The Hausdorff dimension of the limit set of a geometrically finite Kleinian group

by

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A. Introduction

We consider in this paper groups of Möbius transformations of $\bar{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$. Such a group is called *Kleinian* if it acts discontinuously somewhere in $\bar{\mathbf{R}}^n$. The action of G extends to the $(n+1)$-dimensional hyperbolic space $\mathbf{H}^{n+1} = \mathbf{R}^n \times (0, \infty)$ and G is *geometrically finite* if there is a hyperbolic fundamental polyhedron with a finite number of faces for the action of G in $\mathbf{H}^{n+1}$ (for a more precise definition see [15, 1 B]). We prove in this paper that the *Hausdorff dimension* $\dim_{\mathrm{H}} L(G)$ of the limit set $L(G)$ of a geometrically finite Kleinian group G of $\bar{\mathbf{R}}^n$ is less than n (Theorem D).

Our proof of this theorem is based on the following observation. Assume that G is of *compact type*, i.e. if $\bar{\mathbf{H}}^{n+1} = \mathbf{H}^{n+1} \cup \bar{\mathbf{R}}^n$, then $(\bar{\mathbf{H}}^{n+1} \setminus L(G))/G$ is compact. Then there is an integer q such that if we divide any n-cube Q of $\mathbf{R}^n$ into q^n equal subcubes, then at least one of these subcubes does not touch $L(G)$. Let $\mathscr{L}(Q)$ be the family of these subcubes which touch $L(G)$. Then the n-measures of $Q' \in \mathscr{L}(Q)$ do not add up to the n-measure of Q and we get

$$\sum_{Q' \in \mathscr{L}(Q)} d(Q')^\alpha \leq c\, d(Q)^\alpha \tag{A 1}$$

for $\alpha = n$ and $c = 1 - 1/q^n$. Obviously, this remains valid for slightly smaller $\alpha < n$ and slightly bigger $c < 1$. Passing now to the families $\mathscr{L}(Q')$, $Q' \in \mathscr{L}(Q)$, we get an inductive argument which shows that the Hausdorff dimension of $L(G)$ cannot exceed $\alpha < n$.

The existence of such q is based on a compactness argument. If $r = d(Q \cap L(G))/d(Q)$ is small, the existence of such q is clear. On the other hand, if say $r \geq 1/2$, let z_Q be the center of Q and let s_Q be its side length. If $\bar{z}_Q = (z_Q, s_Q) \in \mathbf{H}^{n+1}$, the hyperbolic distance of $\bar{z}_Q$ from the hyperbolic convex hull H_G of $L(G)$ (see Section

B) is bounded. In the present case, H_G/G is compact. Hence there is a compact set $C \subset \mathbf{H}^{n+1}$ such that $z_Q \in GC$. Let β be a similarity of $\mathbf{R}^n \times [0, \infty)$ such that $\beta(Q) = Q_0$ where Q_0 is a given standard cube. Now we factor β as

$$\beta = hg$$

where $g \in G$ and $g(z_Q) \in C$ and h is some Möbius transformation. Thus $h^{-1}(z_{Q_0}) \in C$ and it follows that h varies in a compact set of Möbius transformations. Now

$$\beta(Q \cap L(G)) = Q_0 \cap hg(L(G)) = Q_0 \cap h(L(G)). \tag{A 2}$$

Since every $h(L(G))$ is nowhere dense in $\bar{\mathbf{R}}^n$ and h varies in a compact set we can conclude the existence of the required q.

The argument above gives an easy proof of the fact that $\dim_H L(G) < n$ for groups of compact type. This paper is an elaboration of it to get the result for all geometrically finite Kleinian G. It turns out that, given an n-cube Q, we can define cube families $\mathcal{L}(Q)$ covering $Q \cap (L(G) \setminus (\text{parabolic fixpoints}))$ and that these satisfy (A 1) for some $a < n$ and $c < 1$ if Q satisfies a certain condition and then every $Q' \in \mathcal{L}(Q)$ satisfies this condition. Hence we can again apply an inductive argument to get the theorem.

Incidentally, the above compactness argument can be extended for all geometrically finite G which do not contain parabolic elements of rank n (cf. Section B). Then in (A 2) $h(L(G)) \in \mathcal{C}$ where $\mathcal{C}$ is a family of subsets of $\bar{\mathbf{R}}^n$ which is compact in a suitable topology and such that every $F \in \mathcal{C}$ is nowhere dense ([16]). If G contains parabolic fixpoints of rank n, then $\bar{\mathbf{R}}^n \in \mathcal{C}$ and thus this method cannot be extended to the general case, a new idea is needed. This idea then allows to treat all parabolic fixpoints in the same manner regardless of rank.

It is a consequence of our theorem that the n-measure of the limit set vanishes. However, if one wishes to have only this theorem, then our proof could be simplified since then one does not need Lemma D which is the most complicated result needed in the proof.

