ON CERTAIN FUNCTIONAL SOLUTIONS OF THE SATELLITE PROBLEM OF THREE BODIES.

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of PRAGUE.

Henri Poincaré complains in various passages of his classical Méthodes Nouvelles de la Mécanique Céleste 1) of a serious difficulty we always encounter when trying to apply the theory of periodic orbits to concret astronomical problems.

The fundamental determinant, namely the Hessian—Jacobi—Poincaré 2), disappears identically just in the cases in which celestial mechanics is most interested. I refer especially to the all important example of the general problem of three bodies.

And as a matter of fact the vanishing determinant causes the necessary periodic solutions to remain unattainable, as it renders every possibility of their detection futile.

This makes the very known solutions too scarce and far between. And so it happened that for a long time all theoretical efforts resulted in the general belief that the most needed periodic solutions did not exist at all.

Poincaré himself puts it clearly as follows 3):

With every other law than that of Newton, which uses the second power of the reciprocal distance, we meet with lesser difficulties when trying to solve the problem of three bodies. (Donc avec une loi différente de la loi Newtonienne on ne rencontrerait plus dans la recherche des solutions périodiques du problème des trois corps la difficulté que je viens de signaler.) Many years ago I tried to overcome the aforesaid difficulty 4). With this object in view I generalized a substitution — which although very well known even to Poincaré himself was never rightly appreciated for the purpose in question.

And, indeed, by using this infinitesimal transformation and introducing small parameters I succeeded in attaining another Jacobians. The trial always results in the possibility of suppressing a single zero factor (which represents the small parameter of the disturbing mass) of the determinant.

By this very simple means the original vanishing determinant yields another Jacobian — the later generally remaining distinct from zero. In the following paper I shall call this method — for sake of brevity — "an operation".

By the aforesaid process huge 5) quantities of periodic solutions — spread densely enough throughout all space — are obtained important as it appears just in cases in which theoretical Astronomy is mostly interested.

I tried to apply the method in planetary problems and the investigation has yielded results quite satisfactory for practical use of the Méthodes Nouvelles of Poincaré.

It stands to reason that it is always possible to apply the same process in the case of the motion of the Moon.

But the aforesaid means is not the chief idea that induces me to publish the following paper after so many years.

This time my purpose is to call the attention of geometers to a possibility which appears rather remarkable and even so unexpected.

It consists of the following:

All the authors dealing with the theory of the Moon's motion — from the beginning to the present day: Abul Vefa, Tycho Brahe, Kepler, Newton, Euler, Laplace, Poisson, Pontécoulant, Hansen, Delaunay, Gyldén, v. Oppolzer, Newcomb, J. C. Adams, G. W. Hill, Ernst W. Brown, Andoyer, — all of them faced the following problem:

The Moon being "a planet of the Earth", revolves round the latter in a fixed Keplerian ellipse or in a rotating ellipse or else in a distorted ellipse (periodic orbit of G. W. Hill) and so on. These original intermediary orbits show deviations, Perturbations caused by the Sun, etc. This classical, mathematical standpoint always gives the disturbing parameter (μ, as used by H. Poincaré) of an approximate amount $\frac{1}{400}$, and it is understood, that all the following approximations are to be developed according to the powers of this small quantity. Now the possibility I am putting forward enables us to choose a parameter — ceteris paribus 1 000 times smaller, this being represented by the small mass of the Earth $\frac{1}{350\,000}$.

And, indeed, when trying to solve the satellite problem of the three bodies Sun, Earth, Moon, we can start with another formulation of the question than that which the classics had hitherto used.

Let us imagine two planets of the Sun, the Earth and the Moon (both revolving round the Sun). By entirely neglecting their masses $\mu = 0$, we obtain two heliocentric ellipses round the Sun. — I suppose firstly — for sake of simplification — a circle for the Earth and a slightly excentric ellipse for the second planet (Moon) — both of them moving round the Sun at the same Keplerian speed, and thus keeping the same starting length $M + \tilde{\pi} = M' + \pi'$.

Now when introducing a rotating system of the velocity just mentioned, we immediately obtain a fixed position of the Earth and a small closed curve round it — the path of our Moon.

And, indeed, it is very easy to see, that the original planet has changed into a satellite. Unfortunately this Moon revolves round the Earth which constitutes the centre of its orbit, in a year instead of a month 6). Now the idea immediately presents itself — to study the analytical continuation of this curve and thus obtain the whole complicated motion of the Moon — just the same as the classical theory has studied the analytical continuation of an originally simple or distorted planetary ellipse round the Earth.

If we succeed in this endeavour, we would acquire the enormous advantage of operating — ceteris paribus — with the disturbing parameter $\frac{1}{350\,000}$ instead of $\frac{1}{400}$ of the classical theory.

However, when approaching this so formulated satellite problem of three bodies and choosing the mass of the Earth for a new disturbing parameter which is a thousand time smaller, we are met with two impossibilities within the meaning of the classics, mentioned above.

1. How to pass from the heliocentric to the geocentric orbit so as to change the original planet into a Moon.

2. How to set a planet in motion round the Earth so as to acquire the requisite speed of our real Moon.

Fortunately the first impossibility is reduced merely to a fitting passage from heliocentric to geocentric coordinates.

Lastly the second classical impossibility mentioned above, simply means to apply an "operation" namely to pass from an identically disappearing Jacobian—Hessian to a determinant distinct from zero. This is easily carried out by means of small parameters.

Karl Schwarzschild discussed 7) the convergence of the series used by G. W. Hill in the Lunar theory and ascertained that in the case of the periodic solutions in question this convergence appears to be rather probable, but is not sufficiently guaranteed 8) in the case of a parameter à la Hill—Brown—Poincaré $\mu = \frac{1}{400}$.

I hope in this way an extreme probability of this convergence is gained by the considerable diminution of the amount of the new disturbing parameter $\frac{1}{350\,000}$, at least a thousand times smaller.

In the present paper I am giving an exact demonstration of the theory explained and studying the analytic continuation of the undisturbed problem $\mu = 0$ of the abovesaid two ellipses in the case of the complete problem $\mu > 0$. The result is the accessibility of huge classes (manyfold infinity) of short periodic and of secular particular integrals of the satellite problem formulated herewith.

On the whole the Lunar problem appears to be reducible to the study of analytic continuation of a small non-elliptic closed curve, instead of a strictly elliptic orbit or else a distorted Hill's periodic solution.

In this so formulated Lunar problem the Earth plays the part of the disturbing (third) body, instead of the disturbing Sun of the classical theory 9).

The scope of the harvest of particular solutions obtained herewith appears to be so large that I hope I am not compelled — at least in these preliminary sketches — to numerical computations of the natural phenomena.

I content myself with showing that all the movements of a small Moon revolving round the Earth in the aforesaid curve (this being an ellipse round the Sun in reality), can be freely calculated by our modern methods.

So all the solutions of the problem in question are clearly shown to be within reach.

FIRST PART.

§ 1. Investigations into the theory of movements in the immediate neighbourhood of large planetary masses.

Let us start with the well-known equations of motion, governing the movement of three bodies, Sun and two planets. If we choose rectangular, relative coordinates the equations are as follows 10).

$$\frac{d^2 \xi}{d t^2} = -\frac{k^2 (m + 0)}{\varrho^3} \xi + k^2 m' \left(\frac{x' - \xi}{\Delta^3} - \frac{x'}{r'^3}\right) = -\frac{k^2 m}{\varrho^3} \xi + k^2 m' \frac{\partial}{\partial \xi}\left(\frac{1}{\Delta} - \frac{x' \xi + y' \eta + z' \zeta}{r'^3}\right),$$

$$\frac{d^2 \eta}{d t^2} = -\frac{k^2 m}{\varrho^3} \eta + k^2 m' \frac{\partial}{\partial \eta}\left(\frac{1}{\Delta} - \frac{x' \xi + y' \eta + z' \zeta}{r'^3}\right), \tag{1}$$

$$\frac{d^2 \zeta}{d t^2} = -\frac{k^2 m}{\varrho^3} \zeta + k^2 m' \frac{\partial}{\partial \zeta}\left(\frac{1}{\Delta} - \frac{x' \xi + y' \eta + z' \zeta}{r'^3}\right).$$

The asteroid of zero mass and coordinates ξ, η, ζ (the moon) revolves round the Sun of mass $m = 1$ and coordinates 0, 0, 0 and is disturbed by a planet m' (Earth) x', y', z',

$$\varrho^2 = \xi^2 + \eta^2 + \zeta^2,\ r'^2 = x'^2 + y'^2 + z'^2,\ \Delta^2 = (x' - \xi)^2 + (y' - \eta)^2 + (z' - \zeta)^2, \tag{2}$$

k^2 the konstant of Gauss.

Apparently the kinetic energy of the problem will be given by the expression

$$2\,T = \left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2. \tag{3}$$

Now let us specialize these well-known formulas as follows:

In the present outline, where we shall be concerned only with the first approximations, we are going to suppose the mass of the Moon (asteroid) to be zero and to be moving when undisturbed in an ellipse of excentricity approximately $\varepsilon = \frac{1}{400}$. For the path of the disturbing planet (the Earth) we take simply a circular orbit of zero excentricity so that $r' = a'$ (constant). Further we suppose that the mean lengths l, l'

$$l - l' = M + \tilde{\pi} - M' - \pi' = 0. \tag{4}$$

(M, M' mean anomalies $\tilde{\pi}$ π' longitudes of the perihelions) start with a zero difference in longitude. It remains to point out expressly the chief characteristics of our configuration chosen herewith:

It is supposed that the movements of both the asteroid and the disturbing planet, when $m' = 0$ proceed with the same angular speed $n = n'$.

Now whether we introduce a rotating system with angular velocity n' or not, the orbits hitherto ascertained admit the following description:

The Earth revolves round the Sun with its customary mean speed n' in a circular orbit. It is accompanied by a small satellite of negligible mass. This small body represents a kind of Moon, describing a small closed curve round the Earth as its centre (not focus of the ellipse). But it is important to mention that the speed of this Moon is very slow. It revolves round the Earth with the same velocity as the Earth revolves round the Sun, so that the time of its revolution round the Earth is just one year.

First of all we shall proceed to study the equation of the small curve, closed round the position of the revolving Earth.

Let us introduce for that purpose the usual planetary coordinates, the ecliptic being chosen for the cardinal plane Ξ, Y, the Ξ axis aiming towards the vernal point.

$\tilde{\Omega}$ denotes the length of the node of the orbit of the Moon counting from the ecliptic, $\tilde{\pi}$ that of Moon's perihelion, $\tilde{\omega} = \tilde{\pi} - \tilde{\Omega}$ the distance of the perihelion, ι the inclination of the asteroid-Moon-orbit, ε its excentricity, ψ the excentric anomaly. Let us indicate with dashes, the same signs in the case of the Earth's orbit, and especially a', ψ', π'.

If we take for semi major axes resp a, a' we immediately see that according to the above hypotheses $r'^2 = x'^2 + y'^2 + z'^2 = a'^2$, r' is reduced to a' and we can quote the well-known formulas of the elliptic motion

$$\xi = a(\cos\psi - \varepsilon)(\cos\tilde{\Omega}\cos\tilde{\omega} - \sin\tilde{\Omega}\sin\tilde{\omega}\cos\iota) - \\ - a\sqrt{1-\varepsilon^2}\sin\psi(\cos\tilde{\Omega}\sin\tilde{\omega} + \sin\tilde{\Omega}\cos\tilde{\omega}\cos\iota),$$

$$\eta = a(\cos\psi - \varepsilon)(\sin\tilde{\Omega}\cos\tilde{\omega} + \cos\tilde{\Omega}\sin\tilde{\omega}\cos\iota) - \\ - a\sqrt{1-\varepsilon^2}\sin\psi(\sin\tilde{\Omega}\sin\tilde{\omega} - \cos\tilde{\Omega}\cos\tilde{\omega}\cos\iota),$$

$$\zeta = a(\cos\psi - \varepsilon)\sin\tilde{\omega}\sin\iota + a\sqrt{1-\varepsilon^2}\sin\psi\cos\tilde{\omega}\sin\iota, \tag{5}$$

$$x' = a'\cos\psi'\cos\pi' - a'\sin\psi'\sin\pi' = a'\cos(\psi' + \pi'),$$

$$y' = a'\cos\psi'\sin\pi' + a'\sin\psi'\cos\pi' = a'\sin(\psi' + \pi'),$$

$$z' = 0.$$

We now pass from excentric ψ, to the mean anomaly M, of the Moon, by means of the well known elaborate formulas of Dziobek 11) or Le Verrier 12): Mathematische Theorien der Planetenbewegungen pp. 24, 25, Leipzig, 1888, Annales de l'Observatoire National de Paris, Tome I, and obtain the following explicit result

$$\xi = \alpha_1\tilde{\xi} + \beta_1\tilde{\eta}, \quad \tilde{\xi} = a(\cos\psi - \varepsilon),$$

$$\eta = \alpha_2\tilde{\xi} + \beta_2\tilde{\eta}, \quad \tilde{\eta} = a\sqrt{1-\varepsilon^2}\sin\psi = a\{1 - (1 - \sqrt{1-\varepsilon^2})\}\sin\psi, \tag{6}$$

$$\zeta = \alpha_3\tilde{\xi} + \beta_3\tilde{\eta}, \quad \tilde{\eta} = a(1 - \bar{\eta})\sin\psi, \quad \bar{\eta} = 1 - \sqrt{1-\varepsilon^2}, \quad \tilde{\zeta} = 0.$$

$$\alpha_1 = \cos\tilde{\pi} + \sin^2\frac{\iota}{2}[\cos(2\tilde{\Omega} - \tilde{\pi}) - \cos\tilde{\pi}] = \cos(\tilde{\omega} + \tilde{\Omega}) +$$
$$+ \sin^2\frac{\iota}{2}[\cos\tilde{\Omega} - \tilde{\omega}) - \cos(\tilde{\omega} + \tilde{\Omega})],$$

$$\alpha_2 = \sin\tilde{\pi} + \sin^2\frac{\iota}{2}[\sin(2\tilde{\Omega} - \tilde{\pi}) - \sin\tilde{\pi}], \quad \tilde{\pi} = \tilde{\omega} + \tilde{\Omega},$$

$$\alpha_3 = \sin\iota\sin(\tilde{\pi} - \tilde{\Omega}) = \left(2\sin\frac{\iota}{2} - \sin^3\frac{\iota}{2}\right)\sin(\tilde{\pi} - \tilde{\Omega}),$$

$$\beta_1 = -\sin\tilde{\pi} + \sin^2\frac{\iota}{2}[\sin(2\tilde{\Omega} - \tilde{\pi}) + \sin\tilde{\pi}],$$

$$\beta_2 = \cos\tilde{\pi} - \sin^2\frac{\iota}{2}[\cos(2\tilde{\Omega} - \tilde{\pi}) + \cos\tilde{\pi}],$$

$$\beta_3 = \sin\iota\cos(\tilde{\pi} - \tilde{\Omega}) = 2\sin\frac{\iota}{2}\cos(\tilde{\pi} - \tilde{\Omega}) - \sin^3\frac{\iota}{2}\cos(\tilde{\pi} - \tilde{\Omega}), \tag{6}$$

$$\frac{\tilde{\xi}}{a} = \cos\psi - \varepsilon,$$
$$= \cos M - \frac{3}{2}\varepsilon + \frac{\varepsilon}{2}\cos 2M - \frac{3}{8}\varepsilon^2\cos M + \frac{3\varepsilon^2}{8}\cos 3M + \frac{\varepsilon^3}{3}\cos 4M -$$
$$- \frac{\varepsilon^2}{3}\cos 2M + \cdots$$

$$\frac{\tilde{\eta}}{a} = \sin\psi\sqrt{1 - \varepsilon^2}$$
$$= \sin M + \frac{\varepsilon}{2}\sin 2M - \frac{5}{8}\varepsilon^2\sin M + \frac{3}{8}\varepsilon^2\sin 3M + \frac{\varepsilon^3}{3}\sin 4M - \frac{5\varepsilon^3}{12}\sin 2M + \cdots$$

According to our scheme — just explained — we are able to write down the integral curve of movement of the Moon-asteroid for the undisturbed problem $m' = 0$, in case of our fixed system of relative coordinates.

This integral is given by the set of equations:

$$\begin{aligned}\xi = a\cos(M + \tilde{\pi}) - \frac{3}{2}a\varepsilon\cos\tilde{\pi} + \frac{a\varepsilon}{2}\cos(2M + \tilde{\pi})\\ - \frac{a\varepsilon^2}{2}\cos(M + \tilde{\pi}) + \frac{a\varepsilon^2}{8}\cos(M - \tilde{\pi}) + \frac{3}{8}a\varepsilon^2\cos(3M + \tilde{\pi})\\ - a\sin^2\frac{\iota}{2}\cos(M + \tilde{\pi}) + a\sin^2\frac{\iota}{2}\cos(M + \tilde{\pi} - 2\tilde{\Omega})\\ - \frac{3}{8}a\varepsilon^3\cos(2M + \tilde{\pi}) + \frac{a\varepsilon^3}{24}\cos(2M - \tilde{\pi}) + \frac{a\varepsilon^3}{3}\cos(4M + \tilde{\pi})\end{aligned} \tag{7}$$

$$+\frac{3}{2}\alpha\varepsilon\sin^2\frac{\iota}{2}\cos\tilde{\pi}-\frac{3}{2}\alpha\varepsilon\sin^2\frac{\iota}{2}\cos(2\tilde{\Omega}-\tilde{\pi})$$

$$-\frac{\alpha\varepsilon}{2}\sin^2\frac{\iota}{2}\cos(2M+\tilde{\pi})+\frac{\alpha\varepsilon}{2}\sin^2\frac{\iota}{2}\cos(2M+\tilde{\pi}-2\tilde{\Omega}),$$

$$\eta=\alpha\sin(M+\tilde{\pi})-\frac{3}{2}\alpha\varepsilon\sin\tilde{\pi}+\frac{\alpha\varepsilon}{2}\sin(2M+\tilde{\pi})$$

$$-\frac{\alpha\varepsilon^2}{2}\sin(M+\tilde{\pi})-\frac{\alpha\varepsilon^2}{8}\sin(M-\tilde{\pi})+\frac{3}{8}\alpha\varepsilon^2\sin(3M+\tilde{\pi})$$

$$-\alpha\sin^2\frac{\iota}{2}\sin(M+\tilde{\pi})-\alpha\sin^2\frac{\iota}{2}\sin(M+\tilde{\pi}-2\tilde{\Omega})$$

$$-\frac{3}{8}\alpha\varepsilon^3\sin(2M+\tilde{\pi})-\frac{\alpha\varepsilon^3}{24}\sin(2M-\tilde{\pi})+\frac{\alpha\varepsilon^3}{3}\sin(4M+\tilde{\pi})$$

$$+\frac{3}{2}\alpha\varepsilon\sin^2\frac{\iota}{2}\sin\tilde{\pi}-\frac{3}{2}\alpha\varepsilon\sin^2\frac{\iota}{2}\sin(2\tilde{\Omega}-\tilde{\pi}) \qquad (7)$$

$$-\frac{\alpha\varepsilon}{2}\sin^2\frac{\iota}{2}\sin(2M+\tilde{\pi})-\frac{\alpha\varepsilon}{2}\sin^2\frac{\iota}{2}\sin(2M+\tilde{\pi}-2\tilde{\Omega}),$$

$$\zeta=2\alpha\sin\frac{\iota}{2}\sin(M+\tilde{\pi}-\tilde{\Omega})-3\alpha\varepsilon\sin\frac{\iota}{2}\sin(\tilde{\pi}-\tilde{\Omega})+\alpha\varepsilon\sin\frac{\iota}{2}\sin(2M+\tilde{\pi}-\tilde{\Omega})$$

$$-\alpha\varepsilon^2\sin\frac{\iota}{2}\sin(M+\tilde{\pi}-\tilde{\Omega})-\frac{\alpha\varepsilon^2}{4}\sin\frac{\iota}{2}\sin(M-\tilde{\pi}+\tilde{\Omega})+$$

$$+\frac{3}{4}\alpha\varepsilon^2\sin\frac{\iota}{2}\sin(3M+\tilde{\pi}-\tilde{\Omega})$$

$$-\alpha\sin^3\frac{\iota}{2}\sin(M+\tilde{\pi}-\tilde{\Omega}),$$

However, it is to be pointed out expressly, that the mean anomaly of the Earth must not be introduced for $e'\neq 0$. This would entirely spoil our starting suppositions of the problem restreint. Moreover the new curve of Lunar path would lose its defining meaning and the present study would lead to nothing.

Let us now pass to a new origin of coordinates in the Earth, thus changing our starting heliocentric into a geocentric system. It is understood that the new axes of the geocentric system always remain parallel to the original heliocentric ones.

The final expressions of the geocentric coordinates are the same except for the first terms on the righthandsides of ξ, η, these latter being replaced by

$$a'\cos(M'+\pi')=\alpha\cos(M+\tilde{\pi}),\qquad a'\sin(M'+\pi')=\alpha\sin(M+\tilde{\pi}). \qquad (8)$$

The space curve fixed by the last set of equations is closed in itself and embraces as its centre (not focal point) the movable position of the Earth. Thus it represents — as was explained above — the starting, not disturbed, orbit of the Moon. Only the period of revolution coincides precisely with that of the revolution of the Earth round the Sun and so appears twelwe times shorter then the period of our real Moon. The newly chosen origin as well as the form of the aforesaid starting Moon-space-curve suggest another angle to be chosen for the new distance of the perihelion. This will be best defined as the fixed angle between the two directions, the line parallel to the nodal line of the Moon-planet-ellipse- and the direction from the Earth to the fixed Keplerian perihelion of the Moon Planet ellipse —$\tilde{\omega}$.

When choosing for a moment the geocentric rectangular system of axes, so that ξ runs through the node, we can immediately write down the coordinates of the Earth as

$$\begin{aligned} x_0' &= a' \cos (M_0' + \pi') = a' \cos (M_0' + \tilde{\omega}'), \quad M' = M_0', \quad \Omega_0' = 0, \\ y_0' &= a' \sin (M_0' + \pi') = a' \sin (M_0' + \tilde{\omega}'), \quad M_0' + \tilde{\omega}' = \tilde{\omega}, \quad M_0 = 0, \\ M + \tilde{\pi} &= M' + \pi' = M + \tilde{\omega} + \tilde{\Omega} = M' + \tilde{\omega}' + \Omega', \quad \text{put } \tilde{\Omega} = \Omega', \\ M + \tilde{\omega} &= M' + \tilde{\omega}' \text{ and for } M_0 = 0, \quad M' = M_0', \text{ hence } \tilde{\omega} = M_0' + \tilde{\omega}', \end{aligned} \tag{9}$$

and the coordinates of the Moon-perihelion as

$$\begin{aligned} \xi_0 &= (a - a\varepsilon) \cos \tilde{\omega}, \\ \eta_0 &= (a - a\varepsilon) \sin \tilde{\omega} \cos \iota = a(1 - \varepsilon) \left(1 - 2 \sin^2 \frac{\iota}{2}\right) \sin \tilde{\omega}, \\ \zeta_0 &= (a - a\varepsilon) \sin \tilde{\omega} \sin \iota, \quad \iota \doteq 0{\cdot}000\,225 = \varepsilon i, \quad i = 5^\circ 9', \quad \varepsilon = \frac{1}{400}. \end{aligned} \tag{10}$$

It is to be expressly noted that the meaning of the constant ι is the inclination of the plane of the starting Moon ellipse to the ecliptic, namely the heliocentric inclination.

For the distance Node-Moonperihelion, we easily get the final expression

$$\begin{aligned} \varrho_0^2 &= (\xi_0 - x_0')^2 + (\eta_0 - y_0')^2 + \zeta_0^2 = \xi_0^2 + \eta_0^2 + \zeta_0^2 + x_0'^2 + y_0'^2 - 2\,\xi_0 x_0' - 2\,\eta_0 y_0', \\ \varrho_0^2 &= a^2 + a'^2 - 2a^2\varepsilon + a^2\varepsilon^2 - 2(a - a\varepsilon)a' \cos(\tilde{\omega} - M_0' - \tilde{\omega}') + \\ &\quad + 4(a - a\varepsilon)a \sin^2 \frac{\iota}{2} \sin \tilde{\omega} \sin (M_0' + \tilde{\omega}'), \; \tilde{\omega} - M_0' - \tilde{\omega}' = 0, \quad a = a', \\ \varrho_0^2 &= a^2\varepsilon^2 + 4a^2(1 - \varepsilon) \sin^2 \frac{\iota}{2} \sin^2 \tilde{\omega}. \end{aligned} \tag{11}$$

Taking account of our starting fundamental condition (4)

$$M + \tilde{\pi} - M' - \pi' = 0, \quad \text{we find} \quad M_0 + \tilde{\Omega} + \tilde{\omega} = M_0' + \omega' + \Omega'$$

and as we put the ξ axis into the direction of the node of the Moon-Planet ellipse: $M_0' + \omega' = \tilde{\omega}$. Recalling that we have chosen both the Earth ellipse as well as the Moon ellipse of precisely the same major axes, it will be $a = a'$.

In this way it turns out to be

$$\varrho_0 = a\,\varepsilon \left\{1 + 4\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\right) \sin^2 \tilde{\omega} \sin^2 \frac{\iota}{2}\right\}^{\frac{1}{2}}. \tag{12}$$

We have then to construct the direction cosinus $\cos \bar{\omega}$, by means of (10):

$$\begin{aligned} \xi_0 - x_0' &\equiv - a\,\varepsilon \cos \tilde{\omega} \\ \cos \bar{\omega} = \frac{\xi - x_0'}{\varrho_0} &= -\cos \tilde{\omega} \left\{1 + 4\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\right) \sin^2 \tilde{\omega} \sin^2 \frac{\iota}{2}\right\}^{-\frac{1}{2}} \end{aligned} \tag{13}$$

from which expression we immediately gather that

$$\sin^2 \bar{\omega} = 1 - \cos^2 \bar{\omega} = \sin^2 \tilde{\omega} \left\{1 + 4\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\right) \cos^2 \tilde{\omega} \sin^2 \frac{\iota}{2}\right\}. \tag{14}$$

We easily adjust the signs of the roots, remembering that the two directions of $\tilde{\omega}$ heliocentric and $\bar{\omega}$ geocentric differ by $180°$ and obtain finally

$$\cos \bar{\omega} = -\left\{1 + \frac{1}{2}\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\right) \sin^2 \frac{\iota}{2}\right\} \cos \bar{\omega} + \frac{1}{2}\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\right) \sin^2 \frac{\iota}{2} \cos 3\,\bar{\omega},$$

$$\sin \bar{\omega} = -\left\{1 - \frac{1}{2}\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\right) \sin^2 \frac{\iota}{2}\right\} \sin \bar{\omega} + \frac{1}{2}\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\right) \sin^2 \frac{\iota}{2} \sin 3\,\bar{\omega},$$

and putting

$$h = \left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\right) \sin^2 \frac{\iota}{2}, \qquad p_1 = 1 + \frac{h}{2}, \qquad p_2 = 1 - \frac{h}{2},$$

$$\cos \tilde{\omega} = -p_1 \cos \bar{\omega} + \frac{h}{2} \cos 3\,\bar{\omega}, \tag{15}$$

$$\sin \tilde{\omega} = -p_2 \sin \bar{\omega} + \frac{h}{2} \sin 3\,\bar{\omega},$$

$$\cos(\tilde{\omega} + \tilde{\Omega}) = -\cos(\bar{\omega} + \tilde{\Omega}) - \frac{h}{2} \cos(\bar{\omega} - \tilde{\Omega}) + \frac{h}{2} \cos(3\,\bar{\omega} + \tilde{\Omega}),$$

$$\sin(\tilde{\omega} + \tilde{\Omega}) = -\sin(\bar{\omega} + \tilde{\Omega}) + \frac{h}{2} \sin(\bar{\omega} - \tilde{\Omega}) + \frac{h}{2} \sin(3\,\bar{\omega} + \Omega),$$

$$\cos(\tilde{\omega} + \tilde{\Omega} + L) = -\cos(\bar{\omega} + \tilde{\Omega} + L) - \frac{h}{2}\cos(\bar{\omega} - \tilde{\Omega} - L) + \frac{h}{2}\cos(3\bar{\omega} + \tilde{\Omega} + L),$$
$$\sin(\tilde{\omega} + \tilde{\Omega} + L) = -\sin(\bar{\omega} + \tilde{\Omega} + L) + \frac{h}{2}\sin(\bar{\omega} - \tilde{\Omega} - L) + \frac{h}{2}\sin(3\bar{\omega} + \tilde{\Omega} + L). \tag{15}$$

These expressions are to be substituted into our coordinates (6) written above as well as into the disturbing function which will be fixed here after § 4.

