STATE SPACES OF C^*-ALGEBRAS

BY

ERIK M. ALFSEN, HARALD HANCHE-OLSEN and
FREDERIC W. SHULTZ

University of Oslo, Norway and Wellesley College, Wellesley Mass., U.S.A.

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§ 1. Introduction

The purpose of this paper is to characterize the state spaces of C^*-algebras among the state spaces of all JB-algebras. In a previous paper [6] we have characterized the state spaces of JB-algebras among all compact convex sets. Together, these two papers give a complete geometric characterization of the state spaces of C^*-algebras.

Recall from [6] that the state spaces of JB-algebras will enjoy the *Hilbert ball property*, by which the face $B(\varrho, \sigma)$ generated by an arbitrary pair ϱ, σ of extreme states is (affinely isomorphic to) the unit ball of some real Hilbert space, and that there actually exist such faces of any given (finite or infinite) dimension for suitably chosen JB-algebras. In the present paper we show that for an arbitrary pair ϱ, σ of extreme states of a C^*-algebra, then the dimension of $B(\varrho, \sigma)$ is three or one. This statement, which we term the *3-ball property*, is the first of our axioms for state spaces of C^*-algebras. The second and last axiom is a requirement of orientability: the state space K of a JB-algebra with the 3-ball property is said to be *orientable* if it is possible to make a "consistent" choice of orientations for the 3-balls $B(\varrho, \sigma)$ in the w^*-compact convex set K, the idea being that the orientation shall never be suddenly reversed by passage from one such ball to a neighbouring one. (See § 7 for the precise definition.) Thus we have the following:

MAIN THEOREM. *A JB-algebra A with state space K is (isomorphic to) the self-adjoint part of a C^*-algebra iff K has the 3-ball property and is orientable.*

Note that a C^*-algebra, unlike a JB-algebra, is not completely determined by the affine geometry and the w^*-topology of its state space. However, the state space does determine the Jordan structure, and with this prescribed we have a 1–1 correspondence between C^*-structures and consistent orientations of the state space. Thus, for C^*-algebras the *oriented state space* is a dual object from which we can recapture all relevant structure.

We will now briefly discuss the background for the problem, and then indicate the content of the various sections.

By results of Kadison [24], [26], [29], the self-adjoint part $\mathfrak{A}_{sa}$ of a C^*-algebra $\mathfrak{A}$ with state space K is isometrically order-isomorphic to the space $A(K)$ of all w^*-continuous affine functions on K. More specifically, $\mathfrak{A}_{sa}$ is an *order unit space* (a "function system" in Kadison's terminology), and the order unit spaces A are precisely the $A(K)$-spaces where K is a compact convex subset of a locally convex Hausdorff space; (in fact K can be taken to be the state space of A, formally defined as in the case of a C^*-algebra). Thus, the problem of characterizing the state spaces of C^*-algebras among all compact convex sets, is equivalent to that of characterizing the self-adjoint parts of C^*-algebras among all order unit spaces. This problem is of interest in its own right, and it also gains importance by the applications to quantum mechanics, where the order unit space $\mathfrak{A}_{sa}$ represents bounded observables, while the full C^*-algebra $\mathfrak{A}$ is devoid of any direct physical interpretation. Note in this connection that the Jordan product in $\mathfrak{A}_{sa}$ (unlike the ordinary product in $\mathfrak{A}$) is physically relevant, and that the pioneering work on Jordan algebras by Jordan, von Neumann and Wigner [19] was intended to provide a new algebraic formalism for quantum mechanics (cf. also [30]).

In [25] Kadison proved that the Jordan structure in the order unit space $\mathfrak{A}_{sa}$ is completely determined, in that any unital order automorphism of $\mathfrak{A}_{sa}$ is a Jordan automorphism, and he pointed out the great importance of the Jordan structure for the study of C^*-algebras. An axiomatic investigation of normed Jordan algebras was carried out in [7]. Here the basic notion is that of a *JB-algebra*, which is defined to be a real Jordan algebra with unit 1 which is also a Banach space, and where the Jordan product and the norm are related as follows:

$$\|a \circ b\| \leqslant \|a\|\,\|b\|, \quad \|a^2\| = \|a\|^2, \quad \|a^2\| \leqslant \|a^2+b^2\|. \tag{1.1}$$

These axioms are closely related to those of Segal [32], and the JB-algebras will include the finite dimensional formally real algebras studied by Jordan, von Neumann and Wigner (which can be normed in a natural way), as well as the norm closed Jordan algebras of bounded self-adjoint operators on a Hilbert space (JC-algebras) studied by Topping, Størmer and Effros [41], [37], [39], [18]. The main result of [7] shows that the study of general JB-algebras can be reduced to the study of JC-algebras and the exceptional algebra M_3^8 of all self-adjoint 3×3-matrices over the Cayley numbers. (For related results, see [34].)

The geometric description of the state spaces of JB-algebras involves, in addition to the Hilbert ball property, three more axioms stated in terms of facial structure. (They are quoted in § 8. See [6] for further details.) These axioms relate the geometry of the state space to the projection lattice and the spectral theory of the "enveloping JBW-algebra" (generalizing the enveloping von Neumann algebra of a C^*-algebra). The connection between faces and projections was first noted by Effros and Prosser in their papers on ideals in operator algebras [17], [31]. This connection was the starting point for the development of a non-commutative spectral theory for convex sets [4], [5], which was used extensively in the passage from compact convex sets to Jordan algebras in [6].

The transition from JB-algebras to C^*-algebras presents difficulties of a new kind due to the lack of uniqueness. There is no natural candidate for the C^*-product; it must be chosen, and orientability is needed to make this choice possible. The first time a notion of orientation was used for a similar purpose, was in Connes' paper [14], where he gave a geometric characterization of the cones associated with von Neumann algebras via Tomita-Takesaki theory. Although both the setting and the actual definition are different in the two cases, they are related in spirit. In both cases the orientation serves the same purpose, namely to provide the complex Lie structure when the Jordan product is given. (See also the papers by Bellissard, Iochum and Lima [8], [9], [10].)

In the present paper, § 2 provides the necessary machinery of states and representations for JB-algebras. The results here are for the most part analogues of well known results for C^*-algebras.

In § 3 we go into the classification theory and concentrate on JB-algebras of "complex type". They are shown to be precisely those for which the state space has the 3-ball property.

§ 4 provides a technical result which is also of some independent interest, namely that a JB-algebra of complex type acts reversibly in each concrete representation on a complex Hilbert space.

In § 5 it is shown that each JB-algebra A admits an enveloping C^*-algebra $\mathfrak{A}$ with a universal property relating Jordan and *-homomorphisms. It is shown that if A is of complex type, the pure states of $\mathfrak{A}$ form (except for degeneracy) a double covering of the set of pure states of A.

In § 6 we discuss the orientation of balls in the normal state space of $B(H)$.

§ 7 is a general treatment of orientability for state spaces of JB-algebras of complex type.

§ 8 contains the main theorem.

The prerequisites include standard theory of C^*- and von Neumann algebras plus the theory of JB-algebras as presented in [7]. We will also draw upon the portion of [6] which establishes properties of state spaces of JB-algebras. The rest of [6] (and thus indirectly the work in [4] and [5]) will be used only when the main theorem of the present paper, characterizing state spaces of C^*-algebras among state spaces of JB-algebras, is combined with the main theorem of [6], characterizing state spaces of JB-algebras among all compact convex sets, to give a complete geometric description of the state spaces of C^*-algebras (Corollary 8.6).

§ 2. States and representations for JB-algebras

This section is of preliminary nature, and the results are for the most part analogues of well known results for C^*-algebras.

Note that when we work in the context of Jordan algebras, we will use the word *ideal* to mean a norm closed Jordan ideal. Also if A, B are Jordan algebras and $T: A \to B$ is a bounded linear map, then we denote the adjoint map from B^* into A^* by T^*. Occasionally if $T: A^{**} \to B^{**}$ is a σ-weakly continuous linear map, we will denote the adjoint map from $B^* \to A^*$ by T^*. Recall that a *JBW-algebra* is a JB-algebra which is a Banach dual space, and that the *enveloping JBW-algebra* of a JB-algebra A is A^{**} with the (right = left) Arens product (cf. [34] and [6]).