If $n=1$, then it is known that $\dim_H L(G) < n$ for geometrically finite Kleinian G, that is, for finitely generated Fuchsian groups of the second kind. (Beardon [4], Patterson [8].) If $n=2$ this has been proved by Sullivan [12] whose proof should generalize also for $n>2$. In any case, Sullivan's earlier paper [11, Section 3] implies that $\dim_H L(G) < n$ for Kleinian groups of compact type if one in addition knows that the n-measure of $L(G)$ vanishes. This latter result is classical if $n=1$ and well-known for $n=2$ (Ahlfors [1], Beardon–Maskit [5]). It is true also for $n>2$ although we do not know an explicit reference, apart from the results of our paper.([1]) For instance, it follows from Ahlfors [2, 7.13] since in the geometrically finite case a point $x \in L(G)$ is in the conical limit set unless it is fixed by a parabolic $g \in G$ as follows from the existence of cusp neighbourhoods (cf. Section B). Also, the method of Beardon–Maskit [5] should generalize for all n.

On the other hand, D. Sullivan [13] has shown that there are finitely generated Kleinian groups of $\bar{\mathbf{R}}^2$ for which the Hausdorff dimension of the limit set is 2 (although the areal measure is zero). Thus the condition that the groups are geometrically finite cannot be relaxed. In fact, Ahlfors' famous problem on the vanishing of the areal measure of limit sets of finitely generated Kleinian groups is still open.

In the final section we show that the Poincaré series for a geometrically finite Kleinian group of $\bar{\mathbf{R}}^n$ converges for some exponent $s<n$. Only very slight strengthening of our method is needed to obtain this theorem (Theorem E).

Notation and terminology. We denote the group of Möbius transformations of $\bar{\mathbf{R}}^n$ by Möb(n); it includes also orientation reversing elements. Every $g \in$ Möb(n) has a unique extension to a Möbius transformation of $\bar{\mathbf{H}}^{n+1}$; we do not distinguish between g and its extension. As usual, we classify $g \in$ Möb(n), $g \neq$ id, as *elliptic, parabolic* or *loxodromic* and loxodromic g can be also *hyperbolic,* see e.g. [15, 1 C].

A *Möbius group* of $\bar{\mathbf{R}}^n$ is a subgroup of Möb(n). It is a topological group in the topology of maps of $\bar{\mathbf{R}}^n$ given by uniform convergence in the spherical metric. It is then discrete if and only if its action in $\mathbf{H}^{n+1}$ is discontinuous (see the argument in Ahlfors [2, p. 79]). We denote the *limit set* of G by $L(G)$ which is the accumulation set of Gx for $x \in \mathbf{H}^{n+1}$. We denote the set of points of $\bar{\mathbf{R}}^n$ fixed by some parabolic $g \in G$ by $P(G)$. Then $P(G) \subset L(G)$.

The hyperbolic metric of $\mathbf{H}^{n+1}$ is given by $|dx|/x_{n+1}$ and is denoted by d: the diameter of a set is $d(A)$, the distance of two sets $d(A, B)$. We use this notation also for diameters in the euclidean metric of $\mathbf{R}^{n+1}$. If confusion is possible we say which metric we mean. The euclidean distance of two points is $|x-y|$, the *closed* euclidean ball with center x and radius r is $B^n(x, r)$, $B^n = B^n(0, 1)$ and $B^n(r) = B^n(0, r)$. The boundary of a set (in $\bar{\mathbf{R}}^{n+1}$) is ∂A, the closure is cl$A$ and the interior int$A$ (which may be taken also in $\mathbf{R}^n$). The standard basis of $\mathbf{R}^{n+1}$ is $e_1, \ldots, e_{n+1}$. A *similarity* is a map which multiplies euclidean distances by a constant.

([1]) B. Apanasov's paper "Geometrically finite groups of spatial transformations" (Russian). Siberian Math. J. 23:6 (1982), 16–27, contains this result. His definition of a geometrically finite group is different but leads to the same class of groups. (Note added in proof.)

B. Horoballs at parabolic fixpoints

Let G be a discrete Möbius group of $\bar{\mathbf{R}}^n$. In this section we pay special attention to parabolic elements in G and to the set $P(G)$ of points fixed by some parabolic $g \in G$.

If $v \in P(G)$, let $G_v = \{g \in G: g(v) = v\}$. Then every $g \in G_v$ is non-loxodromic and G_v contains a free abelian subgroup H of finite index ([15, Theorem 2.1]). The rank k of H depends only on v and we say k is also the *rank* of v. If v has rank k, then a *cusp neighbourhood* U of v is a G_v-invariant subset of $\bar{\mathbf{H}}^{n+1} \setminus L(G)$ such that $\{g(U): g \in G\}$ is a disjoint family and which is of the form

$$h(U) = (\bar{\mathbf{H}}^{n+1} \setminus \mathbf{R}^k \times B^{n-k+1}) \setminus \{\infty\} \tag{B 1}$$

for some $h \in \text{Möb}(n)$ with $h(v) = \infty$. Moreover, $h^{-1}(\mathbf{R}^k \times B^{n-k+1}(r))/G_v$ is compact for every $r \geq 0$. If G is geometrically finite, then every $v \in P(G)$ has cusp neighbourhoods ([15, Theorem 2.4]).

