslant y + 7\varepsilon. \tag{3}$$

Some notation and terminology. Given $l_N \in \mathfrak{F}_N$ and $n \leqslant N$, we wish to define a sort of restriction of l_N to $\mathrm{Cl}_W \Omega_n$, to be denoted by $l_N \| \Omega_n$, with the property that $l_N \| \Omega_n \in \mathfrak{F}_n$.

There is a greatest t for which $l_N(t) \in \alpha_{0n}$; call it t_1. Let t_2 be the smallest t for which $l_N(t) \in \partial\Omega_n$ and also $t > t_1$. Then $l_n(t_2)$ is on some contour of β_n (or possibly α_n); call the contour c_2. Set $t_3 =$ greatest t for which $l_N(t) \in c_2$. We continue this way and obtain an even number of *stopping times* $t_1 < t_2 \ldots < t_k$, a sequence of stopping points $l_N(t_1), \ldots, l_N(t_k)$, and a *contour sequence* $\alpha_{0n} = c_1, c_2, \ldots, c_{(k+2)/2} = \alpha_n$ of distinct contours on $\partial\Omega_n$ such that $l_N(t_j) \in c_{[(j+2)/2]}$ $(j = 1, \ldots, k)$. Define $l_N \| \Omega_n$ to be the restriction of l_N to $[t_1, t_2] \cup [t_3, t_4] \cup \ldots \cup [t_{k-1}, t_k]$.

Definition of $\mathfrak{F}'$. A 1-chain l' on W will belong to $\mathfrak{F}'$ if either $l' = l \cap W$ for some $l \in \mathfrak{F}$, or if l' is a continuous map of an open dense subset of (0,1) into W such that:

F'–1. If t_0 is not in the domain dom l' of l' and $0 < t_0 < 1$, then there exist sequences $\{r_n\}$, $\{s_n\}$ in dom l' such that $r_n \nearrow t_0$, $s_n \searrow t_0$ and a point $* \in \beta$ such that $l'(r_n) \to *$, $l'(s_n) \to *$. If $t_0 = 0$ (resp. 1) we require only a sequence $\{s_n\}$ (resp. $\{r_n\}$) from dom l' with $s_n \searrow 0$ (resp. $r_n \nearrow 1$) and $l'(s_n) \to \alpha_0$ (resp. $l'(r_n) \to \alpha$).

F'–2. There is a canonical exhaustion $\{\Omega_n\}$ such that $l' \| \Omega_N \in \mathfrak{F}_N$ for each $N \geqslant 1$.

F'–3. If $t \in$ dom l' then there exists N such that $t \in$ dom $l' \| \Omega_n$ for all $n \geqslant N$.

Proof of (a) *and* (b'). Condition (a) is part of the definition of $\mathcal{F}'$. To prove (b') suppose given an $\varepsilon>0$. By passage to a subsequence of $\{l_n\}$ we may assume that

$$|L^2(\mathcal{F}_n, \varrho)-y|<\varepsilon/2^n \quad (n\geqslant 1). \tag{4}$$

Whenever a subsequence of $\{l_n\}$ is extracted and the notation is unchanged we tacitly agree that $\{\Omega_n\}$ shall refer to the corresponding subsequence of $\{n\}$.

Choose $l_n\in\mathcal{F}_n$ such that

$$|L^2(l_n, \varrho)-y|<\varepsilon/2^n \quad (n\geqslant 1). \tag{5}$$

A subsequence of $\{l_n\}$, after some modification, will be used to construct $l(\varepsilon)$.

The first step is to find a subsequence $\{l_{n_i}\}$ of $\{l_n\}$ such that all $l_{n_i}\|\mathbf{\Omega}_N$ $(n_i\geqslant N)$ have the same contour sequence on $\partial\mathbf{\Omega}_N$. Since there are only a finite number of possible contour sequences on $\partial\mathbf{\Omega}_1$ we may select a first subsequence of $\{l_n\}$, all elements of which have the same contour sequence on $\partial\mathbf{\Omega}_1$. By induction we obtain for each N a subsequence of the preceding one, all of whose elements follow a common contour sequence on $\partial\mathbf{\Omega}_N$. The diagonal process yields a subsequence with the desired property. We shall not change notation, but denote it still by $\{l_n\}$. Note that (5) continues to hold.

The next step will be to modify each l_n so that not only will all $l_n\|\mathbf{\Omega}_N$ $(n\geqslant N)$ follow the same contour sequence but furthermore, $l_n\|\mathbf{\Omega}_{n-1}$ and $l_{n-1}\|\mathbf{\Omega}_{n-1}$ will have the same sequence of stopping points on $\partial\mathbf{\Omega}_{n-1}$. To do this we use the diagonal process to find a preliminary subsequence, again called $\{l_n\}$, with the following property: Suppose l_N has k stopping points on $\partial\mathbf{\Omega}_N$. Then for each $i\leqslant k$ the ith stopping point P_n of $l_n\|\mathbf{\Omega}_N$ $(n\geqslant N)$ gives rise to a convergent sequence of points $\{P_n\}$ on a contour of $\partial\mathbf{\Omega}_N$. Now, we put a topological disk around the limit point of this sequence, the circumference of which has very small ϱ-length. The actual length will be determined below. Note, however, that it can be required to be arbitrarily small. Indeed, the extremal length of all Jordan arcs in a punctured disk which surround a fixed point is zero, and hence for any $\varrho|dz|$ there is such an arc of arbitrarily small ϱ-length.

For each N we have as many disks on $\partial\mathbf{\Omega}_N$ as there are stopping points for $l_n\|\mathbf{\Omega}_N$ (any $n\geqslant N$). Choose their circumferences so small that the total ϱ-length of them is $<\varepsilon/2^N$. By the diagonal process we can achieve a situation were each stopping point of $l_n\|\mathbf{\Omega}_N$ on $\partial\mathbf{\Omega}_N$ is inside its appropriate disk for all n, N with $n\geqslant N$. For each disk pick a point on the intersection of its circumference and the corresponding contour; call such a point a distinguished stopping point. By a modification of l_n we mean the result of replacing part of its path inside a disk by a path on the circumference of the disk. Now modify l_1 so that all its stopping points are distinguished; in general, modify l_n so that the stopping points of $l_n\|\mathbf{\Omega}_{n-1}$ on $\partial\mathbf{\Omega}_{n-1}$ and $l_n\|\mathbf{\Omega}_n$ on $\partial\mathbf{\Omega}_n$ are distinguished. Denote the modified sequence

again by $\{l_n\}$. The ϱ-length of l_n has been increased by no more than $\varepsilon/2^{n-1}+\varepsilon/2^n$ as a result of modification. For the present sequence $\{l_n\}$ equation (5) must be changed to

$$|L(l_n, \varrho)-y| < 4\varepsilon/2^n. \tag{5'}$$

These modifications can be accomplished so that each new l_n remains in $\mathfrak{F}_n$.

By induction, for each n reparametrize l_n so that $\operatorname{dom} l_n \| \mathbf{\Omega}_{n-1} = \operatorname{dom} l_{n-1}$. Then $\operatorname{dom} l_{n-1}$ consists of a finite number of closed intervals $[t_1, t_2] \cup [t_3, t_4] \cup \ldots \cup [t_{k-1}, t_k]$ and $l_{n-1}(t_i) = l_n(t_i)$ $(1 \leqslant i \leqslant k)$.