Moreover it is advisable to introduce — instead of the real heliocentric inclination ι of the two starting Earth and Moon ellipse — an average geocentric inclination i. The meaning of this function is to be understood only as an average constant, or the last taken as a disturbed variable of the whole problem in question.

This new constant replaces so to say, the inclination of the geocentric Moon orbit, although we know from the above, that even the not disturbed quasi osculating Moon path — being a space (and never a plane) curve — does not admit the precise geocentrical meaning of the inclination of a plane curve in relation to the fundamental plane of the ecliptic.

We prefer to put for sake of a suitable choice of geocentric canonical elements

$$\begin{aligned} &\varepsilon = \frac{a}{a'} = \frac{1}{400} = 0{\cdot}0025, \quad \sin\iota = \varepsilon \sin i, \quad \iota = \frac{i}{400}, \\ &i \doteq 5^\circ 9', \quad \sin i = 0{\cdot}09 \doteq i = \frac{\iota}{\varepsilon} \\ &\sin^2\frac{\iota}{2} \doteq \frac{\varepsilon^2 i^2}{4}. \end{aligned} \tag{16}$$

§ 2. Remarks on the starting Moon-space-curve.

When introducing a rotating system $\hat{\xi}, \hat{\eta}, \hat{\zeta}$ with angular velocity $\psi' = M' = n't + M'_0$ whose $\hat{\xi}$ axis points perpetually towards the Earth, we obtain the expressions

$$\begin{aligned} \hat{\xi} &= a(\cos\psi - \varepsilon)[\cos(\tilde{\Omega} - \psi' - \pi')\cos\tilde{\omega} - \sin(\tilde{\Omega} - \psi' - \pi')\sin\tilde{\omega}\cos\iota] - \\ &\quad - a\sqrt{1-\varepsilon^2}\sin\psi[\cos\tilde{\Omega} - \psi' - \pi')\sin\tilde{\omega} + \sin(\tilde{\Omega} - \psi' - \pi')\cos\tilde{\omega}\cos\iota] \\ \hat{\eta} &= a(\cos\psi - \varepsilon)[\sin(\tilde{\Omega} - \psi' - \pi')\cos\tilde{\omega} + \cos(\tilde{\Omega} - \psi' - \pi')\sin\tilde{\omega}\cos\iota] - \\ &\quad - a\sqrt{1-\varepsilon^2}\sin\psi[\sin(\tilde{\Omega} - \psi' - \pi')\sin\tilde{\omega} - \cos(\tilde{\Omega} - \psi' - \pi')\cos\tilde{\omega}\cos\iota] \\ \hat{\zeta} &= a(\cos\psi - \varepsilon)\sin\tilde{\omega}\sin\iota + a\sqrt{1-\varepsilon^2}\sin\psi\sin\iota\cos\tilde{\omega} \end{aligned} \tag{17}$$

these representing the coordinates of the slowly moving Moon round the Earth in our rotating heliocentric system. The same equations can be written in another form, more suitable both for theoretical and calculating studies

$$\begin{aligned}\hat{\xi} = {} & \alpha \cos(\psi - \psi' + \tilde{\pi} - \pi') - \alpha\varepsilon \cos(\psi' + \pi' - \tilde{\pi}) - \frac{\alpha\bar{\eta}}{2} \cos(\psi - \psi' + \tilde{\pi} - \pi') + \\ & + \frac{\alpha\bar{\eta}}{2} \cos(\psi + \psi' + \pi' - \tilde{\pi}) - \alpha \sin^2\frac{\iota}{2} \cos(\psi - \psi' + \tilde{\pi} - \pi') + \\ & + \alpha \sin^2\frac{\iota}{2} \cos(\psi + \psi' + \tilde{\pi} + \pi' - 2\tilde{\Omega}) + \alpha\varepsilon \sin^2\frac{\iota}{2} \cos(\psi' + \pi' - \tilde{\pi}) - \\ & - \alpha\varepsilon \sin^2\frac{\iota}{2} \cos(\tilde{\pi} + \pi' + \psi' - 2\tilde{\Omega}) \\ \hat{\eta} = {} & \alpha \sin(\psi - \psi' + \tilde{\pi} - \pi') + \alpha\varepsilon \sin(\psi' + \pi' - \tilde{\pi}) - \frac{\alpha\bar{\eta}}{2} \sin(\psi - \psi' + \tilde{\pi} - \pi') - \\ & - \frac{\alpha\bar{\eta}}{2} \sin(\psi + \psi' - \tilde{\pi} + \pi') - \alpha \sin^2\frac{\iota}{2} \sin(\psi - \psi' + \tilde{\pi} - \pi') - \\ & - \alpha \sin^2\frac{\iota}{2} \sin(\psi + \psi' + \tilde{\pi} + \pi' - 2\tilde{\Omega}) - \alpha\varepsilon \sin^2\frac{\iota}{2} \sin(\psi' + \pi' - \tilde{\pi}) + \\ & + \alpha\varepsilon \sin^2\frac{\iota}{2} \sin(\tilde{\pi} + \pi' + \psi' - 2\tilde{\Omega}) \\ \hat{\zeta} = {} & \alpha\left(2\sin\frac{\iota}{2} - \sin^3\frac{\iota}{2}\right) \sin(\psi + \tilde{\pi} - \tilde{\Omega}) - 2\alpha\varepsilon \sin\frac{\iota}{2} \sin(\tilde{\pi} - \tilde{\Omega}) - \\ & - \frac{\alpha\bar{\eta}}{1} \sin\frac{\iota}{2} [\sin(\psi + \tilde{\pi} - \tilde{\Omega}) + \sin(\psi - \tilde{\pi} + \tilde{\Omega})],\end{aligned} \tag{18}$$

where we have put as in (6) $\bar{\eta} = 1 - \sqrt{1 - \varepsilon^2}$.

And again we pass from excentric ψ to the mean anomalies of the Moon, by means of the well-known elaborate formulas of Dziobek (see 11) pp. 24, 25) and obtain the following explicit result

$$\begin{aligned}\hat{\xi} &= \hat{\alpha}_1 \tilde{\xi} + \hat{\beta}_1 \tilde{\eta}, \\ \hat{\eta} &= \hat{\alpha}_2 \tilde{\xi} + \hat{\beta}_2 \tilde{\eta}, \\ \hat{\zeta} &= \hat{\alpha}_3 \tilde{\xi} + \hat{\beta}_3 \tilde{\eta},\end{aligned} \tag{19}$$

$$\begin{aligned}\hat{\alpha}_1 &= \cos(\tilde{\pi} - \psi' - \pi') + \sin^2\frac{\iota}{2} [\cos(2\tilde{\Omega} - \tilde{\pi} - \psi' - \pi') - \cos(\tilde{\pi} - \psi' - \pi')], \\ \hat{\alpha}_2 &= \sin(\tilde{\pi} - \psi' - \pi') + \sin^2\frac{\iota}{2} [\sin(2\tilde{\Omega} - \tilde{\pi} - \psi' - \pi') - \sin(\tilde{\pi} - \psi' - \pi')], \\ \hat{\alpha}_3 &= \sin\iota \sin(\tilde{\pi} - \tilde{\Omega}) = \left(2\sin\frac{\iota}{2} - \sin^3\frac{\iota}{2}\right) \sin(\tilde{\pi} - \tilde{\Omega}),\end{aligned} \tag{20}$$

$$
\begin{aligned}
\beta_1 &= -\sin(\tilde{\pi}-\psi'-\pi') + \sin^2\frac{\iota}{2}\left[\sin(2\tilde{\Omega}-\tilde{\pi}-\psi'-\pi') + \sin(\tilde{\pi}-\psi'-\pi')\right],\\
\beta_2 &= \cos(\tilde{\pi}-\psi'-\pi') - \sin^2\frac{\iota}{2}\left[\cos(2\tilde{\Omega}-\tilde{\pi}-\psi'-\pi') + \cos(\tilde{\pi}-\psi'-\pi')\right],\\
\beta_3 &= \sin\iota\cos(\tilde{\pi}-\tilde{\Omega}) = \left(2\sin\frac{\iota}{2} - \sin^3\frac{\iota}{2}\right)\cos(\tilde{\pi}-\tilde{\Omega}),
\end{aligned}
\tag{20}
$$

$$
\tilde{\xi} = \alpha(\cos\psi - \varepsilon), \quad \tilde{\eta} = \alpha\sqrt{1-\varepsilon^2}\sin\psi, \quad \tilde{\zeta} = 0.
$$

If we limit ourselves to the third power of small quantities only (exclusive), we can immediately write down the following expressions for the **rotating** heliocentric coordinates of the Moon

$$
\begin{aligned}
\hat{\xi} = {} & \alpha\cos(M-M'+\tilde{\pi}-\pi') - \frac{3\alpha\varepsilon}{2}\cos(\tilde{\pi}-\pi'-M') + \frac{\alpha\varepsilon}{2}\cos(2M-M'+\tilde{\pi}-\pi') - \\
& - \frac{\alpha\varepsilon^2}{2}\cos(M-M'+\tilde{\pi}-\pi') + \frac{\alpha\varepsilon^2}{8}\cos(M+M'-\tilde{\pi}+\pi') + \\
& + \frac{3}{8}\alpha\varepsilon^2\cos(3M-M'+\tilde{\pi}-\pi') - \alpha\sin^2\frac{\iota}{2}\cos(M-M'+\tilde{\pi}-\pi') + \\
& + \alpha\sin^2\frac{\iota}{2}\cos(M+M'+\tilde{\pi}+\pi'-2\tilde{\Omega}) - \frac{3}{8}\alpha\varepsilon^3\cos(2M-M'+\tilde{\pi}-\pi') + \\
& + \frac{\alpha\varepsilon^3}{24}\cos(2M+M'-\tilde{\pi}+\pi') + \frac{\alpha\varepsilon^3}{3}\cos(4M-M'+\tilde{\pi}-\pi') + \\
& + \frac{3}{2}\alpha\varepsilon\sin^2\frac{\iota}{2}\cos(\tilde{\pi}-\pi'-M') - \frac{3}{2}\alpha\varepsilon\sin^2\frac{\iota}{2}\cos(\tilde{\pi}+\pi'+M'-2\tilde{\Omega}) - \\
& - \frac{\alpha\varepsilon}{2}\sin^2\frac{\iota}{2}\cos(2M-M'+\tilde{\pi}-\pi') + \frac{\alpha\varepsilon}{2}\sin^2\frac{\iota}{2}\cos(2M+M'+\tilde{\pi}+\pi'-2\tilde{\Omega}),\\
\hat{\eta} = {} & \alpha\sin(M-M'+\tilde{\pi}-\pi') - \frac{3\alpha\varepsilon}{2}\sin(\tilde{\pi}-\pi'-M') + \frac{\alpha\varepsilon}{2}\sin(2M-M'+\tilde{\pi}-\pi') - \\
& - \frac{\alpha\varepsilon^2}{2}\sin(M-M'+\tilde{\pi}-\pi') - \frac{\alpha\varepsilon^2}{8}\sin(M+M'-\tilde{\pi}+\pi') + \\
& + \frac{3}{8}\alpha\varepsilon^2\sin(3M-M'+\tilde{\pi}-\pi') - \alpha\sin^2\frac{\iota}{2}\sin(M-M'+\tilde{\pi}-\pi') - \\
& - \alpha\sin^2\frac{\iota}{2}\sin(M+M'+\tilde{\pi}+\pi'-2\tilde{\Omega}) - \frac{3}{8}\alpha\varepsilon^3\sin(2M-M'+\tilde{\pi}-\pi') - \\
& - \frac{\alpha\varepsilon^3}{24}\sin(2M+M'-\tilde{\pi}+\pi') + \frac{\alpha\varepsilon^3}{3}\sin(4M-M'+\tilde{\pi}-\pi') + \\
& + \frac{3}{2}\alpha\varepsilon\sin^2\frac{\iota}{2}\sin(\tilde{\pi}-\pi'-M') + \frac{3}{2}\alpha\varepsilon\sin^2\frac{\iota}{2}\sin(\tilde{\pi}+\pi'+M'-2\Omega) - \\
& - \frac{\alpha\varepsilon}{2}\sin^2\frac{\iota}{2}\sin(2M-M'+\tilde{\pi}-\pi') - \frac{\alpha\varepsilon}{2}\sin^2\frac{\iota}{2}\sin(2M+M'+\tilde{\pi}+\pi'-2\tilde{\Omega}),
\end{aligned}
\tag{21}
$$

$$\hat{\zeta} = 2\alpha \sin\frac{\iota}{2}\sin(M+\pi-\tilde{\Omega}) - 3\alpha\varepsilon\sin\frac{\iota}{2}\sin(\tilde{\pi}-\tilde{\Omega}) + \alpha\varepsilon\sin\frac{\iota}{2}\sin(2M+\tilde{\pi}-\tilde{\Omega}) -$$

$$-\alpha\varepsilon^2\sin\frac{\iota}{2}\sin(M+\tilde{\pi}-\tilde{\Omega}) - \frac{\alpha\varepsilon^2}{4}\sin\frac{\iota}{2}\sin(M-\tilde{\pi}+\tilde{\Omega}) + \tag{21}$$

$$+\frac{3}{4}\alpha\varepsilon^2\sin\frac{\iota}{2}\sin(3M+\tilde{\pi}-\tilde{\Omega}) - \alpha\sin^3\frac{\iota}{2}\sin(M+\tilde{\pi}-\tilde{\Omega}).$$

To simplify our survey it is advantageous to eliminate throughout the following computations in the expressions of $\hat{\xi}$, $\hat{\eta}$, $\hat{\zeta}$ the explicit time $M' = n't + c = n't + M'_0$ by means of the principal condition (4), $M - M' + \tilde{\pi} - \pi' = 0$ which according to our original assumption, lodges both ellipses — the ellipse of the Moon-asteroid and the Earth ellipse — conveniently so that the never escaping Moon changes from the original planet into an ideal, slowly moving satellite. In this manner and only so, we can avoid all delicate questions concerning rotation or libration. And indeed when keeping the Sun as the origin of coordinates, we introduce the rotating system; our ideal Moon always presents itself as a librating Planet, never going round the Sun without the Earth.

In this manner the angle M is never allowed to grow to the full amount of 360° without a parallel growing of M' as a consequence of the aforesaid condition $M - M' + \tilde{\pi} - \pi' = 0$.

However, it appears most important to note that the original meaning of M as a mean anomaly with respect to the ellipse round the Sun, disappears and our new variable $\bar{M}$ signifies quite another angle, marking the revolutions round the Earth.

We carry out this elimination $M - M' + \tilde{\pi} - \pi' = 0$, but at the same time we pass from the starting heliocentric origin of rotating axes to geocentric ones, thus obtaining the following equations which represent the undisturbed Moon-path (orbit) of our study

$$\hat{\xi} = \alpha - \alpha\varepsilon\cos\bar{M} - \frac{\alpha\varepsilon^2}{2} + \frac{\alpha\varepsilon^2}{2}\cos 2\bar{M} - \alpha\sin^2\frac{\iota}{2} + \alpha\sin^2\frac{\iota}{2}\cos(2\bar{M}+2\pi-2\tilde{\Omega}) +$$

$$+\frac{3\alpha\varepsilon^3}{8}\cos 3\bar{M} - \frac{3}{8}\alpha\varepsilon^3\cos\bar{M} + \alpha\varepsilon\sin^2\frac{\iota}{2}\cos\bar{M} - \tag{22}$$

$$-\frac{3}{2}\alpha\varepsilon\sin^2\frac{\iota}{2}\cos(2\tilde{\pi}+\bar{M}-2\tilde{\Omega}) + \frac{\alpha\varepsilon}{2}\sin^2\frac{\iota}{2}\cos(3\bar{M}+2\tilde{\pi}-2\tilde{\Omega}),$$

$$\hat{\eta} = 2\alpha\varepsilon \sin \bar{M} + \frac{\alpha\varepsilon^2}{4} \sin 2\bar{M} + \alpha \sin^2\frac{\iota}{2} \sin(2\tilde{\pi} + 2\bar{M} - 2\tilde{\Omega}) + \frac{7\alpha\varepsilon^3}{24} \sin 3\bar{M} -$$

$$- \frac{3}{8}\alpha\varepsilon^3 \sin\bar{M} - 2\alpha\varepsilon\sin^2\frac{\iota}{2}\sin M + \frac{3}{2}\alpha\varepsilon\sin^2\frac{\iota}{2}\sin(2\tilde{\pi} + \bar{M} - 2\tilde{\Omega}) -$$

$$- \frac{\alpha\varepsilon}{2}\sin^2\frac{\iota}{2}\sin(3\bar{M} + 2\tilde{\pi} - 2\tilde{\Omega}),$$

$$\hat{\zeta} = 2\alpha\sin\frac{\iota}{2}\sin(M + \tilde{\omega}) - 3\alpha\varepsilon\sin\frac{\iota}{2}\sin\tilde{\omega} + \alpha\varepsilon\sin\frac{\iota}{2}\sin(2M + \tilde{\omega}) - \tag{22}$$

$$- \alpha\varepsilon^2\sin\frac{\iota}{2}\sin(M + \tilde{\omega}) - \frac{\alpha\varepsilon^2}{4}\sin\frac{\iota}{2}\sin(M - \tilde{\omega}) + \frac{3}{4}\alpha\varepsilon^2\sin\frac{\iota}{2}\sin(2M + \tilde{\omega}) -$$

$$- \alpha\sin^3\frac{\iota}{2}\sin(M + \tilde{\omega}). \qquad \tilde{\pi} = \tilde{\omega} + \tilde{\Omega}.$$

On the whole in our rotating system of coordinates — the aforesaid path of the slowly moving Moon — appears to be a spacecurve closely rounding the position of the fixed Earth — during the period of one year. It is easy to obtain the equation of the curve in rectangular coordinates by eliminating the time, which enters into the right hand members trough the mean anomalies M and M'.

When judging according the first, most important terms of our rotating coordinates and entirely neglecting $\sin\frac{\iota}{2}$, we are led to the conclusion that the curve in question can best be approximated by a plane ellipse. The excentricity of the ellipse is about 0·87, but it is very important to point out that the Earth occupies its centre and not the focus, as we are always accustomed to suppose. Moreover, for the whole of following theory, it is necessary to express the coordinates of the aforesaid ellipse exclusively by means of the mean anomalies M, M' (not possibly of the excentric ψ, ψ' or else true anomalies v, v'). As we immediately ascertain from the latest developments, the small slowly moving Moon-ellipse has a major axis $2\alpha\varepsilon$, twice as long as the minor one $\alpha\varepsilon$. This is easily inferred from the two starting terms $-\alpha\varepsilon\cos\bar{M}$ in the coordinate $\hat{\xi}$ and $2\alpha\varepsilon\sin M$ in $\hat{\eta}$.

The triangle (end of the smal axis, centre of the ellipse and its focus) gives the relations

$$a = 2\alpha\varepsilon, \quad b = \frac{a}{2} = \alpha\varepsilon, \quad a^2 - b^2 = a^2 - \frac{a^2}{4} = \frac{3}{4}a^2 = a^2e^2, \quad e = \frac{\sqrt{3}}{2} = 0{\cdot}866. \tag{23}$$

§ 3. On a system of canonical elements.

Let us consider the expressions for the fixed rectangular geocentric coordinates (7) of the aforesaid Moon-path as a customary transformation à la Lagrange. We have then to pass to new Lagrangian coordinates, for which we shall choose the three angles $M, \bar{\omega}, \tilde{\Omega}$.

It will be easy to construct the Lagrangian kinetic Energy of our geocentric system and to pass to the Pfaffian differential form.

Now to find out the best canonical elements of the problem in question we have to calculate the Lagrangian impulses (momentum).

These expressions yield manifestly periodic series, proceeding according to multiples of the chosen angles $M, \bar{\omega}, \tilde{\Omega}$. In this manner we are able to write down immediately the total differential form of Pfaff as follows:

$$
\begin{aligned}
&\frac{\partial T}{\partial \dot{M}} d M + \frac{\partial T}{\partial \dot{\bar{\omega}}} d \bar{\omega} + \frac{\partial T}{\partial \dot{\tilde{\Omega}}} d \tilde{\Omega} - F\, d t = d S \\
&F = \dot{M} \frac{\partial T}{\partial \dot{M}} + \dot{\bar{\omega}} \frac{\partial T}{\partial \dot{\bar{\omega}}} + \dot{\tilde{\Omega}} \frac{\partial T}{\partial \dot{\tilde{\Omega}}} - T - V \\
&2 T = \left(\frac{d \xi}{d t}\right)^2 + \left(\frac{d \eta}{d t}\right)^2 + \left(\frac{d \zeta}{d t}\right)^2 \qquad (24) \\
&V = \frac{k^2 \mu_1}{\varrho}, \quad \mu_1 = 1 + m_{☾}, \quad \mu R = k^2 \mu \left(\frac{1}{\varDelta} - \frac{\varrho \cos \sigma}{r'^2}\right), \quad \mu = m_{♁} \\
&r'^2 = x'^2 + y'^2 + z'^2 = a'^2, \quad \varrho^2 = \xi^2 + \eta^2 + \zeta^2, \\
&\varDelta^2 = (\xi - x') + (\eta - y')^2 + \zeta^2, \quad z' = 0,
\end{aligned}
$$

$d S$ signifies an exact differential.

Now the series $\frac{\partial T}{\partial \dot{M}}, \frac{\partial T}{\partial \dot{\bar{\omega}}}, \frac{\partial T}{\partial \dot{\tilde{\Omega}}}$ etc. represent clearly the integral of the simplified not disturbed problem, where $m' = 0$. Consequently they must satisfy the Pfaffian condition term by term, and we can limit ourselves to calculating the simplest term among them. For this we choose the best, the first constant term called secular (in Astronomy). Now the well-known principles of analysis show clearly that we need not even to calculate the whole expression of T as we can isolate the periodic series step by step.

$$\frac{\partial T}{\partial \dot{M}} = \frac{\partial T}{\partial \dot{\xi}}\frac{\partial \dot{\xi}}{\partial \dot{M}} + \frac{\partial T}{\partial \dot{\eta}}\frac{\partial \dot{\eta}}{\partial \dot{M}} + \frac{\partial T}{\partial \dot{\zeta}}\frac{\partial \dot{\zeta}}{\partial \dot{M}} = \dot{\xi}\frac{\partial \xi}{\partial M} + \dot{\eta}\frac{\partial \eta}{\partial M} + \dot{\zeta}\frac{\partial \zeta}{\partial M} = L$$

$$\frac{\partial T}{\partial \dot{\tilde{\omega}}} = \frac{\partial T}{\partial \dot{\xi}}\frac{\partial \dot{\xi}}{\partial \dot{\tilde{\omega}}} + \frac{\partial T}{\partial \dot{\eta}}\frac{\partial \dot{\eta}}{\partial \dot{\tilde{\omega}}} + \frac{\partial T}{\partial \dot{\zeta}}\frac{\partial \dot{\zeta}}{\partial \dot{\tilde{\omega}}} = \dot{\xi}\frac{\partial \xi}{\partial \tilde{\omega}} + \dot{\eta}\frac{\partial \eta}{\partial \tilde{\omega}} + \dot{\zeta}\frac{\partial \zeta}{\partial \tilde{\omega}} = G \tag{25}$$

$$\frac{\partial T}{\partial \dot{\tilde{\Omega}}} = \frac{\partial T}{\partial \dot{\xi}}\frac{\partial \dot{\xi}}{\partial \dot{\tilde{\Omega}}} + \frac{\partial T}{\partial \dot{\eta}}\frac{\partial \dot{\eta}}{\partial \dot{\tilde{\Omega}}} + \frac{\partial T}{\partial \dot{\zeta}}\frac{\partial \dot{\zeta}}{\partial \dot{\tilde{\Omega}}} = \dot{\xi}\frac{\partial \xi}{\partial \tilde{\Omega}} + \dot{\eta}\frac{\partial \eta}{\partial \tilde{\Omega}} + \dot{\zeta}\frac{\partial \zeta}{\partial \tilde{\Omega}} = H,$$

as we can easily prove that according to Lagrange

$$\frac{\partial \dot{\xi}}{\partial \dot{\tilde{\omega}}} = \frac{\partial \xi}{\partial \tilde{\omega}}, \quad \frac{\partial \dot{\eta}}{\partial \dot{\tilde{\omega}}} = \frac{\partial \eta}{\partial \tilde{\omega}}, \quad \frac{\partial \dot{\zeta}}{\partial \dot{\tilde{\omega}}} = \frac{\partial \zeta}{\partial \tilde{\omega}}, \quad \frac{\partial \dot{\xi}}{\partial \dot{M}} = \frac{\partial \xi}{\partial M}, \quad \frac{\partial \dot{\eta}}{\partial \dot{M}} = \frac{\partial \eta}{\partial M}, \quad \frac{\partial \dot{\zeta}}{\partial \dot{M}} = \frac{\partial \zeta}{\partial M},$$

$$\frac{\partial \dot{\xi}}{\partial \dot{\tilde{\Omega}}} = \frac{\partial \xi}{\partial \tilde{\Omega}}, \quad \frac{\partial \dot{\eta}}{\partial \dot{\Omega}} = \frac{\partial \eta}{\partial \Omega}, \quad \frac{\partial \dot{\zeta}}{\partial \dot{\Omega}} = \frac{\partial \zeta}{\partial \Omega}. \tag{26}$$

To simplify a general survey and facilitate computation, I rewrite the previous expressions (6) of the heliocentric fixed coordinates as transformed into the geocentric system, whose axes remain parallel to the original heliocentric fixed axes, — as follows — thereby neglecting all terms of the third order (exclusively) of small quantities i, ε:

$$x' = a' \cos(M' + \pi'), \quad y' = a' \sin(M' + \pi'), \quad \tilde{\pi} = \tilde{\omega} + \tilde{\Omega},$$

$$\begin{aligned}\xi = a \cos(M + \tilde{\pi}) - \frac{3}{2} a \varepsilon \cos \tilde{\pi} + \frac{a\varepsilon}{2} \cos(2M + \tilde{\pi}) - \frac{a\varepsilon^2}{2} \cos(M + \tilde{\pi}) + \\ + \frac{a\varepsilon^2}{8} \cos(M - \tilde{\pi}) + \frac{3}{8} a \varepsilon^2 \cos(3M + \tilde{\pi}) - a \sin^2 \frac{\iota}{2} \cos(M + \tilde{\pi}) + \\ + a \sin^2 \frac{\iota}{2} \cos(M + \tilde{\pi} - 2\tilde{\Omega}),\end{aligned} \tag{27}$$

$$\begin{aligned}\eta = a \sin(M + \tilde{\pi}) - \frac{3}{2} a \varepsilon \sin \tilde{\pi} + \frac{a\varepsilon}{2} \sin(2M + \tilde{\pi}) - \frac{a\varepsilon^2}{2} \sin(M + \tilde{\pi}) - \\ - \frac{a\varepsilon^2}{8} \sin(M - \tilde{\pi}) + \frac{3}{8} a \varepsilon^2 \sin(3M + \tilde{\pi}) - a \sin^2 \frac{\iota}{2} \sin(M + \tilde{\pi}) - \\ - a \sin^2 \frac{\iota}{2} \sin(M + \tilde{\pi} - 2\tilde{\Omega}),\end{aligned}$$

$$\zeta = 2a \sin\frac{\iota}{2} \sin(M + \tilde{\pi} - \tilde{\Omega}) - 3a\varepsilon \sin\frac{\iota}{2} \sin(\tilde{\pi} - \tilde{\Omega}) + a\varepsilon \sin\frac{\iota}{2} \sin(2M + \tilde{\pi} - \tilde{\Omega}).$$

As we have supposed $M + \tilde{\pi} = M' + \pi'$, (see (4) p. 5), and $\alpha = a'$, we can replace the starting terms $a \cos (M + \tilde{\pi})$, $\alpha \sin (M + \tilde{\pi})$ by $a' \cos (M' + \pi')$, $a' \sin (M' + \pi')$ and skip them entirely, as they remain always independent of our variables M, $\tilde{\omega}$, $\tilde{\Omega}$.