A related notion is that of a *horoball* at v. It is an open $(n+1)$-ball $B \subset \bar{\mathbf{H}}^{n+1}$ such that ∂B is tangent to $\bar{\mathbf{R}}^n$ at v. Since $\bar{\mathbf{R}}^n$ is G_v-invariant, also B is then G_v-invariant. This is easy to see if $v = \infty$ since then every $g \in G_v$ is a euclidean isometry. From the expression (B 1) one sees that every cusp neighbourhood of v contains horoballs. We say that B_v, $v \in P(G)$, is a *complete set of horoballs* for G if it is a disjoint family and if $g(B_v) = B_{g(v)}$ for $g \in G$.

Finally, we need the notion of the *hyperbolic convex hull* $H_G \subset \mathbf{H}^{n+1}$ of G. It is the smallest convex and closed subset of $\mathbf{H}^{n+1}$ such that $L(G) \subset \text{cl}\, H_G$. It is well-defined unless $L(G) = \{\text{a point}\}$ in which case we set $H_G = \emptyset$. We then set for $m \geq 0$

$$H_G^m = \{x \in \mathbf{H}^{n+1}: d(x, H_G) \leq m\}. \tag{B 2}$$

The next lemma is the one in this paper in which the assumption that G is geometrically finite is utilized.

LEMMA B. *Let G be a geometrically finite Kleinian group of $\bar{\mathbf{R}}^n$. Then G has a complete set of horoballs and if B_v, $v \in P(G)$, is such a set, then*

$$(H_G^m \setminus (\cup \{B_v: v \in P(G)\}))/G$$

is compact for $m \geq 0$.

Proof. Since G is geometrically finite, we can pick a cusp neighbourhood U_v of v for $v \in P(G)$ ([15, Theorem 2.4]) and, furthermore, we can assume that U_v, $v \in P(G)$, is a disjoint family and that $g(U_v) = U_{g(v)}$ for $v \in P(G)$. Next we pick horoballs $B_v \subset U_v$ and we can do this in such a way that they form a complete set of horoballs for G. Thus G has such a set.

Let then B_v, $v\in P(G)$, be a complete set of horoballs for G. We can assume that each B_v is contained in a cusp neighbourhood U_v as above. Now $((\bar{\mathbf{H}}^{n+1}\setminus L(G))\setminus(\bigcup\{U_v: v\in P(G)\}))/G$ is compact by [15, Theorem 2.4] and hence so is $(H_G^m\setminus(\bigcup\{U_v: v\in P(G)\}))/G$ which is a closed subset of it. If U_v is of the form (B 1) with $h=\mathrm{id}$, then $L(G)\cap\mathbf{R}^n\subset\mathbf{R}^k\times B^{n-k}$ and hence $H_G\subset\mathbf{R}^k\times B^{n-k}\times(0,\infty)$. Consequently $H_G^m\setminus B_v\subset\mathbf{R}^k\times B^{n-k+1}(r)$ for some $r>0$. Remembering that $\mathbf{R}^k\times B^{n-k+1}(r)/G_v$ is compact, we infer that $(H_G^m\cap\mathrm{cl}(U_v\setminus B_v))/G$ is compact and the lemma follows since there are only a finite number of non-conjugate parabolic fixpoints.

Remark. After this paper was completed, I have been informed that B. Apanasov has also proved results similar to Lemma B (using a slightly different definition of a geometrically finite group). See his paper "Geometrically finite hyperbolic structures on manifolds", Annals of Global Analysis and Geometry, 1: 3 (1983), 1–22.

C. The cube lemma

We now give in a precise form the result on the subdivision of cubes which we mentioned in the introduction.

We first fix some notation. We denote the set of n-cubes of $\mathbf{R}^n$ by $\mathcal{K}_n$. If $Q\in\mathcal{K}_n$, we let z_Q be its center and s_Q its side length and set

$$\bar{z}_Q=(z_Q, s_Q)\in\mathbf{H}^{n+1},$$
$$Q_+=Q\times[0,s_Q]\in\mathcal{K}_{n+1}.$$

If $q>0$ is an integer, we let

$$\mathcal{K}(Q,q)\subset\mathcal{K}_n$$

be the family of cubes obtained by subdividing Q into q^n subcubes of side length s_Q/q.