We are now ready to construct $l(\varepsilon)$. On $\operatorname{dom} l_1$ set $l(\varepsilon) = l_1$. In general, if $l(\varepsilon)$ has been defined on $\operatorname{dom} l_{n-1}$ set $l(\varepsilon) = l_n$ on $\operatorname{dom} l_n - \operatorname{dom} l_{n-1}$. Then $l(\varepsilon)$ is a continuous 1-chain on W. Its domain is an open subset of (0,1) which, by reparametrization, we may assume to be dense.

To estimate the ϱ-length of $l(\varepsilon)$ note that the ϱ-length of l_n restricted to $\operatorname{dom} l_n - \operatorname{dom} l_{n-1}$ is $< (6\varepsilon/2^n)^{\frac{1}{2}}$. Indeed,

$$L^2(l_n, \varrho) < y + 4\varepsilon/2^n$$

by (5') and, since $l_n | \operatorname{dom} l_{n-1} = l_n \| \mathbf{\Omega}_{n-1} \in \mathfrak{F}_{n-1}$,

$$L^2(l_n \| \mathbf{\Omega}_{n-1}, \varrho) \geqslant L^2(\mathfrak{F}_{n-1}, \varrho) > y - \varepsilon/2^n$$

by (4). Hence $L^2(l(\varepsilon), \varrho) < L^2(l_1, \varrho) + \Sigma 6\varepsilon/2^n < y + 7\varepsilon$.

It remains to show that $l(\varepsilon) \in \mathfrak{F}'$. We can satisfy F'–1 as follows. Suppose $t_0 \notin \operatorname{dom} l(\varepsilon)$ and $t_0 \neq 0,1$. Consider the stopping times $t_1, \ldots, t_k$ on $l(\varepsilon)$ on $\partial\mathbf{\Omega}_n$. For n sufficiently large t_0 is between two stopping times which correspond to stopping points on a common contour c_n of $\partial\mathbf{\Omega}_n$. Thus we obtain a sequence of contours $\{c_n\}$ with $c_n \subset \partial\mathbf{\Omega}_n$. We cannot assert that these contours tend to a single point of $\hat{W} - W$. However, there is a subsequence which does have a limit point, say $* \in \hat{W} - W$. The corresponding stopping times yield $\{r_n\}$, $\{s_n\}$. The cases $t_0 = 0,1$ can be handled similarly. The checking of F'–2, F'–3 will be omitted.

Proof of (c). If a 1-chain $l' \in \mathfrak{F}'$ can be extended continuously to [0,1] with values in $\hat{W}$ the extension will automatically be an arc in $\mathfrak{F}$. For each Ω_n we consider annular regions A_{ni} around each contour of $\partial\mathbf{\Omega}_n$. We show that if no such annulus is crossed infinitely often by l' then l' can be extended continuously to [0, 1]. This will prove (c) because the extremal length of a family of 1-chains, each 1-chain of which crosses some A_{ni} infinitely often, is ∞.

Given $t_0 \in \operatorname{dom} l'$, $t_0 \neq 0,1$. Let $\{r_n\}$, $\{s_n\}$, $*$ be as in F'–1. We need only prove

$$\lim_{t \to t_0} l'(t) = * \quad (t \in \operatorname{dom} l'). \tag{6}$$

Let G be a neighborhood of ∞ on $\hat{W}$. We wish to find a neighborhood of t_0 whose l'-image is in G, and for this we may assume G is a component of $\hat{W}-\mathbf{\Omega}_N$ for some N. Let A be the annular region around ∂G chosen above; for definiteness assume A and G have intersection ∂G. Now $l'(r_n)$, $l'(s_n)\in G$ for sufficiently large n. If (6) failed we could find for some G, $u_n\in\operatorname{dom} l'$ with $u_n\to t_0$ and $l'(u_n)\notin G\cup A$. A subsequence of $\{u_n\}$ is monotonic so suppose $u_n\nearrow t_0$. Choose r_{i_1} and $u_{i_1}>r_{i_1}$ and $\mathbf{\Omega}_{m_1}\supset\Omega_N$ so that r_{i_1}, $u_{i_1}\in\operatorname{dom} l'\|\mathbf{\Omega}_{m_1}$. Since $l'\|\mathbf{\Omega}_{m_1}\in\mathfrak{F}_{m_1}$ there is a crossing of A during (r_{i_1}, u_{i_1}). Next choose r_{i_2}, u_{i_2}, $\mathbf{\Omega}_{m_2}$ such that $r_{i_1}<u_{i_1}<r_{i_2}<u_{i_2}$, and r_{i_2}, $r_{i_1}\in\operatorname{dom} l'\|\mathbf{\Omega}_{m_2}$. There must be a crossing of A during (r_{i_2}, u_{i_2}). In this way we see that l' crosses A infinitely many times. The cases $t_0=0,1$ can be treated similarly.

III.3. Solution of the extremal length problems. The definitions of $\mathfrak{F}$, $\mathfrak{F}^*$ were given in I.3, and $du(z)=du(z;\alpha_0,\alpha,\beta,\gamma)$ was constructed in II.1.

THEOREM III.3.1. *Let W be an arbitrary open Riemann surface and let du, $\mathfrak{F}$, $\mathfrak{F}^*$ be defined with respect to an admissible partition of the ideal boundary of W. Then*

$$\text{(a)}\quad \lambda(\mathfrak{F})=\|du\|^{-2}$$

$$\text{(b)}\quad \lambda(\mathfrak{F}^*)=\|du\|^{2}.$$

Proof of (a). Using the notation of Chapter II and replacing the pair F and Ω of III.1 by F_i and Ω_{ni} Lemma III.1.1 implies that $\lambda(F_i)=\|du_{ni}\|_{\Omega_{ni}}^{-2}$. Using $\varrho|dz|=|\operatorname{grad} u_n|\,|dz|$ as an admissible metric for the problem $\lambda(\mathfrak{F}_n)$ (see III.2) we find

$$\|du_n\|_{\Omega_n}^{-2}=\left(\int_0^1 du_n\right)^2\|du_n\|_{\Omega_n}^{-2}\leqslant\inf_{c\in\mathfrak{F}_n}\left(\int_c\varrho\,|dz|\right)^2A(\varrho)^{-1}$$

and hence for all i,

$$\|du_n\|_{\Omega_n}^{-2}\leqslant\lambda(\mathfrak{F}_n)\leqslant\lambda(F_i)=\|du_{ni}\|_{\Omega_{ni}}^{-2}.$$

Consequently $\lambda(\mathfrak{F}_n)=\|du_n\|_{\Omega_n}^{-2}$.

Since every curve in $\mathfrak{F}$ contains a curve in $\mathfrak{F}_n$, $\lambda(\mathfrak{F})\geqslant\lambda(\mathfrak{F}_n)$. On the other hand, from Lemma III.2.1 $\|du\|^{-2}=\lim\lambda(\mathfrak{F}_n)\geqslant\lambda(\mathfrak{F})$. The proof of (a) is now complete.

Proof of (b). We write Ω_{ni}, F_i^* for Ω and F^* of III.1. Since every curve in $\mathfrak{F}_n^*$ contains a curve in F_i^* for all i we find

$$\|du_{ni}\|_{\Omega_{ni}}^2=\lambda(F_i^*)\leqslant\lambda(\mathfrak{F}_n^*)$$

and $\|du_n\|_{\Omega_n}^2\leqslant\lambda(\mathfrak{F}_n^*)$. Conversely every level loci $u_n=c$ in $\mathbf{\Omega}_n$ contains a curve in $\mathfrak{F}_n^*$ and application of the Schwarz inequality yields $\lambda(\mathfrak{F}_n^*)\leqslant\|du_n\|_{\Omega_n}^2$. Hence $\lambda(\mathfrak{F}_n^*)=\|du_n\|_{\Omega_n}^2$.