In this way we obtain

$$\begin{aligned}
\xi &= a' \cos (M' + \pi') - \frac{3}{2} \alpha \varepsilon \cos \tilde{\pi} + \frac{\alpha \varepsilon}{2} \cos (2 M + \tilde{\pi}) - \frac{\alpha \varepsilon^2}{2} \cos (M + \tilde{\pi}) + \\
&\quad + \frac{\alpha \varepsilon^2}{8} \cos (M - \tilde{\pi}) + \frac{3}{8} \alpha \varepsilon^2 \cos (3 M + \tilde{\pi}) - \alpha \sin^2 \frac{\iota}{2} \cos (M + \tilde{\pi}) + \\
&\quad + \alpha \sin^2 \frac{\iota}{2} \cos (M + \tilde{\pi} - 2 \tilde{\Omega}), \\
\eta &= a' \sin (M' + \pi') - \frac{3}{2} \alpha \varepsilon \sin \tilde{\pi} + \frac{\alpha \varepsilon}{2} \sin (2 M + \tilde{\pi}) - \frac{\alpha \varepsilon^2}{2} \sin (M + \tilde{\pi}) - \\
&\quad - \frac{\alpha \varepsilon^2}{8} \sin (M - \tilde{\pi}) + \frac{3}{8} \alpha \varepsilon^2 \sin (3 M + \tilde{\pi}) - \alpha \sin^2 \frac{\iota}{2} \sin (M + \tilde{\pi}) - \\
&\quad - \alpha \sin^2 \frac{\iota}{2} \sin (M + \tilde{\pi} - 2 \tilde{\Omega}), \\
\zeta &= 2 \alpha \sin \frac{\iota}{2} \sin (M + \tilde{\pi} - \tilde{\Omega}) - 3 \alpha \varepsilon \sin \frac{\iota}{2} \sin (\tilde{\pi} - \tilde{\Omega}) + \alpha \varepsilon \sin \frac{\iota}{2} \sin (2 M + \tilde{\pi} - \tilde{\Omega})
\end{aligned} \tag{28}$$

and by means of (15)

$$\begin{aligned}
\frac{\xi}{\alpha} &= \frac{3}{2} \varepsilon \cos (\tilde{\omega} + \tilde{\Omega}) + \frac{3 h \varepsilon}{4} \cos (\tilde{\omega} - \tilde{\Omega}) - \frac{3 h \varepsilon}{4} \cos (3 \tilde{\omega} + \tilde{\Omega}) - \frac{\varepsilon}{2} \cos (2 M + \tilde{\omega} + \tilde{\Omega}) - \\
&\quad - \frac{h \varepsilon}{4} \cos (\tilde{\omega} - \tilde{\Omega} - 2 M) + \frac{h \varepsilon}{4} \cos (3 \tilde{\omega} + \tilde{\Omega} + 2 M) + \frac{\varepsilon^2}{2} \cos (M + \tilde{\omega} + \tilde{\Omega}) - \\
&\quad - \frac{\varepsilon^2}{8} \cos (\tilde{\omega} + \tilde{\Omega} - M) - \frac{3}{8} \varepsilon^2 \cos (3 M + \tilde{\omega} + \tilde{\Omega}) + \sin^2 \frac{\iota}{2} \cos (M + \tilde{\omega} - \tilde{\Omega}) - \\
&\quad - \sin^2 \frac{\iota}{2} \cos (M + \tilde{\omega} - \tilde{\Omega}), \\
\frac{\eta}{\alpha} &= \frac{3}{2} \varepsilon \sin (\tilde{\omega} + \tilde{\Omega}) - \frac{3 h \varepsilon}{4} \sin (\tilde{\omega} - \tilde{\Omega}) - \frac{3 h \varepsilon}{4} \sin (3 \tilde{\omega} + \tilde{\Omega}) - \frac{\varepsilon}{2} \sin (2 M + \tilde{\omega} + \tilde{\Omega}) + \\
&\quad + \frac{h \varepsilon}{4} \sin (\tilde{\omega} - \tilde{\Omega} - 2 M) + \frac{h \varepsilon}{4} \sin (3 \tilde{\omega} + \tilde{\Omega} + 2 M) + \frac{\varepsilon^2}{2} \sin (M + \tilde{\omega} + \tilde{\Omega}) - \\
&\quad - \frac{\varepsilon^2}{8} \sin (\tilde{\omega} + \tilde{\Omega} - M) - \frac{3 \varepsilon^2}{8} \sin (3 M + \tilde{\omega} + \tilde{\Omega}) + \sin^2 \frac{\iota}{2} \sin (M + \tilde{\omega} + \tilde{\Omega}) + \\
&\quad + \sin^2 \frac{\iota}{2} \sin (M + \tilde{\omega} - \tilde{\Omega})
\end{aligned} \tag{29}$$

$$\frac{\zeta}{a} = -2 \sin\frac{\iota}{2} \sin(\bar\omega + M) + h \sin\frac{\iota}{2} \sin(\bar\omega - M) + \sin\frac{\iota}{2} h \sin(3\bar\omega + M) -$$
$$- \varepsilon \sin\frac{\iota}{2} \sin(\bar\omega + 2M) + 3\varepsilon \sin\frac{\iota}{2} \sin\bar\omega. \tag{29}$$

From these expressions, we easily form the necessary factors

$$\frac{\dot\xi}{a} = \varepsilon n \sin(2M + \bar\omega + \tilde\Omega) - \frac{h\varepsilon n}{2} \sin(\bar\omega - \tilde\Omega - 2M) - \frac{h\varepsilon n}{2} \sin(2M + \tilde\Omega + 3\bar\omega) +$$
$$+ \frac{n\varepsilon^2}{2} \sin(M + \bar\omega + \tilde\Omega) - \frac{n\varepsilon^2}{8} \sin(M - \bar\omega - \tilde\Omega) + \frac{9n\varepsilon^2}{8} \sin(3M + \bar\omega + \tilde\Omega) -$$
$$- n \sin^2\frac{\iota}{2} \sin(M + \bar\omega + \tilde\Omega) + n \sin^2\frac{\iota}{2} \sin(M + \bar\omega - \tilde\Omega),$$

$$\frac{1}{a}\frac{\partial\xi}{\partial M} = \frac{\dot\xi}{na}, \quad \frac{1}{a}\frac{\partial\eta}{\partial M} = \frac{\dot\eta}{na}, \quad \frac{1}{a}\frac{\partial\zeta}{\partial M} = \frac{\dot\zeta}{na}, \tag{30}$$

$$\frac{\dot\eta}{a} = -\varepsilon n \cos(2M + \bar\omega + \tilde\Omega) - \frac{h\varepsilon n}{2} \cos(\bar\omega - \tilde\Omega - 2M) + \frac{h\varepsilon n}{2} \cos(3\bar\omega + \tilde\Omega + 2M) +$$
$$+ \frac{n\varepsilon^2}{2} \cos(M + \bar\omega + \tilde\Omega) + \frac{n\varepsilon^2}{8} \cos(M - \bar\omega - \tilde\Omega) - \frac{9n\varepsilon^2}{8} \cos(3M + \bar\omega + \tilde\Omega) +$$
$$+ n \sin^2\frac{\iota}{2} \cos(M + \bar\omega + \tilde\Omega) + n \sin^2\frac{\iota}{2} \cos(M + \bar\omega - \tilde\Omega),$$

$$\frac{\dot\zeta}{a} = -2n \sin\frac{\iota}{2} \cos(M + \bar\omega) - hn \sin\frac{\iota}{2} \cos(\bar\omega - M) + hn \sin\frac{\iota}{2} \cos(3\bar\omega + M) -$$
$$- 2n\varepsilon \sin\frac{\iota}{2} \cos(2M + \bar\omega).$$

Similarly we find out by simple derivations the expressions necessary for the moment $\dfrac{\partial T}{\partial \dot{\bar\omega}}$ as:

$$\frac{1}{a}\frac{\partial\dot\xi}{\partial\bar\omega} = \frac{1}{a}\frac{\partial\xi}{\partial\bar\omega} = \frac{\varepsilon}{2} \sin(2M + \bar\omega + \tilde\Omega) + \frac{h\varepsilon}{4} \sin(\bar\omega - \tilde\Omega - 2M) -$$
$$- \frac{3h\varepsilon}{4} \sin(3\bar\omega + \tilde\Omega + 2M) - \frac{\varepsilon^2}{2} \sin(M + \bar\omega + \tilde\Omega) + \frac{\varepsilon^2}{8} \sin(\bar\omega + \tilde\Omega - M) +$$
$$+ \frac{3\varepsilon^2}{8} \sin(3M + \bar\omega + \tilde\Omega) - \sin^2\frac{\iota}{2} \sin(M + \bar\omega + \tilde\Omega) + \sin^2\frac{\iota}{2} \sin(M + \bar\omega - \tilde\Omega)$$

$$\frac{1}{a}\frac{\partial\dot{\eta}}{\partial\dot{\bar{\omega}}}=\frac{1}{a}\frac{\partial\eta}{\partial\bar{\omega}}=-\frac{\varepsilon}{2}\cos(2M+\bar{\omega}+\tilde{\Omega})+\frac{h\varepsilon}{4}\cos(\bar{\omega}-\tilde{\Omega}-2M)+ \tag{31}$$

$$+\frac{3h\varepsilon}{4}\cos(3\bar{\omega}+\tilde{\Omega}+2M)+\frac{\varepsilon^2}{2}\cos(M+\bar{\omega}+\tilde{\Omega})-\frac{\varepsilon^2}{8}\cos(\bar{\omega}+\tilde{\Omega}-\mathrm{M})-$$

$$-\frac{3\varepsilon^2}{8}\cos(3M+\bar{\omega}+\tilde{\Omega})+\sin^2\frac{\iota}{2}\cos(M+\bar{\omega}+\tilde{\Omega})+\sin^2\frac{\iota}{2}\cos(M+\bar{\omega}-\tilde{\Omega}),$$

$$\frac{1}{a}\frac{\partial\dot{\zeta}}{\partial\dot{\bar{\omega}}}=\frac{1}{a}\frac{\partial\zeta}{\partial\bar{\omega}}=-2\sin\frac{\iota}{2}\cos(\bar{\omega}+M)+h\sin\frac{\iota}{2}\cos(\bar{\omega}-M)-$$

$$-3\sin\frac{\iota}{2}h\cos(3\bar{\omega}+M)-\varepsilon\sin\frac{\iota}{2}\cos(2M+\bar{\omega}).$$

And for the momentum $\frac{\partial T}{\partial\dot{\tilde{\Omega}}}$, it turns out to be

$$\frac{1}{a}\frac{\partial\dot{\xi}}{\partial\dot{\tilde{\Omega}}}=\frac{1}{a}\frac{\partial\xi}{\partial\tilde{\Omega}}=\frac{\varepsilon}{2}\sin(2M+\bar{\omega}+\tilde{\Omega})-\frac{h\varepsilon}{4}\sin(\bar{\omega}-\tilde{\Omega}-2M)-$$

$$-\frac{h\varepsilon}{4}\sin(3\bar{\omega}+\tilde{\Omega}+2M)-\frac{\varepsilon^2}{2}\sin(M+\bar{\omega}+\tilde{\Omega})+\frac{\varepsilon^2}{8}\sin(\bar{\omega}+\tilde{\Omega}-M)+$$

$$+\frac{3\varepsilon^2}{8}\sin(3M+\bar{\omega}+\tilde{\Omega})-\sin^2\frac{\iota}{2}\sin(M+\bar{\omega}+\tilde{\Omega})-\sin^2\frac{\iota}{2}\sin(M+\bar{\omega}-\tilde{\Omega}),$$

$$\frac{1}{a}\frac{\partial\dot{\eta}}{\partial\dot{\tilde{\Omega}}}=\frac{1}{a}\frac{\partial\eta}{\partial\tilde{\Omega}}=-\frac{\varepsilon}{2}\cos(2M+\bar{\omega}+\tilde{\Omega})-\frac{h\varepsilon}{4}\cos(\bar{\omega}-\tilde{\Omega}-2M)+ \tag{32}$$

$$+\frac{h\varepsilon}{4}\cos(3\bar{\omega}+\tilde{\Omega}+2M)+\frac{\varepsilon^2}{2}\cos(M+\bar{\omega}+\tilde{\Omega})-\frac{\varepsilon^2}{8}\cos(\bar{\omega}+\tilde{\Omega}-M)-$$

$$-\frac{3}{8}\varepsilon^2\cos(3M+\bar{\omega}+\tilde{\Omega})+\sin^2\frac{\iota}{2}\cos(M+\bar{\omega}+\tilde{\Omega})-\sin^2\frac{\iota}{2}\sin(M+\bar{\omega}-\tilde{\Omega})$$

$$\frac{1}{a}\frac{\partial\dot{\zeta}}{\partial\dot{\tilde{\Omega}}}=\frac{1}{a}\frac{\partial\zeta}{\partial\tilde{\Omega}}=0.$$

After this short Lagrangian algebra, we are able to isolate — from the final product — periodic series — the first, namely the constant (secular) terms, the latter fixing the canonical elements of our Lunar problem

$$L = x_1 = n\alpha^2\varepsilon^2 + \frac{49}{32}n\alpha^2\varepsilon^4 + 2n\alpha^2\sin^2\frac{\iota}{2} + \frac{h^2\varepsilon^2 n\alpha^2}{2} + 2n\alpha^2\sin^4\frac{\iota}{2} + $$
$$+ n\alpha^2 h^2 \sin^2\frac{\iota}{2} + 2n\alpha^2\varepsilon^2\sin^2\frac{\iota}{2}$$
$$G = x_2 = \frac{n\alpha^2\varepsilon^2}{2} + \frac{h^2\varepsilon^2 n\alpha^2}{4} + \frac{21}{32}n\alpha^2\varepsilon^4 + 2n\alpha^2\sin^2\frac{\iota}{2} + 2n\alpha^2\sin^4\frac{\iota}{2} + \tag{33}$$
$$+ n\alpha^2 h^2\sin^2\frac{\iota}{2} + n\alpha^2\varepsilon^2\sin^2\frac{\iota}{2}$$
$$H = x_3 = \frac{n\alpha^2\varepsilon^2}{2} + \frac{h^2\varepsilon^2 n\alpha^2}{4} + \frac{21}{32}n\alpha^2\varepsilon^4.$$

It probably appears useless to mention that the constant terms looked for thereby are easily picked up from two factors of equal arguments, and where these do not exist by passing from powers of trigonometrical functions to the multiples (namely the doubles).

Now the canonical elements just computed represent the set of scalar variables corresponding to the chosen angular quantities M, $\bar{\omega}$, $\tilde{\Omega}$ both sets joining together through the existence of the perfect differential of the Pfaffian form, namely

$$dS = L\,dM + G\,d\bar{\omega} + H\,d\tilde{\Omega} - F\,dt. \tag{24}$$

But for further investigation it appears more advantageous, if not necessary, to choose for one angular variable, instead of $\tilde{\Omega}$, the linear combination $M - M' + \tilde{\Omega}$.

When passing so to the new angular variables M, $\bar{\omega}$, $M - M' + \tilde{\Omega}$, we have to transcribe our Pfaffian form into

$$dS = (L - H)\,dM + G\,d\bar{\omega} + H\,d(M - M' + \tilde{\Omega}) - (F - Hn')\,dt, \tag{34}$$

finding in this simple way a new canonical system of elements

$$\begin{aligned} x_1' &= x_1 - x_3 = L - H = \Lambda, & y_1' &= M, \\ x_2' &= x_2 = G, & y_2' &= \bar{\omega}, \\ x_3' &= x_3 = H, & y_3' &= M - M' + \tilde{\Omega}, \end{aligned} \tag{35}$$

$$\frac{\partial\alpha}{\partial\Lambda} \doteq -\frac{2 + \frac{21}{4}\varepsilon^2}{\varepsilon^4\frac{7}{16}\frac{\partial n\alpha^2}{\partial\alpha}}, \qquad \frac{\partial\varepsilon}{\partial\Lambda} \doteq \frac{1 + \frac{21}{16}\varepsilon^2}{\varepsilon^3\frac{7}{16}n\alpha^2} \quad \text{etc.} \tag{35 a}$$

with the equations of movement defined by

$$\frac{d x_i'}{d t} = \frac{\partial F'}{\partial y_i'}, \quad \frac{d y_i'}{d t} = -\frac{\partial F'}{\partial x_i'}, \quad i = 1, 2, 3, \quad F' = -F + H n'. \tag{36}$$

If we drop the dashes, we gain the system

$$\frac{d x_i}{d t} = \frac{\partial F}{\partial y_i}, \quad \frac{d y_i}{d t} = -\frac{\partial F}{\partial x_i}. \tag{37}$$

§ 4. Development of the disturbing function.

With the view of finding out the final form of the differential and integral equations of the problem and of solving them qualitatively, we are bound to look for a suitable development of the disturbing function.

This qualitative development although convergent strongly enough is intended to simplify explanation of our method and for a closer approach to the point of view of Poincaré's theory.

However, I should like to point out that for quantitative purposes and especially for numerical computation another, far more convergent development may be chosen. And indeed for the sake of computation we may even try to regularize the shock point ($\Delta = 0$) of the problem. So firstly for qualitative purpose let us consider the customary, unchanged planetary disturbing function

$$\mu R = k^2 \mu \left(\frac{1}{\Delta} - \frac{\varrho \cos \sigma}{r'^2} \right), \quad \mu = m_{♁}. \tag{24}$$

It will be noticed that in our satellite case the indirect part namely

$$-\frac{\varrho \cos \sigma}{r'^2} k^2 \mu$$

becomes far and away the most important, in consequence of our transfer of the origin of coordinates from the Sun to the Earth.

In this way we are able to understand, that just this indirect part yields the main secular and critical (commensurable terms) of the trigonometrical development. This appears to be the more comprehensible and natural, as, unlike the usual planetary theory, the path of the Moon-planet always embraces the movable position of the disturbing planet — the Earth — as its centre.

Let us start first with the expressions (7) p. 7, and construct by means of these the mutual distance of Earth-Moon, namely (24) p. 16,

$$\varDelta^2 = (\xi - x')^2 + (\eta - y')^2 + \zeta^2.$$

As the original amounts (see the first terms of (7), (4) p. 7, (27) (28) pp. 17, 18),

$$a \cos (M + \tilde{\pi}) = a' \cos (M' + \pi'), \quad a \sin (M + \tilde{\pi}) = a' \sin (M + \pi'),$$

remain the same all through the computations of the present paper according to our chief condition and lodging of both starting ellipses $M + \tilde{\pi} - M' - \pi' = 0$, (4) p. 5, — it appears clear that the aforenamed starting terms cancel out, and the whole distance becomes quite independent of the coordinates of the disturbing body (the Earth). The chief consequence of this important fact is evidently that no critical term of the commensurability of mean movements n, n' of the Moon-Planet and of the Earth is to be obtained from the direct Lagrangian part $\frac{1}{\varDelta}$ of the disturbing function.

On the contrary many such critical commensurable terms remain contained in the indirect Lagrangian part of the disturbing function, which thus becomes the most important.

An indeed the indirect part may be written, as $z' = 0$:

$$-\frac{\varrho \cos \sigma}{r'^2} = -\frac{\varrho}{r'^2}\left(\frac{\xi}{\varrho}\frac{x'}{r'} + \frac{\eta}{\varrho}\frac{y'}{r'}\right) = -\frac{\xi x' + \eta y'}{r'^3}, \quad \varrho^2 = \xi^2 + \eta^2 + \zeta^2, \quad r' = a', \tag{38}$$

and we have simply to introduce on the righthand side the explicite expressions (7) pp. 7, 8, thus obtaining the final development:

$$\begin{aligned}
-\frac{\varrho \cos \sigma}{a'^2} = -& \\
-\frac{a}{a'^2}&\left(1 - \frac{\varepsilon^2}{2} - \sin^2\frac{\iota}{2}\right)\cos(M - M' + \tilde{\pi} - \pi') + \frac{3}{2}\frac{a}{a'^2}\left(\varepsilon - \varepsilon \sin^2\frac{\iota}{2}\right)\cos(\tilde{\pi} - \pi' - M') - \\
-\frac{a}{a'^2}&\left(\frac{\varepsilon}{2} - \frac{3\varepsilon^3}{8} - \frac{\varepsilon \sin^2\frac{\iota}{2}}{2}\right)\cos(2M - M' + \tilde{\pi} - \pi') \\
-\frac{a\varepsilon^2}{8a'^2}&\cos(M + M' + \pi' - \tilde{\pi}) - \frac{3}{8}\frac{a\varepsilon^2}{a'^2}\cos(3M - M' + \tilde{\pi} - \pi') - \\
&- \frac{a \sin^2\frac{\iota}{2}}{a'^2}\cos(M + M' + \tilde{\pi} + \pi' - 2\tilde{\Omega})
\end{aligned} \tag{39}$$

$$
\begin{aligned}
&-\frac{\alpha\,\varepsilon^3}{24\,a'^2}\cos\,(2\,M + M' - \tilde{\pi} + \pi') - \frac{\alpha\,\varepsilon^3}{3\,a'^2}\cos\,(4\,M - M' + \tilde{\pi} - \pi')\\
&+\frac{\alpha\,\varepsilon}{2\,a'^2}\sin^2\frac{\iota}{2}\cos\,(2\,M - M' + \tilde{\pi} - \pi') + \frac{3}{2}\,\frac{\alpha\,\varepsilon}{a'^2}\sin^2\frac{\iota}{2}\cos\,(2\,\tilde{\Omega} - \tilde{\pi} - \pi' - M') \qquad (39)\\
&-\frac{\alpha\,\varepsilon}{2\,a'^2}\sin^2\frac{\iota}{2}\cos\,(2\,M + M' + \tilde{\pi} + \pi' - 2\,\tilde{\Omega}).
\end{aligned}
$$

Moreover it is to be noted that our critical terms start even with such terms, which appear not multiplied by small factors containing the excentricity ε or inclination $\iota\,(i)$. This is very important for reaching the necessary critical terms of the periodic solutions in question.

From this result it can immediately be seen that no special development of this chief part of the disturbing function appears necessary, except for the well-known Lagrange Bessel series for purely Keplerian elliptic motion.

As to the aforementioned direct Lagrangian part $\frac{1}{\Delta}$ the final expression of Δ^2 is easily found to be

$$
\begin{aligned}
\frac{(\xi - x')^2 + (\eta - y')^2 + \zeta^2}{a^2} &= \frac{5}{2}\varepsilon^2 + 2\sin^2\frac{\iota}{2} + 2\sin^4\frac{\iota}{2} + 5\,\varepsilon^2\sin^2\frac{\iota}{2} + \frac{13}{32}\varepsilon^4 + h^2\sin^2\frac{\iota}{2} +\\
&+\frac{5}{4}h^2\varepsilon^2 - \frac{3\,\varepsilon^2}{2}\cos 2\,M + \varepsilon^3\cos M - \varepsilon^3\cos 3\,M + 2\,\varepsilon\sin^2\frac{\iota}{2}\cos M +\\
&+3\,\varepsilon\sin^2\frac{\iota}{2}\cos\,(2\,\bar{\omega} + M) - \varepsilon\sin^2\frac{\iota}{2}\cos\,(2\,\bar{\omega} + 3\,M) - 2\sin^2\frac{\iota}{2}\cos\,(2\,\bar{\omega} + 2\,M) -\\
&-4\,\varepsilon\sin^2\frac{\iota}{2}\cos M + 2\,h\sin^2\frac{\iota}{2}\cos\,(2\,M + 4\,\bar{\omega}) - 2\,h\sin^2\frac{\iota}{2}\cos 2\,M - \qquad (40)\\
&-\frac{h^2\sin^2\frac{\iota}{2}}{2}\cos\,(2\,\bar{\omega} - 2\,M) - \frac{h^2\sin^2\frac{\iota}{2}}{2}\cos\,(6\,\bar{\omega} + 2\,M) + h^2\varepsilon^2\cos\bar{\omega} -\\
&-\frac{3}{4}h^2\varepsilon^2\cos 2\,M + \frac{3}{8}h^2\varepsilon^2\cos\,(2\,M + 4\,\bar{\omega}) + \frac{3}{8}h^2\varepsilon^2\cos\,(2\,M - 4\,\bar{\omega}) + \cdots
\end{aligned}
$$

$$
h = \left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\right)\sin^2\frac{\iota}{2}.
$$

the whole of this development can be clearly summed up by three representative terms, which run thus:

$$\frac{5}{2}\,\bar{\varepsilon}^2 - \frac{3}{2}\,\varepsilon^2 \cos 2M + \delta \tag{41}$$

$$\frac{5}{2}\bar{\varepsilon}^2 = \frac{5}{2}\varepsilon^2 + 2\sin^2\frac{\iota}{2} + 2\sin^4\frac{\iota}{2} + 5\varepsilon^2\sin^2\frac{\iota}{2} + \frac{13}{32}\varepsilon^4 + h^2\sin^2\frac{\iota}{2} + \frac{5}{4}h^2\varepsilon^2,$$

$$\bar{\varepsilon}^2 = \varepsilon^2\left\{1 + \frac{4}{5}\,\frac{\sin^2\frac{\iota}{2}}{\varepsilon^2} + \frac{4}{5}\,\frac{\sin^4\frac{\iota}{2}}{\varepsilon^2} + 2\sin^2\frac{\iota}{2} + \frac{13}{80}\varepsilon^2 + \frac{2h^2}{5\varepsilon^2}\sin^2\frac{\iota}{2} + \frac{h^2}{2}\right\}$$

the aforementioned small function δ being clearly of order $\frac{1}{400} = \varepsilon$.

$$\begin{aligned}\delta = &-2\sin^2\frac{\iota}{2}\cos(2\bar{\omega} + 2M) + \varepsilon^3\cos M - \varepsilon^3\cos 3M + 2\varepsilon\sin^2\frac{\iota}{2}\cos M + \\ &+ 3\varepsilon\sin^2\frac{\iota}{2}\cos(2\bar{\omega} + M) \\ &-\varepsilon\sin^2\frac{\iota}{2}\cos(2\bar{\omega} + 3M) - 4\varepsilon\sin^2\frac{\iota}{2}\cos M + 2h\sin^2\frac{\iota}{2}\cos(2M + 4\bar{\omega}) - \\ &-2h\sin^2\frac{\iota}{2}\cos 2M - \\ &-\frac{h^2\sin^2\frac{\iota}{2}}{2}\cos(2\bar{\omega} - 2M) - \frac{h^2\sin^2\frac{\iota}{2}}{2}\cos(6\bar{\omega} + 2M) + h^2\varepsilon^2\cos\bar{\omega} - \\ &-\frac{3}{4}h^2\varepsilon^2\cos 2M + \frac{3}{8}h^2\varepsilon^2\cos(2M + 4\bar{\omega}) + \frac{3}{8}h^2\varepsilon^2\cos(2M - 4\bar{\omega})\end{aligned} \tag{42}$$

Then put $\delta = 0$ in the expression Δ^2. When trying to develop

$$\frac{1}{\Delta} = \frac{1}{a'}\left(\frac{5}{2}\,\bar{\varepsilon}^2 - \frac{3}{2}\varepsilon^2\cos 2M + \delta\right)^{-\frac{1}{2}}, \quad a = a'. \tag{43}$$

it appears that we cannot take much advantage out of the Laplacian 14) transcendents of the classical theory. Instead we can use a well known formula of Euler, whose convergence strongly overshadows all hypergeometric coëfficients.