LEMMA C. *Let G be a geometrically finite Kleinian group of $\bar{\mathbf{R}}^n$ and let B_v, $v\in P(G)$, be a complete set of horoballs for G. Let $C\subset\mathbf{H}^{n+1}$ be compact. Then there is an integer $q>0$ such that if $Q\in\mathcal{K}_n$ and if $\bar{z}_Q\in B_v$ for no $v\in P(G)$, then, for at least one $Q'\in\mathcal{K}(Q,q)$, Q'_+ does not touch $L(G)\cup GC$.*

Proof. We first show that there is $q'>0$ such that if $Q\in\mathcal{K}_n$, then there is $Q'\in\mathcal{K}(Q,q')$ such that

$$Q' \cap L(G) = \varnothing. \tag{C 1}$$

Let $r = d(Q \cap L(G))/d(Q)$. If $r \leq 1/2$, then clearly for big q' there is such Q' for which (C 1) is true. Hence we can assume that $r > 1/2$. Pick $x, y \in Q \cap L(G)$ such that $|x-y| > d(Q)/2$. Let L be the hyperbolic line with endpoints x and y. Then the hyperbolic distance $d(\bar{z}_Q, L) < c = \log 2 + \log n + \sqrt{n}$. Now $L \subset H_G$ and hence $\bar{z}_Q \in H_G^c$, H_G^c as in (B 2). In view of Lemma B there is a compact set $C' \subset \mathbf{H}^{n+1}$ such that $z \in GC'$ if $z \in H_G^c$ and $z \in B_v$ for no $v \in P(G)$. Hence $\bar{z}_Q \in GC'$.

Fix a standard cube $Q_0 \in \mathcal{K}_n$. Let β be a similarity of $\mathbf{R}^n \times [0, \infty)$ such that $\beta(Q) = Q_0$. Let M be the set of Möbius transformations of $\bar{\mathbf{H}}^{n+1}$ such that $h^{-1}(\bar{z}_{Q_0}) \in C'$ if $h \in M$. Then M is compact. Now there is $g \in G$ and $h \in \text{Möb}(n)$ such that $\beta = hg$ and $g(\bar{z}_Q) \in C'$. Thus $h \in M$ and

$$\beta(Q \cap L(G)) = Q_0 \cap hg(L(G)) = Q_0 \cap h(L(G))$$

since $L(G)$ is G-invariant. Given $h \in M$, there is q_h such that, for at least one $Q' \in \mathcal{K}(Q_0, q_h)$, Q' does not touch $h(L(G))$ and (C 1) now follows by compactness.

If $Q \in \mathcal{K}_n$ and $r > 0$, set

$$B_Q^r = \{x \in \bar{\mathbf{H}}^{n+1} : x \neq \infty \quad \text{and} \quad |x - z_Q| \leq s_Q/r\}.$$

If $Q' \in \mathcal{K}_n$ and $Q' \cap L(G) = \varnothing$, then $B_{Q'}^r \cap L(G) = \varnothing$ for $r \geq 1 - d_Q$ for some $d_Q > 0$. Since $\mathbf{H}^{n+1} \setminus B_{Q'}^r$ is hyperbolically convex, it follows that $B_{Q'}^r \cap H_G = \varnothing$ for $r \geq 1$. Now $d(\partial B_{Q'}^1 \cap H^{n+1}, B_{Q'}^r \cap H^{n+1}) = \log r$ for $r \geq 1$. Thus, when H_G^m is as in (B 2),

$$H_G^m \cap B_{Q'}^r = \varnothing \tag{C 2}$$

for $r \geq e^m$ whenever $Q' \cap L(G) = \varnothing$.

Suppose that $H_G \neq \varnothing$. Then $GC \subset H_G^m$ for some $m > 0$, and (C 1) and (C 2) now imply the lemma.

If $H_G = \varnothing$, then either $L(G) = \varnothing$ or $L(G) = \{x\}$. In the first case G is finite and hence $GC \subset \mathbf{H}^{n+1}$ is compact. Obviously the lemma is then true. In the second case there is a G-invariant horoball B at x such that $GC \subset B$ and the lemma is again obvious.

Remark. A further analysis shows that it would suffice in Lemma C to assume that $\bar{z}_Q \in B_v$ for no parabolic fixpoint of rank n, cf. [16]. This follows since if $d(Q \cap L(G))/d(Q) \geq 1/2$ and if $\bar{z}_Q$ is in a small horoball B_v of a parabolic fixpoint of rank k, then $\beta(L(G))$ is near some $h(\bar{\mathbf{R}}^k)$, $h \in \text{Möb}(n)$.