Since every curve in $\mathcal{F}_n^*$ is a curve in $\mathcal{F}^*$, $\lambda(\mathcal{F}^*) \leqslant \lambda(\mathcal{F}_n^*)$ for all n. Therefore $\lambda(\mathcal{F}^*) \leqslant \|du\|^2$. It remains to prove that $\|du\|^2 \leqslant \lambda(\mathcal{F}^*)$.

Set $du_{ni} = 0$ in $\mathbf{\Omega}_n - \Omega_{ni}$ and consider the sequence of linear densities $\varrho_{ni}|dz| = |\operatorname{grad}(u_n - u_{ni})||dz|$ in $\mathbf{\Omega}_n$. According to Lemma I.2 there exists a subclass $\mathcal{F}_n'^*$ of $\mathcal{F}_n^*$ with $\lambda(\mathcal{F}_n'^*) = \lambda(\mathcal{F}_n^*)$ and a sequence of numbers $\{i\}$ such that for all curves $c \in \mathcal{F}_n'^*$,

$$\lim_{i\to\infty} \left| \int du_n^* - du_{ni}^* \right| \leqslant \lim_{i\to\infty} \int_c |du_n - du_{ni}| = 0.$$

Here $\int_c du_{ni}^* = \int_{c \cap \Omega_{ni}} du_{ni}^* = \|du_{ni}\|_{\Omega_{ni}}^2$ as can be seen most easily by replacing c by a simplicial approximation to c in a triangulation of W in which $\partial\Omega_{ni}$ is a cycle. We conclude that

$$\int_c du_n^* = \|du_n\|_{\Omega_n}^2$$

for all $c \in \mathcal{F}'_{n*}$.

For each $c \in \mathcal{F}^*$, there exists a number $N(c)$, depending on c, such that $c \subset \mathbf{\Omega}_n$ for all $n \geqslant N(c)$, that is $c \in \mathcal{F}_n^*$ for all $n \geqslant N(c)$. Otherwise c would have limit points on α_0, α, or β. Since the countable union of classes of curves of infinite extremal length has infinite extremal length, there exists a subclass $\mathcal{F}'^*$ of $\mathcal{F}^*$ with $\lambda(\mathcal{F}'^*) = \lambda(\mathcal{F}^*)$ such that $c \in \mathcal{F}'^*$ if and only if $c \in \mathcal{F}_n'^*$ for all $n \geqslant N(c)$.

Set $du_n = 0$ in $W - \mathbf{\Omega}_n$ and apply Lemma I.2 to the sequence of linear densities $\varrho_n|dz| = |\operatorname{grad}(u - u_n)||dz|$. There exists a subclass $\mathcal{F}''^*$ of $\mathcal{F}'^*$ with $\lambda(\mathcal{F}''^*) = \lambda(\mathcal{F}'^*)$ and a sequence of numbers $\{m\}$ such that for all curves $c \in \mathcal{F}''^*$,

$$\lim_{m\to\infty} \left| \int_c du^* - \int_c du_m^* \right| \leqslant \lim_{m\to\infty} \int_c \varrho_m |dz| = 0.$$

We have shown above that $\int_c du_m^* = \|du_m\|_{\Omega_m}^2$ when $m \geqslant N(c)$. Again we may conclude that $\int_c du^*$ exists and that

$$\int_c du^* = \|du\|^2 \quad \text{for all } c \in \mathcal{F}''^*.$$

Now using $\varrho|dz| = |\operatorname{grad} u||dz|$ as an admissible linear density for $\lambda(\mathcal{F}''^*)$, we find

$$\|du\|^2 = \inf_{c\in\mathcal{F}''^*} \int_c du^* \leqslant \inf_{c\in\mathcal{F}''^*} \int_c \varrho |dz|.$$

Consequently, $$\|du\|^2 \leqslant \lambda(\mathcal{F}''^*) = \lambda(\mathcal{F}^*),$$

and the theorem is proved.

III.4. Some consequences. Properties of u can be discovered by extremal length considerations.

COROLLARY III.4.1. *(Uniqueness theorem for $u(z;\ \alpha_0,\ \alpha,\ \beta,\ \gamma)$.) If v is harmonic in W with $\|dv\| \leqslant \|du\|$ and $\int_c dv \geqslant 1$ for almost all $c \in \mathfrak{J}$, then $v = u + a$ constant. The same conclusion is true if $\|dv\| \geqslant \|du\|$ and $\int_c dv^* \geqslant \|dv\|^2$ for almost all $c \in \mathfrak{J}^*$.*

Proof. We will prove only the first statement; the proof of the second is similar. The hypotheses imply that $\varrho_1 |dz| = |\text{grad } v|\,|dz|$ is the extremal metric for a subclass $\mathfrak{J}_1$ of $\mathfrak{J}$ with $\lambda(\mathfrak{J} - \mathfrak{J}_1) = \infty$. We already know that $\varrho |dz| = |\text{grad } u|\,|dz|$ is the extremal metric for a subclass $\mathfrak{J}'$ of $\mathfrak{J}$ with $\lambda(\mathfrak{J} - \mathfrak{J}') = \infty$. Setting $\mathfrak{J}_2 = \mathfrak{J}_1 \cap \mathfrak{J}'$ we have that $\lambda(\mathfrak{J}_2) = \lambda(\mathfrak{J})$ since $\mathfrak{J} = \mathfrak{J}_2 \cup (\mathfrak{J} - \mathfrak{J}_1) \cup (\mathfrak{J} - \mathfrak{J}')$. Hence $\varrho_1 |dz|$ and $\varrho |dz|$ are both extremal metrics for the class $\mathfrak{J}_2$ and thus $\varrho_1 \equiv \varrho$ or $|\text{grad } v| \equiv |\text{grad } u|$. This implies that $dv = (\cos\theta)du - (\sin\theta)du^*$ for some constant θ. Let d be a cycle that separates α_0 from α, then $\int_d du^* = \pm \|du\|^2$. Hence because $\int_c du = 1$ and $\int_c dv = 1$ for almost all $c \in \mathfrak{J}$, $\theta = 0$.

COROLLARY III.4.2. *The Dirichlet integral $\|du(z;\ \alpha_0,\ \alpha,\ \beta,\ \gamma)\|$* a) *increases (or is unchanged) when components in β or γ are placed in α_0, α or β;* b) *decreases (or is unchanged) when components in α_0, α or β are placed in β or γ.*

Next, we will formulate the corresponding results for $v(z;\ p,\ \alpha,\ \beta,\ \gamma)$. Let z be a local parameter about p, $z(p) = 0$, and

$$v(z;\ p,\ \alpha,\ \beta,\ \gamma) = \log|z| + c_z(p,\ \alpha,\ \beta,\ \gamma) + o(1)$$

be the expansion of v in z. For small $r > 0$ set $\alpha_0 = \{|z| = r\}$ and let $\mathfrak{J}_r$ be the family of curves $\mathfrak{J}_r \equiv \mathfrak{J}(\alpha_0,\ \alpha,\ \beta,\ \gamma)$. Similarly, let $\mathfrak{J}_r^*$ be the conjugate class of curves $\mathfrak{J}_r^* = \mathfrak{J}^*(\alpha_0,\ \alpha,\ \beta, \gamma)$. An elementary inequality gives for $r' < r$

$$\lambda(\mathfrak{J}_{r'}) \geqslant \lambda(\mathfrak{J}_r) + (\log r/r')/2\pi,$$

and hence $2\pi\lambda(\mathfrak{J}_r) + \log r$ is increasing as r decreases and has a limit $\leqslant +\infty$.