We now consider two JBW-algebras M_1 and M_2 and a homomorphism $\varphi: M_1 \to M_2$ which is σ-weakly continuous (i.e. continuous in the w^*-topology determined by the unique preduals of M_1 and M_2). By the same argument as for von Neumann algebras [33; Prop. 1.16.2], the unit ball of $\varphi(M_1)$ is σ-weakly compact. Hence $\varphi(M_1)$ is σ-weakly closed in M_2, and so it is a JBW-algebra. In other words: *A σ-weakly continuous homomorphic image of a JBW-algebra in a JBW-algebra is a JBW-algebra.*

We next relate homomorphisms of JB-algebras to σ-weakly continuous homomorphisms of their enveloping JBW-algebras. Here the results and proofs for C^*-algebras [33; Prop. 1.17.8 and 1.21.13] can be transferred without significant change. Specifically: *If $\varphi: A \to M$ is a homomorphism from a JB-algebra A into a JBW-algebra M, then there* *exists a unique σ-weakly continuous homomorphism* $\tilde{\varphi}: A^{**} \to M$ *which extends* φ; *moreover* $\tilde{\varphi}(A^{**})$ *is the σ-weak closure of* $\varphi(A)$ *in* M. (When no confusion is likely to arise, we will denote the extended homomorphism by φ instead of $\tilde{\varphi}$.)

We will now provide Jordan analogues of the basic notions in the representation theory of C^*-algebras. Since a JB-algebra might not have any (non-zero) representations into $B(H)_{sa}$, these notions can not be carried over directly. However, it is reasonable to replace $B(H)$ by any JBW-factor of type I when we work with general JB-algebras. (Recall that the JBW-factors of type I are the JBW-algebras with trivial center which contain minimal idempotents, and that they have been completely classified [7; Th. 8.6] and [37; Th. 5.2]. We return to this classification in § 3.) Note that two representations $\varphi_i: \mathfrak{A} \to B(H_i)$ $(i=1,2)$ of a C^*-algebra $\mathfrak{A}$ are unitarily equivalent iff there exists a *-isomorphism Φ from $B(H_1)$ onto $B(H_2)$ such that $\varphi_2 = \Phi \circ \varphi_1$ [15; Cor. III.3.1]. Observe also that a representation $\varphi: \mathfrak{A} \to B(H)$ of a C^*-algebra $\mathfrak{A}$ is irreducible iff $\varphi(\mathfrak{A})$ is weakly $(=\sigma$-weakly) dense in $B(H)$ [33; Prop. 1.21.9]. This motivates the following:

Definitions. A *representation* of a JB-algebra A is a homomorphism $\varphi: A \to M$ into a type I JBW-factor M. We say φ is a *dense representation* if $\varphi(A)^- = M$ (σ-weak closure). Two representations $\varphi_i: A \to M_i$ $(i=1,2)$ are said to be *Jordan equivalent* if there exists an isomorphism Φ of M_1 onto M_2 such that $\varphi_2 = \Phi \circ \varphi_1$.

LEMMA 2.1. *Let A be a JB-algebra with state space K and let $\varphi_i: A \to M_i$ $(i=1,2)$ be dense representations. Then φ_1 and φ_2 are Jordan equivalent iff the unique σ-weakly continuous extensions $\tilde{\varphi}_i: A^{**} \to M_i$ satisfy* $\ker \tilde{\varphi}_1 = \ker \tilde{\varphi}_2$.

Proof. Suppose that φ_1 and φ_2 are equivalent, and let Φ be a Jordan isomorphism of M_1 onto M_2 such that $\varphi_2 = \Phi \circ \varphi_1$. Since Φ is σ-weakly continuous, we also have $\tilde{\varphi}_2 = \Phi \circ \tilde{\varphi}_1$ and so $\ker \tilde{\varphi}_1 = \ker \tilde{\varphi}_2$.

Conversely, suppose $\ker \tilde{\varphi}_1 = \ker \tilde{\varphi}_2$. Note that $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are surjective. Thus we can define $\Phi: M_1 \to M_2$ by $\Phi(\tilde{\varphi}_1(a)) = \tilde{\varphi}_2(a)$ for all $a \in A^{**}$. This Φ determines a Jordan equivalence of φ_1 and φ_2. □

We will now relate the representations of a JB-algebra A to the state space K. As usual, the extreme points of K are called *pure states*, and the set of pure states is denoted $\partial_e K$. We recall from [7] how one can associate with any pure state ϱ on A a dense representation $\varphi_\varrho: A \to A_\varrho$ with $A_\varrho = c(\varrho) \circ A^{**}$ and $\varphi_\varrho(a) = c(\varrho) \circ a$ for $a \in A$, where $c(\varrho)$ is the *central support* of ϱ, i.e. the smallest central idempotent of A^{**} such that $\langle c(\varrho), \varrho \rangle = 1$. (See [7; § 5] for the existence of $c(\varrho)$, and see [7; Prop. 5.6 and Prop. 8.7] for the demonstration that

φ_ϱ is a dense representation.) Recall also that a face F of K is said to be *split* if it admits a, necessarily unique, complementary face F' such that K is direct convex sum of F and F' (cf. [1; § 6]).

PROPOSITION 2.2. *Let A be a JB-algebra with state space K. If $\varphi: A \to M$ is a dense representation, then there exists $\varrho \in \partial_e K$ such that φ is Jordan equivalent with φ_ϱ; moreover, φ^* maps the normal state space of M injectively onto the smallest split face of K containing ϱ. Two such dense representations $\varphi_i: A \to M_i$ $(i=1, 2)$ are Jordan equivalent iff the corresponding split faces coincide.*

Proof. Since $\ker \tilde{\varphi}$ is a σ-weakly closed ideal in A^{**}, there exists a central idempotent $c \in A^{**}$ such that $\ker \tilde{\varphi} = (1-c) \circ A^{**}$ [34; Lem. 2.1]. Let $P, Q: A^{**} \to A^{**}$ be the two σ-weakly continuous projections defined by $Pa = c \circ a$, $Qa = (1-c) \circ a$ for $a \in A^{**}$. Clearly the dual projections $P^*, Q^*: A^* \to A^*$ satisfy $P^*\varphi^* = \varphi^*$, $Q^*\varphi^* = 0$. Hence φ^* maps the normal state space of M onto $F = K \cap \operatorname{im} P^*$. Since $\tilde{\varphi}$ is surjective, φ^* will be injective. Clearly $P + Q = I$, from which it easily follows that $F = K \cap \operatorname{im} P^*$ is a split face of K with complementary face $F' = K \cap \operatorname{im} Q^*$.

We will show F is a minimal split face. To this end we consider an arbitrary split face G such that $F \cap G \neq \emptyset$, and we will prove $F \subseteq G$. Let G' be the face complementary to G. By linear algebra, there exists a unique bounded affine function on K which takes the value 1 on G and vanishes on G'. Let d be the corresponding element of A^{**}_{sa}, i.e. $\langle d, \sigma \rangle = 1$ for all $\sigma \in G$ and $\langle d, \sigma \rangle = 0$ for all $\sigma \in G'$. Since d is seen to be an extreme point of the positive part of the unit ball of A^{**}, it must be an idempotent. (The standard argument for C^*-algebras applies.) To show that d is central, it suffices by [7; Lemma 4.5] to verify the inequality $U_d a \leqslant a$ for all $a \geqslant 0$, $a \in A^{**}$. (Recall that $U_d a = \{dad\}$ where the brackets denote the Jordan triple product, and also that $U_d: A^{**} \to A^{**}$ is a positive linear map by [7; Prop. 2.7].) For given $a \in A^{**}$, $a \geqslant 0$ and for each $\sigma \in G'$ we have

$$0 \leqslant \langle U_d a, \sigma \rangle \leqslant \|a\| \langle U_d 1, \sigma \rangle = \|a\| \langle d, \sigma \rangle = 0.$$

Hence $U_d a$ vanishes on G'. Applying the same argument with $1-d$ in place of d and using [7; Cor. 2.10], we conclude that $U_d a$ coincides with a on G. But by linear algebra there can only be one such affine function on K, and this function is nowhere greater than a. Hence $U_d a \leqslant a$, which proves that d is a central idempotent. By assumption $F \cap G \neq \emptyset$, which implies $c \circ d \neq 0$. Since $\tilde{\varphi}$ is injective on $c \circ A^{**}$, we also have $\tilde{\varphi}(d) \neq 0$. Since M is a factor we must have $\tilde{\varphi}(d) = 1$, hence $c \leqslant d$, which in turn implies $F \subseteq G$.