(See for ex. Lobatto: 15) Lessen over hoogere algebra p. 232, II. edition, Studnička: 16) O počtu integrálním, Praha, 1871, p. 76.)

$$\begin{aligned}-\frac{1}{2}\log\text{nat}\,(1 + a^2 - 2a\cos\varphi) &= -\frac{1}{2}\log(1 - a e^{i\varphi}) - \frac{1}{2}\log(1 - a e^{-i\varphi}) \\ &= a\cos\varphi + \frac{a^2}{2}\cos 2\varphi + \frac{a^3}{3}\cos 3\varphi + \frac{a^4}{4}\cos 4\varphi + \cdots \\ &= -\frac{1}{2}\log(1 + a^2) - \frac{1}{2}\log\left(1 - \frac{2a}{1 + a^2}\cos\varphi\right).\end{aligned} \tag{44}$$

Now, if we put

$$n = \frac{2a}{1+a^2},$$

$$-\frac{1}{2}\log(1-n\cos\varphi) = \frac{1}{2}\log(1+a^2) + a\cos\varphi + \frac{a^2}{2}\cos 2\varphi + \frac{a^3}{3}\cos 3\varphi + \cdots$$

$$= \frac{1}{2}\log\frac{2a}{n} + a\cos\varphi + \frac{a^2}{2}\cos 2\varphi + \frac{a^3}{3}\cos 3\varphi + \cdots \qquad (45)$$

In our case

$$\frac{1}{\varDelta} = \frac{1}{a'}\left(\frac{5}{2}\bar{\varepsilon}^2 - \frac{3}{2}\varepsilon^2\cos 2M + \delta\right)^{-\frac{1}{2}}, \quad a = a'$$

$$\frac{1}{\varDelta} = \frac{1}{a'}\frac{\sqrt{2}}{\sqrt{5}}\frac{1}{\bar{\varepsilon}}\left(1 - \frac{3}{5}\frac{\varepsilon^2}{\bar{\varepsilon}^2}\cos 2M + \frac{2\delta}{5\bar{\varepsilon}^2}\right)^{-\frac{1}{2}}. \qquad (46)$$

Put then

$$\frac{3}{5}\frac{\varepsilon^2}{\bar{\varepsilon}^2} = \frac{2a}{1+a^2}, \quad 1 + a^2 - 2a\,\frac{5}{3}\frac{\bar{\varepsilon}^2}{\varepsilon^2} = 0, \quad a \doteq \frac{1}{3}$$

$$\frac{1}{\varDelta} = \frac{\sqrt{1+a^2}}{a'\bar{\varepsilon}}\frac{\sqrt{2}}{\sqrt{5}}\frac{1}{\left[1 + a^2 - 2a\left(\cos 2M + \frac{2\delta}{5\bar{\varepsilon}^2}\frac{1+a^2}{2a}\right)\right]^{\frac{1}{2}}} \doteq$$

$$\doteq \frac{2}{3\bar{\varepsilon}a'}(1 + a^2 - 2a\cos 2M)^{-\frac{1}{2}}.$$

It is easy to be seen, that for $\delta = 0$

$$\frac{1}{\varDelta} \doteq \frac{2}{3a'\bar{\varepsilon}}\, e^{\frac{1}{3}\cos 2M + \frac{1}{18}\cos 4M + \frac{1}{81}\cos 6M + \cdots} \qquad (47)$$

When introducing the Besselian functions by means of the definition $i = \sqrt{-1}$, (17)

$$e^{xi\cos\varphi} = J_0(x) - 2J_2(x)\cos 2\varphi + 2J_4(x)\cos 4\varphi \cdots + i[2J_1\cos\varphi - 2J_3(x)\cos 3\varphi + \cdots]$$

$$J_n(x) = \frac{\left(\frac{x}{2}\right)^n}{\underline{|n}}\left\{1 - \frac{\left(\frac{x}{2}\right)^2}{1.\,(n+1)} + \frac{\left(\frac{x}{2}\right)^4}{1.\,2.\,(n+1)(n+2)} - \cdots\right\} \qquad (48)$$

we replace x by $-xi$ thus obtaining

$$e^{x\cos\varphi} = H_0(x) + 2H_1(x)\cos\varphi + 2H_2(x)\cos 2\varphi + 2H_3(x)\cos 3\varphi + \cdots,$$

$$\varphi = 2M,$$

we then put

$$J_{2n}(-xi) = (-1)^n H_{2n}(x), \quad J_{2n+1}(-xi) = (-1)^{n+1} i H_{2n+1}(x),$$

$$H_n(x) = \frac{\left(\frac{x}{2}\right)^n}{\underline{|n}}\left\{1 + \frac{\left(\frac{x}{2}\right)^2}{1.\,(n+1)} + \frac{\left(\frac{x}{2}\right)^4}{1.\,2.\,(n+1)(n+2)} + \cdots\right\} \qquad (49)$$

$$e^{\frac{a^k}{k}\cos\varphi} \cong e^{\frac{a^k}{k}\cos 2kM} \cong e^{\frac{\left(\frac{1}{3}\right)^k}{k}\cos 2kM},$$

and finally

$$= H_0\left(\frac{1}{3}\right) + 2H_1\left(\frac{1}{3}\right)\cos 2M + 2H_2\left\{\frac{\left(\frac{1}{3}\right)^2}{2}\right\}\cos 4M + 2H_3\left\{\frac{\left(\frac{1}{3}\right)^3}{3}\right\}\cos 6M + \cdots$$

$$\Delta^{-1} = \frac{2}{3a'\bar{\varepsilon}}\, e^{\frac{1}{3}\cos 2M + \frac{1}{18}\cos 4M + \frac{1}{81}\cos 6M + \cdots}$$

$$= \frac{2}{3a'\bar{\varepsilon}}\prod_{k=1}^{\infty}\sum_{l=0}^{\infty} 2H_l\left\{\frac{\left(\frac{1}{3}\right)^k}{k}\right\}\cos 2lkM, \quad 2H_0 \cong H_0. \tag{50}$$

We have then to take into account the various powers of the increment (series) δ.

In order to compute the influence of these terms successfully it suffices to recall that the result hithertoo obtained is somewhat a kind of a power series in $\cos 2M$. And indeed we are always able to pass from the multiples of the arguments of the cosinus to the powers, by means of the well-known formulas

$$\begin{aligned} \cos 2\varphi &= 2\cos^2\varphi - 1, \\ \cos 3\varphi &= 4\cos^2\varphi - 3\cos\varphi, \\ \cos 4\varphi &= 8\cos^4\varphi - 8\cos^2\varphi + 1, \\ \cos 5\varphi &= 16\cos^5\varphi - 20\cos^3\varphi + 5\cos\varphi, \\ \cos 6\varphi &= 32\cos^6\varphi - 48\cos^4\varphi + 18\cos^2\varphi - 1. \quad 18). \end{aligned} \tag{51}$$

It is then very easy to insert into these various powers of the cosinus their increments δ, and after multiplying the diverse cosinus factors to repass to the multiples of the angular arguments lM etc.

In this manner we get the final result in the form of a cosinus series with multiple arguments

$$\omega'' = j_1 M + j_2\tilde{\omega} + j_3\tilde{\Omega}.$$

These can always be adjusted to our choice of canonical elements (35)

$$M,\ \tilde{\omega},\ M - M' + \tilde{\Omega},$$

in the form

$$\omega''' = j_1' M + j_2'\tilde{\omega} + j_3'(M - M' + \tilde{\Omega}).$$

Lastly we have to carry out the change of the resulting angular variables by means of the formulae (15) into $M, \bar{\omega}, M - M' + \tilde{\Omega}$ thus obtaining the definite development with the general multiple argument

$$\omega' = i_1 M + i_2 \bar{\omega} + i_3 (M + \tilde{\Omega} - M'). \tag{52 a}$$

Now in the following study we shall be interested chiefly in the transformed indirect Lagrangian part $\frac{\xi x' + \eta y'}{a'^3}$ see (38) p. 23, of the disturbing function, which exclusively contains all secular, critical terms, becoming constant in consequence of the commensurability of the mean movements $n = n'$.

This important part of the development is obtained without any special calculus as a result of multiplying together the well-known Besselian series for the Keplerian elliptic motion.

The main terms result in

$$\begin{aligned} - \frac{\varrho \cos \sigma}{a'^2} = \frac{a}{a'^2} \left(1 - \frac{\varepsilon^2}{2} - \sin^2 \frac{\iota}{2}\right) \cos (M - M' + \bar{\omega} + \tilde{\Omega} - \pi') - \\ - \frac{a}{a'^2} \frac{h}{2} \cos (\bar{\omega} - M + M' - \tilde{\Omega} + \pi') - \frac{a}{a'^2} \frac{h}{2} \cos (3 \bar{\omega} + M - M' + \tilde{\Omega} - \pi') \\ - \frac{3}{2} \frac{a}{a'^2} \varepsilon \cos (\bar{\omega} + \tilde{\Omega} - \pi' - M') + \frac{a}{a'^2} \frac{\varepsilon}{2} \cos (2 M - M' + \bar{\omega} + \tilde{\Omega} - \pi') \\ + \frac{a \varepsilon^2}{8 a'^2} \cos (M + M' + \pi' - \bar{\omega} - \tilde{\Omega}) + \frac{3}{8} \frac{a \varepsilon^2}{a'^2} \cos (3 M - M' + \bar{\omega} + \tilde{\Omega} - \pi') + \\ + \frac{a}{a'^2} \sin^2 \frac{\iota}{2} \cos (M + M' + \bar{\omega} + \pi' - \tilde{\Omega}) \end{aligned} \tag{52}$$

$$h = \left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\right) \sin^2 \frac{\iota}{2}, \quad \tilde{\pi} = \tilde{\omega} + \tilde{\Omega}.$$

SECOND PART.

In the previous first Part we succeeded in obtaining another formulation of the satellite problem unlike any hitherto dealt with. The chief characteristics of the formulation offer two advantages.

Firstly, the disturbing function of the problem appears, developed into a periodic series, proceeding according to multiple arguments $M, \bar{\omega}, \tilde{\Omega}, M'$ composed, as usual, of angular variables.

However, the mean movements n, n' which occur as a rule in two distinct amounts, appear this time reduced to a unique mean movement $n = n'$. This leads necessarily to the commensurability $\frac{n}{n'} = \frac{1}{1}$, n representing the angular velocity of the Moon, n' that of the Earth. Further, the disturbing mass and consequently all the disturbing forces of the problem become diminished to the amount of the very small mass of the Earth.

We have now to pass to the integration of the problem.

With this in view, we first introduce another independent variable

$$t = (1 + \varkappa)(1 + \vartheta)(1 + \eta)\tau,$$

strongly influenced by small parameters, and change the whole aspect of the differential problem into suitable integral equations.

In the following investigation, we shall be interested chiefly in the form of the development of the disturbing function, or rather, more particularly in the critical terms of the development. These are build up by trigonometric expressions, whose arguments degenerate into sums independent of the mean movements n, n' and of the time t. We shall have to deal with the commensurability of the mean movements n of the Moon and of the Earth n' — both of the latter concurring into one and the same amount $n = n'$.

The finding and isolating of the aforesaid critical terms is an extremely simple matter. We have only to start with the form of the disturbing function given at the end of the previous first Part, § 4, and remember that the critical arguments arise merely by superposition of two term factors of the same argument. These are to be found exclusively in the indirect Langrangian part of the development. They turn out to be of zero order in small quantities ε, ι, which is in complete agreement with the well-known classical theory of Laplace and Le Verrier.

Chapter I.

The Restricted Problem of Three Bodies.

Before going any further I must explain the method we shall be using all through the present paper.

For the sake of clearness I limit myself at first to two degrees of freedom. However, it is to be pointed out that this restriction does not lessen the generality of the method in any way; it involves not only the problem restreint treated herewith, but governs even the most general case of the problem of three bodies, in which the Hessian—Jacobi—Poincaré disappears identically. Let us first imagine the problem of H. Poincaré, dealing with the lengthening of the original undisturbed period of movement 19).

Karl Schwarzschild 20) uses the substitution of H. Poincaré successfully in case of the problem restreint, where the time does not appear explicitly in the right hand members of the equations of movement, Unfortunately he fails, or rather neglects a thorough geometrical interpretation of the transformation. And so it happens that even in the simplest case, where the excentricity of the orbit of the disturbing planet becomes distinct from zero (namely in the case of the so-called asteroidal elliptical problem restreint) and the explicit time starts appearing in the developments within the trigonometrical functions, he abandons the whole substitution and not seeing its far-reaching consequences, has recourse to Delaunay transformations and to the neighbouring cases of high number-commensurabilities.

In this respect Schwarzschild 7) AN 147, pp. 289—98 conforms to an erroneous reasoning of Poincaré Méc. cél. I, pp. 89, 90, and indeed Poincaré did not succeed in avoiding this, as he says, difficulty. It was probable that even A. Wilkens an Klose 21) were misled under the influence of this classical paper by Schwarzschild.

Both of the authors named clearly saw the cause of the failure, but were unable to overcome it.

And as the Hessian disappears identically in the general problem, it long remained impossible to penetrate further.

The case of K. Schwarzschild is well illustrated by his paper 20) quoted above or by Charlier 22), Mechanik d. Himmels II, pp. 249—281. Let us be guided by Charlier II, p. 248 using the same notation as l. c.

Charlier starts with the well-known Delaunay elements of the restreint with two degrees of freedom namely

$$\frac{d x_k}{d t} = \frac{\partial F}{\partial y_k}, \quad \frac{d y_{k'}}{d t} = -\frac{\partial F}{\partial x_k}, \quad k = 1, 2, \tag{1}$$

mean anomaly

$$x_1 = \sqrt{a}, \quad y_1 = l,$$
$$x_2 = \sqrt{a(1-e^2)}, \quad y_2 = \pi \tag{2}$$

$$F_0 = \frac{1}{2 x_1^2}, \quad F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots = F_0 + \mu F''. \tag{3}$$

Charlier puts for the constant of Gauss $k = 1$.

This original problem has manifestly a disappearing Hessian

$$\frac{\partial\left(\frac{\partial F_0}{\partial x_1},\ \frac{\partial F_0}{\partial x_2}\right)}{\partial(x_1,\ x_2)} \equiv 0. \tag{4}$$

Now according to the prescription of Poincaré, in this very special case the difficulty is easily overcome by simply introducing a rotating system, moving with the same angular velocity as the disturbing planet.

And indeed the explicite time within the arguments of the trigonometrical functions appears exclusively in the combination $-M'-\pi'+\tilde{\pi}$. So the radical remedy is easily reached by introducing other angular variables

$$\begin{aligned} x_1 &= \sqrt{a}, & y_1 &= l = nt + c, \quad c = M_0 \\ x_2 &= \sqrt{a(1-e^2)}, & y_2 &= \pi - n't - M_0' - \pi' = -n't + g, \quad g = -M_0' - \pi' + \pi. \end{aligned} \tag{5}$$

In order to preserve the canonical form of the equations, we must add to the original F_0 the term $n'x_2$, so that this time

$$F_0'' = \frac{1}{2x_1^2} + n'x_2, \tag{6}$$

but we write F_0 again.

Schwarzschild studies now the case of the small number commensurability of the mean movements n, n'

$$\frac{n}{n'} = \frac{p+q}{p},$$

where p, q signify two integer numbers, relative prim. The undisturbed period of movement in the two starting Keplerian ellipses for $\mu = 0$ (disturbing mass) is given by

$$T_1 = \frac{2\pi(p+q)}{n} = \frac{2\pi p}{n'}. \tag{7}$$

Supposing now $\mu > 0$ he changes the initial constants into

$$\begin{aligned} &\sqrt{a} + \beta_1, \quad nt + c + \gamma_1, \\ &\sqrt{a(1-e^2)} + \beta_2, \quad -n't + g + \gamma_2, \quad t = 0, \end{aligned} \tag{8}$$

passing therewith to the new variables and at the same time prolonging the original period by the the substitution $t = (1+\varkappa)\tau$

$$\tilde{x}_1 = \sqrt{a} + \beta_1 + \varphi_1, \qquad \tilde{y} = n t + c + \gamma_1 + \psi_1,$$
$$\tilde{x}_2 = \sqrt{a(1-e^2)} + \beta_2 + \varphi_2, \quad \tilde{y}_2 = -n' t + g + \gamma_2 + \psi_2, \tag{9}$$

$\varphi_1, \varphi_2, \psi_1, \psi_2$ are unknown integral functions including small deviations of the order μ and namely of $\varkappa$. The period of the disturbed movement will be given by $T = (1+\varkappa)\frac{2\pi p}{n'}$. The integral equations serving for the definition of the aforesaid functions φ, ψ are given by the inversion of the original differential equations

$$\frac{d\varphi_1}{d\tau} = \frac{\partial F}{\partial \psi_1}, \quad \frac{d\psi_1}{d\tau} = -\frac{\partial F}{\partial \varphi_1},$$
$$\frac{d\varphi_2}{d\tau} = \frac{\partial F}{\partial \psi_2}, \quad \frac{d\psi_2}{d\tau} = -\frac{\partial F}{\partial \varphi_2}, \tag{10}$$

The meanings of $\varphi_1, \varphi_2, \psi_1, \psi_2$ can be made completely clear by summing up of the whole mathematical process of Poincaré: He considers a Keplerian undisturbed periodic motion in two ellipses, gives to all the starting constants as well as to the time t a displacement of the order μ (disturbing parameter, mass), and investigates the new disturbed movement, asking under what conditions it could remain periodic, even with a prolonged period.

Evidently the disturbed, displaced movement remains periodic in case the very complicated integral functions $\varphi_1 = \varphi_2 = \psi_1 = \psi_2 = 0$ disappear after the lapse of the whole prolonged period.

The meaning of these functions clearly appears to be "the functional deviations of the displaced coordinates, during and towards the close of the new prolonged period".

In this manner we are able to write down immediately the integral equations (inverted differential equations) as follows:

$$\frac{\Phi_1}{\mu} = (1+\varkappa)\int_0^{T_1} \frac{\partial F''}{\partial y_1} d\tau, \quad \frac{\Phi_2}{\mu} = (1+\varkappa)\int_0^{T_1} \frac{\partial F''}{\partial y_2} d\tau,$$
$$\Psi_1 = +\varkappa n T_1 + (1+\varkappa)\int_0^{T_1} \frac{d\tau}{(x_0+\beta_1+\psi_1)^3} - \mu(1+\varkappa)\int_0^{T_1} \frac{\partial F''}{\partial x_1} d\tau, \tag{11}$$
$$\Psi_2 = -\varkappa n' T_1 - \mu(1+\varkappa)\int_0^{T_1} \frac{\partial F''}{\partial x_2} d\tau.$$

Φ_1, Φ_2, Ψ_1, Ψ_2 signify the values, that the functions φ_1, φ_2, ψ_1 ψ_2 assume towards the close of the total period $T = (1 + \varkappa) T_1$ 20).

These very complicated integral equations can easily be changed into simple Taylor series. And indeed the whole method of solving them consists in a clever application of the existence theorem of implicit functions (Cauchy, Weierstrass).

However before performing this simple task, we must explain the meaning and geometrical significance of the parameter $\varkappa$.

We started putting (9)

$$\tilde{y}_2 = -n'(1 + \varkappa)\tau + g + \gamma_2 + \cdots$$
$$\tilde{y}_2 = -n'\tau + g + \gamma_2 + \psi_2,$$

including the small term $\varkappa n$ into ψ_2 or rather destroying it with $\psi_2 = 0$, before it could give rise to the explicit time $n \varkappa \tau$.

Let us suppose now, as will turn out later on, that we succeed in annulling the functional deviations (11) $\varphi_1 = \varphi_2 = \psi_1 = \psi_2 = 0$. Then we have only to return to the original signification of the variable y_2, by subtracting $-n' t - M'_0 - \pi' + \pi$

$$\pi - n' t + n' t = \pi - \frac{n' t}{1 + \varkappa} + n' t = \pi + \frac{\varkappa n' t}{1 + \varkappa}$$
$$t = (1 + \varkappa)\tau, \quad \tau = \frac{t}{1 + \varkappa}, \tag{12}$$

in order to see that the meaning of $-\frac{\varkappa n'}{1 + \varkappa}$ indicates the "birth" of a new mean movement or, as one says in Astronomy, of a new secular movement.

Now let us carry out the quadratures of the complicated integral equations. For this purpose we recall that the Hamiltonian function F appears in the form of a trigonometric development according to multiples of the two angular variables

$$F = \sum A \cos (i_1 y_1 + i_2 y_2) \tag{13}$$

thereby the coefficients A constitute the functions of the scalar variables x_1, x_2.

Now the integrals of our integral equations can easily be carried out term by term if we distinguish only three categories of arguments:

1. Simple short-periodic terms of the argument

$$i_1 y_1 + i_2 y_2, \quad y_1 = n t + c, \quad c = M_0, \quad y_2 = g - n' t, \quad g = -M'_0 - \pi' + \pi.$$

i_1, i_2 representing whole numbers both positive and negative.

The quadrature changes only every particular $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ into $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}$ and gives the same result in the upper as well as in the lower limit. These two amounts being of opposite signs cancel mutually and give zero.

So these short-periodic terms cannot contribute in any way to the final value of the coefficients in the implicit functions in question.

2. Poincaré further selects from the development of the disturbing function the so-called critical terms, corresponding to the above supposed commensurability of the mean movements $\frac{n}{n'} = \frac{p+q}{p}$, p, q integer numbers for which

$$i_1 = s\,p, \quad i_2 = s\,(p+q) \tag{14}$$

s, integer number positive or negative.

In this case the argument of the particular term of the disturbing function gives

$$\begin{aligned} i_1 y_1 + i_2 y_2 = i_1 (n t + c) - i_2 n' t + i_2 g = \frac{n'}{p} \left[(p+q)\, p\, s - p\,(p+q)\, s \right] (1+\varkappa)\, \tau + \\ + s\,p\,c + s\,(p+q)\, g = + s\,\{p\,c + (p+q)\, g\}. \end{aligned} \tag{15}$$

So it happens that the coefficient of the time τ disappears.

These terms are to be integrated as a constant, independent of τ.

The result of the quadrature is manifestly distinct from zero, at least in the upper limit, giving the value

$$\frac{1}{T_1} \int\limits_0^{T_1} d\tau \text{ const} = \text{const}.$$

These critical terms are extremely important, as they guarantee the existence of the coefficients in the development of our implicit functions.

3. Secular terms, well known from the classical theory, in which $i_1 = i_2 = 0$.

These terms equally yield a result, distinct from zero, being integrated as a constant of the previous case (no. 2).

Their existence gives a contribution to the desired coefficients of the implicit functions as in case (no. 2).

When putting for the sake of brevity for the average time value

$$\frac{1}{T}\int_0^T F\,d\tau = [F] = \sum A_{i_1 i_2} \cos(i_1 c + i_2 g) = \sum_{s=0}^{\infty} A_{sp,\,s(p+q)} \cos s\{pc + (p+q)g\}, \tag{16}$$

summation being taken over the integer numbers s, (14)

$$i_1(p+q) - i_2 p = 0$$

we finally obtain the following four conditions admitting the possibility of the periodicity of the whole movement.

$$\begin{aligned}
\Phi_1 &= 0 = \frac{\partial [F]}{\partial c} = -p \sum_{s=0}^{\infty} s A_{sp,\,s(p+q)} \sin s\{pc + (p+q)g\},\\
\Phi_2 &= 0 = \frac{\partial [F]}{\partial g} = -(p+q) \sum_{s=0}^{\infty} s A_{sp,\,s(p+q)} \sin s\{pc + (p+q)g\},\\
\Psi_1 &= 0 = \varkappa \frac{\partial F_0}{\partial x_1} + \frac{\partial^2 F_0}{\partial x_1^2}\beta_1 + \frac{\partial^2 F_0}{\partial x_1 \partial x_2}\beta_2 + \mu \frac{\partial [F_1]}{\partial x_1} + \cdots\\
\Psi_2 &= 0 = \varkappa \frac{\partial F_0}{\partial x_2} + \frac{\partial^2 F_0}{\partial x_1 \partial x_2}\beta_1 + \frac{\partial^2 F_0}{\partial x_2^2}\beta_2 + \mu \frac{\partial [F_1]}{\partial x_2} + \cdots\\
F_0 &= \frac{1}{2x_1^2} + n' x_2.
\end{aligned} \tag{17}$$

And indeed only by the disappearance of the functional displacements as defined by (11), the same amounts for the coordinates and their velocities at the end and at the beginning of the movement can be secured. The last four conditions in the form of power series can define five of the implicit functions $\varkappa, \beta_1, \beta_2, \gamma_1, \gamma_2$. If their determinant is distinct from zero, then this important fact includes the possibility of developing the unknown function according to the powers of the small parameter μ.