D. The Hausdorff dimension of the limit set

In this section we prove our first main

THEOREM D. *The Hausdorff dimension of the limit set of a geometrically finite Kleinian group G of $\bar{\mathbf{R}}^n$ is less than n.*

First some notation. We abbreviate $L(G)$ as L and $P(G)$ as P. Since Möbius transformations do not change the Hausdorff dimension, we can assume that $L\subset\mathbf{R}^n$. If a cube $Q\in\mathscr{K}_n$ has side length s_Q and $t\geqslant 1$, we set

$$\begin{aligned}\tilde{Q} &= Q\times\{s_Q\}\subset\mathbf{H}^{n+1},\\ Q_+ &= Q\times[0,s_Q]\in\mathscr{K}_{n+1},\\ Q_t &= Q\times[s_Q/t,s_Q]\subset\mathbf{H}^{n+1}.\end{aligned}\tag{D 1}$$

We fix a complete set of horoballs B_v, $v\in P$, for G as in Section B. Having chosen B_v, we then define a smaller horoball $B'_v\subset B_v$ tangent to $\bar{\mathbf{R}}^n$ at v such that the hyperbolic distance

$$d(B'_v,\partial B_v\setminus\{v\})=\log 2+2\sqrt{n}.\tag{D 2}$$

Then also B'_v, $v\in P$, is a complete set of horoballs for G.

We also fix some $y\in\mathbf{H}^{n+1}$ such that $Gy\cap B_v=\emptyset$ for all $v\in P$. This point is needed in the next section; for Theorem D we could replace the condition $Q'_+\cap(L\cup Gy)\neq\emptyset$ in (D 3) by $Q\cap L\neq\emptyset$.

Fix now an integer $q>1$ such that if $Q\in\mathscr{K}_n$ and $\tilde{Q}\cap B'_v=\emptyset$ for all $v\in P$, then for at least one $Q'\in\mathscr{K}(Q,q)$, $Q'_+\cap(L\cup Gy)=\emptyset$. By Lemma C, there is such a q. Using this q, we define for $Q\in\mathscr{K}_n$

$$\begin{aligned}\mathscr{L}(Q)=\{&Q'\in\mathscr{K}(Q,2^iq): i\geqslant 0,\ Q'_+\cap(L\cup Gy)\neq\emptyset \text{ and } \tilde{Q}'\cap B'_v=\emptyset \text{ for all } v\in P\\ &\text{and that this is true for no } Q''\supset Q',\ Q''\in\mathscr{K}(Q,2^jq) \text{ with } 0\leqslant j<i\}.\end{aligned}\tag{D 3}$$

Then obviously

$$\tilde{Q}'\cap B'_v=\emptyset\tag{D 4}$$

for all $v\in P$ and $Q'\in\mathscr{L}(Q)$. Let $V_Q=\{v\in P: B_v\cap Q\times\{s_Q/q\}\neq\emptyset\}$. Then

$$\bigcup_{Q'\in\mathscr{L}(Q)}Q'_+\supset Q\times[0,s_Q/q]\cap(Gy\cup(L\setminus V_Q)).\tag{D 5}$$

To prove (D 5), pick x from the right-hand side of (D 5). Let $z=x$ if $x\in Gy$. If $x\in L\setminus V_Q$, let z be a point of the form $z=(x,a)$ where $a\in(0,s_Q/q]$ and $z\in B_v$ for no $v\in P$; obviously there is such z. Thus in all cases $z\notin B_v$ if $v\in P$. Pick now $Q'\in\mathcal{H}(Q,2^i q)$, $i\geq 0$, such that $z\in Q'_2$. If $z'\in\bar{Q}'$, then $d(z,z')\leq\log 2+\sqrt{n}$ and hence $\bar{Q}'\cap B'_v=\emptyset$ for all $v\in P$ by (D 2). It follows that there is $Q''\in\mathcal{H}(Q,2^j q)$, $0\leq j\leq i$, such that $Q''\supset Q'$ and $Q''\in\mathcal{L}(Q)$. Then $z\in Q''_+$, and (D 5) follows. In particular,

$$\mathcal{L}(Q) \quad \text{covers} \quad Q\cap(L\setminus V_Q). \tag{D 6}$$

If, in addition, $\bar{Q}\cap B'_v=\emptyset$ for all $v\in P$, then, considering the n-measure, we get by the choice of q,

$$\sum_{Q'\in\mathcal{L}(Q)} d(Q')^n \leq (1-1/q^n)\, d(Q)^n. \tag{D 7}$$

We now construct inductively cube families covering $L\setminus P$. Pick first some cube $Q_0\in\mathcal{H}_n$ such that $(Q_0)_+\supset L\cup Gy$ and that $\bar{Q}_0\cap B'_v=\emptyset$ for all $v\in P$. Since $\infty\notin L$, there is such Q_0. We then define inductively cube families $\mathcal{L}_i$, $i\geq 0$, and $\mathcal{L}$ by

$$\begin{aligned} &\mathcal{L}_0=\{Q_0\},\\ &\mathcal{L}_{i+1}=\bigcup_{Q\in\mathcal{L}_i}\mathcal{L}(Q) \quad \text{for} \quad i>0, \quad \text{and}\\ &\mathcal{L}=\bigcup_{i\geq 0}\mathcal{L}_i. \end{aligned} \tag{D 8}$$