Next we claim that the minimal split face F must contain pure states. In fact, the normal state space of the type I JBW-factor M contains pure states (cf. e.g. [6; p. 159]), therefore F also does. Let $\varrho \in F \cap \partial_e K$ be arbitrary. By the minimality, F is the smallest split face containing ϱ. Also $c(\varrho)=c$; for if $c(\varrho)<c$ then the same argument as above would provide a split face strictly contained in F. Now $\ker \tilde{\varphi}_\varrho = \ker \tilde{\varphi}$, and by Lemma 2.1, φ_ϱ and φ are Jordan equivalent.

Finally we consider two dense representations $\varphi_i: A \to M_i$ $(i=1, 2)$. Note that by the above definition of P, the split face $F=K \cap \operatorname{im} P^*$ is the annihilator of $\ker \tilde{\varphi}=(1-c)\circ A^{**}$, and vice versa. Hence the split faces corresponding to φ_1 and φ_2 coincide iff $\ker \tilde{\varphi}_1 = \ker \tilde{\varphi}_2$, and by Lemma 2.1 this equality holds iff φ_1 and φ_2 are Jordan equivalent. □

It follows from Proposition 2.2 that for every pure state ϱ of a JB-algebra there exists a smallest split face containing ϱ. We will denote this split face F_ϱ. (Note that our notation differs from that of [2] where F_ϱ denotes the smallest w^*-closed split face containing ϱ.)

Two pure states ϱ, σ of a JB-algebra will be called *equivalent* (or "non-separated by a split face") if $F_\varrho = F_\sigma$. By Proposition 2.2, ϱ and σ are equivalent iff the representations φ_ϱ and φ_σ are Jordan equivalent; hence the terminology.

Recall the brief notation $B(\varrho, \sigma)=\text{face}\,\{\varrho, \sigma\}$ used for any pair ϱ, σ of pure states.

PROPOSITION 2.3. *Let ϱ, σ be pure states of a JB-algebra A. If ϱ and σ are equivalent then $B(\varrho, \sigma)$ is a Hilbert ball of dimension at least two. If ϱ and σ are not equivalent, then $B(\varrho, \sigma)$ reduces to the line segment $[\varrho, \sigma]$.*

Proof. If ϱ and σ are equivalent, then it follows from the proof of [6; Th. 3.11] that $B(\varrho, \sigma)$ is the state space of a certain spin factor, and so it is a Hilbert ball. This ball must be of dimension at least two, since every spin factor is of dimension at least three.

If ϱ and σ are not equivalent, then it follows from [6; Prop. 3.1] that $B(\varrho, \sigma)=[\varrho, \sigma]$. □

By a *concrete representation* of a JB-algebra A on a complex Hilbert space H, we shall mean a Jordan homomorphism $\pi: A \to B(H)_{sa}$ with $\pi(1)=1$. Note that there exist JB-algebras without any non-zero concrete representation. (An example is M_3^8. See [7; § 9].)

A standard argument for C^*-algebras can be applied to show that if a concrete representation $\pi: A \to B(H)_{sa}$ is dense, then it is *irreducible*, i.e. there is no proper invariant subspace of H. The converse is false in general. (We shall return to this question in § 3.)

We say that two concrete representations $\pi_i: A \to B(H_i)_{sa}$ $(i=1, 2)$ of a JB-algebra A are *unitarily equivalent* (*conjugate*) if there exists a complex linear (conjugate linear) isometry u from H_2 onto H_1 such that

$$\pi_2(a) = u^* \pi_1(a) u \quad \text{for all } a \in A. \tag{2.1}$$

PROPOSITION 2.4. *If two dense concrete representations $\pi_i: A \to B(H_i)$ $(i=1,2)$ are Jordan equivalent, then they are either unitarily equivalent or conjugate. The only case in which π_1 and π_2 are both unitarily equivalent and conjugate at the same time, is when* $\dim H_1 = \dim H_2 = 1$.

Proof. By the assumptions there exists a Jordan isomorphism Φ from $B(H_1)_{sa}$ onto $B(H_2)_{sa}$ such that $\pi_2 = \Phi \circ \pi_1$. By a known theorem (see [25]) there exists an isometry $u: H_2 \to H_1$ which is either complex linear or conjugate linear such that $\Phi(b) = u^*bu$ for all $b \in B(H_1)_{sa}$. Now (2.1) is satisfied.

Assume now that $u: H_2 \to H_1$ is a complex linear isometry and that $v: H_2 \to H_1$ is a conjugate linear isometry such that $\Phi(b) = u^*bu = v^*bv$ for all $b \in B(H_1)_{sa}$. Then the two complex linear maps $a \mapsto u^*au$ and $a \mapsto v^*a^*v$ from $B(H_1)$ onto $B(H_2)$ must coincide. But the former of the two is a *-isomorphism while the latter is a *-anti-isomorphism. This is possible only if both algebras are commutative, i.e. $\dim H_1 = \dim H_2 = 1$. □

An *involution* of a complex Hilbert space H is a conjugate linear isometry $j: H \to H$ of period two; an example is $j: \sum \lambda_\gamma \xi_\gamma \mapsto \sum \bar{\lambda}_\gamma \xi_\gamma$ where $\{\xi_\gamma\}$ is any orthonormal basis. If $j: H \to H$ is an involution then uj is a conjugate linear isometry for each unitary u, and every conjugate linear isometry v is of this form. (Write $u = vj$ and note that $uj = v$ since $j^2 = 1$.)

To a given involution $j: H \to H$ we associate a *transpose map* $a \mapsto a^t$ from $B(H)$ onto itself by writing $a^t = ja^*j$. Clearly the transpose map is a *-anti-automorphism of order two for the C^*-algebra $B(H)$. If $\pi: A \to B(H)_{sa}$ is a concrete representation of a JB-algebra A, then the *transposed representation* $\pi^t: a \mapsto \pi(a)^t$ (w.r.t. to j) will be conjugate to π.

Thus, in the study of dense concrete representations of JB-algebras we encounter two natural equivalence relations: Jordan equivalence and unitary equivalence. Except for the one-dimensional case, each Jordan equivalence class splits in two (mutually conjugate) unitary equivalence classes.

§ 3. *JB*-algebras of complex type

The following classification theorem is essentially contained in [37] and [7; § 8]. Recall from [42] and [7; § 7] that a *spin factor* is, by definition, $H \oplus \mathbf{R}$ where H is a real Hilbert space of dimension at least two. Here Jordan multiplication is defined so that $1 \in \mathbf{R}$ acts as a unit and $a \circ b = (a|b)1$ where $a, b \in H$.

THEOREM 3.1. *The type I JWB-factors can be divided into the following classes (up to isomorphism):*

(i) $B(H)_s$, *the symmetric bounded operators on a real Hilbert space* H;

(ii) $B(H)_{sa}$, *the self-adjoint bounded operators on a complex Hilbert space* H;

(iii) $B(H)_{sa}$, *the self-adjoint bounded operators on a quaternionic Hilbert space* H;

(iv) *the spin factors;*

(v) *the exceptional algebra* M_3^8 *of self-adjoint* 3 *by* 3 *matrices over the Cayley numbers.*

Moreover, these classes are mutually disjoint, with the exceptions that the matrix algebras $M_2(\mathbf{R})_s$, $M_2(\mathbf{C})_{sa}$, *and* $M_2(\mathbf{H})_{sa}$ *are all spin factors.*

Note. The algebra M_2^8 of self-adjoint 2 by 2 matrices over the Cayley numbers is also seen to be a spin factor, see the proof of Prop. 3.2 below.

Proof. Let M be a type I JBW-factor. Assume that M is not isomorphic to M_3^8 or a spin factor. By [7, Th. 8.6 and Prop. 7.1], [34; Cor. 2.4], and [37; Th. 5.1] we may assume that M is concretely represented as an irreducible JW-algebra of type I_n on a complex Hilbert space H.

Let $\mathcal{R}(M)$ be the norm-closed real subalgebra of $B(H)$ generated by M. We claim that M is the self-adjoint part of $\overline{\mathcal{R}(M)}$, where the bar denotes σ-weak closure.