Now the aforesaid four conditions are not independent. There exists in our case (of the problem restreint) the Jacobi-Integral of Energy

$$F - C = 0. \tag{18}$$

From this algebraic equation we can immediately draw the important conclusion, that one of our conditions, for example $\Phi_2 = 0$, will always be a consequence of the other three conditions

$$\Phi_1 = \Psi_1 = \Psi_2 = 0.$$

And indeed this follows from the integral of energy written in differential form

$$F(\Phi_i, \Psi_i) - F(0, 0) = C - C \equiv 0$$

$$\frac{\partial F}{\partial x_1} \Phi_1 + \frac{\partial F}{\partial x_2} \Phi_2 + \frac{\partial F}{\partial y_1} \Psi_1 + \frac{\partial F}{\partial y_2} \Psi_2 + \cdots = 0. \tag{19}$$

The first factor of the second term $\frac{\partial F}{\partial x_2} = n' \gtrless 0$ is without doubt distinct from zero. If then $\Phi_1 = \Psi_1 = \Psi_2 = 0$ it is clear that Φ_2 must equally disappear. Summing up, we can show the existence of the five implicit functions $\beta_1, \beta_2, \gamma_1, \gamma_2, \varkappa$ as soon as we prove that the determinant of the coefficients of the first powers of the unknown quantities remains distinct from zero.

Now the equation $\Phi_1 = 0$ can be replaced by $\frac{\Phi_1}{\mu} = 0$ because the development of the lefthand side reduces to μF_1, see (3), (6),

$$F = \frac{1}{2 x_1^2} + n' x_2 + \mu F_1 + \cdots = F_0' + \mu F_1 + \cdots$$

and

$$\frac{\Phi_1}{\mu} = \frac{\partial [F_1]}{\partial y_1} + \frac{\partial^2 [F_1]}{\partial x_1 \partial y_1} \beta_1 + \frac{\partial^2 [F_1]}{\partial x_2 \partial y_1} \beta_2 + \frac{\partial^2 [F_1]}{\partial y_1^2} \gamma_1 + \frac{\partial^2 [F_1]}{\partial y_1 \partial y_2} \gamma_2 + \cdots \tag{20}$$

$$\frac{\partial F_0}{\partial y_1} = 0,$$

and we gather from this that instead of $\Phi_1 = 0$ it appears sufficient to satisfy (20). Consequently we can leave out the factor μ. The other two equations for $\Psi_1 = 0$, and $\Psi_2 = 0$ can be written as follows

$$\begin{aligned} \Psi_1 &= -\frac{\varkappa}{x_1^3} + \frac{3 \beta_1}{x_1^4} + \mu \frac{\partial [F_1]}{\partial x_1} + \cdots = 0, \\ \Psi_2 &= n' \varkappa + \mu \frac{\partial [F_1]}{\partial x_2} + \cdots = 0. \end{aligned} \tag{21}$$

And indeed we easily find that

$$\begin{aligned} &\frac{\partial F_0}{\partial x_1} = -\frac{1}{x_1^3}, \quad \frac{\partial F_0}{\partial x_2} = n', \quad \frac{\partial^2 F_0}{\partial x_1^2} = \frac{3}{x_1^4}, \\ &\frac{\partial^2 F_0}{\partial x_2^2} = \frac{\partial^2 F_0}{\partial x_1 \partial x_2} = 0. \end{aligned} \tag{22}$$

Now our aim is to obtain conditions defining the implicit functions of μ namely $\varkappa$, β_1, β_2, γ_1, γ_2 which would dissappear with μ.

According to the theory of implicit functions developed by Cauchy and Weierstrass, this is possible only when the defining functions do not involve the term independent of μ. We gather from this that the term of the starting equations not containing $\varkappa$, β_1, β_2, γ_1, γ_2 as factor must disappear, and so we find the well-known conditions of symmetric conjunction and opposition.

And indeed the expression $\frac{\partial [F_1]}{\partial y_1} = 0$ being constituted exclusively by a series of sinus of the critical terms

$$A_{sp,(p+q)s} \sin s \{pc + (p+q) g\}, \quad c = M_0, \quad g = -M_0' + \pi - \pi' \tag{16}$$

we can satisfy our condition by butting

$$\begin{aligned} c &= 0, & g &= 0, \\ c &= 180^\circ, & g &= 0, \\ c &= 0, & g &= 180^\circ, \\ c &= 180^\circ, & g &= 180^\circ. \end{aligned} \tag{23}$$

We have three independent equations containing five unknown quantities $\varkappa$, β_1, β_2, γ_1, γ_2. Two of them can be freely disposed of provided they remain within the limits of convergence of the development of our disturbing function. Schwarzschild simply chooses $\beta_2 = 0$ with the view to obtaining new orbits with movable perihelion $\varkappa \gtrless 0$, and finds three equations for the determination of $\varkappa$, β_1, γ_1, $\gamma_2 = 0$

$$\begin{aligned} \Psi_1 &= 0 = -\frac{\varkappa}{x_1^3} + \frac{3\beta_1}{x_1^4} + \mu \frac{\partial [F_1]}{\partial x_1} + \cdots \\ \Psi_2 &= 0 = n'\varkappa + \mu \frac{\partial [F_1]}{\partial x_2} + \cdots \\ \frac{\Phi_1}{\mu} &= 0 = \frac{\partial^2 [F_1]}{\partial x_1 \partial y_1} \beta_1 + \frac{\partial^2 [F_1]}{\partial x_2 \partial y_1} \beta_2 + \frac{\partial^2 [F_1]}{\partial y_1^2} \gamma_1 + \frac{\partial^2 [F_1]}{\partial y_1 \partial y_2} \gamma_2 + \cdots \end{aligned} \tag{24}$$

The determinant, namely the Hessian—Jacobi—Poincaré, is manifestly distinct from zero

$$\begin{gathered} \Delta = \Delta_1 \cdot \Delta_2, \quad \Delta_1 = \frac{\partial \left(F_0, \frac{\partial F_0}{\partial x_1}\right)}{\partial (x_1, x_2)} = -\frac{3n'}{x_1^4} \gtrless 0, \\ \frac{\partial^2 [F_1]}{\partial x_1 \partial y_1} = \frac{\partial^2 [F_1]}{\partial x_2 \partial y_1} = 0, \quad \Delta_2 = \frac{\partial^2 [F_1]}{\partial y_1^2} \gtrless 0. \end{gathered} \tag{25}$$

In this manner the existence of new solutions with rotating perihelion $\varkappa$ is proved. But the way chosen by Schwarzschild is far from unique. I mention only that we can just as easily secure a Jacobian distinct from zero simply by putting

$$\varkappa = -\frac{\mu}{n'}\frac{\partial [F_1]}{\partial x_2}$$

instead of $\beta_2 = 0$, F_1 proving to be a function of one free parameter e.

And indeed our secular part of the function $[F]$ fixed above involves solely the coefficients $A_{sp,s(p+q)}$ (see (14), (17)), these latter being built up exclusively of the scalar Keplerian elements of the starting ellipses a, e, a', $e' = 0$.

It is to be noticed that only the critical terms of the second and third categories mentioned above guarantee the existence of a real $\varkappa$, as they contain exclusively the cosinus series distinct from zero term by term. The starting development $[F]$ consists of a cosinus series which leads, after two differentiations of the arguments, again only to a cosinus series. And according to our choice of a symmetrical conjunction or opposition, all the cosinuses are reduced to 1, namely to coefficients build up solely by scalar Keplerian elements a, e.

The same remark concerns the coefficients $\dfrac{\partial^2 [F_1]}{\partial y_i \partial y_k}$ in (24)

When considering the three new equations, we get

$$\Psi_1 = 0, \quad \frac{\Psi_2}{\mu} = 0, \quad \frac{\Phi_1}{\mu} = 0 \tag{26}$$

clearly showing a determinant distinct from zero

$$\begin{aligned}
\Psi_1 = 0 &= \frac{3\beta_1}{x_1^4} + \mu\frac{\partial [F_1]}{\partial x_1} + \frac{\mu}{n'}\frac{\partial [F_1]}{\partial x_1}\frac{1}{x_1^3} + \cdots, \\
\frac{\Psi_2}{\mu} = 0 &= \frac{\partial^2 [F_1]}{\partial x_1 \partial x_2}\beta_1 + \frac{\partial^2 [F_1]}{\partial x_2^2}\beta_2 + \mu\frac{\partial [F_2]}{\partial x_2} + \cdots, \\
\frac{\Phi_1}{\mu} = 0 &= \frac{\partial^2 [F_1]}{\partial x_1 \partial y_1}\beta_1 + \frac{\partial^2 [F_1]}{\partial x_2 \partial y_1}\beta_2 + \frac{\partial^2 [F_1]}{\partial y_1^2}\gamma_1 + \frac{\partial^2 [F_1]}{\partial y_1 \partial y_2}\gamma_2 + \cdots + \\
&\qquad + \frac{\partial [F_1]}{\partial y_1} + \mu\frac{\partial [F_2]}{\partial y_1} + \cdots
\end{aligned} \tag{27}$$

where $\dfrac{\partial [F_1]}{\partial y_1} = 0$ as previously, while $\dfrac{\partial^2 [F_1]}{\partial x_1 \partial y_1} = \dfrac{\partial^2 [F_1]}{\partial x_2 \partial y_1} = 0$, as mere sinus series of critical terms for the symmetric conjunction or opposition (23). The corresponding solutions coincide with those of Schwarzschild, but this is not always the case in more general problems:

And indeed, unfortunately, the general equations of dynamics bear the following character: F has the form (see Poincaré, Méthodes nouvelles de la Méc. cél. I, p. 133. 1))

$$F = F_0 + F_1 \mu + F_2 \mu^2 + \cdots$$
$$F_0 = f_1(x_1) + f_2(x_2) + f_3(x_3) + f_4(x_4), \tag{28}$$

while at the same time $f_2(x_2) = f_4(x_4) \equiv 0$. Now these parts furnish the undisturbed mean movements

$$n_1 = -\frac{\partial F_0}{\partial x_1}, \quad n_3 = -\frac{\partial F_0}{\partial x_3}, \quad n_2 = n_4 = \frac{\partial F_0}{\partial x_2} = \frac{\partial F_0}{\partial x_4} \equiv 0 \tag{29}$$

which circumstance causes the Hessian—Jacobi—Poincaré to disappear identically

$$\frac{\partial\left(\frac{\partial F_0}{\partial x_1}, \frac{\partial F_0}{\partial x_2}, \frac{\partial F_0}{\partial x_3}, \frac{\partial F_0}{\partial x_4}\right)}{\partial(x_1, x_2, x_3, x_4)} = \frac{\partial\left(\frac{\partial F_0}{\partial x_1}, 0, \frac{\partial F_0}{\partial x_3}, 0\right)}{\partial(x_1, x_2, x_3, x_4)} \equiv 0. \tag{30}$$

Still we are always endeavouring to get this determinant distinct from zero, as it happens to coincide with our determinant as given above and proving the existence of the implicit functions, and admits the possibility and reality of a periodic solution.

Chapter II.

The General Problem of Three Bodies in a Plane.

The secular solutions treated in this II. chapter — intended for the use of Planetary Orbits — bear the following description:

Let us imagine the problem of three bodies: Two planets of relatively small mass revolving about a big central body in the same plane. Let us suppose the undisturbed mean movements of the two planets to be commensurable. Then it is shown below:

In the neighbourhood of every type of commensurability of the mean movements there always exists a multiple infinity of secular solutions. The orbits of both planets in question are given approximately by two rotating ellipses.

The rotational speed of one of the two apsides can outrun possibly the other apside thus far, that during one period only of the whole movement, this apside catches up with the other — after having described one, or more full angles of 360° degrees.

The synodical period of the perihelia is not confined to the accurate and at the start $\mu = 0$ prescribed length of the original undisturbed period of the aforesaid ellipses, but can differ from it even within the scope of the size of the small mass μ.

To facilitate explanation let us take up the case of the general problem of three bodies in a plane, and start with the equations of Charlier, volume II, p. 216, using the same notation only slightly modified, and to follow up the conclusions of my previous paper Publ. de la faculté des sciences de l'Université Charles No. 15, 1923, Nouvelles classes des solutions séculaires du problème général des trois corps 4).

$$\frac{d x_i}{d t} = \frac{\partial F}{\partial y_i}, \quad \frac{d y_i}{d t} = -\frac{\partial F}{\partial x_i}, \quad i = 1, 2, 3,$$

$$\begin{aligned} x_1 &= \beta_1'' \sqrt{a}, & y_1 &= M + \pi - \pi', \quad \beta = \mu \beta_1'', \\ x_2 &= \beta_1'' \sqrt{a}\,(1 - \sqrt{1 - e^2}), & y_2 &= -\pi + \pi', \quad \beta' = \mu \beta_2'', \\ x_3 &= \beta_2'' \sqrt{a'}, & y_3 &= M' \end{aligned}$$

$$F_0 = \frac{k^2 \alpha_1'' m_c}{2a} + \frac{k^2 \alpha_2'' m_c}{2a'}, \quad m_1 = m = \alpha_1'' \mu = m_b, \quad m_2 = m' = \alpha_2'' \mu = m_a \tag{31}$$

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots, \quad F_1 = k^2 \left(\frac{1}{\Delta} - \frac{r \cos \sigma}{r'^2}\right),$$

$$-\frac{\partial F_0}{\partial x_1} = n_1 = n = \frac{\beta}{\varrho} a^{-\frac{3}{2}}, \quad -\frac{\partial F_0}{\partial x_3} = n_3 = n' = \frac{\beta'}{\varrho'} a'^{-\frac{3}{2}},$$

$$\varrho = \frac{m_b m_c}{m_b + m_c}, \quad \beta = \frac{k m_b m_c}{\sqrt{m_b + m_c}}, \quad \varrho' = \frac{m_a (m_b + m_c)}{m_a + m_b + m_c}, \quad \beta' = \frac{k m_a \sqrt{m_c (m_b + m_c)}}{\sqrt{m_a + m_b + m_c}}.$$

Let us study the case of the commensurability of the mean movements (p q relative prime numbers)

$$\frac{n}{n'} = \frac{p + q}{p}, \tag{32}$$

the undisturbed period of the movement in two ellipses $\mu = 0$, is

$$T_0 = \frac{2\pi p}{n_0'} = \frac{2\pi (p + q)}{n_0}. \tag{33}$$

To help in clearing up the formulation of the problem let us recall that, in the right-hand sides of the equations of motion, the Hamiltonian function F and especially the part $F_1 \mu + F_2 \mu^2 + \cdots = F''$ appears to be a suitably developed periodical series. The general argument of the trigonometrical cos terms found by Laplace and Le Verrier has the form $\omega'' = i_1 y_1 + i_2 y_2 + i_3 y_3$ where i_k represent integer numbers both positive and negative.

The Hessian of the problem disappears identically, as the coefficient of t in y_2 is missing. In order to avoid this inconvenience of which Poincaré complains in vain Méth. nouv. I. p. 68, p. 119, p. 133, II. p. 47, I first introduce other angular variables by putting

$$t = (1 + \varkappa)(1 + \vartheta)\tau \tag{34}$$

$$\begin{aligned} y_1' &= y_1 + \frac{k_1 n_0 t + l_1 n_0' t}{1+\vartheta} = y_1 + n_1'' t, \qquad & n_1'' &= \frac{k_1 n_0 + l_1 n_0'}{1+\vartheta}, \\ y_2' &= y_2 + \frac{k_2 n_0 t + l_2 n_0' t}{1+\vartheta} = y_2 + n_2'' t, \qquad & n_2'' &= \frac{k_2 n_0 + l_2 n_0'}{1+\vartheta}, \\ y_3' &= y_3 + \frac{k_3 n_0 t + l_3 n_0' t}{1+\vartheta} = y_3 + n_3'' t, \qquad & n_3'' &= \frac{k_3 n_0 + l_3 n_0'}{1+\vartheta}. \end{aligned} \tag{35}$$

$k_1, l_1, k_2, l_2, k_3, l_3$ representing quite arbitrary integer numbers both positive and negative, thereby supposing that we are dealing with the commensurability $\frac{n_0}{n_0'} = \frac{p+q}{p}$, and n_0, n_0', represent fixed constants equal to the osculating functions $n = n_0$, $n' = n_0'$. This only means urging the appearance of explicit time within the trigonometrical arguments of the development of F as we have to replace every y_k by

$$y_i' - \frac{k_i n_0 t + l_i n_0' t}{1+\vartheta},$$

$$\begin{aligned} y_1 &= y_1' - \frac{k_1 n_0 t + l_1 n_0' t}{1+\vartheta} = y_1' - k_1 n_0 (1+\varkappa)\tau - l_1 n_0'(1+\varkappa)\tau = y_1' - n_1''(1+\varkappa)(1+\vartheta)\tau, \\ y_2 &= y_2' - \frac{k_2 n_0 t + l_2 n_0' t}{1+\vartheta} = y_2' - k_2 n_0 (1+\varkappa)\tau - l_2 n_0'(1+\varkappa)\tau = y_2' - n_2''(1+\varkappa)(1+\vartheta)\tau, \\ y_3 &= y_3' - \frac{k_3 n_0 t + k_3 n_0' t}{1+\vartheta} = y_3' - k_3 n_0 (1+\varkappa)\tau - l_3 n_0'(1+\varkappa)\tau = y_3' - n_3''(1+\varkappa)(1+\vartheta)\tau \end{aligned} \tag{36}$$

$$n_1'' = \frac{k_1 n_0 + l_1 n_0'}{1+\vartheta}, \quad n_2'' = \frac{k_2 n_0 + l_2 n_0'}{1+\vartheta}, \quad n_3'' = \frac{k_3 n_0 + l_3 n_0}{1+\vartheta}.$$

Of course we have then to take special care that this explicit time remains compatible with the periodicity of the resulting disturbed periodic solution (see p. 47).

In order to preserve the original canonical form of the equations of motion the three complementary terms must be added to the function F_0

$$-\frac{k_1 n_0 + l_1 n_0'}{1+\vartheta} x_1 - \frac{k_2 n_0 + l_2 n_0'}{1+\vartheta} x_2 - \frac{k_3 n_0 + l_3 n_0'}{1+\vartheta} x_3. \tag{37}$$

We shall put then $t = (1+\varkappa)(1+\vartheta)\tau$ and try to annul the expressions of the functional increments $\varphi_1, \varphi_2, \varphi_3, \psi_1, \varphi_2, \psi_3$ thereby starting with the values

$$\begin{aligned}
x_1 &= \beta_1'' \sqrt{a_0} + \beta_1 + \varphi_1, & y_1' &= M_0 + \pi - \pi' + n\tau + (k_1 n_0 + l_1 n_0')\tau + \\
&& &\quad + \gamma_1 + \psi_1 \\
x_2 &= \beta_1'' \sqrt{a_0}\,(1 - \sqrt{1-e^2}) + \beta_2 + \varphi_2, & y_2' &= -\pi + \pi' + (k_2 n_0 + l_2 n_0')\tau + \gamma_2 + \psi_2, \\
x_3 &= \beta_2'' \sqrt{a_0'} + \beta_3 + \varphi_3, & y_3' &= M_0' + n'\tau + (k_3 n_0 + l_3 n_0')\tau + \gamma_3 + \psi_3, \\
&& M &= M_0 + nt.
\end{aligned} \tag{38}$$

It is to be expressly noted that in the argumental part of y_i' namely ψ_i are contained the terms $\varkappa(n_0 k_i + n_0' l_i)$, these being intended to distroy by annulling the final functional deviations see (42) and (43). On the contrary in the general trigonometrical argument of the development of F'' namely,

$$k^2 A \cos(i_1 y_1 + i_2 y_2 + i_3 y_3)$$

after the transformation (36) and putting $t = (1+\varkappa)(1+\vartheta)\tau$,

$$k^2 A \cos[i_1 y_1' + i_2 y_2' + i_3 y_3' - (i_1 n_1'' + i_2 n_2'' + i_3 n_3'')t] \tag{39}$$

the two terms

$$(1+\vartheta)\{(i_1 n_1'' + i_2 n_2'' + i_3 n_3'')\tau - (i_1 n_1'' + i_2 n_2'' + i_3 n_3'')\tau\}$$

cancel out mutually, whereas $-\varkappa(n_1'' i_1 + n_2'' i_2 + n_3'' i_2)(1+\vartheta)\tau$ remains as the newly appearing explicit time.

Two planets, defined by the undisturbed elements $a, e, M, \pi, a', e', M', \pi'$, describe two strictly elliptical Keplerian starting paths for $\mu = 0$.

Let us imagine the same movement many times (r) repeated, so that

$$T_1 = \frac{2\pi p r}{n'} = \frac{2\pi(p+q)r}{n}, \tag{40}$$

p, q being integral numbers.

When displacing the original positions by the small amounts β_i, γ_i to be determined, we are trying at the same time to displace, namely to prolong, the period by the quantities $\varkappa, \vartheta$ in putting $t = (1+\varkappa)(1+\vartheta)\tau$, so that the final disturbed period goes over into

$$T = (1+\varkappa)(1+\vartheta)T_1. \tag{41}$$

It is now our chief purpose to destroy the complicated functions φ_i, ψ_i, expressing the functional displacement (final deviation), thereby defining the suitable starting amounts of the implicit functions $x_i(\beta_i)$, $y_i(\gamma_i)$.

The differential equations for φ_i, ψ_i, as well as their inversion in form of integral equations are easily obtained as follows

$$\psi_1 = 0 = -\beta_1 \int_0^{T_1} \frac{\partial^2 F_0}{\partial x_1^2} d\tau + n\varkappa T_1 + n\vartheta T_1 + n\varkappa\vartheta T_1 + (k_1 n_0 + l_1 n_0')\varkappa T_1 - \mu \int_0^{T_1} \frac{\partial F_1}{\partial x_1} d\tau - \cdots$$

$$\psi_2 = 0 = + (k_2 n_0 + l_2 n_0')\varkappa T_1 - \mu \int_0^{T_1} \frac{\partial F_1}{\partial x_2} d\tau - \cdots$$

$$\psi_3 = 0 = -\beta_3 \int_0^{T_1} \frac{\partial^2 F_0}{\partial x_3^2} d\tau + n'\varkappa T_1 + n'\vartheta T_1 + n'\varkappa\vartheta T_1 + (k_3 n_0 + l_3 n_0')\varkappa T_1 - \mu \int_0^{T_1} \frac{\partial F_1}{\partial x_3} d\tau - \cdots \tag{42}$$

$$\frac{\varphi_1}{\mu} = 0 = \beta_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_1 \partial y_1} d\tau + \beta_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_2 \partial y_1} d\tau + \beta_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_3 \partial y_1} d\tau + \gamma_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1^2} d\tau + \gamma_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1 \partial y_2} d\tau + \gamma_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1 \partial y_3} d\tau + \int_0^{T_1} \frac{\partial F_1}{\partial y_1} d\tau + \mu \int_0^{T_1} \frac{\partial F_2}{\partial y_1} d\tau + \cdots$$

$$\frac{\varphi_2}{\mu} = 0 = \beta_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_1 \partial y_2} d\tau + \beta_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_2 \partial y_2} d\tau + \beta_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_3 \partial y_2} d\tau + \gamma_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1 \partial y_2} d\tau + \gamma_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_2^2} d\tau + \gamma_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_2 \partial y_3} d\tau + \int_0^{T_1} \frac{\partial F_1}{\partial y_2} d\tau + \mu \int_0^{T_1} \frac{\partial F_2}{\partial y_2} d\tau + \cdots \tag{43}$$

$$\frac{\varphi_3}{\mu} = 0 = \beta_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_1 \partial y_3} d\tau + \beta_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_2 \partial y_3} d\tau + \beta_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_3 \partial y_3} d\tau + \gamma_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1 \partial y_3} d\tau + \gamma_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_2 \partial y_3} d\tau + \gamma_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_3^2} d\tau + \int_0^{T_1} \frac{\partial F_1}{\partial y_3} d\tau + \mu \int_0^{T_1} \frac{\partial F_2}{\partial y_3} d\tau + \cdots$$

As a consequence of the well-known Jacobian integral of energy, the fourth and sixth of our equations are not independent of each other, and indeed it is easy to show that $p\varphi_3 - (p+q)\varphi_1 = 0$, because the right hand sides of (43) are reduced merely to the critical terms of argument ζ as we shall see hereafter pp. 46—54 ad 1, 2, 3, 4, 5 and then it is clear that

$$p \frac{\partial^2 [F_1]}{\partial y_1 \partial y_k} = (p+q) \frac{\partial^2 [F_1]}{\partial y_3 \partial y_k}. \tag{44}$$

Just as in the previous asteriodal case all terms in $\varphi_i = 0$ independent of the small increments β_i, γ_i must disappear.

Thus we have the three conditions

$$\frac{\partial [F_1]}{\partial y_i} = 0, \qquad i = 1, 2, 3. \tag{45}$$

thereby putting for the sake of brevity

$$[F_1] = \frac{1}{T_1} \int_0^{T_1} F_1 \, d\tau.$$

Now the left hand sides are formed by series of the sinus of critical terms such as (see hereafter ad 2. p. 48)

$$\begin{aligned} k^2 A_{-sp,\,-s(p+q),\,s(p+q)} \sin s\{(p+q)M' - pM - (p+q)(\pi - \pi')\} = \\ = k^2 A \sin s\{-qy_2 - py_1 + (p+q)y_3\} \end{aligned} \tag{46}$$

and these can best disappear term by term if we accept:

$$M_0', M_0, \pi - \pi' = 0,\ 180^\circ \tag{47}$$

which conditions are interpreted as the starting symmetrical conjunction or opposition of both planets.

The development of the disturbing Hamiltonian function is constituted by cosinus series of the structure $F = k^2 \sum A \cos(i_1 y_1 + i_2 y_2 + i_3 y_3)$,$A$ containing solely the scalar elements x_k. Consequently all expressions including odd numbers of derivations in the angular elements y_k such as for instance $\frac{\partial^2 [F_1]}{\partial y_k \partial x_k}$ will be composed of sinus series, whose arguments vanish owing to the supposed symmetrical conjunction and opposition.

Thus it immediately appears clear that our integral equations when putting

$$[F_1] = \frac{1}{T_1} \int_0^{T_1} F_1, d\tau, \text{ etc.,}$$

will be reduced to the following form:

$$n\varkappa + n\vartheta + (k_1 n_0 + l_1 n_0')\varkappa - \beta_1 \frac{\partial^2 F_0}{\partial x_1^2} + \cdots \qquad - \mu \frac{\partial [F_1]}{\partial x_1} + n\varkappa\vartheta + \cdots = 0.$$

$$+ (k_2 n_0 + l_2 n_0')\varkappa \qquad - \mu \frac{\partial [F_1]}{\partial x_2} + \cdots \qquad = 0.$$

$$n'\varkappa + n'\vartheta + (k_3 n_0 + l_3 n_0')\varkappa \qquad - \beta_3 \frac{\partial^2 F_0}{\partial x_3^2} - \mu \frac{\partial [F_1]}{\partial x_3} + n'\varkappa\vartheta + \cdots = 0. \quad (48)$$

$$\gamma_1 \frac{\partial^2 [F_1]}{\partial y_1^2} + \gamma_2 \frac{\partial^2 [F_1]}{\partial y_1 \partial y_2} + \gamma_3 \frac{\partial^2 [F_1]}{\partial y_1 \partial y_3} + \frac{\partial [F_2]}{\partial y_1}\mu + \cdots = 0.$$

$$\gamma_1 \frac{\partial^2 [F_1]}{\partial y_1 \partial y_2} + \gamma_2 \frac{\partial^2 [F_1]}{\partial y_2^2} + \gamma_3 \frac{\partial^2 [F_1]}{\partial y_2 \partial y_3} + \frac{\partial [F_2]}{\partial y_2}\mu + \cdots = 0.$$

$$n = -\frac{\partial F_0}{\partial x_1}, \quad n' = -\frac{\partial F_0}{\partial x_3}.$$

The coefficients of these equations are built up by critical terms, which just provide the necessary elements of the solving determinant Hessian-Jacobi-Poincaré.