We note the following properties of these cube families, which follow from (D 4)–(D 6).

$$\mathcal{L}_i \quad \text{covers} \quad L\setminus P \quad \text{for all } i, \tag{D 9}$$

$$\text{if} \quad Q\in\mathcal{L}, \quad \text{then} \quad \bar{Q}\cap B'_v=\emptyset \quad \text{for all} \quad v\in P, \tag{D 10}$$

$$\bigcup_{Q\in\mathcal{L}} Q_q\supset Gy. \tag{D 11}$$

We give some explanation of only (D 11). Note that if $z\in Gy$, then, by (D 5), either $z\in(Q_0)_q$ or there is $Q'\in\mathcal{L}(Q_0)\subset\mathcal{L}$ such that $z\in Q'_+$. In the last case, again by (D 5), either $z\in Q'_q$ or there is $Q''\in\mathcal{L}(Q')\subset\mathcal{L}$ such that $z\in Q''_+$. This process cannot go on indefinitively, and we eventually find $Q\in\mathcal{L}$ such that $z\in Q_q$, implying (D 11) which is needed in the next section.

If we can show that for some $\alpha\in(0,n)$ and $c\in(0,1)$

$$\sum_{Q\in\mathcal{L}_{i+1}} d(Q)^\alpha \leq c \sum_{Q\in\mathcal{L}_i} d(Q)^\alpha \tag{D 12}$$

then it follows immediately that $\dim_H L=\dim_H L\setminus P\leq\alpha<n$. Furthermore, then the α-dimensional Hausdorff measure of $L\setminus P$ is zero and so is that of L. By (D 7), (D 12) is true for $\alpha=n$ and $c=1-1/q^n$. Thus, to get the result that the n-measure of L is zero, we need proceed no further.

The validity of (D 12) for some $\alpha<n$ and $c<1$ is an immediate consequence of

LEMMA D. *There is* $\alpha_0\in(0,n)$ *and* $c\in(0,1)$ *such that if* $Q\in\mathcal{K}_n$ *and if* $\bar{Q}\cap B'_v=\emptyset$ *for all* $v\in P$, *then for all* $\alpha\geq\alpha_0$,

$$\sum_{Q'\in\mathcal{S}(Q)} d(Q')^\alpha \leq c d(Q)^\alpha. \tag{D 13}$$

Proof. By (D 7), we know that this is true for $\alpha=n$ and $c=1-1/q^n$. Starting from this fact, we now prove this improvement of (D 7). Since the lemma is unaffected by a change of scale, we can assume that

$$d(Q)=1.$$

Let

$$V=\{v\in P: B'_v\cap Q\times\{s_Q/q\}\neq\emptyset\}.$$

Since $\bar{Q}\cap B'_v=Q\times\{s_Q\}\cap B'_v=Q\times\{1/\sqrt{n}\}\cap B'_v=\emptyset$ for all $v\in P$, it follows that $1/\sqrt{n}\,q\leq d(B'_v)\leq 2/\sqrt{n}$ if $v\in V$. Since $d(B_v)=2e^{2\sqrt{n}}d(B'_v)$ by (D 2), the euclidean diameters satisfy

$$1/\sqrt{n}\,q\leq d(B'_v)<d(B_v)\leq 4e^{2\sqrt{n}}/\sqrt{n} \tag{D 14}$$

for $v\in V$. It follows that there is $\varepsilon=\varepsilon(q,n)>0$ such that $|v-v'|\geq\varepsilon$ if $v,v'\in V$ and $v\neq v'$. Thus the number of points in V

$$\operatorname{card} V\leq N \tag{D 15}$$

for some $N=N(q,n)$.

Let

$$\mathcal{L}'=\mathcal{S}(Q)\setminus\mathcal{K}(Q,q).$$

If $Q'\in\mathscr{L}'$, then $Q'\in\mathscr{H}(Q,2^iq)$ for some $i>0$. Hence there is $Q''\in\mathscr{H}(Q,2^{i-1}q)$ such that $Q''\supset Q'$. Then $Q''\notin\mathscr{L}(Q)$ by (D 3) and hence $\bar{Q}''\cap B'_v\neq\emptyset$ for some $v\in P$ since in any case $Q''_+\cap(Gy\cup L)\supset Q'_+\cap(Gy\cup L)\neq\emptyset$. Pick $z''\in\bar{Q}''\cap B'_v$. Then $d(z'',z')<\log 2+2\sqrt{n}$ for all $z'\in\bar{Q}'$. It follows by (D 2) that

$$\bar{Q}'\subset B_v\setminus B'_v. \tag{D 16}$$

If $Q'\in\mathscr{L}'$, we denote by $v_{Q'}$ the point of V satisfying (D 16) (which is obviously unique). We show that there is $m=m(q,n)>1$ such that