Indeed, let x be a self-adjoint element of $\overline{\mathcal{R}(M)}$. Then x is a σ-weak limit of sums of terms of the form $y_1 \dots y_n$, where each $y_j \in M$. Since $x = x^*$, $x = \frac{1}{2}(x + x^*)$ is a σ-weak limit of sums of terms $\frac{1}{2}(y_1 \dots y_n + (y_1 \dots y_n)^*) = \frac{1}{2}(y_1 \dots y_n + y_n \dots y_1)$, where each $y_j \in M$. By [37; Lemma 3.1] M is reversible, that is $y_1 \dots y_n + y_n \dots y_1 \in M$. Since M is σ-weakly closed $x \in M$, and the claim is proved.

If M is the self-adjoint part of a von Neumann algebra, then this algebra, being irreducible, equals $B(H)$. Then we have case (ii).

Otherwise, according to [37; Lemma 6.1] we have $\mathcal{R}(M) \cap i\mathcal{R}(M) = \{0\}$. Using Lemma 2.3 and Theorem 2.4 of [39], we get the direct sum decomposition

$$B(H) = \overline{\mathcal{R}(M)} \oplus i\overline{\mathcal{R}(M)}.$$

Thus we can define a σ-weakly continuous mapping Φ: $B(H) \to B(H)$ by setting

$$\Phi(x + iy) = (x - iy)^* = x^* + iy^* \quad (x, y \in \overline{\mathcal{R}(M)}).$$

Φ is easily seen to be a *-anti-automorphism of $B(H)$, and $\Phi^2 = I$.

In [38] it is proved that there exists a conjugate linear isometry j: $H \to H$ such that

$$\Phi(x) = j^{-1} x^* j \quad (x \in B(H)).$$

Since M is the self-adjoint part of $\overline{R(M)}$ we find that $x \in M$ iff x is self-adjoint and $\Phi(x) = x$, i.e.,

$$M = \{x \in B(H)_{sa} \mid xj = jx\}. \tag{3.1}$$

Because $\Phi^2 = I$, j^2 is a scalar multiple of the identity, say $j^2 = \lambda 1$, where $|\lambda| = 1$. Since j commutes with j^2, we find $j\lambda = \lambda j$. But, since j is conjugate linear, $j\lambda = \bar{\lambda} j$. This implies $\lambda = \bar{\lambda}$, so we have $j^2 = \pm 1$.

First, assume $j^2 = 1$. Let $K = \{\xi \in H: j\xi = \xi\}$. Then K is a real Hilbert space, $H = K \oplus iK$, and $J(\xi + i\eta) = \xi - i\eta$ whenever $\xi, \eta \in K$. By (3.1) $x \in M$ iff x is self-adjoint and leaves K invariant, that is, $x \in M \Leftrightarrow x|_K \in B(K)_s$. Since any $x \in B(H)$ is determined by its restriction to K, $M \cong B(K)_s$ follows. Thus we have case (i).

Next, assume $j^2 = -1$. Define $k = ij$. It is easily verified that i, j, k satisfy the multiplication table of the unit quaternions, so H may be considered a quaternionic vector space. Also, i, j and k are isometries and skew symmetric with respect to the real part of the inner product in H. Thus H is a quaternionic Hilbert space with the inner product

$$(\xi \mid \eta)_{\mathbf{H}} = \operatorname{Re}(\xi \mid \eta) - (\operatorname{Re}(i\xi \mid \eta))\, i - (\operatorname{Re}(j\xi \mid \eta))\, j - (\operatorname{Re}(k\xi \mid \eta))\, k.$$

By (3.1) the elements of M are exactly the self-adjoint $\mathbf{H}$-linear operators. Thus we have case (iii), and we have proved that M falls into one of the classes mentioned.

Next we prove that $M_2(\mathbf{R})_s$, $M_2(\mathbf{C})_{sa}$ and $M_2(\mathbf{H})_{sa}$ are spin factors. Note that, by definition, a finite dimensional spin factor admits a basis $1, s_1, \ldots, s_N$ where each s_m is a symmetry ($s_m^2 = 1$), and $s_m \circ s_n = 0$ if $m \neq n$. (Namely, let $s_1, \ldots, s_N$ be an orthonormal basis in H.) Defining

$$\begin{aligned} &s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \\ &s_4 = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \quad s_5 = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}, \end{aligned} \tag{3.2}$$

we find that $1, s_1, s_2$ (resp. $1, s_1, s_2, s_3$ resp. $1, s_1, \ldots, s_5$) is such a basis for $M_2(\mathbf{R})_s$ (resp. $M_2(\mathbf{C})_{sa}$ resp. $M_2(\mathbf{H})_{sa}$).

Finally, the disjointness of the isomorphism classes, with the stated exceptions, follows by considering orthogonal minimal idempotents e, f in M and noting that $\{(e+f)M(e+f)\}$ is isomorphic to $M_2(\mathbf{R})_s$, $M_2(\mathbf{C})_{sa}$, $M_3(\mathbf{H})_{sa}$, M and M_2^3 in the respective cases. $\square$

Definition. A type I JBW-factor is said to be *real* (resp. *complex* resp. *quaternionic*) if it is isomorphic to $B(H)_s$ for some real Hilbert space H (resp. $B(H)_{sa}$ for some complex resp. quaternionic Hilbert space).

In [6; § 3] the normal state spaces of type I JBW-factors are characterized geometrically. In particular, if ϱ, σ are distinct extreme points of the normal state space, the face $B(\varrho, \sigma)$ they generate is an exposed face affinely isomorphic to a Hilbert ball (the unit ball in a Hilbert space). The different types of JBW-factors can be distinguished by the dimension of this ball:

PROPOSITION 3.2. *Let M be a type I JBW-factor, and N its normal state space. If ϱ, σ are distinct extreme points of N, we have:*

(i) *If M is real, then* $\dim B(\varrho, \sigma) = 2$.
(ii) *If M is complex, then* $\dim B(\varrho, \sigma) = 3$.
(iii) *If M is quaternionic, then* $\dim B(\varrho, \sigma) = 5$.
(iv) *If M is a spin factor, then* $\dim B(\varrho, \sigma) = N$.
(v) *If $M \cong M_3^8$, then* $\dim B(\varrho, \sigma) = 9$.

Proof. Let $\check{\varrho}$ be the (non-central) support projection of ϱ. Then $B(\varrho, \sigma)$ is isomorphic to the normal state space of $\{(\check{\varrho} \vee \sigma) M (\varrho \vee \check{\sigma})\}$ (see the proof of [6; Th. 3.11]). Also, by [6; Lemma 3.6] $\check{\varrho} \vee \check{\sigma} = e + f$ for some pair of minimal orthogonal projections e, f of M. Thus, if M is real $\{(\check{\varrho} \vee \check{\sigma}) M (\check{\varrho} \vee \check{\sigma})\} \cong M_2(\mathbf{R})_s$ and so, counting dimensions, we find $\dim B(\varrho, \sigma) = \dim M_2(\mathbf{R})_s - 1 = 2$. The complex and quaternionic cases are treated similarly.

Next, assume $M \cong M_3^8$, and let $e_{ij} \in M_3^8$ be the matrix units ($i, j = 1, 2, 3$). Since M is of type I_3, $1 - e - f \sim e_{33}$ via a symmetry, so we may as well assume $e + f = e_{11} + e_{22}$. Thus

$$\{(e+f) M (e+f)\} \cong M_2^8,$$

so $\dim B(\varrho, \sigma) = \dim M_2^8 - 1 = 9$.

Finally, if M is a spin factor, N is a Hilbert ball, so (iv) follows trivially. $\square$

Definition. The state space K of a JB-algebra is said to have the 3-*ball property* if, for any pair ϱ, σ of distinct extreme points of K, the ball $B(\varrho, \sigma)$ has dimension 1 or 3. (By Proposition 2.3 $\dim B(\varrho, \sigma) = 3$ iff ϱ and σ are equivalent.)

Definition. A JB-algebra A is said to be of *complex type* if all its dense representations are into a type I factor isomorphic to $B(H)_{sa}$ where H is a complex Hilbert space. Similarly, we may define JB-algebras of real, quaternionic, spin, and purely exceptional types.

COROLLARY 3.3. *A JB-algebra is of complex type iff its state space has the 3-ball property.*

Proof. The JB-algebra A is of complex type iff A_ϱ is complex for all pure states ϱ.

Since the normal state space of A_ϱ is isomorphic to F_ϱ, the corollary follows directly from Theorem 3.1 and Proposition 3.2. $\square$

The relevance of the above discussions for our purpose stems from the following lemma. As will be seen later on, its converse is false.