Let us examine the new mean motions holding true throughout the resulting periodical (secular) movement. These are easily found by mere analysis of the infinitesimal substitution in question.

As we have replaced the original time t by

$$t = (1 + \varkappa)(1 + \vartheta)\tau \text{ and } y_i \text{ by } y_i' = y_i + \frac{n_0 k_i + n_0' l_i}{1 + \vartheta} t \quad (36)$$

we have to carry out the quadratures of the integral equations in such a way that the new variables y_i' may yield, at the beginning and at the close of the period, deviations which disappear entirely, save for the finite amount of the full angles $2\pi = 360°$.

Every mean movement, even one produced artificially through the introduction of the new dependent (primed) variables y_i', will be preserved by our suitable choice of the starting conditions for the periodic or secular solutions.

In the case fixed above our integral equations give the result

$$y_i' = M_0 + \pi - \pi' + n\tau + (n_0 k_1 + n_0' l_1)\tau + \gamma_1 = M_0 + \pi - \pi' + \frac{n + k_1 n_0 + l_1 n_0'}{(1+\varkappa)(1+\vartheta)} t + \gamma_1 + (\psi_1 = 0),$$

$$y_2' = -\pi + \pi' + (n_0 k_2 + n_0' l_2)\tau + \gamma_2 = -\pi + \pi' + \frac{k_2 n_0 + l_2 n_0'}{(1+\varkappa)(1+\vartheta)} t + \gamma_2 + (\psi_2 = 0), \quad (49)$$

$$y_0' = M_0' + n'\tau + (n_0 k_3 + n_0' l_3)\tau + \gamma_3 = M_0' + \frac{n' + n_0 k_3 + n_0' l_3}{(1+\varkappa)(1+\vartheta)} t + \gamma_3 + (\psi_3 = 0),$$

Returning now to the original unprimed variables by subtraction of

$$-\frac{k_i n_0 + l_i n_0'}{1+\vartheta} t$$

we immediately find for the mean movements of the disturbed resulting motion

$$\text{in } M+\pi-\pi',\ f_1 = \frac{n + k_1 n_0 + l_1 n_0'}{(1+\varkappa)(1+\vartheta)} - \frac{k_1 n_0 + l_1 n_0'}{1+\vartheta} = \frac{n}{(1+\vartheta)(1+\varkappa)} - \frac{\varkappa(k_1 n_0 + l_1 n_0')}{(1+\vartheta)(1+\varkappa)},$$

$$\text{in } -\pi+\pi',\ f_2 = \frac{k_2 n_0 + l_2 n_0'}{(1+\varkappa)(1+\vartheta)} - \frac{k_2 n_0 + l_2 n_0'}{1+\vartheta} = -\frac{\varkappa(k_2 n_0 + l_2 n_0')}{(1+\vartheta)(1+\varkappa)}, \tag{50}$$

$$\text{in } M',\ f_3 = \frac{n' + k_3 n_0 + l_3 n_0'}{(1+\varkappa)(1+\vartheta)} - \frac{k_3 n_0 + l_3 n_0'}{1+\vartheta} = \frac{n'}{(1+\vartheta)(1+\varkappa)} - \frac{\varkappa(k_3 n_0 + l_3 n_0')}{(1+\vartheta)(1+\varkappa)}.$$

It appears most important to known the accurate amounts of these angular velocities thoroughly, whose origin has been forced by the introduction of our new (primed) variables and by annulling of the integral conditions $\varphi_i = \psi_i = 0$.

And indeed these quantities always figure in the development of the disturbing function being combined with the old variables $M+\pi-\pi'$, $-\pi+\pi'$, M'. In this way the explicit time appears inside the trigonometric terms see (39) and we have to take special care that the periodicity of the movement is not spoiled.

Let us pass to the study of the individual terms of the disturbing function thus approaching the next task, namely that reckening of the quadratures of our integral equations term by term. As previously we have to examine the terms of three categories. The Hamiltonian function has the form: 23), 24), 5), pp. 14—16.

$$F_1 = k^2 \sum A \cos(i_1 y_1 + i_2 y_2 + i_3 y_3), \tag{39}$$

$$F_0 = \frac{k^2 \alpha_1'' \beta_1''^2}{2 x_1^2} + \frac{k^2 \alpha_2'' \beta_2''}{2 x_3^2} - \frac{k_1 n_0 + l_1 n_0'}{1+\vartheta} x_1 - \frac{k_2 n_0 + l_2 n_0'}{1+\vartheta} x_2 - \frac{k_3 n_0 + l_3 n_0'}{1+\vartheta} x_3, \tag{37}$$

the coefficients A being the functions exclusively of the scalar elements x_k:

1. Simple short periodic terms of the argument

$$i_1 y_1 + i_2 y_2 + i_3 y_3$$

$$y_1 = nt + c + \pi - \pi', \quad y_2 = -\pi + \pi', \quad y_3 = n't + c', \quad c = M_0, \quad c' = M_0'.$$

After introducing the new primed variables and the distorted time τ we immediately state the false appearance of explicit time τ (or t) obtaining for $t = (1+\varkappa)(1+\vartheta)\tau$ in the argument in question

$$i_1 y_1 + i_2 y_2 + i_3 y_3 = i_1 y_1' + i_2 y_2' + i_3 y_3' - i_1 \frac{k_1 n_0 + l_1 n_0'}{1+\vartheta} t - i_2 \frac{k_2 n_0 + l_2 n_0'}{1+\vartheta} t +$$
$$+ i_3 \frac{k_3 n_0 + l_3 n_0'}{1+\vartheta} t = i_1 n \tau - i_1 (k_1 n_0 + l_1 n_0') \varkappa \tau + (M_0 + \pi - \pi' + \gamma_1) i_1 + i_1 \psi_1 +$$
$$+ - i_2 (k_2 n_0 + l_2 n_0') \varkappa \tau + (-\pi + \pi' + \gamma_2) i_2 + i_2 \psi_2 + i_3 n' \tau - \tag{35}$$
$$- i_3 (k_3 n_0 + l_3 n_0') \varkappa \tau + (M_0' + \gamma_3) i_3 + i_3 \psi_3.$$

The false appearance of time must remain compatible with the total period of the motion, which circumstance gives rise to three conditions

$$T_1 = \frac{2\pi p r}{n_0'} = \frac{2\pi r (p+q)}{n_0}, \quad T = T_1 (1+\vartheta)(1+\varkappa) = \frac{2\pi p r}{n_0'} (1+\vartheta)(1+\varkappa) =$$
$$= \frac{2\pi r (p+q)}{n_0} (1+\varkappa)(1+\vartheta). \tag{51}$$
$$f_1 T = \pm 2\pi s_1, \quad f_2 T = \pm 2\pi s_2, \quad f_3 T = \pm 2\pi s_3$$

s_k representing integral numbers.

These conditions expressed by means of our f_k fixed above (50) lead to the result:

$$f_1 T = \frac{n - \varkappa (k_1 n_0 + l_1 n_0')}{(1+\varkappa)(1+\vartheta)} \frac{2\pi}{n'} p r (1+\varkappa)(1+\vartheta) = \pm 2\pi s_1 = 2\pi (p+q) r \pm$$
$$\pm 2\pi \frac{(p+q) k_1 + p l_1}{(p+q) k_2 + p l_2} s_2,$$
$$f_2 T = \pm 2\pi s_2 = - \frac{\varkappa (k_2 n_0 + l_2 n_0')}{(1+\varkappa)(1+\vartheta)} \frac{2\pi}{n'} p r (1+\varkappa)(1+\vartheta) =$$
$$= - \frac{2\pi p r}{n'} (k_2 n_0 + l_2 n_0') \varkappa = - \frac{2\pi (p+q) r \varkappa}{n} (k_2 n_0 + l_2 n_0')$$

this leads to

$$\varkappa = \mp \frac{s_2}{p r} \frac{n'}{k_2 n_0 + l_2 n_0'} = \mp \frac{s_2}{r(p+q)} \frac{n}{k_2 n_0 + l_2 n_0'} = \mp \frac{s_2}{r[(p+q) k_2 + p l_2]}. \tag{52}$$

$$f_3 T = \pm 2\pi s_3 = \frac{n' - \varkappa (k_3 n_0 + l_3 n_0')}{(1+\varkappa)(1+\vartheta)} \frac{2\pi}{n'} p r (1+\varkappa)(1+\vartheta) =$$
$$= 2\pi p r \pm 2\pi \frac{(p+q) k_3 + p l_3}{(p+q) k_2 + p l_2} s_2.$$

Hence if all s_k are to remain integral numbers we find beside the condition

$$\varkappa = \mp \frac{s_2}{r[(p+q) k_2 + p l_2]}, \tag{52}$$

the necessity that

$$(p + q) k_2 + p l_2 = \pm s_2, \tag{53}$$

or its divisor.

This statement enables us to succeed in carrying out all quadratures of the shortperiodic terms in question. And indeed we have proved that all arguments of the form

$$i_1 y_1 + i_2 y_2 + i_3 y_3 = i_1 y_1' + i_2 y_2' + i_3 y_3' - \frac{k_1 n_0 + l_1 n_0'}{1 + \vartheta} t - \frac{k_2 n_0 + l_2 n_0'}{1 + \vartheta} t - \frac{k_3 n_0 + l_3 n_0'}{1 + \vartheta} t, \tag{35}$$

give the same result save for the whole number of $2\pi = 360°$.

The quadrature changes only every particular $\frac{\cos}{\sin}$ into $\frac{\sin}{\cos}$ and gives the same value in the upper as well as in the lower limit, that is: at the start and at the close of the whole period. These two amounts being of opposite signs cancel mutually and give zero. We immediately conclude from that:

1. The short periodic terms cannot contribute anything to the final value of the coefficients in the implicit functions in question.

2. In the second place let us examine the critical terms in the development of the disturbing function F. These are terms which become independent of time by the supposed commensurability of the mean movements $\frac{n}{n'} = \frac{p + q}{p}$, p, q signifying integral numbers. In order to isolate these terms we have to select all i_1, i_2, i_3 for which $i_1 = - s p$, $i_2 = s q$, $i_3 = s (p + q)$, s signifying integral numbers both positive and negative 7 a).

It is necessary to ascertain that these critical terms really exist, for only their existence secures the possibility of solving the integral equations and satisfying them by means of real starting increments β_i, γ_i. Thus for example the case of the non existence of the aforesaid terms, the sinus terms in our developments disappear, and the determinant of the integral equations contains a subdeterminant of a zero amount.

In short, in case the critical terms are entirely missing, every proof of any real existence of the integrals in question would become meaningless.

The argument of these particular terms of the disturbing function leads to

$$i_1 y_1 + i_2 y_2 + i_3 y_3 = s \zeta = s \{(p + q) M' - p M - (p + q)(\pi - \pi')\} = s \{-p y_1 + q y_2 + (p + q) y_3\}. \tag{54}$$

When passing from the original (31) to the primed angular-variables (35) we get new right-hand sides which show the false explicit time inside the trigonometric terms

$$\begin{aligned} y_1 &= M + \pi - \pi' = y_1' - k_1 n_0 (1+\varkappa)\tau - l_1 n_0' (1+\varkappa)\tau = y_1' - n_1''(1+\varkappa)(1+\vartheta)\tau, \\ y_2 &= -\pi + \pi' \quad = y_2' - k_2 n_0 (1+\varkappa)\tau - l_2 n_0'(1+\varkappa)\tau = y_2' - n_2''(1+\varkappa)(1+\vartheta)\tau, \\ y_3 &= M' \quad = y_3' - k_3 n_0 (1+\varkappa)\tau - l_3 n_0'(1+\varkappa)\tau = y_3' - n_3''(1+\varkappa)(1+\vartheta)\tau \end{aligned} \tag{55}$$

$$n_1'' = \frac{k_1 n_0 + l_1 n_0'}{1+\vartheta}, \quad n_2'' = \frac{k_2 n_0 + l_2 n_0'}{1+\vartheta}, \quad n_3'' = \frac{k_3 n_0 + l_3 n_0'}{1+\vartheta}.$$

The primed coordinates y_k' are finally obtained in the form

$$\begin{aligned} y_1' &= M_0 + \pi - \pi' + n\tau + (k_1 n_0 + l_1 n_0')\tau + \gamma_1 + \psi_1, \\ y_2' &= -\pi + \pi' \qquad + (k_2 n_0 + l_2 n_0')\tau + \gamma_2 + \psi_2, \\ y_3' &= M_0' \qquad + n'\tau + (k_3 n_0 + l_3 n_0')\tau + \gamma_3 + \psi_3. \end{aligned} \tag{56}$$

The functions $\psi_1 = \psi_2 = \psi_3 = 0$ as final deviations of the starting configuration become zero, if only the conditional integral equations for the increments $\varkappa$, ϑ, β_i, γ_i turn out to be solvable. In this case clearly the whole movement is governed by the same conditions at the start as well as towards the close of the period $T = T_1(1+\varkappa)(1+\vartheta)$ and becomes periodic.

Now our principal aim is so to prove the solubility of the conditions (48) or, which is the same thing, to find out a suitable non-disappearing Hessian—Jacobi—Poincaré.

For this purpose it first appears necessary to make sure that all the factors of the apparently explicit time mentioned above disappear in (36) entirely from the critical arguments, $s\zeta$ in question. And indeed it is only in this case that a corresponding critical term changes from a trigonometrical term to be integrated into a simple constant h, whose integration is carried out by adjoining the factor τ and inserting the upper and lower limit, namely $hT - h.0 = hT$ etc. Indeed it is only through the existence of these constant terms, the possibility of non-disappearing elements of the determinant, and the solution of the conditional equations for $\varkappa$, ϑ, β_i, γ_i can be guaranteed. It is only in this way the existence of a not disappearing Hessian can be reached.

Now the coefficients of the time τ become annulled by the choice of the starting increments $\varkappa$, ϑ, β_i, γ_i, and by annulment of the ψ_1, ψ_2, ψ_3 as a consequence of the supposed commensurabilities of f_1, f_2, f_3.

When constructing the critical argument we have to pay special attention to the cancelling out of the apparently explicit time-terms contained in the additive part.

This means we must choose all arbitrary numbers k_1, k_2, k_3, l_1, l_2, l_3 in such a way that the whole factor of $\varkappa\tau$ disappears. We have to build up the whole argument containing the artificially appearing explicit time $\tau = \frac{t}{(1+\varkappa)(1+\vartheta)}$.

With this in view we insert the expressions y_i' in (34), (35) into the original coordinates y_i (56), $s\zeta$. Then all terms not containing $\varkappa\tau$ namely

$$\begin{aligned}
&\text{in } y_1, \quad + p(k_1 n_0 + l_1 n_0')\tau - p(k_1 n_0 + l_1 n_0')\tau,\\
&\text{in } y_2, \quad + q(k_2 n_0 + l_2 n_0')\tau - q(k_2 n_0 + l_2 n_0')\tau,\\
&\text{in } y_3, \quad + (p+q)(k_3 n_0 + l_3 n_0')\tau - (p+q)(k_3 n_0 + l_3 n_0')\tau,
\end{aligned}$$

cancel out, the ψ_k likewise containing the factors $\varkappa n$, $\varkappa n'$ disappear according to the supposed fulfilment of our integral equations (42), (43), (48) and the explicit time expressions $-(k_1 n_0 + l_1 n_0')\varkappa\tau$, $-(k_2 n_0 + l_2 n_0')\varkappa\tau$, $-(k_3 n_0 + l_3 n_0')\varkappa\tau$ remain within y_1, y_2, y_3 respectively. Thus we have to pay special attention that these parts disappear through a suitable choice of the hitherto free integer numbers k_1, k_2, k_3, l_1, l_2, l_3 which appear only in the combination

$$s\zeta = s(-py_1 + qy_2 + (p+q)y_3) - \{-p(k_1 n_0 + l_1 n_0') + q(k_2 n_0 + l_2 n_0') + \\ + (p+q)(k_3 n_0 + l_3 n_0')\}\varkappa\tau s$$

which leads to the simple condition

$$n_0\{pk_1 - qk_2 - (p+q)k_3\} + n_0'\{pl_1 - ql_2 - (p+q)l_3\} = 0$$

or by means of our supposed commensurability $n_0 p = n_0'(p+q)$,

$$(p+q)pk_1 - (p+q)qk_2 - (p+q)^2 k_3 + p^2 l_1 - pql_2 - p(p+q)l_3 = 0 \tag{57}$$

and this fundamental condition is to be supplemented by our previous result (53):

$$(p+q)k_2 + pl_2 = \pm 1, \ \pm s_2. \tag{58}$$

Let us state two particular cases, for the sake of an instructive example:

1) $k_1 = k_2 = k_3 = l_3 = 0, \quad l_1 = l_2 = 1, \quad p = q = 1, \quad n_1' = n_0', \quad n_2' = n_0', \quad n_3' = 0.$
2) $l_1 = l_2 = -1, \quad l_3 = 1, \quad k_1 = 1, \quad k_2 = k_3 = 0, \quad n_1' = n_0 - n_0', \quad n_2' = -n_0', \quad n_3' = n_0'.$

On summing up, we have got as yet two diophantine equations for the admissible choice of our six arbitrary numbers k_1, k_2, k_3, l_1, l_2, l_3 both positive and negative, which give the structure of a suitable rotating system of coordinates. The resulting rotational velocities are obtained in the form

$$f_1 = \frac{n}{(1+\vartheta)(1+\varkappa)} - \frac{\varkappa(k_1 n_0 + l_1 n_0')}{(1+\vartheta)(1+\varkappa)}, \quad f_2 = -\frac{\varkappa(k_2 n_0 + l_2 n_0')}{(1+\vartheta)(1+\varkappa)},$$
$$f_3 = \frac{n'}{(1+\varkappa)(1+\vartheta)} - \frac{\varkappa(k_3 n_0 + l_3 n_0')}{(1+\vartheta)(1+\varkappa)}. \tag{50}$$

In finding the two diophantine relations (57), (58) we only carry out two guiding principal rules:

(a) that the final deviations of the coordinates $\varphi_1, \varphi_2, \varphi_3, \psi_1, \psi_2, \psi_3$, disappear separately, which conveys the disappearance of the time increment of the whole critical argument,

(b) we destroy the appearance of the explicit time by a suitable choice of the arbitrary integral numbers both positive and negative $k_i, l_i, i = 1, 2, 3$.

It is only in this manner that terms of the second category (2) turn out to be really secular, to be constant, without a trigonometric, periodical part ready for an integration.

It is just these terms which constitute the real, non-disappearing coefficients of the conditional equations, and only they maintain the existence of a new, non-disappearing Hessian—Jacobi—Poincaré determinant, thus entirely avoiding the difficulty which for so many years baffled all the efforts of Poincaré and his disciples.

3. Now let us pass to the third category of critical terms of argument (see Kempinski 7 a))

$$\zeta' = (p+q)M' - pM - p(\pi - \pi') = -p y_1 + (p+q) y_3. \tag{59}$$

One may be inclined to try to annul in a similar manner the false part of explicit time clearly appearing even in the latter. And, indeed, if we succeeded in this endeavour, these terms would equally yield only further constant (secular) critical parts as in no. 2. But it can be shown that this is impossible. When superimposing a similar condition as in no. 2 with this view we easily get a further diophantine equation

$$(p+q)p k_1 - (p+q)^2 k_3 + p^2 l_1 - (p+q) p l_3 = 0. \tag{60}$$

But unfortunately we can never satisfy both of these conditions (57), (60) without spoiling and annulling our chief purpose, namely the newly arising mean movement. And in fact when substracting both of the last conditions from each other, we are led to the relation

$$-(p+q) q k_2 - l_2 p q \equiv 0, \quad \text{or} \quad \frac{k_2}{l_2} = -\frac{p}{p+q},$$

and

$$k_2 n_0 + l_2 n_0' = 0, \tag{61}$$

which clearly expresses the disappearance of the whole mean movement aimed at. Thus only the first argument of the structure ζs leads to a really critical, constant, secular terms, while $s\zeta'$ remains periodical even after the quadrature.

Such a periodical trigonometric term naturally shows a long period, being affected moreover by a small divisor, as a result of the integration of the original periodic term in question.

In our conditional integral equations (42), (48) these terms disappear entirely, contributing nothing to the real elements of our determinant aimed at. They appear only in the final periodical development of the coordinates, causing the well-known long periodical perturbations of the classical theory.

4. An almost identical result is obtained for the terms of the form $m_1\zeta + m_2\zeta'$, m_1, m_2 being fixed particular integral numbers. They contribute in no way to the constant coefficients of the conditional integral equations (42), but unlike the real critical terms they appear only in the periodical part of the final development as terms of long period.

It is to be expressly noted that the wide freedom of choice among the admissible solutions of the two diophantine equations (57), (58) allows a great variety of corresponding periodic orbits.

Last but not least, instead of $s\zeta$ we can just as well choose the diophantine equation (60), thus taking $s\zeta'$ or even $m_1\zeta + m_2\zeta'$ instead of $s\zeta$ for real critical, constant terms. Then just the remaining possible arguments $s\zeta$ and $m_1\zeta + m_2\zeta$; resp. $s\zeta$, $s\zeta'$ yield only long-periodic terms. In the third eventuality, the conditional diophantine equation is naturaly to be suitably modified as

$$\begin{aligned} & m_1\{(p+q)[k_1 p - k_2 q - k_3(p+q)] + p[l_1 p - l_2 q - l_3(p+q)]\} \\ + & m_2\{(p+q)[k_1 p - k_3(p+q)] \qquad + p[l_1 p - l_3(p+q)]\} \equiv 0. \end{aligned} \tag{62}$$

It is to be borne in mind that one argument only can always remain constant and really critical, while the other two kinds of arguments become long-periodical.

5. Finally let us consider the secular terms within the meaning of the classics.

These terms are formed

(a) by really constant terms (without trigonometrical argument) whose structure is built up exclusively of scalar elements a, e, a', e'.

They are the only accessory terms, contributing to the constant coefficients of the conditional integral equations. As a rule one includes them in the whole series of critical terms ζs.

But the question arises, whether their existence alone may be sufficient for proving the existence of the new periodic orbits in question.

However it can easily be seen that such a proof without the ζ terms never can be given.

And indeed the derivation for the sake of obtaining the necessary coefficients

$$\frac{\partial^2 F}{\partial x_i \partial y_k} = \frac{\partial^2 F}{\partial y_k \partial y_i} = 0.$$

as the corresponding cosinus and sinus terms, according to our supposition, do not contain the necessary y_k at all, always yield zero. This makes the Jacobian show a subdeterminant

$$\left\{\begin{matrix} 0 & 0 & 0 & \dfrac{\partial^2 F}{\partial y_1^2}, & \dfrac{\partial^2 F}{\partial y_1 \partial y_2} \\ 0 & 0 & 0 & \dfrac{\partial^2 F}{\partial y_1 \partial y_2}, & \dfrac{\partial^2 F}{\partial y_2^2} \end{matrix}\right\} = 0$$

disappearing identically — see p. 45, and (48) of this paper.

Thus the same calamity of which Poincaré was complaining l. c. occurs in another way Mét. nouv. I. p. 68, 119, 133.

(b) by secular terms of arguments $ee' \cos(\pi - \pi')$.

It is easy to see that these terms can only lead to new long-periodical terms of the categories 3 and 4.

It is clear that the argument can be written down as:

$$y_2 = -\pi + \pi' = y_2' + \psi_2 - k_2 n_0 (1 + \varkappa)\tau - l_2 n_0' (1 + \varkappa)\tau$$

and no possibility remains save exceptionally to annul or get rid of the false time $\tau\varkappa$, by a suitable choice of the numbers k_2, l_2, as we have already disposed of these numbers by solving our diophantine equations (57), (58) and abolishing the false parts containing the time τ in the terms of 2. p. 48.

Naturally the result just explained concerned with the various terms of the categories 1, 2, 3, 4, and 5, has given merely the final contribution to the non-disappearing coefficients of our conditional integral equations (42), (43), (48).

As to the formation of the real periodical developments of the integrals, this is obtained without inserting both integral limits.

For this purpose only the critical ζ terms no. 4 prove to be constant, whereas all the other terms of categories 1, 3, 4, and 5 lead to short periodical or else to secular long-periodical terms 3, 4, and 5. But they contribute nothing at all to the coefficients of our integral equations.

On summing up we must bear in mind that for the construction of real, non-disappearing elements of the determinant in question solely the terms of the second category are to be respected.

Now let us proceed to our main task, the solution of the conditional integral equations which determine the starting increments of the initial non-disturbed Keplerian elements $x_1, x_2, x_3, y_1', y_2', y_3'$ namely $\beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \varkappa, \vartheta$.

All through this intricate consideration, we must be clearly aware of the large freedom by which the supernumerary unknown quantities can always be disposed of. And indeed the number of the unknown is always greater than that of the independent equations.

Moreover the coefficients of the left-hand sides of our conditional integral equations (42), (43) are built up entirely of scalar elements, which constitute constant starting parameters, otherwise freely eligible.

In our planetary case let us consider the commensurability of the mean movements $\frac{n}{n'} = \frac{p+q}{p}$, which partly fixes the Keplerian starting axes of the two ellipses $a, e, a' e'$. Then we are free to choose among the increments $\beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \varkappa, \vartheta$ and the parameters e, e' which form the aforesaid coefficients of the disturbing function and the integral numbers both positive and negative $k_1, k_2, k_3, l_1, l_2, l_3, r, s_1, s_2, s_3$.

So for example after disposing of $s_2 = 1$, we have

$$(p+q)k_2 + p\,l_2 = 1$$

$$\varkappa = \frac{1}{r} = \frac{s_2}{r\{(p+q)k_2 + p\,l_2\}} = \frac{\mu}{k_2 n_0 + l_2 n_0'} \frac{\partial [F_1(e, e')]}{\partial x_2}. \tag{52}$$

The two starting excentricities can always be chosen so as to satisfy the last relation, whose left-hand side appears to be fixed and given.