$$|x-v_{Q'}|^2/m\leqslant s_{Q'}\leqslant m|x-v_{Q'}|^2 \tag{D 17}$$

when $x\in Q'$ and $Q'\in\mathscr{L}'$. Now, we have $s_{Q'}\leqslant 1/2q\sqrt{n}\leqslant d(B'_v)/2<d(B_v)/2$ by (D 14) where we have set $v=v_{Q'}$. The sets $\partial B_{v'}$ and ∂B_v are n-spheres and considering the equation of the lower hemispheres of them we find $m>1$ such that

$$\bar{Q}'=Q'\times\{s_{Q'}\}\subset\{(x,t)\in\mathbf{R}^n\times[0,\infty):|x-v|^2/m\leqslant t\leqslant m|x-v|^2\}.$$

In view of (D 14), $m=m(q,n)$ and (D 17) follows.

To get (D 13), we use (D 17). Let $\alpha>n/2$. Then

$$d(Q')^\alpha=n^{\alpha/2}s^\alpha_{Q'}=n^{\alpha/2}\int_{Q'}s^{\alpha-n}_{Q'}\,d\mu$$

where μ is the Lebesgue measure of $\mathbf{R}^n$. Thus for $r\geqslant 0$

$$\int_{|z-v|\leqslant r}|z-v|^{2(\alpha-n)}\,d\mu(z)=\omega_n\int_0^r x^{2(\alpha-n)+(n-1)}\,dx=\omega_n r^{2\alpha-n}/(2\alpha-n)\geqslant c_0\sum d(Q')^\alpha \tag{D 18}$$

where ω_n is the $(n-1)$-measure of S^{n-1} and $c_0=m^{n-\alpha}n^{-\alpha/2}=c_0(n,q)$ and where the sum is taken over $Q'\in\mathscr{L}'$ such that $v=v_{Q'}$ and that $Q'\subset B^n(v,r)=\{z\in\mathbf{R}^n:|z-v|\leqslant r\}$.

If $Q'\in\mathscr{L}(Q)$ and $d(Q')<1/q$, then $Q'\in\mathscr{L}'$. Hence $v_{Q'}$ is defined and, by (D 17),

$$Q'\subset B^n(v_{Q'},(m\varepsilon')^{1/2})$$

if $Q'\in\mathscr{L}(Q)$ and $d(Q')\leqslant\varepsilon'<1/q$ and where $v_{Q'}\in V$. Suppose that $\varepsilon'<1/q$ and that $\alpha\geqslant 2n/3$. Then (D 18), (D 15) and the above inclusion imply

$$\sum_{Q'\in\mathscr{L}(Q),\,d(Q')\leqslant\varepsilon'}d(Q')^\alpha\leqslant c_1\,\varepsilon'^{n/6} \tag{D 19}$$

where $c_1=3N\omega_n m^{n/6}/c_0 n=c_1(q,n)$. Fix now $\varepsilon'=\varepsilon'(q,n)\in(0,1/q)$ such that $c_1\varepsilon'^{n/6}\leq 1/3q^n$. This done, we can then find $\alpha_0=\alpha_0(q,n)\in[2n/3,n)$ such that $d(Q')^\alpha\leq(1+1/3q^n)\,d(Q')^n$ for $\alpha\geq\alpha_0$ if $Q'\in\mathscr{L}(Q)$ and $d(Q')\geq\varepsilon'$. Hence by (D 19) and (D 7)

$$\sum_{Q'\in\mathscr{L}(Q)} d(Q')^\alpha\leq 1/3q^n+(1+1/3q^n)(1-1/q^n)<(1-1/3q^n)\,d(Q)^\alpha.$$

and the lemma is true for this α_0 and $c=1-1/3q^n$.

Finally, we note that by (D 12) the sum

$$\sum_{Q\in\mathscr{L}} d(Q)^\alpha<\infty \tag{D 20}$$

for $\alpha\geq\alpha_0$. Our proof of the convergence of the Poincaré series g_s of G for $s\geq\alpha_0$ in the next section is an almost direct consequence of (D 20).

Remark. If G is non-elementary, i.e. if the limit set of G contains more than two points, then $\dim_{\mathrm{H}} L(G)>0$. This follows from Beardon [3, Theorem 13] since we can find a Schottky subgroup $G'\subset G$ and a quasiconformal f of $\bar{\mathbf{R}}^n$ such that $fG'f^{-1}$ is Fuchsian (cf. [14, pp. 334–335]). Now f maps sets of positive Hausdorff dimension on sets of positive Hausdorff dimension (Gehring–Väisälä [6]). See also Sullivan [11, Section 3].