LEMMA 3.4. *The self-adjoint part of a C^*-algebra is a JB-algebra of complex type.*

Proof. Let A be the self-adjoint part of a C^*-algebra, and let $\pi: A \to M$ be a dense representation. As in the proof of Proposition 2.2 we can find a central idempotent $c \in A^{**}$ such that $\tilde{\pi}: c \circ A^{**} \to M$ is a surjective isomorphism. Since A^{**} is the self-adjoint part of a von Neumann algebra, M must be isomorphic to the self-adjoint part of a type I von Neumann factor, i.e. to $B(H)_{sa}$ for some complex Hilbert space H. $\square$

PROPOSITION 3.5. *A JB-algebra is of complex type iff it is special and all its irreducible concrete representations are dense.*

Proof. Assume that A is of complex type. By [7, § 9] A is special. Now let $\pi: A \to B(H)_{sa}$ be an irreducible concrete representation. Then $\pi(A)^-$ is an irreducible JW-algebra, hence [39; Th. 4.1] it is a type I JBW-factor, and therefore is a complex factor. The proof of Theorem 3.1 shows that actually $\pi(A)^- = B(H)_{sa}$, i.e. π is a dense representation.

Conversely, assume A is a special JB-algebra all of whose irreducible representations are dense. Let $\varphi: A \to M$ be a dense representation. Since A is special, M is not isomorphic to M_3^8. Then M can be represented as an irreducible JW-algebra [37; Th. 51], say $M \subseteq B(H)_{sa}$. (That this is also true when M is a spin factor, is seen by first representing M on a Hilbert space, and then choosing an irreducible representation of the C^*-algebra generated by M.) But then φ, viewed as a map into $B(H)_{sa}$, is an irreducible representation, and hence it is dense. Thus $M = \overline{\varphi(A)} = B(H)_{sa}$, and A is of complex type. $\square$

Definition. Let A be a JB-algebra of complex type. We say an irreducible concrete representation $\pi: A \to B(H)_{sa}$ is *associated with* $\varrho \in \partial_e K$ if there exists a (unit) vector $\xi \in H$ such that

$$\langle a, \varrho \rangle = (\pi(a)\xi \mid \xi) \quad \text{for all} \quad a \in A. \tag{3.3}$$

Note that the unit vector ξ of (3.3) is uniquely determined up to scalar multiples (of modulus one) by virtue of the density of $\pi(A)$ in $B(H)_{sa}$. We will say that this vector ξ *represents* ϱ *w.r. to* π.

PROPOSITION 3.6. *Let A be a JB-algebra of complex type. Then for each pure state $\varrho \in \partial_e K$ there is associated at least one irreducible concrete representation. Two irreducible* *concrete representations associated with the same $\varrho \in \partial_e K$ are either unitarily equivalent or conjugate; both happen iff the representations are one-dimensional. Furthermore, if an irreducible concrete representation π of A is associated with $\varrho \in \partial_e K$, then the set of pure states with which π is associated, is precisely $\partial_e F_\varrho = F_\varrho \cap \partial_e K$.*

Proof. By Proposition 3.5, an irreducible concrete representation of A is the same as a dense concrete representation. By Proposition 2.2, such a representation $\pi: A \to B(H)_{sa}$ is Jordan equivalent to φ_ϱ iff π^* maps the normal state space of $B(H)$ bijectively onto F_ϱ. Since the pure normal states of $B(H)$ are the vector states, this is equivalent to π being associated with ϱ. This proves the final statement of the proposition, and also the first since to each $\varrho \in \partial_e K$ is associated the dense representation $\varphi_\varrho: A \to A_\varrho$ and A_ϱ is isomorphic to $B(H)_{sa}$ by assumption.

Finally, the second statement is a direct consequence of Proposition 2.4, since two irreducible representations associated with ϱ both are Jordan equivalent to φ_ϱ, and therefore to each other. □

§ 4. Reversibility

Following Størmer [36; p. 439] we will say that a JC-algebra A is *reversible* if

$$a_1 a_2 \dots a_n + a_n a_{n-1} \dots a_1 \in A \tag{4.1}$$

whenever $a_1, \dots, a_n \in A$. Note that the left hand side of (4.1) is the Jordan triple product for $n=3$. Thus, (4.1) always holds for $n=3$, but it is worth noting that it can fail already for $n=4$. (In fact, $n=4$ is the critical value; if (4.1) holds for $n=4$, then it holds for all $n>4$ as shown by P. M. Cohn [13].)

For a given JC-algebra $A \subseteq B(H)_{sa}$ we denote by $\mathcal{R}_0(A)$ the real subalgebra of $B(H)$ generated by A, and we denote by $\mathcal{R}(A)$ the norm closure of $\mathcal{R}_0(A)$. We observe that $\mathcal{R}_0(A)$ is closed under the *-operation since $(a_1 a_2 \dots a_n)^* = a_n a_{n-1} \dots a_1$ for $a_1, \dots, a_n \in A$. From this it follows that $\mathcal{R}(A)$ is a norm closed real *-algebra of operators on H. (Such an algebra is sometimes called a "real C^*-algebra".) If A is reversible and $b = a_1 a_2 \dots a_n$ where $a_1, \dots, a_n \in A$, then the self-adjoint part $b_n = \frac{1}{2}(b + b^*)$ will be in A. From this it follows that A is reversible iff $\mathcal{R}_0(A)_{sa} = A$.

Assume now that A is reversible and consider an element $b \in \mathcal{R}(A)_{sa}$, say $b = b^*$ and $b = \lim_n b_n$ where $b_n \in \mathcal{R}_0(A)$ for $n = 1, 2, \dots$ (norm limit). Then $b = \lim_n (b_n)_{sa} \in A$ since A is closed. From this it follows that A is reversible iff $\mathcal{R}(A)_{sa} = A$.

By definition, reversibility is a spatial notion involving the non-commutative multiplication of Hilbert space operators. In general it is not an isomorphism invariant; it is possible for a reversible and a non-reversible JC-algebra to be isomorphic. This situation is illustrated by the spin-factors. A spin factor $A \subseteq B(H)_{sa}$ is always reversible when $\dim A = 3$ or 4, non-reversible when $\dim A \neq 3$, 4 or 6, and it can be either reversible or non-reversible when $\dim A = 6$, even though all spin factors of the same dimension are isomorphic. Of these results we will prove only the one with $\dim A = 4$, since we shall not need the others.

Recall that the Hilbert norm of a spin factor is equivalent with the JB-algebra norm, and that the two coincide on $N = \{1\}^{\perp}$ (cf. [41]). It follows that every spin factor is a Banach dual space, hence a JBW-algebra. It is easily verified that the center of any spin factor is trivial, hence it is a factor (which justifies the terminology). In fact, the spin factors are precisely the JBW-factors of type I_2 (see [7; § 7] for definition and proof).

If S is a spin factor, then the hyperplane $N = \{1\}^{\perp}$ consists of all elements λs where $\lambda \in \mathbf{R}$, $s \neq \pm 1$, and s is a *symmetry*, i.e. $s^2 = 1$. Note also that two elements of N are orthogonal iff their Jordan product is zero. Thus, if $\{s_\alpha\}$ is an orthonormal basis in S such that $s_{\alpha_0} = 1$ for some index α_0, then all the other basis-elements are symmetries satisfying $s_\alpha \circ s_\beta = \delta_{\alpha,\beta} 1$. For later references we observe that the orthogonal components of an element $a \in S$ with respect to such a basis, can be expressed in terms of the Jordan product. In fact, if $a = \lambda_0 + \sum_{\alpha \neq \alpha_0} \lambda_\alpha s_\lambda$, then for each $\alpha \neq \alpha_0$:

$$(a \circ s_\alpha) \circ s_\alpha = (\lambda_0 s_\alpha + \lambda_\alpha 1) \circ s_\alpha = \lambda_0 1 + \lambda_\alpha s_\alpha;$$

hence for any index $\beta \neq \alpha_0$ distinct from α:

$$(((a \circ s_\alpha) \circ s_\alpha) \circ s_\beta) \circ s_\beta = \lambda_0 1; \tag{4.2}$$

moreover:

$$(a - \lambda_0 1) \circ s_\alpha = \lambda_\alpha 1. \tag{4.3}$$

Simple examples of spin factors are the Jordan algebra $M_2(\mathbf{R})_s$ of all symmetric 2×2-matrices over $\mathbf{R}$ and the Jordan algebra $M_2(\mathbf{C})_{sa}$ of all self-adjoint 2×2-matrices over $\mathbf{C}$. For these algebras, orthonormal bases are respectively (s_0, s_1, s_2) and (s_0, s_1, s_2, s_3), where s_0 is the unit matrix and s_1, s_2, s_3 are the elementary spin matrices (cf. (3.2)).

