CURVES ON 2-MANIFOLDS: A COUNTEREXAMPLE

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In [1] R. Baer proved that if two simple closed curves on a closed orientable 2-manifold M of genus greater than one are homotopic, but not homotopic to zero, then they are isotopic. It is well-known that this theorem is true without restriction on M (see [2] for example). One might be tempted to assert the stronger: if two simple closed curves are homotopic keeping a basepoint fixed, then they are isotopic keeping the basepoint fixed. In this paper we show that the stronger result is not true in general. The counterexample has the property that each simple closed curve is the boundary of a Möbius band in M. In [2] it is proved that this is the only type of counterexample.

If $f: X \times I \to Y \times I$ is a level preserving imbedding, we say that $f_0, f_1: X \to Y$ are *isotopic.*

THEOREM 1. *Let $f_0, f_1: S^1, * \to M, *$ be imbeddings of simple closed curves which bound disks with opposite orientations. (Note that we can define orientations in a neighbourhood of the basepoint, even if M is non-orientable.) Then f_0 is isotopic to f_1 if and only if M is a 2-sphere.*

Proof. Suppose $M \neq S^2$ and f_0 is isotopic to f_1. We shall deduce a contradiction. If M is non-orientable, let M' be the orientable double cover and let $f_0', f_1': S^1, * \to M', *$ be liftings of f_0, f_1. Let $\tau: M' \to M'$ be the covering translation. An isotopy between f_0 and f_1, keeping the basepoint fixed, can be lifted to an isotopy between f_0' and f_1' in $M' - \tau *$. Since $M' - \tau * \neq S^2$, there is no loss of generality in assuming that M is orientable.

Let $F_t: S^1, * \to M, *$ be the isotopy between f_0 and f_1. Since $f_t \simeq 0$, $f_t S^1$ bounds a disk D_t. Since $M \neq S^2$, $f_t S^1$ bounds only one disk. $f_t S^1$ assigns an orientation to D_t and hence to M for each t. It is easy to see that this orientation is unchanged by a small change in t. (For example remove a point p from $\operatorname{int} D_t$ and a point q from $M - D_t$. Then f_t gives a homology class in $H_1(M - p - q)$, which determines the orientation of M.) It follows that $f_0 S^1$ and $f_1 S^1$ bound disks with the same orientation, which contradicts our hypothesis.

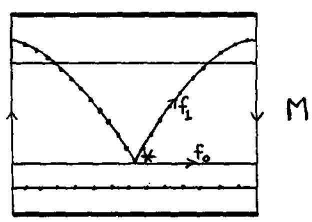

Diagram 1.

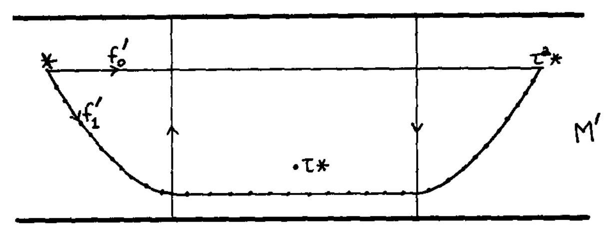

Diagram 2.

THEOREM 2. *Let $f_0: S^1, * \to \operatorname{int} M, *$ bound a Möbius band in M. Then there is a homotopy $f_t: S^1, * \to \operatorname{int} M, *$ such that f_1 is an imbedding, which is not isotopic to f_0 keeping the basepoint fixed.*

Remark. Conversely, it is proved in [2], Theorem 4.1, that if $f_0 S^1$ does not bound a Möbius band or a disk, then every imbedding, which is homotopic to f_0 keeping the basepoint fixed, is isotopic to f_0 keeping the basepoint fixed.

Proof. Diagram 1 shows a slightly larger Möbius band than that bounded by $f_0 S^1$. The imbedding f_1 shown in the diagram is homotopic to f_0, by a homotopy which changes only the vertical coordinate in the diagram.

To show that f_0 and f_1 are isotopic, we examine two cases. Suppose first that M is a projective plane. Then f_0 and f_1 bound disks with opposite orientations and we apply Theorem 1. Suppose M is not a projective plane.

Then every multiple of f_0 is non trivial in $\pi_1(M, *)$. Lifting to the universal cover M', we see that each component of the inverse image of the Möbius band is an infinite strip. Let τ be the covering translation corresponding to a generator of the fundamental group of the Möbius band. Let $f_0', f_1': I, 0 \to M', *$ be liftings of f_0, f_1.

Suppose we had an isotopy $f_t: S^1, * \to M, *$. Then this would lift to an isotopy

$$f_t': I, 0, 1 \to (M' - \tau *), *, \tau^2 *.$$

In particular the simple closed curve obtained by going first along $f_0' I$ and then along $f_1' I$ would be homotopic to zero, and would therefore bound a disk in $M' - \tau *$.

References

[1]. BAER, R., Isotopie von Kurven auf orienteierbaren geschlossenen Flächen und ihr Zusammenhang mit der topologischen Deformation der Flächen. *J. reine angew. Math.*, 159 (1928), 101–111.

[2]. EPSTEIN, D. B. A., Curves on 2-manifolds and isotopies. *Acta Math.*, 115 (1966), 83–107.

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