LEMMA III.4.3. $1 - c_z(p, \alpha, \beta, \gamma) = \lim_{r \to 0} 2\pi\lambda(\mathfrak{J}_r) + \log r.$

Proof. We first consider the effect of changing the local parameter about p from z to w where $w = az + o(z)$ or $\log|w| = \log|z| + \log|a| + o(1)$ (as $z \to 0$). Choose r_1 so that the disk $|w| \leqslant r_1$ is the largest such disk that fits inside the disk $|z| \leqslant r$. Then

$$2\pi\lambda(\mathfrak{J}_r) + \log r \leqslant 2\pi\lambda(\mathfrak{J}_{r_1}) + \log r_1 - \log|a| + o(1)$$

and

$$\lim_{r \to 0} (2\pi\lambda(\mathfrak{J}_r) + \log r) \leqslant \lim_{r_1 \to 0} (2\pi\lambda(\mathfrak{J}_{r_1}) + \log r_1) - \log|a|.$$

By considering the smallest disk $|w| \leqslant r_2$ which contains $|z| \leqslant r$ the opposite inequality can be established and hence, using an obvious notation,

$$\lim_{r\to 0} (2\pi\lambda_z(\mathfrak{F}_r) + \log_z r) + \log\left|\frac{dw}{dz}\right|_{z=0} = \lim_{r\to 0} (2\pi\lambda_w(\mathfrak{F}_r) + \log_w r).$$

Now consider $v = v(z;\, p,\, \alpha,\, \beta,\, \gamma)$ and let w be the local parameter

$$w = e^{v+iv^*} = ze^{cz(p,\,\alpha,\,\beta,\,\gamma)} + o(z),$$

where v^* is the harmonic conjugate to v. Setting $v_1 = (v - \log r)/(1 - \log r)$ and $\alpha_0 = \{|w| = r\}$ we see that $v_1 = u(w;\, \alpha_0,\, \alpha,\, \beta,\, \gamma)$ and hence $\|dv_1\|^{-2} = \lambda_w(\mathfrak{F}_r)$. A simple computation gives $\|dv_1\|^2 = (1 - \log r)^{-2}\|dv\|^2 = 2\pi(1 - \log r)^{-1}$ where the Dirichlet integrals are extended over $W - \{|w| \leqslant r\}$. Consequently

$$\lim_{r\to 0} (2\pi\lambda_w(\mathfrak{F}_r) + \log r) = 1$$

and the validity of the lemma is clear.

COROLLARY III.4.4. *(Uniqueness theorem for $v(z;\, p,\, \alpha,\, \beta,\, \gamma)$.) Let w be harmonic in $W - \{p\}$ with the expansion $w = \log|z| + c + o(1)$ at p in the local parameter z $(z(p) = 0)$. Set $A = \lim_{\Omega\to W} \int_{\partial\Omega} w\, dw^*$ where $\{\Omega\}$ is an exhaustion of W and set $\alpha_0(r) = \{z : v(z) = \log r\}$. If for some decreasing sequence $r = r(n)$, $\lim r(n) = 0$, we have*

$$\int_s dw \geqslant 1 - \log r, \quad \textit{almost all } s \in \mathfrak{F}_r = \mathfrak{F}(\alpha_0(r),\, \alpha,\, \beta,\, \gamma) \textit{ and all } r = r(n),$$

and

$$c - c_z(p,\, \alpha,\, \beta,\, \gamma) \geqslant (A/2\pi) - 1,$$

then

$$w \equiv v(z;\, p,\, \alpha,\, \beta,\, \gamma) + a \textit{ constant}.$$

Proof. We note the condition above is independent of the choice of local parameter z and that if $w \equiv v$ then $A = 2\pi$.

Choose the local parameter $\zeta = \exp(v + iv^*) = z \exp c_z + o(z)$ about p, where v^* is the conjugate harmonic function to v. Let Δ be the disk $|\zeta| \leqslant r$ and set $w' = w/(1 - \log r)$ in $W - \Delta$. Corollary III.4.1 implies that

$$\|dw'\|^{-2} = (1 - \log r)^2\|dw\|^{-2} \leqslant \lambda_\zeta(\mathfrak{F}_r),$$

where the Dirichlet integrals are extended over $W - \Delta$. We then find that, setting $d = c - c_z$,

$$2\pi(1 - \log r)^2\ (A - 2\pi d - 2\pi \log r)^{-1} + \log r \leqslant 2\pi\lambda_\zeta(\mathfrak{F}_r) + \log r.$$

As $r = r(n)$ approaches 0 the left side decreases and the right side increases. We obtain on using the lemma that

$$2 + d - A/2\pi \leqslant 1.$$

By assumption equality must hold. Hence $\lambda_\zeta(\mathfrak{F}_r) = \|dw'\|^{-2}$ for all $r = r(n)$, and from Corollary III.4.1,

$$w' = u(\zeta;\ \alpha_0(r),\ \alpha,\ \beta,\ \gamma) + \text{a constant (depending on } r).$$

But we have previously observed that $u(\zeta) = (v(z) - \log r)/(1 - \log r)$ and we conclude that $w \equiv v +$ a constant.

Remark. The hyptheses of Corollary III.4.4, which are satisfied by $w +$ a constant if they are satisfied by w, can be replaced by the following conditions. Set $\alpha_0'(r) = \{z:\ w(z) = \log r\}$. Then require

$$\int_s dw \geqslant 1 - \log r, \quad \text{almost all } s \in \mathfrak{F}_r' = \mathfrak{F}(\alpha_0'(r), \alpha, \beta, \gamma)$$

and

$$c - c_z(p, \alpha, \beta, \gamma) \leqslant 1 - A/2\pi.$$

These conditions are not necessarily satisfied if a constant is added to w but it now follows, exactly as above, that $w \equiv v$. Indeed, let $\zeta = \exp(w + iw^*)$ and $w' = (w - \log r)/(1 - \log r)$. The lemma implies that $\lim (2\pi\lambda(\mathfrak{F}_r') + \log r) = 1 - c_z + c$ and we conclude that $w' = u(\zeta;\ \alpha_0',\ \alpha,\ \beta,\ \gamma)$. But then (in $W - \Delta$) $\|d(w - v)\|^2 = -\int_{\partial\Delta}(w - v)d(w - v)^* = 0$.

COROLLARY III.4.5. *Notation as in Corollary* III.4.4. *Assume*

$$\int_\tau dw^* \geqslant 2\pi, \quad \textit{almost all } \tau \in \mathfrak{F}_r^*,$$

and

$$c - c_z(p, \alpha, \beta, \gamma) \geqslant (A/2\pi) - 1.$$

Then

$$w \equiv v(z;\ p, \alpha, \beta, \gamma) + \textit{a constant}.$$

COROLLARY III.4.6. *The constant* $c_z(p, \alpha, \beta, \gamma)$ a) *decreases (or remains unchanged) when components in* $\beta \cup \gamma$ *are placed in* $\alpha \cup \beta$, b) *increases (or remains unchanged) when components in* $\alpha \cup \beta$ *are placed in* $\beta \cup \gamma$.