Then the conditional integral equations (48) may be rewritten in the final form after leaving out in the second, forth and fifth equation the constant factor μ:

$$-\frac{\partial F_0}{\partial x_1}\vartheta - \beta_1\frac{\partial^2 F_0}{\partial x_1^2} + 0 + 0 + 0 + 0 + 0 - \mu\,\frac{\partial [F_1]}{\partial x_1} + $$
$$+ \frac{n(1+\vartheta)+k_1 n_0 + l_1 n_0'}{k_2 n_0 + l_2 n_0'}\mu\frac{\partial [F_1]}{\partial x_2} + \cdots = 0$$

$$-\beta_1\frac{\partial^2 [F_1]}{\partial x_1 \partial x_2} - \beta_2\frac{\partial^2 [F_1]}{\partial x_2^2} - \beta_3\frac{\partial^2 [F_1]}{\partial x_2 \partial x_3} - \mu\frac{\partial [F_2]}{\partial x_2} = 0 \tag{63}$$

$$-\frac{\partial F_0}{\partial x_3}\vartheta + 0 + 0 - \beta_3\frac{\partial^2 F_0}{\partial x_3^2} - \mu\frac{\partial [F_1]}{\partial x_3} + \frac{n'(1+\vartheta)+k_3 n_0 + l_3 n_0'}{k_2 n_0 + l_2 n_0'}\mu\frac{\partial [F_1]}{\partial x_2} + \cdots = 0$$

$$\gamma_1\frac{\partial^2 [F_1]}{\partial y_1^2} + \gamma_2\frac{\partial^2 [F_1]}{\partial y_1 \partial y_2} + \gamma_3\frac{\partial^2 [F_1]}{\partial y_1 \partial y_3} + \mu\frac{\partial [F_2]}{\partial y_1} + \cdots = 0$$

$$\gamma_1\frac{\partial^2 [F_1]}{\partial y_1 \partial y_2} + \gamma_2\frac{\partial^2 [F_1]}{\partial y_2^2} + \gamma_3\frac{\partial^2 [F_1]}{\partial y_2 \partial y_3} + \mu\frac{\partial [F_2]}{\partial y_2} + \cdots = 0.$$

This last form of the conditions gives a very clear survey of the coefficients of all unknown quantities, thereby showing the structure of elements of the fundamental Hessian—Jacobi—Poincaré. We have shortened the second of them by making use of the relation $\varkappa n_2'' = \frac{\partial [F_1]}{\partial x_2}$ and dividing the whole equation by μ. Our chief purpose is to survey especially all coefficients of the first powers of the Cauchy development.

Thus we have five conditions for seven unknowns. The last equation namely

$$\gamma_1\frac{\partial^2 [F_1]}{\partial y_1 \partial y_3} + \gamma_2\frac{\partial^2 [F_1]}{\partial y_2 \partial y_3} + \gamma_3\frac{\partial^2 [F_1]}{\partial y_3^2} + \frac{\partial [F_2]}{\partial y_3}\mu + \cdots = 0$$

can be suppressed, through its being a consequence of the two foregoing, which circumstance has been shown above p. 36, (19) by the existence of the energy integral.

As both the coefficients

$$-\frac{\partial^2 F_0}{\partial x_1^2} = -3\frac{k^2 \alpha_1''}{x_1^4}\beta_1''^2, \quad -\frac{\partial^2 F_0}{\partial x_3^2} = -3\frac{k^2 \alpha_2''}{x_3^4}\beta_2''^2$$

are distinct from zero two of the seven unknown quantities $\vartheta, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$ are freely eligible. And if we choose for example $\beta_1 = 0$, $\gamma_3 = 0$ we finally obtain five equations for five unknowns $\vartheta, \beta_2, \beta_3, \gamma_1, \gamma_2$, showing a functional determinant clearly distinct from zero.

Chapter III.

On the Problem of the Moon.

This second general case has led us very far from our original satellite problem. Still I was obliged to explain it most thoroughly as I was aiming to expound the whole subject of an "operation" clearly and step by step. See p. 2 of this paper.

After these explanatory remarks let us return to our Moon-problem (see previous p. 29). It can easily be shown that, to reach a complete solution, we have only to apply the same principles, namely a similar "operation" to this particular case of commensurability of the mean movements $n = n'$.

In order to approach this most interesting question with success, we shall start with the equations of the ordinary planetary problem which we formulated and adjusted near the end of the previous First Part § 3, (37).

In the right-hand sides the equations (37), First Part, contain the development intended for dealing with two planetary Keplerian ellipses round the Sun.

The path-ellipse of the disturbing Earth appears reduced to a circular orbit (namely an ellipse of zero excentricity) while the Moon-Planet describes a second ellipse of small excentricity (1/400). The speed of movement along both these ellipses is presumed to be the same for the two bodies.

When introducing a rotating system of the velocity just mentioned, we immediately obtain in this rotating system a fixed point, marking the position of the Earth and a small closed curve round it constituting the path of our Moon. In this manner we have succeeded in changing the Moon-Planet into a satellite.

Unfortunately this "Moon" revolves very slowly round the Earth, as its centre (not its focus), taking a whole year for its circuit round the Earth. And our chief purpose is to bring about a new mean movement quickening the slowly turning satellite into our real Moon, that appears twelve times as quick.

With this in view and copying the previous case of two planets pp. 39, 40 I apply the same operation and put forward the following canonical variables (35), p. 21:

$$\begin{aligned} x_1' &= \Lambda = L - H, & y_1' &= M, \\ x_2' &= G, & y_2' &= \bar{\omega}, \\ x_3' &= H, & y_3' &= M - M' + \tilde{\Omega}, \end{aligned} \tag{64}$$

I drop the dashes $x_i' = x_i$ and try to annul the expression of the functional increments φ_i, ψ_i thereby starting with the original values for $\mu > 0$.

$$\begin{array}{ll} \Lambda_0 + \beta_1 + \varphi_1, & M + \gamma_1 + \psi_1, \\ G_0 + \beta_2 + \varphi_2, & \bar{\omega} + \gamma_2 + \psi_2, \\ H_0 + \beta_3 + \varphi_3, & -n't + M + \tilde{\Omega} - M' + \gamma_3 + \psi_3, \end{array} \tag{65}$$

which means adding to the F_0 of $+2n'H$.

The orbits of our two planets are fixed by the canonical elements found out in the previous First Part p. 21, (35).

The equations of movement are this time given by:

$$\begin{aligned} &\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}, \quad i = 1, 2, 3, \\ &F = -\dot{M}\Lambda - \dot{\bar{\omega}} G - \dot{\tilde{\Omega}} H + \tilde{T} + V\mu + 2n'H, \quad F_1 = k^2 \left(\frac{1}{\Delta} - \frac{r \cos\sigma}{a'^2}\right), \\ &V = k^2 \left(\frac{1}{\Delta} - \frac{\varrho \cos\sigma}{r'^2}\right), \quad \tilde{T} \text{ energy, see p. 16, (24).} \end{aligned} \tag{66}$$

Δ, ϱ, r' signifying the distances between resp. Earth and Moon, Sun and Moon, Earth and Sun. At the start the two planets, Earth and Moon, are describing two Keplerian undisturbed paths $\mu = 0$.

The period of their movement appears to be

$$\begin{aligned} &T_1' = \frac{2\pi p}{n'}, \quad \frac{n}{n'} = \frac{p+q}{p} = 1, \quad q = 0. \\ &T_1 = r T_1'. \end{aligned} \tag{67}$$

When as usual displacing the original starting positions of the elements by small amounts $\beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$ to be determined, we shall at the same time try to distort the independent variable t into

$$t = (1 + \varkappa)\tau. \tag{68}$$

This means we prolong algebraically the period into

$$T = (1 + \varkappa) T_1. \tag{69}$$

Through this change of variables the equations of movement are converted into:

$$\frac{dx_i}{d\tau} = (1+\varkappa)\frac{\partial F}{\partial y_i}; \quad \frac{dy_i}{d\tau} = -(1+\varkappa)\frac{\partial F}{\partial x_i},$$

or else,

$$\frac{d\varphi_i}{d\tau} = (1+\varkappa)\frac{\partial F}{\partial y_i}, \quad \frac{d\psi_i}{d\tau} = -(1+\varkappa)\frac{\partial F}{\partial x_i}, \quad i = 1, 2, 3. \tag{70}$$

Evidently the disturbed and displaced movement remains periodic only in case these very complicated functions $\varphi_1, \varphi_2, \varphi_3, \psi_1, \psi_2, \psi_3$ disappear entirely towards the close of the whole prolonged period.

The differential equations (51) can immediately be converted into the integral equations

$$\frac{\varphi_1}{\mu} = (1+\varkappa)\int\limits_0^{T_1} \frac{\partial F_1}{\partial y_1} d\tau, \quad \frac{\varphi_2}{\mu} = (1+\varkappa)\int\limits_0^{T_1} \frac{\partial F_1}{\partial y_2} d\tau, \quad \frac{\varphi_3}{\mu} = (1+\varkappa)\int\limits_0^{T_1} \frac{\partial F_1}{\partial y_3} d\tau,$$

$$\psi_1 = n\varkappa + (1+\varkappa)\int\limits_0^{T_1} \frac{d\tau}{\alpha^{3/2}(\Lambda, G, H)} + \cdots - \mu(1+\varkappa)\int\limits_0^{T_1} \frac{\partial F_1}{\partial x_1} d\tau,$$

$$\psi_2 = \quad -\beta_1 \frac{\partial^2 F_0}{\partial x_1 \partial x_2} - \beta_2 \frac{\partial^2 F_0}{\partial x_2^2} - \beta_3 \frac{\partial^2 F_0}{\partial x_2 \partial x_3} - \mu \frac{(1+\varkappa)}{T_1} \int\limits_0^{T_1} \frac{\partial F_1}{\partial x_2} d\tau + \cdots \tag{71}$$

$$\psi_3 = -n'\varkappa - \beta_1 \frac{\partial^2 F_0}{\partial x_1 \partial x_3} - \beta_2 \frac{\partial^2 F_0}{\partial x_2 \partial x_3} - \beta_3 \frac{\partial^2 F_0}{\partial x_3^2} - \mu \frac{(1+\varkappa)}{T_1} \int\limits_0^{T_1} \frac{\partial F_1}{\partial x_3} d\tau + \cdots$$

These do not show the false time mentioned above, and we are led to new periodical satellite orbits analogous to the well-known restreint $e' = 0$, asteroidal case of Schwarzschild—Poincaré (I sorte), see Charlier, Mech. d. Himmels II. p. 251, in our foregoing analysis Chapter I. (This time three degrees of freedom.) Now although these orbits have the advantage of movable-nodes they do not satisfy our claims to ascertain orbits wherein the satellite moves round with the desirable speed of our real Moon, and indeed our satellite has not yet changed the length of its period of revolution. Should there be any change the period always remains the same as the slow period of the nodes $\frac{n'\varkappa}{1+\varkappa}$

$$-n'\tau + n't = -\frac{n't}{1+\varkappa} + n't = \frac{n'\varkappa}{1+\varkappa}\, t. \tag{72}$$

Moreover the existence of this new genus of satellite orbits appears limited by the condition

$$\frac{\partial\,[F_1(\varepsilon, e', \iota)]}{\partial x_2} \equiv 0. \tag{73}$$

But in order to obtain very many classes of particular integrals suitable for the use of celestial mechanics and Lunar theory, it would appear necessary to pass to another set of variables, combined with a more general change of the independent time t.

Returning again to a simple copy of the general case treated above Chapter II, let us imagine the same movement many times (r) repeated we shall this time put

$$t-(1+\varkappa)(1+\vartheta)(1+\eta)\tau.$$

The original starting elements for $\mu=0$, are:

$$\Lambda,\ G,\ H,\ M,\ \bar{\omega},\ -n't+M-M'+\tilde{\Omega}, \tag{74}$$

and the same elements for $\mu>0$ are:

$$\begin{aligned}
&\Lambda+\beta_1+\varphi_1, \quad n\tau+M_0+\gamma_1+\psi_1\\
&G+\beta_2+\varphi_2, \quad \bar{\omega}+\gamma_2+\psi_2,\\
&H+\beta_3+\varphi_3, \quad -n'\tau+M_0-M_0'+\tilde{\Omega}+\gamma_3+\psi_3.
\end{aligned} \tag{75}$$

$$F=\frac{k^2}{2\alpha}+2Hn'+\mu k^2\left(\frac{1}{\Delta}-\frac{\varrho\cos\sigma}{a'^2}\right), \tag{76}$$

$$\begin{aligned}
&x_1=\Lambda=L-H, \quad y_1=M, && \frac{dx_i}{dt}=\frac{\partial F}{\partial y_i}, \quad i=1,2,3.\\
&x_2=G, \quad y_2=\bar{\omega}, &&\\
&y_3=H, \quad y_3=-n't+M-M'+\tilde{\Omega}, && \frac{dy_i}{dt}=-\frac{\partial F}{\partial x_i}.
\end{aligned}$$

We are studying the case of the commensurability of the mean movement

$$\frac{n}{n'}=\frac{p+q}{p}, \quad q=0,\ p=1.$$

The period of the movement in the two undisturbed ellipses is

$$T_1=\frac{2\pi p}{n'}r=rT_1', \quad T=(1+\varkappa)(1+\vartheta)(1+\eta)T_1. \tag{77}$$

When passing to the disturbed case of $\mu>0$, we find the starting amounts of variables by means of the following integral equations

$$\begin{aligned}
\psi_1=0&=\quad n\,(\varkappa+\eta+\vartheta)-\beta_1\frac{\partial^2F_0}{\partial x_1^2}-\beta_2\frac{\partial^2F_0}{\partial x_1\partial x_2}-\beta_3\frac{\partial^2F_0}{\partial x_1\partial x_3}-\frac{\mu}{T_1}\int_0^{T_1}\frac{\partial F_1}{\partial x_1}d\tau+\cdots\\
\psi_2=0&=\qquad\qquad -\beta_1\frac{\partial_2F_0}{\partial x_1\partial x_2}-\beta_2\frac{\partial^2F_0}{\partial x_2^2}-\beta_3\frac{\partial^2F_0}{\partial x_2\partial x_3}-\frac{\mu}{T_1}\int_0^{T_1}\frac{\partial F_1}{\partial x_2}d\tau+\cdots\\
\psi_3=0&=-n'(\varkappa+\eta+\vartheta)-\beta_1\frac{\partial^2F_0}{\partial x_1\partial x_3}-\beta_2\frac{\partial^2F_0}{\partial x_2\partial x_3}-\beta_3\frac{\partial^2F_0}{\partial x_3^2}-\frac{\mu}{T_1}\int_0^{T_1}\frac{\partial F_1}{\partial x_3}d\tau+\cdots
\end{aligned} \tag{78}$$

$$\frac{\varphi_1}{\mu} = 0 = \beta_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_1 \partial y_1}\, d\tau + \beta_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_2 \partial y_2}\, d\tau + \beta_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_3 \partial y_1}\, d\tau + \gamma_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1^2}\, d\tau +$$

$$+ \gamma_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1 \partial y_2}\, d\tau + \gamma_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1 \partial y_3}\, d\tau + \mu \int_0^{T_1} \frac{\partial F_2}{\partial y_1}\, d\tau + \cdots$$

$$\frac{\varphi_2}{\mu} = 0 = \beta_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_1 \partial y_2}\, d\tau + \beta_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_2 \partial y_2}\, d\tau + \beta_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_3 \partial y_2}\, d\tau + \gamma_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1 \partial y_2}\, d\tau +$$

$$+ \gamma_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_2^2}\, d\tau + \gamma_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_2 \partial y_3}\, d\tau + \mu \int_0^{T_1} \frac{\partial F_2}{\partial y_2}\, d\tau + \cdots$$

$$\frac{\varphi_3}{\mu} = 0 = \beta_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_1 \partial y_3}\, d\tau + \beta_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_2 \partial y_3}\, d\tau + \beta_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial x_3 \partial y_3}\, d\tau + \gamma_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1 \partial y_3}\, d\tau +$$

$$+ \gamma_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_2 \partial y_3}\, d\tau + \gamma_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_3^2}\, d\tau + \mu \int_0^{T_1} \frac{\partial F_2}{\partial y_3}\, d\tau + \cdots$$

These show the same determinant as before p. 58.

In order to avoid this fatal inconvenience much complained of by H. Poincaré, we must try to give rise to new mean motions big and small enough, as the case may be, to satisfy our claims concerning the real Moon.

Moreover we must try to isolate the increased mean angular velocity in M and prove that its relatively large amount $12\,n'$ is attainable by our parametrical means.

For this purpose I first introduce other angular variables by putting

$$\begin{aligned} y_1'' &= M + \frac{n' t}{(1+\vartheta)(1+\varkappa)}, \\ y_2'' &= \tilde{\omega} + \frac{n' t}{(1+\eta)(1+\varkappa)}, \\ y_3'' &= M - M' + \tilde{\Omega} - \frac{n' t}{(1+\eta)(1+\varkappa)}, \end{aligned} \tag{79}$$

at the same time changing the independent t into τ,

$$t = (1+\varkappa)(1+\vartheta)(1+\eta)\,\tau. \tag{80}$$

This only means introducing explicit time within the trigonometric arguments of the development of F. And as a matter of fact we have to replace every y_k by y_k''

$$
\begin{aligned}
y_1 &= y_1'' - \frac{n' t}{(1+\vartheta)(1+\varkappa)} = y_1'' - n'(1+\eta)\tau, \\
y_2 &= y_2'' - \frac{n' t}{(1+\eta)(1+\varkappa)} = y_2'' - n'(1+\vartheta)\tau, \\
y_3 &= y_3'' + \frac{n' t}{(1+\eta)(1+\varkappa)} = y_3'' + n'(1+\vartheta)\tau.
\end{aligned} \tag{81}
$$

We have then to take special care that the false explicit time remains compatible with the periodicity of the resulting disturbed periodic solution (see the analysis of terms of the category 1, p. 47).

For this purpose we have only to recall the explicit development of the disturbing function, proved in detail for the present satellite purpose in the previous First Part § 4.

$$
\begin{aligned}
&F = F_0 + \mu F'', \quad F'' = k^2 \sum A \cos \omega'' = k^2 \sum A \cos (i_1 y_1 + i_2 y_2 + i_3 y_3), \\
&\omega'' = i_1 y_1'' + i_2 y_2'' + i_3 y_3'' - i_1 n'(1+\eta)\tau - i_2 n'(1+\vartheta)\tau + i_3 n'(1+\vartheta)\tau
\end{aligned} \tag{82}
$$

compare (55)

$$
\frac{d x_i}{d\tau} = (1+\varkappa)(1+\vartheta)(1+\eta)\frac{\partial F}{\partial y_i''}, \quad \frac{d y_i''}{d\tau} = -(1+\varkappa)(1+\vartheta)(1+\eta)\frac{\partial F}{\partial x_i}, \quad i = 1, 2, 3. \tag{83}
$$

Further, for preserving of the canonical form of the starting equations of motion, the last change of variables requires adding the following three complementary terms to the original function F_0

$$
\begin{aligned}
F_0'' &= F_0 - \frac{n' x_1}{(1+\vartheta)(1+\varkappa)} - \frac{n' x_2}{(1+\eta)(1+\varkappa)} + \frac{n' x_3}{(1+\eta)(1+\varkappa)}, \\
F_0 &= \frac{k^2}{2\alpha} + n' x_3.
\end{aligned} \tag{84}
$$

Now we shall try to annul the functional expressions $\varphi_1, \varphi_2, \varphi_3, \psi_1, \psi_2, \psi_3$ by applying suitable parameters $\varkappa, \eta, \vartheta$ introduced through the time changing substitution

$$
t = (1+\varkappa)(1+\vartheta)(1+\eta)\tau. \tag{80}
$$

This time the starting values of the elements appear to be

$$
\begin{aligned}
x_1 &= \Lambda + \beta_1 + \varphi_1, \quad y_1'' = M_0 + \gamma_1 + \psi_1'' + n\tau + n'\tau + \\
x_2 &= G + \beta_2 + \varphi_2, \quad y_2'' = \tilde{\omega} + \gamma_2 + \psi_2'' + n'\tau \\
x_3 &= H + \beta_3 + \varphi_3, \quad y_3'' = M_0 - M_0' + \tilde{\Omega}_0 + \gamma_3 + \psi_3'' + n\tau - n'\tau - n'\tau
\end{aligned} \tag{85}
$$

thereby involving the small increments such as $n'\eta$, $n'\vartheta$, $-n'\vartheta$ resp. into ψ_1'', ψ_2'', ψ_3''.

Our chief purpose is to destroy the complicated functions φ_i, ψ_i'' giving the functional displacement, final deviation, thereby defining the suitable starting amounts of the explicit functions $x_i(\beta_i)$, $y_i(\gamma_i)$.

The differential equations for ψ_i'', φ_i as well as their inversion in the form of integral equations are easily obtained as follows $\overline{F} = F_0'' + \mu F_1$

$$\frac{d\varphi_i}{d\tau} = (1+\varkappa)(1+\vartheta)(1+\eta)\frac{\partial \overline{F}}{\partial y_i}, \quad \frac{d\psi_i''}{d\tau} = -(1+\varkappa)(1+\vartheta)(1+\eta)\frac{\partial \overline{F}}{\partial x_i}, \quad i = 1,\ 2,\ 3. \tag{86}$$

From these equations the final deviations at the end of the total period $T = (1+\varkappa)(1+\vartheta)(1+\eta)\,T_1$ are found to be:

$$\psi_1'' = 0 = n(\varkappa + \vartheta + \eta)\,T_1 + n'\eta\,T_1 -$$
$$- \beta_1 \int_0^{T_1} \frac{\partial^2 F_0}{\partial x_1^2}\,d\tau - \beta_2 \int_0^{T_1} \frac{\partial^2 F_0}{\partial x_1\,\partial x_2}\,d\tau - \beta_3 \int_0^{T_1} \frac{\partial^2 F_0}{\partial x_1\,\partial x_3}\,d\tau - \mu \int_0^{T_1} \frac{\partial F_1}{\partial x_1}\,d\tau + \cdots$$

$$\psi_2'' = 0 = + n'\vartheta\,T_1 -$$
$$- \beta_1 \int_0^{T_1} \frac{\partial^2 F_0}{\partial x_1\,\partial x_2}\,d\tau - \beta_2 \int_0^{T_1} \frac{\partial^2 F_0}{\partial x_2^2}\,d\tau - \beta_3 \int_0^{T_1} \frac{\partial^2 F_0}{\partial x_2\,\partial x_3}\,d\tau - \mu \int_0^{T_1} \frac{\partial F_1}{\partial x_2}\,d\tau + \cdots$$

$$\psi_3'' = 0 = - n'\vartheta\,T_1 -$$
$$- \beta_1 \int_0^{T_1} \frac{\partial^2 F_0}{\partial x_1\,\partial x_3}\,d\tau - \beta_2 \int_0^{T_1} \frac{\partial^2 F_0}{\partial x_2\,\partial x_3}\,d\tau - \beta_3 \int_0^{T_1} \frac{\partial^2 F_0}{\partial x_3^2}\,d\tau - \mu \int_0^{T_1} \frac{\partial F_1}{\partial x_3}\,d\tau + \cdots$$

$$\frac{\varphi_1}{\mu} = 0 = \gamma_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1^2}\,d\tau + \gamma_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1 \partial y_2}\,d\tau + \gamma_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1 \partial y_3}\,d\tau +$$
$$+ \int_0^{T_1} \frac{\partial F_1}{\partial y_1}\,d\tau + \mu \int_0^{T_1} \frac{\partial F_2}{\partial y_1}\,d\tau + \cdots \tag{87}$$

$$\frac{\varphi_2}{\mu} = 0 = \gamma_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1 \partial y_2}\,d\tau + \gamma_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_2^2}\,d\tau + \gamma_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_2 \partial y_3}\,d\tau +$$
$$+ \int_0^{T_1} \frac{\partial F_1}{\partial y_2}\,d\tau + \mu \int_0^{T_1} \frac{\partial F_2}{\partial y_2}\,d\tau + \cdots$$

$$\frac{\varphi_3}{\mu} = 0 = \gamma_1 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_1 \partial y_3}\,d\tau + \gamma_2 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_2 \partial y_3}\,d\tau + \gamma_3 \int_0^{T_1} \frac{\partial^2 F_1}{\partial y_3^2}\,d\tau +$$
$$+ \int_0^{T_1} \frac{\partial F_1}{\partial y_3}\,d\tau + \mu \int_0^{T_1} \frac{\partial F_2}{\partial y_3}\,d\tau + \cdots$$

As a consequence of the well-known Jacobian integral of energy, the fourth and sixth of our equations (87) are not independent of each other. And it is easily seen that $\varphi_1 = \varphi_3$ as the right hand sides of the last three equations (87) are reduced merely to the critical terms of argument ζ (see the study of the particular terms of various categories carried out hereafter ad 1, 2, 3, 4, 5, pp. 67—70) and as such show clearly that

$$\frac{\partial^2 [F_1]}{\partial y_1 \partial y_k} = \frac{\partial^2 [F_1]}{\partial y_3 \partial y_k}. \tag{88}$$

Just as in all previous cases every term independent of the small increments β_i, γ_i must disappear namely

$$\int_0^{T_1} \frac{\partial F_1}{\partial y_1} d\tau = \int_0^{T_1} \frac{\partial F_1}{\partial y_2} d\tau = \int_0^{T_1} \frac{\partial F_1}{\partial y_3} d\tau \equiv 0. \tag{89}$$

Otherwise neither β_i nor γ_i could be annulled with the disappearing μ. As usual these equations lead to the conditions of a symmetrical conjunction and opposition. For more details see pp. 37, 44 of the present paper. Moreover all terms containing one single derivation in y_k namely $\frac{\partial^2 F_1}{\partial y_k \partial x_k}$ are composed of sinus series, whose arguments vanish owing to the supposed symmetrical conjunction and opposition. In this way it is immediately found that our integral equations are reduced to the following form:

Let us put for the sake of brevity, as previously in case II (general case of three bodies) p. 44, Chapter II, Part II, for the average value

$$\frac{1}{T_1} \int_0^{T_1} F_1 d\tau = [F_1], \text{ etc.} \tag{90}$$

$$\begin{aligned}
n\varkappa + (n+n')\eta + n\vartheta - \beta_1 \frac{\partial^2 F_0}{\partial x_1^2} - \beta_2 \frac{\partial^2 F_0}{\partial x_1 \partial x_2} - \beta_3 \frac{\partial^2 F_0}{\partial x_1 \partial x_3} - \mu \frac{\partial [F_1]}{\partial x_1} + \cdots &= 0,\\
+ n'\vartheta - \beta_1 \frac{\partial^2 F_0}{\partial x_1 \partial x_2} - \beta_2 \frac{\partial^2 F_0}{\partial x_2^2} - \beta_3 \frac{\partial^2 F_0}{\partial x_2 \partial x_3} - \mu \frac{\partial [F_1]}{\partial x_2} + \cdots &= 0,\\
- n'\vartheta - \beta_1 \frac{\partial^2 F_0}{\partial x_1 \partial x_3} - \beta_2 \frac{\partial^2 F_0}{\partial x_2 \partial x_3} - \beta_3 \frac{\partial^2 F_0}{\partial x_3^2} - \mu \frac{\partial [F_1]}{\partial x_3} + \cdots &= 0,\\
\gamma_1 \frac{\partial^2 [F_1]}{\partial y_1^2} + \gamma_2 \frac{\partial^2 [F_1]}{\partial y_1 \partial y_2} + \gamma_3 \frac{\partial^2 [F_1]}{\partial y_1 \partial y_3} + \mu \frac{\partial [F_2]}{\partial y_1} + \cdots &= 0, \qquad (91)\\
\gamma_1 \frac{\partial^2 [F_1]}{\partial y_1 \partial y_2} + \gamma_2 \frac{\partial^2 [F_1]}{\partial y_2^2} + \gamma_3 \frac{\partial^2 [F_1]}{\partial y_2 \partial y_3} + \mu \frac{\partial [F_2]}{\partial y_2} + \cdots &= 0,\\
\gamma_1 \frac{\partial^2 [F_1]}{\partial y_1 \partial y_3} + \gamma_2 \frac{\partial^2 [F_1]}{\partial y_2 \partial y_3} + \gamma_3 \frac{\partial^2 [F_1]}{\partial y_3^2} + \mu \frac{\partial [F_2]}{\partial y_3} + \cdots &= 0.
\end{aligned}$$

As will be shown later (see critical terms of category 2), the coefficients of these equations, which define our unknown implicit functions φ_i, ψ_i are build up of trigonometrical terms, which just provide the necessary elements of the resolving determinant Hessian—Jacobi—Poincaré.