It follows from the above discussion that two spin factors of the same dimension must be isomorphic. In particular, every spin factor of dimension three is isomorphic to $M_2(\mathbf{R})_s$, and every spin factor of dimension four is isomorphic to $M_2(\mathbf{C})_{sa}$.

LEMMA 4.1. *The four-dimensional spin factor $M_2(\mathbf{C})_{sa}$ is reversible in every concrete representation.*

Proof. Let $M \subseteq B(H)_{sa}$ be a concrete spin factor of dimension four. Let $1, s_1, s_2, s_3$ be a basis for M, where $s_i^2 = 1$ and $s_i \circ s_j = 0$ for $i \neq j$.

By multilinearity, it suffices to prove that $x = a_1 \ldots a_n + a_n \ldots a_1 \in M$, whenever the a_j's belong to the above basis. Using the relations $s_i^2 = 1$, $s_i s_j = -s_j s_i$ when $i \neq j$, we may permute the a_j's (possibly reversing a sign in the expression for x) and cancel terms until we find $x = \pm(b_1 \ldots b_m + b_m \ldots b_1)$, where $m \leq 3$. Thus $x \in M$. (If $m = 3$, this expression is the Jordan triple product $\{b_1 b_2 b_3\} = (b_1 \circ b_2) \circ b_3 + (b_2 \circ b_3) \circ b_1 - (b_1 \circ b_3) \circ b_2$.) $\square$

We will reduce the problem of reversibility for a given JC-algebra to the same problem for its weak closure in an appropriate representation. Then we are in a setting where the structure theory for JW-algebras applies. Recall in this connection that any given JW-algebra $A \subseteq B(H)_{sa}$ can be written as

$$A = A_1 \oplus A_2 \oplus \ldots \oplus A_\infty \oplus B, \tag{4.4}$$

where A_1 is an abelian JW-algebra, A_j is of type I_j for $j = 2, 3, \ldots, \infty$, and B is the nontype I summand. (See [41; Theorems 5 & 16] for precise definitions and proofs, but note in particular that the direct sum (4.4) is given by orthogonal central idempotents $z_1, z_2, \ldots, z_\infty$, $w \in A$ such that $z_j A = A_j$ for $j = 1, 2, \ldots, \infty$ and $wA = B$.)

We will see later that the I_2-summand is the key to reversibility. Therefore we will now study JW-algebras of type I_2. We begin by two technical lemmas.

LEMMA 4.2. *For each integer $n \geq 1$ there exists a Jordan polynomial P_n in $n+2$ variables such that for any spin factor S and an arbitrary pair s, t of orthogonal symmetries in S we have $P_n(s, t, a_1, \ldots, a_n) = 0$ iff $a_1, \ldots, a_n \in S$ are linearly dependent.*

Proof. By the well known Gram criterion for spaces with an inner product, n elements $a_1, \ldots, a_n$ of a spin factor S will be linearly dependent iff $\det\{(a_i | a_j)\}_{i,j=1}^n = 0$. Since the Jordan multiplication in S reduces to scalar multiplication in $\mathbf{R}1 \subseteq S$, we can rewrite this condition as

$$Q_n((a_1|a_1)1, (a_1|a_2)1, \ldots, (a_n|a_n)1) = 0, \tag{4.5}$$

where Q_n is an appropriate Jordan polynomial in n^2 variables.

Assume now that s, t are two arbitrary (but fixed) orthogonal symmetries in S. For any set $(a_1, \ldots, a_n)$ of n elements of S we decompose each a_j as $a_j = \alpha_j 1 + n_j$ where $n_j \in N = \{1\}^\perp$. For given i, j the multiplication rules for spin factors give:

$$(a_i|a_j)1 = \alpha_i\alpha_j 1 + (n_i|n_j)1 = (\alpha_i 1) \circ (\alpha_j 1) + n_i \circ n_j = (\alpha_i 1) \circ (\alpha_j 1) + (a_i - \alpha_i 1) \circ (a_j - \alpha_j 1).$$

It follows from (4.2) that $(a_i|a_j)1$ can be expressed as a Jordan polynomial in s, t, a_i, a_j for $i, j = 1, 2, \ldots, n$. Substituting these polynomials into Q_n, we obtain a Jordan polynomial P_n in the $n+2$ variables $s, t, a_1, \ldots, a_n$, which will have the desired property. Clearly, P_n is independent of the spin factor S and the choice of s and t. $\square$

Observe for later applications that if A is a JW-algebra of type I_2 and if $\varphi: A \to M$ is a dense representation, then M must be a spin factor. In fact, if p and q are exchangeable abelian projections in A with sum 1, then $\varphi(p)$ and $\varphi(q)$ are exchangeable abelian projections in M with sum 1, so M is an I_2-factor, i.e. a spin factor.

For the next lemma we also need some new terminology: Two elements a, b of a JB-algebra are said to be *J-orthogonal* if $a \circ b = 0$. Clearly this generalizes the orthogonality of symmetries in a spin factor. Note also that if A is concretely represented as a JC-algebra, then a, b are J-orthogonal iff the operator ab is skew. For a given idempotent p in a JB-algebra A we say that an element $s \in A$ is a *p-symmetry* if $s^2 = p$.

LEMMA 4.3. *If a projection p in a JW-algebra A of type I_2 admits two J-orthogonal p-symmetries, then p is central.*

Proof. Let s, t be two J-orthogonal p-symmetries in A, and define $q = \frac{1}{2}(p+s)$, $r = \frac{1}{2}(p-s)$. Then $q + r = p$, and q, r are exchangeable projections; in fact the symmetry $u = (1-p)+t$ satisfies $uqu = r$, so it exchanges q and r.

Note that the central covers $c(p)$, $c(q)$, $c(r)$ are all equal. We assume for contradiction that $p \neq c(p)$. Then the central covers of q and of $c(p)-p$ will not be orthogonal, so by [41; Lemma 18] there will exist exchangeable non-zero projections $x \leqslant q$, $y \leqslant c(p)-p$. Defining $z = uxu$, we get $z \leqslant uqu = r$.

Now x, y, z are non-zero orthogonal projections with x, y exchangeable and x, y exchangeable. Then any homomorphism which annihilates one of the projections x, y, z, will annihilate the other two. Thus there exists a dense representation $\varphi: A \to M$ which does not annihilate any of the three projections x, y, z (cf. [7; Cor. 5.7]). By the remark preceding this lemma, M must be a spin factor. But a spin factor cannot contain a set of three non-zero orthogonal projections. This contradiction completes the proof. □

The next lemma is crucial.

LEMMA 4.4. *If $A \subseteq B(H)_{sa}$ is a JC-algebra of complex type, then every dense representation of the I_2-summand of $\tilde{A}$ is onto a spin factor of dimension at most four.*

Proof. Let z be the central projection in $\tilde{A}$ such that the I_2-summand of $\tilde{A}$ is equal to $z\tilde{A}$, and let $\varphi: z\tilde{A} \to M$ be a dense representation. As remarked earlier, M must be a spin factor.

Note that $M_0 = \varphi(zA)$ will be a norm closed Jordan subalgebra of M containing the identity. It is not difficult to verify that such a subalgebra is itself a spin factor unless it is of dimension less than three. In the latter case M_0 will be associative (in fact $M \cong \mathbf{R}$ or $M \cong \mathbf{R} \oplus \mathbf{R}$). In the former case the spin factor M_0 satisfies $M_0 \cong B(H_0)_{sa}$ for some finite or infinite Hilbert space H_0; but $B(H_0)_{sa}$ is a spin factor only if H_0 is of (complex) dimension 2, in which case $B(H_0)_{sa}$ is of (real) dimension 4. Hence dim $M_0 = 1$, 2 or 4.