COROLLARY III.4.7. a) *If* α_0 *is a curve imbedded in a surface* W, $\|du(z;\ \alpha_0, \alpha, \beta, \gamma)\| = 0$ *implies* $\|du(z;\ \alpha_0, \alpha, \beta, \gamma)\| = 0$ *for any other curve* α_0; b) $c_z(p, \alpha, \beta, \gamma) = -\infty$ *for some* $p \in W$ *implies* $c_z(p, \alpha, \beta, \gamma) = -\infty$ *for all points* $p \in W$.

IV. Plane regions

IV.1. Definition and extremal properties of the slit mappings. Assume that W is a plane region and that α_0 are each single ideal boundary components. Furthermore, we will assume that

$$\|du(z;\ \alpha_0, \alpha, \beta, \gamma)\| > 0, \quad c_z(p, \alpha, \beta, \gamma) > -\infty.$$

Then $du(z;\alpha_0,\alpha,\beta,\gamma)^*$ has a single period equal to $\|du\|^2$ along dividing cycles separating α_0 from α, and $dv(z;p,\alpha,\beta,\gamma)^*$ has a single period of 2π around cycles bounding a region containing p. Hence the functions

$$f(z;\alpha_0,\alpha,\beta,\gamma)=\exp[2\pi(u+iu^*)/\|du\|^2]$$
$$f(z;p,\alpha,\beta,\gamma)=\exp(v+iv^*)$$

are determined to a multiplicative factor k, $|k|=1$. By using a standard approximation argument, it is seen that $f(z;\alpha_0,\alpha,\beta,\gamma)$ and $f(z;p,\alpha,\beta,\gamma)$ are univalent.

Definition IV.1.1. Assume A is a plane region with a partition $(\alpha_0,\alpha,\beta,\gamma)$ of its boundary subject to the conditions of I.3 and such that α_0 and α consist of single components. If A is contained in the annulus $1<|z|<R$ and $u(z;\alpha_0,\alpha,\beta,\gamma)\equiv\log|z|/\log R$ then A (or $A(R)$) is referred to as an *extremal slit annulus.*

Assume A is a plane region with a partition (α,β,γ) of its boundary subject to the conditions of II.4 and such that α consists of a single component. If A is contained in the unit disk $|z|<1$ and contains the point $z=0$ such that $v(z;0,\alpha,\beta,\gamma)\equiv\log|z|+1$, then A is referred to as an *extremal slit disk.*

It is easily seen that if $\{|z|=r\}$ is contained in the interior of A, then A is an extremal slit disk if and only if $A\cap\{|z|>r\}$ is an extremal slit annulus (after the mapping $z\to z/r$). Indeed the necessity was seen in II.4. Assume conversely that $A'=A\cap\{|z|>r\}$ is an extremal slit annulus (after the mapping $z\to z/r$) with $|z|=r$ an isolated boundary component. Set $\alpha_0=\{|z|=r\}$, α, β, and γ as given with A,

$$u=u(z;\alpha_0,\alpha,\beta,\gamma)=\log|z/r|/\log r^{-1},\quad\text{and}\quad v=v(z;0,\alpha,\beta,\gamma).$$

We find after using a suitable exhaustion that

$$\|d(u-v)\|^2_{A'}=-\int_{|z|=r}(u-v)d(u-v)^*=-\|d(u-v)\|^2_{\{|z|\leqslant r\}}.$$

Hence $u=v$.

Using the preceding result as justification we will deal only with extremal slit annuli. These can be characterized geometrically as follows.

THEOREM IV.1.2. *Suppose A is a region contained* in $\{1<|z|<R\}$ *with admissible boundary partition* $(\alpha_0,\alpha,\beta,\gamma)$. *Set* $\varrho=(|z|\log R)^{-1}$. *The following conditions are equivalent.*

(*a*) *A is an extremal slit annulus $A(R)$*

(*b*) $\int_c\varrho|dz|\geqslant 1$ *and* $\int_d\varrho|dz|\geqslant 2\pi/\log R$ *for almost all* $c\in\mathcal{J}$, $d\in\mathcal{J}^*$.

(*c*) $\lambda(\mathcal{J})\leqslant(\log R)/2\pi$ *and* $\int_c\varrho|dz|\geqslant 1$ *for almost all* $c\in\mathcal{J}$.

(*d*) $\lambda(\mathcal{J})\geqslant(\log R)/2\pi$ *and* $\int_d\varrho|dz|\geqslant 2\pi/\log R$ *for allmost all* $d\in\mathcal{J}^*$.

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Proof that (b) *implies* (a). Using ϱ as a competing density for $\lambda(\mathfrak{F})$, $\lambda(\mathfrak{F}^*)$,

$$(\log R)/2\pi \leqslant A(\varrho)^{-1} \leqslant \lambda(\mathfrak{F}) = \lambda(\mathfrak{F}^*)^{-1} \leqslant (A(\varrho)\log^2 R)/4\pi^2 \leqslant (\log R)/2\pi.$$

Now apply Corollary III.4.1 to $v = \varrho$.

Remark. Suppose for example that $\alpha_0 = \{|z| = 1\}$ and $\alpha = \{|z| = R\}$. Then if $\beta = \emptyset$ or $\gamma = \emptyset$ the conditions (c) and (d) can be replaced by the requirement $\lambda(\mathfrak{F}) = (\log R)/2\pi$. If $\beta \neq \emptyset$ and $\gamma \neq \emptyset$ this single requirement is not sufficient.

Definition IV.1.3. Let A be an extremal slit annulus with respect to the partition $(\alpha_0, \alpha, \beta, \gamma)$ and let π be a collection of boundary components of A not containing α_0 or α. If the new region $A \cup \{\pi\}$ is an extremal slit annulus with respect to the partition $\alpha_0' = \alpha_0$, $\alpha' = \alpha$, $\beta' = \beta - \beta \cap \pi$, $\gamma' = \gamma - \gamma \cap \pi$ then π is said to be *removable.*

THEOREM IV.1.4. *Let J be an analytic Jordan curve in the interior of an extremal slit annulus A which does not separate α_0 and α. Denote by π the collection of all those boundary components of A which are separated from α by J. Then π is removable.*

Proof. The proof depends on the fact that $u = u(z; \alpha_0, \alpha, \beta, \gamma) = \log|z|/\log R$ which was defined with respect to A is also harmonic in $A' = A \cup \{\pi\}$. Let $\alpha_0' = \alpha_0$, $\alpha' = \alpha$, $\beta' = \beta - \beta \cap \pi$, $\gamma' = \gamma - \gamma \cap \pi$ and set $v = u(z; \alpha_0', \alpha', \beta', \gamma')$, the extremal function for A'. Denote by Δ the region in A' bounded by J. Using a suitable exhaustion of A and the induced exhaustion of A' together with the corresponding approximations to u and v, one computes, borrowing the result of Theorem IV.2.1a,

$$\|d(u-v)\|_A^2 = -\int_J (u-v)d(u-v)^* + \|d(u-v)\|_\Delta^2 = 0,$$

and hence $u \equiv v$.

COROLLARY IV.1.5. *In the notation of the Theorem above, if π is contained in β or in γ and if π_1 is any collection of components in π, then π_1 is removable.*

The extension of Theorem IV.1.4 to general non-isolated sets π is fraught with difficulties and is certainly not always possible. Therefore we will make the following definition.

Definition IV.1.6. Let σ be a closed, connected subset of some boundary component of A. Consider collections π of components such that $\beta - \pi \cap \beta$ (or $\gamma - \pi \cap \gamma$) has no accumulation point on σ. If there exists such a π which is removable, then σ is called *β-isolated* (or *γ-isolated*, respectively).