With the view not to interrupt the most intricate analysis of the particular elements of the Hessian—Jacobi—Poincaré, I shall postpone the full explanation of solving these latter conditional equations till a more suitable moment (see pp. 70, 71).

So let us first admit that we have succeeded in solving these equations à la Cauchy—Weierstrass, defining our implicit functions β_1, β_2, β_3, γ_1, γ_2, γ_3 etc. through whose medium only the existence of periodic solutions can be guaranteed.

Then we can proceed to examine the mean motions holding true through the resulting periodic or secular motion.

These are easily found as previously in Chapter I and II by mere analysis of the infinitesimal substitution in question.

As we have replaced the original time by $t = (1+\varkappa)(1+\vartheta)(1+\eta)\tau$ and y_k by resp.:

$$\begin{aligned} y_1'' &= y_1 + \frac{n' t}{(1+\vartheta)(1+\varkappa)} = y_1 + n'(1+\eta)\tau, \\ y_2'' &= y_2 + \frac{n' t}{(1+\eta)(1+\varkappa)} = y_2 + n'(1+\vartheta)\tau, \qquad (79)\ (81) \\ y_3'' &= y_3 - \frac{n' t}{(1+\eta)(1+\varkappa)} = y_3 - n'(1+\vartheta)\tau, \end{aligned}$$

we must carry out the quadratures of our integral equations in such a way that the new variables y_k'' may yield deviations at the beginning and at the close of the period $T = (1+\varkappa)(1+\vartheta)(1+\eta)T_1$ which disappear entirely except for the finite amount of the full angles 2π, 360°.

Every mean movement, even one produced artificially through the introduction of the new dependent (primed) variables y_k'', will remain preserved by a suitable choice of the starting conditions for the periodic or secular solution, which we have just admitted.

In the case dealt with above our integral equations give the result

$$y_1'' = M + \frac{n' t}{(1+\vartheta)(1+\varkappa)} = M_0 + n\tau + n'\tau + \gamma_1 + \psi_1'' = M_0 + \qquad (92)$$

$$+ \frac{n t}{(1+\vartheta)(1+\eta)(1+\varkappa)} + \frac{n' t}{(1+\vartheta)(1+\eta)(1+\varkappa)} + \gamma_1,$$

$$y_2'' = \bar{\omega} + \frac{n' t}{(1+\eta)(1+\varkappa)} = \bar{\omega} + n' \tau + \gamma_2 + \psi_2'' = \bar{\omega} \qquad + \frac{n' t}{(1+\vartheta)(1+\eta)(1+\varkappa)} + \gamma_2,$$

$$y_3'' = M - M' + \tilde{\Omega} - \frac{n' t}{(1+\eta)(1+\varkappa)} = - M_0 - \pi' + M_0 + \tilde{\Omega} - n' \tau + \gamma_3 + \psi_3'' =$$

$$= M_0 - M_0' - \pi' + \tilde{\Omega} - \frac{n' t}{(1+\vartheta)(1+\eta)(1+\varkappa)} + \gamma_3,$$

and returning now to the original unprimed variables by subtraction of resp. in y_1, y_2, y_3:

$$- \frac{n' t}{(1+\varkappa)(1+\vartheta)}, \qquad - \frac{n' t}{(1+\varkappa)(1+\eta)}, \qquad + \frac{n' t}{(1+\varkappa)(1+\eta)}$$

we immediately find out for the movements of the disturbed resulting motion,

$$\begin{aligned} f_1 &= \frac{n}{(1+\varkappa)(1+\vartheta)(1+\eta)} - \frac{n' \eta}{(1+\varkappa)(1+\vartheta)(1+\eta)}, \\ f_2 &= \qquad\qquad - \frac{\vartheta n'}{(1+\varkappa)(1+\vartheta)(1+\eta)}, \\ f_3 &= \qquad\qquad + \frac{\vartheta n'}{(1+\varkappa)(1+\vartheta)(1+\eta)}. \end{aligned} \tag{93}$$

As to the secular movement in the starting not primed variable $y_3 = M + \tilde{\Omega} - M'$ we have ascertained it to be

$$\frac{\vartheta n'}{(1+\varkappa)(1+\vartheta)(1+\eta)} \tag{94}$$

and at the same time in

$$y_1 = M, \quad \frac{n - \eta n'}{(1+\varkappa)(1+\vartheta)(1+\eta)}, \tag{95}$$

from which we gather that the resulting mean secular movement in the angle $\tilde{\Omega} - M'$ will be

$$\frac{n'(\vartheta + \eta)}{(1+\varkappa)(1+\vartheta)(1+\eta)} - \frac{n'}{(1+\varkappa)(1+\vartheta)(1+\eta)}, \tag{96}$$

consequently in the longitude of the node itself $\tilde{\Omega}$

$$\frac{n'(\vartheta + \eta)}{(1+\varkappa)(1+\vartheta)(1+\eta)} \tag{97}$$

and in the original starting length of the perigee $\bar{\pi} = \bar{\omega} + \tilde{\Omega}$

$$\frac{(\vartheta + \eta - \vartheta) n'}{(1+\varkappa)(1+\vartheta)(1+\eta)} = + \frac{n' \eta}{(1+\varkappa)(1+\vartheta)(1+\eta)}. \tag{98}$$

By this most simple method we have succeeded in finding three angular velocities (mean movements) namely f_1, f_2, f_3.

1. The first of them f_1 consists of two parts. And it clearly appears that even the first part $\frac{n}{(1+\varkappa)(1+\vartheta)(1+\eta)}$ can easily reach just the necessary speed of our real Moon.

This amount, about twelve times as large as the mean movement of the Earth $12\,n'$, can always be attained by a suitable choice of the free parameters (see the last page of this paper, the small divisor in (35 a) p. 21). On the contrary the amount η can always be chosen so as to correspond to the small secular motion of the Lunar node $\tilde{\Omega}$.

2. The angular velocity f_2 in $\tilde{\omega}$ gives the secular motion of the Lunar perigee.

3. The total mean movement in the variable y_3'', namely f_3, must disappear entirely, and it is only in this way that we are able to secure the existence of the most important critical terms (see later on, the terms of second category p. 68), and it is only in this way that the necessary critical cos coefficients of the Cauchy—Weierstrass conditions can be supplied. Let us postpone the full explanation till the analysis of the particular periodical terms on pp. 67—70.

Still I should like to point out that, although the total final amount of rotational velocity in $y_3'' + y_2'' = M - M' + \tilde{\omega} + \tilde{\Omega}$ namely $f_2 + f_3$ disappears identically, still we have easily gathered that the resulting secular movement in π appears to be fixed by

$$+\frac{n'\eta}{(1+\varkappa)(1+\vartheta)(1+\eta)}. \tag{99}$$

It appears most important to know the accurate amounts of these angular velocities thoroughly, whose arising has been urged by the introduction of our new (primed) variables y_k'', and by satisfying the integral equations $\varphi_i = \psi_i = 0$.

And indeed these quantities always figure in the development of the right-hand sides through the disturbing function, being unseparably combined with the old non-primed variables. In this way the explicit time appears inside all trigonometrical terms, where any y_k is contained.

We then have to take special care that the periodicity of the resulting movement is not spoiled.

With this in view let us pass to the study of all individual trigonometrical terms of the disturbing function, thus approaching the next task, namely the reckoning of the quadratures of our integral equations term by term.

As previously in chap. II. we shall have to examine the terms of three categories of F

$$F_1 = k^2 \sum A \cos(i_1 y_1 + i_2 y_2 + i_3 y_3), \quad F = F_0'' + \overline{F}_1 \mu + \cdots \tag{100}$$

i_k integers

$$F_0'' = -\frac{k^2}{2\alpha} - \frac{n' x_1}{(1+\varkappa)(1+\vartheta)} - \frac{n' x_2}{(1+\varkappa)(1+\eta)} + \frac{n' x_3}{(1+\varkappa)(1+\eta)}. \tag{101}$$

$$y_1 = M, \quad y_2 = \bar{\omega}, \quad y_3 = M - M' + \tilde{\Omega}. \qquad (74)\ (76)$$

1. Simple short periodic terms of the argument $i_1 y_1 + i_2 y_2 + i_3 y_3$.

After introducing the new primed variables and the distorted time, we immediately state the false explicit time τ (or t) obtaining by $t = (1+\varkappa)(1+\vartheta)(1+\eta)\tau$ in the argument in question

$$\begin{aligned} i_1 y_1 + i_2 y_2 + i_3 y_3 &= i_1 y_1'' + i_2 y_2'' + i_3 y_3'' - \frac{i_1 n' t}{(1+\vartheta)(1+\varkappa)} - \frac{i_2 n' t}{(1+\eta)(1+\varkappa)} + \\ &\quad + \frac{i_3 n' t}{(1+\eta)(1+\varkappa)} \qquad (102) \\ &= i_1(n\tau + n'\tau + M_0 + \gamma_1 + \psi_1) + i_2(\bar{\omega} + n'\tau + \gamma_2 + \psi_2) + \\ &\quad + i_3(M_0 - M_0' - \pi' + \Omega_0 - n'\tau + \gamma_3 + \psi_3). \end{aligned}$$

Now the false time must remain compatible with the total period of the motion, which circumstance gives three conditions, see (77)

$$n = n',\ T_1' = \frac{2\pi p}{n'}, \quad T_1 = r T_1', \quad T = (1+\varkappa)(1+\vartheta)(1+\eta)\, T_1 \tag{103}$$

$$T f_1 = 2\pi s_1, \quad T f_2 = 2\pi s_2, \quad T f_3 = 2\pi s_3, \tag{104}$$

r, s_k representing integral numbers.

These expressions by means of our f_k found above lead to the result

$$\begin{aligned} f_1 T &= \frac{2\pi p r}{n'}(1+\vartheta)(1+\varkappa)(1+\eta)\frac{n - n'\eta}{(1+\varkappa)(1+\vartheta)(1+\eta)} = 2\pi p r - 2\eta\pi r p = \pm 2\pi s_1, \\ f_2 T &= \frac{2\pi p r}{n'}(1+\vartheta)(1+\varkappa)(1+\eta)\frac{-\vartheta n'}{(1+\varkappa)(1+\vartheta)(1+\eta)} = -2\pi p r\vartheta \qquad = \pm 2\pi s_2, \qquad (105) \\ f_3 T &= \frac{2\pi p r}{n'}(1+\vartheta)(1+\varkappa)(1+\eta)\frac{\vartheta n'}{(1+\varkappa)(1+\vartheta)(1+\eta)} = 2\pi p r\vartheta \qquad = \pm 2\pi s_3. \end{aligned}$$

Hence if all s_k are to remain integers, we find the conditions limiting the choice of the small parameters ϑ and η (see later p. 72).

This statement enables us to carry out successfully all quadratures of the short-periodic terms in question. We have found that all arguments of the form

$$i_1 y_1 + i_2 y_2 + i_3 y_3 = \text{whole number of } 2\pi = 360^\circ \tag{106}$$

give the same result in the upper and lower limit of integration, except for a whole number of 2π for every cosinus term.

So the quadrature changes only every particular $\begin{matrix}\cos\\ \sin\end{matrix}$ into $\begin{matrix}-\sin\\ \cos\end{matrix}$ giving the same value in the upper as well in the lower limit, that is at the beginning and at the end of the whole period. These two amounts being of opposite sings, cancel mutually and give zero.

Thus we find that the short periodic terms of the first category cannot contribute anything to the final value of the coefficient in the integral equations in question.

2. Now let us examine the critical terms. These are terms which become independent of the time by the supposed commensurability of the mean movements

$$\frac{n}{n'} = \frac{p+q}{p} = 1, \quad q = 0.$$

In order to isolate these terms, we must pick up all arguments for which $\zeta = y_2'' + y_3''$, and their multiples $s\zeta$, (s integer positive or negative).

When passing from the original to the primed angular variables, we get new right-hand sides which show the false explicit time inside the trigonometric terms

$$\begin{aligned} y_1 &= y_1'' - n'\eta\tau = y_1'' - \frac{n'\eta t}{(1+\varkappa)(1+\vartheta)(1+\eta)}, \\ y_2 &= y_2'' - n'\vartheta\tau = y_2'' - \frac{n'\vartheta t}{(1+\varkappa)(1+\vartheta)(1+\eta)}, \\ y_3 &= y_3'' + n'\vartheta\tau = y_3'' + \frac{n'\vartheta t}{(1+\varkappa)(1+\vartheta)(1+\eta)}. \end{aligned} \tag{81}$$

Now the primed coordinates y_k'' are finally obtained from our integral equations in the form

$$\begin{aligned} y_1'' &= n\tau + n'\tau + M_0 + \gamma_1 + \psi_1'' = \frac{nt}{(1+\varkappa)(1+\eta)(1+\vartheta)} + \frac{n't}{(1+\varkappa)(1+\eta)(1+\vartheta)} + \\ &\qquad + M_0 + \gamma_1 + \psi_1'', \\ y_2'' &= n'\tau + \bar{\omega} + \gamma_2 + \psi_2'' = \frac{n't}{(1+\varkappa)(1+\eta)(1+\vartheta)} + \bar{\omega} + \gamma_2 + \psi_2'', \\ y_3'' &= -n'\tau + M_0 - M_0' - \pi' + \tilde{\Omega} + \gamma_3 + \psi_3'' = -\frac{n't}{(1+\varkappa)(1+\eta)(1+\vartheta)} + \\ &\qquad + M_0 - M_0' - \pi' + \tilde{\Omega} + \gamma_3 + \psi_3'', \end{aligned} \tag{107}$$

so that the mean movement in the original y_i is found to be in

$$y_1 - \frac{n'\eta}{(1+\varkappa)(1+\vartheta)(1+\eta)},$$
$$y_2 - \frac{n'\vartheta}{(1+\varkappa)(1+\vartheta)(1+\eta)}, \tag{108}$$
$$y_3 + \frac{\vartheta n'}{(1+\varkappa)(1+\eta)(1+\vartheta)}.$$

The functions $\psi_1'' = \psi_2'' = \psi_3'' = 0$, $\varphi_i = 0$, giving the final deviations of the starting configuration, become zero if only the integral equations prove to be solvable for the starting increments $\varkappa$, ϑ, η, β_1, β_2, β_2, γ_1, γ_2, γ_3 of the implicit functions in question. In this case clearly, and only in this case, the whole movement is ruled by the same conditions at the start as well as towards the close of the prolonged period and the disturbed motion becomes periodic.

Our principal aim is to prove the solubility of the conditions (91) p. 63) or, which is the same, to find out a suitable non-disappearing Hessian—Jacobi—Poincaré.

For this purpose it first appears necessary to ascertain whether all the factors of the false time mentioned above disappear out of the critical argument ζ. It is only in this way that a corresponding critical term changes from a trigonometrical term to be integrated into a simple constant h, whose integration is carried out by adding the factor t and inserting the upper and lower limit, namely $h\,T - h\,0 = h\,T$, etc. Only through the existence of these constant terms, the possibility of non-disappearing elements of the determinant and of a solution of the integral equations for the implicit functions $\varkappa$, ϑ, η, β_i, γ_i, can be guaranteed. The aforesaid elements are constituted by critical cosinus terms of zero argument.

The existence of a non-disappearing Hessian can be proved only in this way

$$s\,\zeta = s\,(y_2'' + y_3'') - s\,n'\,\vartheta\,\tau + s\,n'\,\vartheta\,\tau. \tag{109}$$

This time, fortunately, we are no longer obliged to satisfy special diophantine equations as int he previous case Chapter II, p. 50, this being brought about by the simplicity of the structure of our disturbing function. And indeed the latter, in our satellite case, proceeds according to multiples of the two mean anomalies M, M', $n\,t$, $n'\,t$, which are reduced because $n = n'$ to one simple anomaly and a Fourier series of one time argument only.

Just in view of these simplifying circumstances the fundamental transformation (79), (80), (81), has been constructed.

On the whole we see that this time it is solely our choice of suitable variables which guarantees our aim, the entire disappearance of time from all critical arguments.

3 and 4. Judging from our previous study Chapter II we now have to examine the terms analogue to ζ' and $m_1 \zeta + m_2 \zeta'$ pp. 51, 52. However, the whole structure of the present satellite case shows clearly that these small divisor-terms do not exist at all.

5. Finally we have to consider the secular terms within the meaning of the classics.

These terms are formed by

a) really constant terms, without trigonometrical argument, build up exclusively from scalar elements α, ε, a', e'.

These terms clearly contribute to the constant coefficients of the integral equations. However, their mere existence would not suffice to prove new periodic orbits by the same reason as explained in the previous case Chapter II, no. 5, p. 53.

As was ascertained therein the mere derivation for the sake of obtaining the necessary coefficients, elements of the determinant

$$\frac{\partial^2 F}{\partial y_i \partial x_k} = \frac{\partial^2 F}{\partial y_i \partial y_k} \equiv 0, \tag{110}$$

yield zero everywhere, as according to our supposition (a) they do not contain angular arguments y_i, at all.

Consequently the most important part of the Hessian disappears identically and thus the calamity which Poincaré complains is brought about, in another way, viz. the Hessian is zero.

(b) terms of arguments $e e' \cos(\pi - \pi')$, non critical and still independent of M, see in the expression (42), First Part, § 4, p. 25 namely $\delta = \cdots + h^2 \varepsilon^2 \cos \bar{\omega} + \cdots$ etc.

It can easily be seen that these terms lead only to new long-periodical terms. The arguments can clearly be written

$$y_2 = \bar{\omega} = y_2'' - n'(1 + \vartheta) t = y_2'' - \frac{n' t}{(1+\eta)(1+\varkappa)} = n' \tau + \psi_2'' - n' \tau - n' \vartheta = - n' \vartheta.$$

and there is no possibility to destroy or get rid of the false time τ. These long-periodical terms caused by small divisors never cease to exist.

Now let us pass to the solution of our integral equations (91) which determine the unknown implicit functions and their starting increments β_i, γ_i, $\varkappa$, ϑ, η, and that has been postponed hitherto for the sake of clearness (see p. 64). Again as in the previous instructive case Chapter II, pp. 54, 55 we must be fully aware of the large freedom by which the supernumerary unknown quantities can always be disposed of.

And indeed the number of the unknown is always greater than that of the independent equations. Morover the coefficients of the aforesaid equations prove to be composed entirely of scalar elements, which represent further variable starting parameters, freely eligible.

In our satellite problem, where we are concerned with the commensurability of the mean movements $\frac{n}{n'} = 1$, we have full liberty to choose among the parameters $\alpha = a'$, ε, e' and the increments β_i, γ_i, $\varkappa$, ϑ, η, with the restriction of five integral equations and three further equations fixing up the parameters $\varkappa$, ϑ, η. There still remain $12-(5+3)=4$ free quantities.

But it appears more recommendable to clear up these large points of vantage by explaining the top important configurations thus attainable.

Let us return to our concrete system, considering our starting ellipse of the Moon-Planet and the circle of the Sun (namely to our integral equations written down above pp. 62, 63). — I first remember that in consequence of the existence of the Jacobian Integral of Energy the last and the fourth equation $\varphi_1 = 0$, $\varphi_3 = 0$ become dependent one on another. By this fact we are enabled simply to skip the aforesaid fifth equation and to retain only the five remaining ones namely

$$\psi_1'' = \psi_2'' = \psi_3'' = \frac{\varphi_1}{\mu} = \frac{\varphi_2}{\mu} = 0. \tag{87}$$

If written down explicitly, including all parameters $\varkappa$, ϑ, η, they clearly show the possibility of solution in case $\varkappa = \vartheta = \eta = 0$. Even so in the more general case treated herewith, where $\varkappa \gtrless \vartheta \gtrless \eta \gtrless 0$, the Hessian—Jacobi—Poincaré has a form which clearly shows its value, although most of the elements of the determinant resp. matrix

$$\frac{\partial(\psi_1'', \psi_2'', \psi_3'')}{\partial(\beta_1, \beta_2, \beta_3)}, \tag{111}$$

$$\left\{\begin{matrix} n & n+n' & n & -\frac{\partial^2 F_0}{\partial x_1^2} & -\frac{\partial^2 F_0}{\partial x_1 \partial x_2} & -\frac{\partial^2 F_0}{\partial x_1 \partial x_3} \\ 0 & 0 & n' & -\frac{\partial^2 F_0}{\partial x_1 \partial x_2} & -\frac{\partial^2 F_0}{\partial x_2^2} & -\frac{\partial^2 F_0}{\partial x_2 \partial x_3} \\ 0 & 0 & -n' & -\frac{\partial^2 F_0}{\partial x_1 \partial x_3} & -\frac{\partial^2 F_0}{\partial x_2 \partial x_3} & -\frac{\partial^2 F_0}{\partial x_3^2} \end{matrix}\right\} \tag{112}$$

disappear.

In order to approach our aim I destroy the generally non-disappearing terms of the second and third equations $\frac{\partial [F_1]}{\partial x_2}$, $\frac{\partial [F_1]}{\partial x_3}$, by putting

$$n'\vartheta = \mu \frac{\partial [F_1(\varepsilon, e', \iota)]}{\partial x_2^0}, \tag{113}$$

$$-n'\vartheta = \mu \frac{\partial [F_1]}{\partial x_3^0}$$

and choosing $\vartheta = \frac{1}{pr}\cdot$

The right-hand sides of these equations, being functions dependent upon the three starting elements (freely eligible) α, ε, ι', can always bring about the possibility of these two equalities; there always remains a certain freedom in the corresponding choice of ι.

Our system of integral equations changes to the remarkable form

$$\begin{aligned} \psi_1'' = 0 &= \varkappa n + \eta(n'+n) + n\vartheta - \beta_1 \frac{\partial^2 F_0}{\partial x_1^2} - \beta_2 \frac{\partial^2 F_0}{\partial x_1 \partial x_2} - \beta_3 \frac{\partial^2 F_0}{\partial x_1 \partial x_3} - \mu \frac{\partial [F_1]}{\partial x_1} + \cdots \\ \psi_2'' = 0 &= - \beta_1 \frac{\partial^2 F_0}{\partial x_1 \partial x_2} - \beta_2 \frac{\partial^2 F_0}{\partial x_2^2} - \beta_3 \frac{\partial^2 F_0}{\partial x_2 \partial x_3} - \mu^2 \frac{\partial [F_2]}{\partial x_2} + \cdots \\ \psi_3'' = 0 &= - \beta_1 \frac{\partial^2 F_0}{\partial x_1 \partial x_3} - \beta_2 \frac{\partial^2 F_0}{\partial x_2 \partial x_3} - \beta_3 \frac{\partial^2 F_0}{\partial x_3^2} - \mu^2 \frac{\partial [F_2]}{\partial x_3} + \cdots \end{aligned} \tag{114}$$

$$\begin{aligned} \frac{\varphi_1}{\mu} &= \gamma_1 \frac{\partial^2 [F_1]}{\partial y_1^2} + \gamma_2 \frac{\partial^2 [F_1]}{\partial y_1 \partial y_2} + \gamma_3 \frac{\partial^2 [F_1]}{\partial y_1 \partial y_3} + \mu \frac{\partial [F_2]}{\partial y_1} + \cdots = 0, \\ \frac{\varphi_2}{\mu} &= \gamma_1 \frac{\partial^2 [F_1]}{\partial y_1 \partial y_2} + \gamma_2 \frac{\partial^2 [F_1]}{\partial y_2^2} + \gamma_3 \frac{\partial^2 [F_1]}{\partial y_2 \partial y_3} + \mu \frac{\partial [F_2]}{\partial y_2} + \cdots = 0. \end{aligned} \tag{115}$$

In this way a new Hessian—Jacobi—Poincaré reappears, and moreover we get 5 equations for determining eight unknowns, namely $\varkappa$, η, β_1, β_2, β_3, γ_1, γ_2, γ_3. As we choose the customary $\gamma_3 = 0$, we are able to put $\beta_1 = 0$, thus obtaining five equations for six unknown functions, which run thus

$$\begin{aligned} \psi_1'' &= \varkappa n + \eta(n'+n) + n\vartheta - \beta_2 \frac{\partial^2 F_0}{\partial x_1 \partial x_2} - \beta_3 \frac{\partial^2 F_0}{\partial x_1 \partial x_3} - \mu \frac{\partial [F_1]}{\partial x_1} + \cdots = 0, \\ \frac{\psi_2''}{\mu} &= - \beta_2 \frac{\partial^2 [F_1]}{\partial x_2^2} - \beta_3 \frac{\partial^2 [F_1]}{\partial x_2 \partial x_3} - \mu \frac{\partial [F_2]}{\partial x_2} + \cdots = 0, \\ \psi_3'' &= - \beta_2 \frac{\partial^2 F_0}{\partial x_2 \partial x_3} - \beta_3 \frac{\partial^2 F_0}{\partial x_3^2} - \mu^2 \frac{\partial [F_2]}{\partial x_3} + \cdots = 0, \\ \frac{\varphi_1}{\mu} &= \gamma_1 \frac{\partial^2 [F_1]}{\partial y_1^2} + \gamma_2 \frac{\partial^2 [F_1]}{\partial y_1 \partial y_2} + \cdots + \mu \frac{\partial [F_2]}{\partial y_1} + \cdots = 0, \\ \frac{\varphi_2}{\mu} &= \gamma_1 \frac{\partial^2 [F_1]}{\partial y_1 \partial y_2} + \gamma_2 \frac{\partial^2 [F_1]}{\partial y_2^2} + \cdots + \mu \frac{\partial [F_2]}{\partial y_2} + \cdots = 0. \end{aligned} \tag{116}$$

Their determinant being manifestly distinct from zero, the problem is solved.

I do not intend to enter into more details in these preliminary outlines. I will only add some further explanatory remarks concerning the most propable increase of the angular speed of our ideal Moon caused by all previous proceedings. With this in view I should like to call the attention of astronomers to the remarkable set of small divisors ascertained by the construction of the canonical elements studied at the end of the previous First Part. The necessary derivations of the chosen canonical elements carried out in detail see (35 a) I p. 21 show clearly the strong increase of $\varkappa$ in agreement with the well-known reckoning in the case of asteroidal Hecuba movement of perihelion and nodes affected by small divisors l.c. e (excentricity of the asteroidal orbit) etc. (see Schwarzschild A.N. 160, p. 395. Heinrich A.N. 192.). Further we shall recall that the necessary critical (secular) terms of the disturbing function appear exclusively in the indirect Lagrangian part namely $-\frac{r\cos\sigma}{a'^2}$, and as such, are not diminished by small factors ε, e', $\sin\frac{\iota}{2}$, but manifest the full zero (finite) order. Furthermore the free choice mentioned above admits always to put $\varkappa n \doteq 12\, n'$.

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