We will next show that dim $M \leqslant 4$. Let p, q be exchangeable abelian projections in $z\bar{A}$ with $p+q=z$. Then there exists a z-symmetry $s \in z\bar{A}$ such that $sps=q$. Now $ps=sq$, so $s(p-q)=(q-p)s$. Thus the elements s and $t=p-q$ are symmetries in the Jordan algebra $z\bar{A}$ satisfying $s \circ t = 0$. Consider now an arbitrary dense representation ψ of $z\bar{A}$. By the above argument (with ψ in place of φ), ψ is a spin factor representation and dim $\psi(zA) \leqslant 4$. By Lemma 4.2 we have

$$\psi(P_5(s, t, za_1, ..., za_5)) = P_5(\psi(s), \psi(t), \psi(za_1), ..., \psi(za_5)) = 0$$

for any set of five elements $a_1, ..., a_5 \in A$. Since the dense representations separate points [7; Cor. 5.7 and Prop. 8.7], it follows that

$$P_5(s, t, za_1, ..., za_5) = 0, \quad \text{all } a_1, ..., a_5 \in A. \tag{4.6}$$

By the Kaplansky density theorem for JC-algebras [18; p. 314], the unit ball of zA is strongly dense in the unit ball of $z\bar{A}$. Hence it follows from (4.6) that $P_5(s, t, x_1, ..., x_5) = 0$ for all $x_1, ..., x_5 \in z\bar{A}$. Applying φ, we get

$$P_5(\varphi(s), \varphi(t), \varphi(x_1), ..., \varphi(x_5)) = 0 \quad \text{all } x_1, ..., x_5 \in z\bar{A}.$$

By Lemma 4.2 $\varphi(x_1), ..., \varphi(x_5)$ is a linearly dependent set of elements of M for any set of five elements $x_1, ..., x_5 \in z\bar{A}$. Hence dim $\varphi(z\bar{A}) \leqslant 4$, and by σ-weak density, dim $M \leqslant 4$. □

It follows from the next result that the dense spin factor representations of Lemma 4.4 have dimension precisely four.

LEMMA 4.5. *Let $A \subseteq B(H)_{sa}$ be a JC-algebra of complex type and let s_0 be the central projection in $\bar{A}$ such that $s_0\bar{A}$ is the I_2-summand of $\bar{A}$. Then $s_0\bar{A}$ contains a subalgebra $M = \mathrm{lin}_{\mathbf{R}}(s_0, s_1, s_2, s_3)$ which is a four dimensional spin factor with s_1, s_2, s_3 J-orthogonal s_0-symmetries. Moreover each $b \in s_0\bar{A}$ can be uniquely expressed as:*

$$b = \sum_{j=0}^{3} f_j s_j, \tag{4.7}$$

where f_j is in the center Z of $s_0\bar{A}$ for $j=0, 1, 2, 3$.

Proof. Let $\{p_\alpha\}$ be a maximal orthogonal set of central projections in $s_0\bar{A}$ with the property that each p_α admits three J-orthogonal p_α-symmetries, say $s_{1\alpha}, s_{2\alpha}, s_{3\alpha}$ and let $p=$

$\sum_\alpha p_\alpha$. A priori, there may not exist any such p_α, in which case the summation over the empty set of indices would give $p=0$. However, we shall see that this eventuality cannot occur; in fact we will prove that $p=s_0$.

Assume that $p\neq s_0$. Now we will first show that every dense representation of $(s_0-p)\tilde{A}$ is of dimension at most three, then we will see that this leads to a contradiction. By Lemma 4.4 all dense representations of $(s_0-p)\tilde{A}$ are onto spin factors of dimension at most four (since each extends to the I_2-summand of $\tilde{A}$). Now if φ: $(s_0-p)\tilde{A}\to M$ is a four-dimensional spin factor representation, then by [34; Lemma 3.6] we can find orthogonal symmetries s_1, s_2, s_3 in M, an idempotent $q\in(s_0-p)\tilde{A}$, and J-orthogonal q-symmetries t_1, t_2, t_3 mapping onto s_1, s_2, s_3, respectively. Note that $(s_0-p)\tilde{A}$ is of type I_2, and so by Lemma 4.3, q is a central idempotent. This contradicts the maximality of $\{p_\alpha\}$, so we conclude that all dense representations of $(s_0-p)\tilde{A}$ are onto spin factors of dimension three. Now, as in the proof of Lemma 4.4, all such representations restricted to $(s_0-p)A$ have associative range. Thus $(s_0-p)\tilde{A}$ must be associative (i.e. abelian). But this is impossible, so $p=s_0$ as claimed.

Define $s_j=\sum_\alpha s_{j\alpha}$ for $j=1, 2, 3$. Then s_1, s_2, s_3 are J-orthogonal s_0-symmetries, and $M=\operatorname{lin}_{\mathbf{R}}(s_0, s_1, s_2, s_3)$ is a spin factor of dimension four.

It remains to establish the decomposition (4.7). For a given $b\in s_0\tilde{A}$ we define

$$f_0=(((b\circ s_1)\circ s_1)\circ s_2)\circ s_2, \tag{4.8}$$

$$f_j=(b-f_0)\circ s_j \quad \text{for } j=1, 2, 3. \tag{4.9}$$

Consider now a dense representation ψ of $s_0\tilde{A}$. Since ψ is a dense spin factor representation of dimension at most four (by Lemma 4.4), and since $\psi(s_1), \psi(s_2), \psi(s_3)$ are orthogonal symmetries, we have a decomposition

$$\psi(b)=\sum_{j=0}^{3}\lambda_j\psi(s_j)=\sum_{j=0}^{3}(\lambda_j 1)\circ\psi(s_j),$$

where the coefficients λ_j are given as in the formulas (4.2) and (4.3). Comparing these formulas with (4.8) and (4.9) (with $a=\psi(b)$), we conclude that

$$\lambda_j 1=\psi(f_j), \quad \text{for } j=0, 1, 2, 3, \tag{4.10}$$

and therefore

$$\psi(b)=\psi\left(\sum_{j=0}^{3} f_j\circ s_j\right). \tag{4.11}$$

By (4.10) and (4.11), ψ will map the elements f_0, f_1, f_2, f_3 onto central elements and the element $b-\sum_{j=0}^{3} f_j\circ s_j$ onto zero. Since the dense representations separate points, it follows that $f_0, f_1, f_2, f_3\in Z$ and that $b-\sum_{j=0}^{3} f_j\circ s_j=0$. The uniqueness follows from (4.8) and (4.9). $\square$

Remark. Note that Lemma 4.5, equation (4.7), implies that the I_2-summand of $\bar{A}$ is isomorphic to $C(X, M_2(\mathbf{C})_{sa})$ where X is a hyperstonean space such that $C(X)$ is isomorphic to the center of $s_0 \bar{A}$.

The next theorem is the main result of this section.

THEOREM 4.6. *Any JC-algebra A of complex type is reversible.*

Proof. 1. Let $\mathfrak{A}$ be the C^*-algebra generated by $A \subseteq B(H)_{sa}$ and let $\pi\colon \mathfrak{A} \to B(H')$ be the universal representation of $\mathfrak{A}$. Since reversibility of A only depends on the embedding of A in $\mathfrak{A}$, we can, and shall, identify $\mathfrak{A}$ and $\pi(\mathfrak{A})$. First we will show that $\bar{A}$ is reversible in this representation.

By [37; Th. 6.4 & Th. 6.6] it suffices to show that the I_2-summand of $\bar{A}$ is reversible. Let this summand be $s_0 \bar{A}$ where s_0 is a central projection in $\bar{A}$, and let s_1, s_2, s_3 be J-orthogonal s_0-symmetries with the properties explained in Lemma 4.5. Consider now an arbitrary finite set of elements

$$b_i = \sum_{j=0}^{3} f_{ij} s_j \in s_0 \bar{A}, \quad i = 1, \ldots, n,$$

where the coefficients f_{ij} are in the center of $s_0 \bar{A}$. By Lemma 4.1 the spin factor $M = \mathrm{lin}_{\mathbf{R}}\, (s_0, s_1, s_2, s_3)$ is reversible. Hence

$$b_1 \ldots b_n + b_n \ldots b_1 = \sum_{(j_1, \ldots, j_n)} f_{1j_1} \ldots f_{nj_n} (s_{j_1} \ldots s_{j_n} + s_{j_n} \ldots s_{j_1}) \in s_0 \bar{A}.$$

This shows that $s_0 \bar{A}$ is reversible, and thus $\bar{A}$ is reversible.

2. We now show that A is reversible. Suppose $a_1, \ldots, a_n \in A$; by reversibility of $\bar{A}$

$$x = a_1 a_2 \ldots a_n + a_n a_{n-1} \ldots a_1 \in \bar{A}.$$

But x is also in $\mathfrak{A}$, and so it lies in $\mathfrak{A} \cap \bar{A}$. We are done if we show $\mathfrak{A} \cap \bar{A} = A$.