Sufficient conditions that σ be β- or γ-isolated are readily obtainable using the following technique. Suppose for example that π is a countable collection of boundary com-

ponents with $\pi \subset \beta$ such that $\pi = \lim \pi_n$ where $\{\pi_n\}$ is a countable sequence of isolated collections π_n. Set

$$\mathfrak{F} = \mathfrak{F}(\alpha_0, \alpha, \beta, \gamma), \quad \mathfrak{F}_n = \mathfrak{F}(\alpha_0, \alpha, \beta - \pi_n, \gamma), \quad \mathfrak{F}' = \mathfrak{F}(\alpha_0, \alpha, \beta - \pi, \gamma).$$

We need to prove that $\lambda(\mathfrak{F}') = \lim \lambda(\mathfrak{F}_n)$ $(= \lambda(\mathfrak{F}))$. To do this we may assume $\mathfrak{F}' \subset \mathfrak{F}_n$ so that it is sufficient to prove $\lim \lambda(\mathfrak{F}_n - \mathfrak{F}') = \infty$. The simplest situation in which this is true occurs when π consists of a countable number of points. Other conditions can be given for more general situations in terms of the geometry of the configuration. We will not pursue this matter further.

IV.2. The geometry of an extremal slit annulus.

THEOREM IV.2.1. *Let $A(R)$ be an extremal slit annulus with boundary partition $(\alpha_0, \alpha, \beta, \gamma)$.*

(a) *The two dimensional Lebesgue area of the boundary of A is zero.*

(b) *Assume σ is a connected, closed subset (not a point) of* (i) α_0, (ii) α, (iii) *a component $\tau \in \beta$, and suppose that σ is γ-isolated. Then for* all *points $z \in \sigma$,* (i) $|z| = 1$, (ii) $|z| = R$, (iii) $|z| = k$ *for some constant k, $1 < k < R$; furthermore, the same k must be used for all such σ contained in τ.*

(c) *If σ is a component in γ, there exists a constant k such that for* all $z \in \sigma$, arg $z = k$.

(d) *If a component $\sigma \in \beta$ (or $\sigma = \alpha_0$ or α) is β-isolated, then σ is a circular slit with radial incisions (including the possibility of a single radial or circular slit).*

(e) *Suppose $\tau = \alpha_0$, $\tau = \alpha$ or τ is a component in β, and assume the class Γ of arcs in $\hat{A} - \gamma$ tending to τ has finite extremal length. Let r be the number, $1 \leqslant r \leqslant R$, such that*

$$\int_c du = (\log r / \log R) - u(p)$$

for almost all $c \in \Gamma$, where p is the initial point of c (if $p \in \alpha_0$ or $p \in \alpha$ take $u(p) = 0$ or 1, respectively). Then each component of $(\tau - \{|z| = r\} \cap \tau)$ which is β-isolated is a radial slit.

Proof. (a) The mapping $z = f(w; \alpha_0, \alpha, \beta, \gamma)$ is the identity. The area of $f(A)$ in the density $|z|^{-1}\,|dz|$ is equal to the area of A in the density $(2\pi/\|du\|^2)\,|\text{grad } u|\,|dw|$ where $u = u(w; \alpha_0, \alpha, \beta, \gamma) = \log|f|$. This latter area is $4\pi^2\|du\|^{-2} = 2\pi \log R$, or the same as the area of the annulus $1 < |z| < R$ in the density $|z|^{-1}|dz|$.

(b) We will prove case (i); the proofs for the other cases are similar. Because of the hypothesis that σ is γ-isolated, we may assume that the components in γ have no accumulation point on σ.

Let σ_1 be a connected subset of σ, not a point, which is not a radial (circular) slit. Then

there exists a family C of radial (circular) slits in $\hat{A}-\gamma$ with (i) initial point in A and terminal point on σ_1, and (ii) which project onto an interval of positive length on $|z|=1$ (on arg $z=0$, respectively). Hence each curve in C may be parametrized in the form c_θ where θ is the projection of the radial (circular) slit c_θ, $a<\theta<b$. Because of (a), except for a set S of linear measure zero, the closed set $c_\theta \cap \beta$ must have linear measure zero.

That this exceptional set S in θ has measure zero ($m(S)=0$) implies that the corresponding class c_θ, $\theta \in S$, has infinite extremal length: use the density $\varrho(z)=1$ for $z \in c_\theta, \theta \in S$, and $\varrho(z)=0$ otherwise. The extremal length of the class $C=\{c_\theta: \theta \notin S\}$ is finite for if ϱ is any linear density and if, say, c_θ are radii with maximum logarithmic length φ, the Schwarz inequality yields $L(\varrho, C)^2 \leqslant A(\varrho)\varphi/b-a$. Because u is continuous in the closure of $c_\theta \cap A$ in $1 \leqslant |z| \leqslant R$ and the closure is obtained by adding a point set of measure zero, $\theta \notin S$, we have (see I.2)

$$\int_{c_\theta} du = \lim u(p) - u(c_\theta(0))$$

as $p \to c_\theta$ (1) where $c_\theta(1)$ and $c_\theta(0)$ are the end point and initial point of c_θ, respectively ($u=\log|z|/\log R$).

Now apply Theorem II.3.2; for almost all curves $\tau \in \hat{A}-\gamma$ traveling from the initial point $\tau(0)$ of τ to σ,

$$\int_\tau du = 0 - u(\tau(0)),$$

and hence we find

$$\lim u(p)=0, \text{ as } p \to c_\theta(1) \text{ along } c_\theta.$$

Therefore there are some points z on σ for which $|z|=1$. If $|z_0| \neq 1$ for some $z_0 \in \sigma$ there exists a connected piece σ_1 of σ, not a point, which contains z_0 and on which $|z| \neq 1$. Again the above argument shows that this is impossible.

(c) The function u^*, the harmonic conjgate of u, is single valued in a neighborhood of any component σ of γ. The reasoning of II.3 can then be applied to show the existence of a constant h such that

$$\int_\tau du^* = h - u^*(\tau(0))$$

along almost all curves (initial point $\tau(0)$ in A) in $\hat{A}-\beta$ which tend to σ. A reasoning analogous to that of (b) now yields the result since σ is isolated from β.

(d) Let π be a removable collection of boundary components such that $\beta_1=\beta-\pi \cap \beta$ has no accumulation points on σ and set $\gamma_1=\gamma-\pi \cap \gamma$. Consider the class $\mathfrak{F}_1=\mathfrak{F}(\alpha_0, \alpha, \beta_1, \gamma_1)$; by hypothesis $\lambda(\mathfrak{F})=\lambda(\mathfrak{F}_1)$ (here $\mathfrak{F}=\mathfrak{F}(\alpha_0, \alpha, \beta, \gamma)$). Suppose first that the class of curves

in $\mathfrak{F}_1$ which meet σ has infinite extremal length. Then, since we may define $\beta_2=\beta_1-\sigma$, $\gamma_2=\gamma_1\cup\sigma$, $\mathfrak{F}_2=\mathfrak{F}(\alpha_0, \alpha, \beta_2, \gamma_2)$ we find $\lambda(\mathfrak{F}_2)=\lambda(\mathfrak{F}_1)$ and hence $A_1=A\cup\pi$ is the extremal slit annulus for the partition $(\alpha_0, \alpha, \beta_2, \gamma_2)$. From (c) it follows that σ is a radial slit.