E. The convergence of the Poincaré series

Let G be a discrete Möbius group of $\bar{\mathbf{R}}^n$. Following Sullivan [11], we define the *absolute Poincaré series* g_s of G for $s\geq 0$ by

$$g_s(x,y)=\sum_{g\in G} e^{-sd(x,g(y))}$$

and where $x,y\in\mathbf{H}^{n+1}$. If it converges for one pair $x,y\in\mathbf{H}^{n+1}$, then it converges for all. The convergence of g_s is equivalent to the convergence of

$$\sum_{g\in G}|g'(z)|^s$$

for $z\in\mathbf{H}^{n+1}$ (Ahlfors [2, p. 93]). This is the form in which Poincaré introduced the series for Fuchsian groups [9, p. 194].

Using (D 20) we can now easily establish that g_s converges for $s \geq \alpha_0 \in (0, n)$ if G is geometrically finite and Kleinian.

THEOREM E. *Let G be a geometrically finite Kleinian group of $\bar{\mathbf{R}}^n$. Then the Poincaré series g_s of G converges for $s \geq \alpha_0 \in (0, n)$ where α_0 is as in* (D 20).

Proof. We use the notation Q_+ and Q_q and as in (D 1) for cubes $Q \in \mathcal{K}_n$. Let now G be a discrete Möbius group of $\bar{\mathbf{R}}^n$, $t>1$, and $y \in \mathbf{H}^{n+1}$.

Then for any $Q \in \mathcal{K}_n$, the number of $g \in G$ for which $g(y) \in Q_q$,

$$\operatorname{card}\{g \in G: g(y) \in Q_q\} \leq N \tag{E 1}$$

for some $N=N(G, y, q)$. Since the hyperbolic diameter $d(Q_q) \leq \log q+(q+1)\sqrt{n}=r$ we can take N to be the number of $g \in G$ such that $d(g(y), y) \leq r$. Another fact which we need is that

$$d(e_{n+1}, y) \geq \log 1/s_Q = \log (d(Q)/\sqrt{n}) \tag{E 2}$$

if $y \in Q_q$ for any $q \geq 1$ and $Q \in \mathcal{K}_n$ such that $s_Q \leq 1$. This follows since $y=(x, u) \in \mathbf{R}^n \times (0, \infty)$ where $u \leq s_Q$ and since $d(e_{n+1}, e_{n+1}/s_Q)=\log 1/s_Q$. It follows that, if $z=e_{n+1}$, then for every $Q \in \mathcal{K}_n$ the sum

$$\sum_{g \in G,\ g(y) \in Q_q} e^{-sd(z, g(y))} \leq N d(Q)^s \tag{E 3}$$

where $N=N(G, y, q)$.

This is valid for any discrete G. Assume now that G is geometrically finite and Kleinian. We can assume that the situation is as in Section D. Let the point $y \in \mathbf{H}^{n+1}$, the integer $q>1$ and the cube $Q_0 \in \mathcal{K}_n$ be as there. We can assume that $s_{Q_0}=1$. Define the cube families $\mathcal{L}_i$ and $\mathcal{L}$ by (D 8). Then every point of Gy is in some Q_q, $Q \in \mathcal{L}$, by (D 11). By (D 20), $\Sigma_{Q \in \mathcal{L}} d(Q)^s$ converges for $s \geq \alpha_0$. Then (E 3) implies that g_s converges for $s \geq \alpha_0$ and the theorem is proved.

Remarks. Beardon [4] proved Theorem E for $n=1$ (i.e. for finitely generated Fuchsian groups of the second kind) and Patterson [8] proved it for groups not containing parabolic elements (when $n=1$). Sullivan [12] proved it for $n=2$ and his proof probably generalizes also for $n>2$.

If G is any discrete Möbius group of $\bar{\mathbf{R}}^n$, then the Poincaré series converges for

$s>n$ (Ahlfors [2, p. 84], originally Poincaré [9] for $n=1$) and if G is Kleinian, then it converges for $s\geqslant n$ (see Lehner [7, p. 178] whose proof can be generalized also for $n>1$).

We then comment on the relation of $\dim_{\mathrm{H}} L(G)$ to the *exponent of convergence* $\delta(G)$ of G which is the infimum of numbers s for which g_s converges. Thus $\delta(G)\leqslant n$, as we observed above. One can show that the Hausdorff dimension of the so-called conical (or radial) limit set of G does not exceed $\delta(G)$ (Sullivan [11] by a method due to Beardon–Maskit [5]). In the geometrically finite case, a point $x\in L(G)$ is in the conical limit set unless it is a parabolic fixpoint which points form a countable set. Hence $\dim_{\mathrm{H}} L(G)\leqslant\delta(G)$ for geometrically finite G. One suspects that in fact then $\dim_{\mathrm{H}} L(G)=\delta(G)$. At least this is so in many cases, see Sullivan [11, 12] who proved this if $n\leqslant 2$ or if G is of compact type by considerations involving a canonical measure in the limit set originally constructed by Patterson [8] (where this was proved for many groups of $\bar{\mathbf{R}}^1$).

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