Now assume that the class of curves in $\mathfrak{F}_1$ which meet σ has finite extremal length. Since σ is isolated from β_1, a curve tending to σ in $\hat{A}-\gamma_1$ contains a curve tending to σ in A. Therefore Theorem II.3.2 implies that there exists a constant r, $1<r<R$, such that $\lim u=\log r/\log R$ along almost all curves in $\hat{A}-\gamma_1$ which tend to σ (here $u=u(z; \alpha_0, \alpha, \beta_1, \gamma_1)=\log|z|/\log R$). Let $\mathfrak{F}_1'$ be the class of curves in $\mathfrak{F}_1$ which do not meet any component of $\sigma-\{|z|=r\}\cap\sigma$; then $\lambda(\mathfrak{F}_1')=\lambda(\mathfrak{F}_1)$.

Let σ_1 be a component of $\sigma-\{|z|=r\}\cap\sigma$ and suppose that $|z|>r$ for $z\in\sigma_1$. Choose numbers r_1, r_2 with $r<r_1<r_2$ and r_2 so close to r that there are points $z\in\sigma_1$ with $|z|>r_2$. We can find an open set $N\subset\{r_1<|z|<r_2\}$ such that (i) N is the union of a finite number of regions with smooth boundaries, (ii) $\sigma_1\cap\{r_1<|z|<r_2\}\subset N$, and (iii) $\mathrm{Cl}(N)\cap\beta_1=\mathrm{Cl}(N)\cap\sigma$.

Denote by $\boldsymbol{\sigma}$ those components of $\sigma-\sigma\cap N$ for which $|z|\neq r$, all $z\in\boldsymbol{\sigma}$; $\boldsymbol{\sigma}$ contains at least a piece of σ_1. Set $A_2=A_1\cup N$, $\beta_2=\beta_1-\beta_1\cap N-\boldsymbol{\sigma}$, $\gamma_2=(\gamma_1-\gamma_1\cap N)\cup\boldsymbol{\sigma}$, and $\mathfrak{F}_2=\mathfrak{F}(\alpha_0, \alpha, \beta_2, \gamma_2)$. Since those curves in $\mathfrak{F}_1$ which meet $\boldsymbol{\sigma}$ and $\beta_1\cap N=\sigma\cap N$ form a class of infinite extremal length, $\lambda(\mathfrak{F}_2)\leqslant\lambda(\mathfrak{F}_1)$.

We claim that A_2 is an extremal slit annulus, that is $\lambda(\mathfrak{F}_2)=\lambda(\mathfrak{F}_1)$. To prove this we need only show that $\left|\int_c du\right|\geqslant 1$ for almost all $c\in\mathfrak{F}_2$. It is enough to show this for those $c\in\mathfrak{F}_2$ which cross the boundary ∂N of N. By a slight deformation of c we may always assume c does not contain an interval on ∂N. The class of curves in $\mathfrak{F}_2$ which cross ∂N infinitely often have infinite extremal length, as is not hard to show. The class of curves which cross ∂N at a point of σ also have infinite extremal length. Indeed each such curve contains a curve in $\hat{A}_1-\gamma_1$ which tends toward $\sigma-\sigma\cap\{|z|=r\}$. Let then $\mathfrak{F}_2(\partial N)$ be the class of curves in $\mathfrak{F}_2$ which cross ∂N a finite number of times and only at points in the *interior* of A_1. We must prove that $\left|\int_c du\right|\geqslant 1$ for almost all $c\in\mathfrak{F}_2(\partial N)$.

We will now briefly outline an application of the method developed in III.3. In A_1, let u_n be the approximation to u in $\boldsymbol{\Omega}_n$, as constructed in II. Define the linear density $\varrho|dz|$ in A_1 by

$$\varrho=\begin{cases}|d(u-u_n)| \text{ in } \boldsymbol{\Omega}_n-\boldsymbol{\Omega}_n\cap N\\ 0 \text{ in the remainder of } A_1.\end{cases}$$

Then for almost all $c\in\mathfrak{F}_2(\partial N)$ there exists a sequence $\{n(\varrho)\}$ such that (for a suitable interpretation)

$$\lim\left|\int_{c\cap A_1'} du-du_n\right|=0, \quad A_1'=A_1-A_1\cap N.$$

Denote the points of intersection of c with ∂N as c runs from α_0 to α by p_1 (first point), $p_2, \ldots, p_{2n}$ (last point). We find

$$\int_{c \cap A'_1} du = 1 - \sum_{k=1}^{n} [u(p_{2k}) - u(p_{2k-1})]$$

since $u_n(z) \to u(z)$ at interior points of A_1. Since $c \cap N$ must connect the points p_i and u is harmonic in N we must have $\int_c du = 1$. This is true for almost all $c \in \mathfrak{F}_2(\partial N)$ thereby proving that $\lambda(\mathfrak{F}_2) = \lambda(\mathfrak{F}_1)$.

Since A_2 is an extremal slit annulus, the components in σ are radial slits; in particular the components of $\sigma_1 - \sigma_1 \cap N$ on which $|z| > r_2$ are radial slits. Since r_2 can be taken arbitrarily close to r, σ_1 must be a radial slit, and this is what we had to prove. We have assumed that $\sigma \in \beta$, but the same proof holds for $\sigma = \alpha_0$ or $\sigma = \alpha$ so long as σ is β-isolated.

(e) The statement here almost implies the result of (d) and the proof is essentially the same. Using the notation above, we need only observe that the extremal length of the class of curves in $\mathfrak{F}_1$ that meet $\tau - \{|z| = r\} \cap \tau$ must be infinite.

As a consequence of Theorem IV.2.1 we obtain the existence of circular and radial slit mappings. For other general methods of derivation see Reich and Warschawski [8, 9], Reich [7], Strebel [12,13], and Ahlfors–Sario [2]. (See also [4], [11] for two less general but older treatments.)

COROLLARY IV. 2.2. (a) *If γ is empty the boundary components of A are circular slits and $|z| = 1$, $|z| = R$. The projection of the circular slits of positive length onto a radius has measure* 0.

(b) *If β is empty, the boundary components of A are radial slits, $|z| = 1$, and $|z| = R$ with possible radial incisions eminating from a set of measure zero along $|z| = 1$ and $|z| = R$. The projection of the radial slits of positive length onto a concentric circle has measure zero.*

Proof. Part (b) follows easily from the theorem but there is a simpler proof that $|z| = 1$ and $|z| = R$ may have radial slits eminating from them. Namely for $1 < r < R$, consider $A_1 = A \cup \{1 < |z| < r\}$ and set $\gamma_1 = \gamma \cup (\alpha \cap \{1 < |z| < r\}$ and $\mathfrak{F}_1 = \mathfrak{F}(\alpha_0, \alpha, \gamma_1)$. By the comparison principle, $2\pi/\log R \leqslant \lambda(\mathfrak{F}_1) \leqslant \lambda(\mathfrak{F})$ and hence $\lambda(\mathfrak{F}_1) = \lambda(\mathfrak{F})$. Consequently A_1 is an extremal slit disk. That is, $\alpha \cap \{1 < |z| < r\}$ are radial slits.

The determination of β- and γ-isolation is closely related to the determination of the boundary. This is illustrated in the following corollary.

COROLLARY IV.2.3. *A component $\tau \in \beta$ is γ-isolated if and only if τ is a circular slit.*

Proof. We have seen above that if τ is γ-isolated, τ is a circular slit. Conversely, let J

be a analytic Jordan curve in the interior of A which separates τ from α_0 and α. If π denotes those boundary components of A, including τ, which are surrounded by J, then π is removable. Now replace τ after removing π. Since τ is an isolated circular slit, an easy computation in the style of Theorem IV.1.4 shows that the resulting region is an extremal slit annulus.

An example. Put a circular slit with a radial incision in the annulus A: $1<|z|<R$; this will be β. Put countably many radial slits in A (call the collection of these slits γ) in such a way that the extremal length of the class of curves in $A-\gamma$ tending to the radial incision in β, is infinite. An easy application of Theorem IV.1.2 shows that this slit annulus is an extremal slit annulus. A similar example shows the existence of an extremal radial slit annulus in $1<|z|<R$ with radial incisions from $|z|=R$.

IV.3. Extremal properties. In this section we present results which have been used in the approaches to extremal radial and circular slit disks by purely classical methods (see [7], [8], [2]).

Let W be a plane region, p a point in W and (α, β, γ) a partition of the boundary of W so that α consists of one boundary component (and, of course, β is closed in the compactification of W). Write $v(z; p, \alpha, \beta, \gamma)=\log|z-p|+c_p(\alpha, \beta, \gamma)+o(1)$ about $z=p$.

THEOREM IV.3.1. (a) *Suppose $\gamma=\emptyset$ and $f(z)$ is univalent in W with $f(p)=0$ and $|f(z)|\leqslant 1$. Then*

$$|f'(p)|\leqslant c_p(\alpha, \beta)$$

with equality if and only if $f=\exp(v+i\,v^*)$.

(b) *Suppose $\beta=\emptyset$ and $f(z)$ is univalent in W with $f(p)=0$ and $\lim|f(z)|\geqslant 1$ as $z\to\alpha$ along almost all paths in W which approach α. Set $B=\lim_{\pi\to\alpha}\int_\pi h\,dh^*$ where $h=\log|f|$ and $\{\pi\}$ is a collection of simple closed curves in W approaching α. Then*

$$|f'(p)|+\left(\frac{B}{2\pi}-1\right)\geqslant c_p(\alpha, \gamma)$$

with equality if and only if $f=\exp(v+i\,v^*)$. (In the case that α is isolated from all γ at least, $B=2\pi$.)

Proof. (a) follows upon setting $h=\log|f|$ and computing $\|d(v-h)\|^2_{W-\Delta}$ where Δ is a small disk about p. We also use the fact that, in the sense of an approximation,

$$\int_\beta h\,dh^*\leqslant 0 \quad \text{while} \quad \int_\alpha h\,dh^*\leqslant 2\pi.$$

(b) follows from our uniqueness theorem (Corollary III.4.1).

Finally we will describe the following interesting result which was derived using a different method by Reich and Warschawski [8]. Suppose for example that $\gamma = \emptyset$. Fixing a boundary component α_0, use Theorem II.5.1 to find another boundary component $\tilde{\alpha}$ so that

$$\|du_{\tilde{\alpha}}\| \geq \|du_\alpha\|, \quad \text{all } \alpha \neq \alpha_0.$$

(Notation as given in II.5.) Realizing W by $\exp 2\pi(u_{\tilde{\alpha}} + iu^*_{\tilde{\alpha}})/\|du_{\tilde{\alpha}}\|^2$ as an extremal circular slit annulus A, it follows that each circular slit of A subtends an angle $< \pi$.

For the proof set $R = \exp 2\pi/\|du_{\tilde{\alpha}}\|^2$. Then $\varrho\,|dz| = (|z|\log R)^{-1}|dz|$ is the extremal metric for the class $\mathcal{F}^*(\tilde{\alpha}, \beta)$ in A (or more precisely, for almost all curves of $\mathcal{F}^*(\tilde{\alpha}, \beta)$). Choose any $\alpha \neq \tilde{\alpha}$ where α is realized as a circular slit of A and use $\varrho\,|dz|$ as a competing density for the problem $\mathcal{F}^*(\alpha, \beta)$ in A. By assumption we must have

$$\inf_c \int_c \varrho\,|dz| < 2\pi/\log R = \|du_{\tilde{\alpha}}\|^2,$$

where $c \in \mathcal{F}^*(\alpha, \beta)$. Equality cannot hold since the extremal metrics for $\mathcal{F}^*(\tilde{\alpha}, \beta)$, $\mathcal{F}^*(\alpha, \beta)$ are known to be different. On examining the geometry of the situation, it follows easily that α, as realized in A, must subtend an angle $< \pi$.

The same statement holds for an extremal circular slit disk.

V. A dual problem

We will briefly consider in this chapter the case of a partition $(\alpha_0, \alpha, \beta, \gamma)$ of the ideal boundary such that α_0, α, $\alpha_0 \cup \alpha \cup \gamma$ are closed in the compactification $\hat{W}$ of W (α_0 and α cannot be empty). The corresponding classes $\mathcal{F}$ and $\mathcal{F}^*$ were defined in I.3.

Let $\{\Omega_n\}$ be an exhaustion of W such that all components of $\partial\Omega_n$ are piecewise analytic dividing cycles. Define a partition α_{0n}, α_n, β_n, γ_n of $\partial\Omega_n$ as follows: a) α_{0n} consists of those components of $\partial\Omega_n$ which have ideal boundary components contained in α_0 as derivations; b) α_n consists of those components which have ideal boundary components in α as derivations; c) β_n consists of those components which have *only* ideal boundary components in β as derivations; d) γ_n consists of the remaining components. Denote by $\boldsymbol{\Omega}_n$ the region obtained by adjoining to Ω_n the non-compact subregions of $W - \Omega_n$ which are bounded by α_{n0}, α_n and β_n. The construction of II.1 can now be repeated to show the existence of a harmonic function $u(z; \alpha_0, \alpha, \beta, \gamma)$ in W with boundary behavior as described in Theorem II.3.2.

Consider the classes $\mathcal{F}(\alpha_0, \alpha, \beta, \gamma)$ and $\mathcal{F}^*(\alpha_0, \alpha, \beta, \gamma)$. Let $\mathcal{F}_n$ be the class corresponding to $\mathcal{F}$ in $\boldsymbol{\Omega}_n$. Since every $c \in \mathcal{F}_n$ is a curve in $\mathcal{F}$, we find $\|du\|^{-2} \geq \lambda(\mathcal{F})$. Upon using $|\operatorname{grad} u|\,|dz|$ as an admissible density we see that $\lambda(\mathcal{F}) = \|du\|^{-2}$.

The proof that $\lambda(\mathcal{F}^*) = \|du\|^2$ is much harder. Indeed the comparison principle imlies that $\lambda(\mathcal{F}^*) \geqslant \|du\|^2$ since each $c \in \mathcal{F}^*$ contains a curve in the corresponding class in. $\mathbf{\Omega}_n$ To prove equality, we need an analogue of the continuity lemma of III.2; such a lemma can be proven in a similar manner.

In the case that W is a plane region, the function u may be used to construct a mapping onto a corresponding extremal slit annulus B. The boundary of B can be described in a fashion entirely analogous to that in IV.2. Thus the β are all circular slits and the γ are, in many cases, radial slits with circular incisions. The nature of α_0 and α is no different than that considered in IV.2.

There are also other extremal length problems that may be solved by the methods used here. For example β may be decomposed into subsets β_i which are not necessarily points. Then $\hat{W}$ can be replaced by the quotient topological space obtained by identifying all points contained in the same β_i. The corresponding classes $\mathcal{F}$, $\mathcal{F}^*$ can then be defined.

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