Recall that $\bar{\mathfrak{A}}$ can be identified with $\mathfrak{A}^{**}$. The weak and σ-weak closures of A will coincide [39; Lemma 4.2], so $\bar{A}$ is also the σ-weak closure of A (i.e. the closure in $w(\mathfrak{A}^{**}, \mathfrak{A}^{*})$). Now $\bar{A} \cap \mathfrak{A}$ is obtained by intersecting $\mathfrak{A}$ with the intersection of all $w(\mathfrak{A}^{**}, \mathfrak{A}^{*})$-closed hyperplanes containing A; but these hyperplanes are of the form $\varphi^{-1}(0)$ where $\varphi \in \mathfrak{A}^{*}$, and thus $\bar{A} \cap \mathfrak{A} = A$ since A is norm closed. This completes the proof. □

§ 5. The enveloping C^*-algebra

Two JC-algebras, even if they are isomorphic, may act on their respective Hilbert spaces in quite different ways; in fact, even the C^*-algebras they generate may be non-isomorphic. In this section we prove the existence, for any special JB-algebra, of a "largest"

C^*-algebra generated by it, such that in any concrete representation, the C^*-algebra generated by the given JB-algebra is a quotient of the "largest" one. Then we specialize to JB-algebras of complex type.

THEOREM 5.1. *Let A be a JB-algebra. There exists a C^*-algebra $\mathfrak{A}$ and a Jordan homomorphism $\psi: A \to \mathfrak{A}$ such that $\mathfrak{A}$ is generated by $\psi(A)$ and such that for any Jordan homomorphism $\theta: A \to \mathcal{B}_{sa}$ where $\mathcal{B}$ is a C^*-algebra, there exists a *-homomorphism $\bar{\theta}: \mathfrak{A} \to \mathcal{B}$ satisfying $\theta = \bar{\theta} \circ \psi$.*

Proof. Let $A^{\mathbf{C}} = A \oplus iA$ be the complexification of A. It is a Jordan *-algebra, i.e. a complex Jordan algebra with an involution satisfying $(a \circ b)^* = a^* \circ b^*$. ($A^{\mathbf{C}}$ can be normed to become a "JB^*-algebra" or "Jordan C^*-algebra" [44], but we will not need this.)

Let $u: A^{\mathbf{C}} \to \mathcal{U}$ be the "unital universal associative specialization" of $A^{\mathbf{C}}$ [21; p. 65]. Here $\mathcal{U}$ is a unital complex associative algebra and $u: A^{\mathbf{C}} \to \mathcal{U}$ is a Jordan homomorphism with roughly the universal property stated in the theorem, only with the C^*-algebras replaced by associative algebras.

We briefly indicate how $\mathcal{U}$ is constructed: $\mathcal{U}$ is the tensor algebra of $A^{\mathbf{C}}$ factored by the ideal generated by all elements of the form

$$a \circ b - \tfrac{1}{2}(a \otimes b + b \otimes a), \quad a, b \in A^{\mathbf{C}},$$

and the difference between the unit of the tensor algebra and that of $A^{\mathbf{C}}$. The details are found in [21; p. 65].

Next we note the existence of a unique involution on $\mathcal{U}$ such that u is a *-map. This is done by defining an involution on the tensor algebra by

$$(a_1 \otimes \ldots \otimes a_n)^* = a_n^* \otimes \ldots \otimes a_1^*, \quad a_1, \ldots, a_n \in A^{\mathbf{C}},$$

and noting that this involution preserves the above-mentioned ideal.

It is easily seen that $u|_A: A \to \mathcal{U}$ satisfies the property of the theorem, with C^*-algebras replaced by *-algebras.

Next, we define a seminorm on $\mathcal{U}$ by

$$\|x\| = \sup \{\|\pi(x)\|: \pi: \mathcal{U} \to B(H) \quad \text{is a *-representation}\}. \tag{5.1}$$

We have to prove that $\|x\| < \infty$ for $x \in \mathcal{U}$. Since $u(A)$ generate $\mathcal{U}$ as an algebra and $\|\cdot\|$ is clearly sub-additive, it is enough to prove this for x of form $x = u(a_1) \ldots u(a_n)$, where $a_1, \ldots, a_n \in A$.

But, whenever $\pi: \mathcal{U} \to B(H)$ is a *-representation, $\pi \circ u|_A$ is a Jordan representation of A, and is consequently of norm 1. Thus

$$\|\pi(x)\| = \|\pi\circ u(a_1) \dots \pi\circ u(a_n)\| \leqslant \|\pi\circ u(a_1)\| \dots \|\pi\circ u(a_n)\| \leqslant \|a_1\| \dots \|a_n\|,$$

so $\|x\| \leqslant \|a_1\| \dots \|a_n\| < \infty$.

Letting

$$N = \{x \in \mathcal{U}: \|x\| = 0\}, \tag{5.2}$$

we obtain a C^*-norm on $\mathcal{U}/N$. $\mathfrak{A}$ is defined to be the completion of $\mathcal{U}/N$. Let

$$\psi(a) = u(a) + N, \quad a \in A. \tag{5.3}$$

Obviously, $\mathfrak{A}$ is generated as a C^*-algebra by $\psi(A)$. To complete the proof, let $\mathcal{B}$ be a C^*-algebra and $\theta: A \to \mathcal{B}_{sa}$ a Jordan homomorphism. Then θ factors through $\mathcal{U}$, i.e. there exists a *-homomorphism $\pi: \mathcal{U} \to \mathcal{B}$ with $\theta = \pi\circ u$. Assume $\mathcal{B}$ is faithfully represented on a Hilbert space. Then π is a *-representation of $\mathcal{U}$, and therefore it annihilates N, and by definition of the norm induces a *-representation of $\mathcal{U}/N$ of norm 1. Its continuous extension $\bar{\theta}$ to $\mathfrak{A}$ is easily seen to satisfy the conditions of the theorem. □

Remark. Note that the *-homomorphism $\bar{\theta}$ in Theorem 5.1 is necessarily unique. The theorem states that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\bar{\theta}"] & \mathcal{B} \\
A \arrow[u, "\psi"] \arrow[r, "\theta"'] & \mathcal{B}_{sa} \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

By abstract nonsense, the pair ψ, $\mathfrak{A}$ is uniquely determined (in the obvious sense).

If A is special, then by definition A can be faithfully represented on a Hilbert space. Factoring such a representation through $\mathfrak{A}$, we conclude that, in this case, ψ is injective. Then we identify A with its image $\psi(A)$ in $\mathfrak{A}$, and call $\mathfrak{A}$ the *enveloping C^*-algebra* of A. We can then rephrase the above results as follows: *A is a JB-subalgebra of $\mathfrak{A}_{sa}$, and generates $\mathfrak{A}$ as a C^*-algebra. Any Jordan homomorphism $\theta: A \to \mathcal{B}_{sa}$ where $\mathcal{B}$ is a C^*-algebra extends uniquely to a *-homomorphism $\bar{\theta}: \mathfrak{A} \to \mathcal{B}$.*

In the general case, the kernel of ψ is easily seen to be the "exceptional ideal" of A defined in [7; § 9]. Then $\mathfrak{A}$ is the enveloping C^*-algebra of $A/\ker\psi$.

The fact that "Jordan multiplication knows no difference between left and right" is reflected in the following:

COROLLARY 5.2. *If A is special, there exists a unique *-anti-automorphism Φ of $\mathfrak{A}$ leaving A pointwise invariant. Also, $\Phi^2 = I$.*

Proof. Let $\mathfrak{A}^{\circ}$ denote the opposite C^*-algebra of $\mathfrak{A}$. Then Φ is the *-homomorphism $\mathfrak{A}\to\mathfrak{A}^{\circ}$ extending the Jordan homomorphism $\psi\colon A\to\mathfrak{A}^{\circ}$. Also, Φ^2 is a *-automorphism of $\mathfrak{A}$ leaving A pointwise invariant, and is therefore the indentity. □

Throughout the rest of this chapter, A will be a JB-algebra of complex type, and K its state space. $\mathfrak{A}$ is its enveloping C^*-algebra, with state space $\mathcal{K}$. The restriction map from $\mathcal{K}$ onto K will be denoted by r. (It is the dual of the embedding $\psi\colon A\to\mathfrak{A}$.) Obviously, $r\circ\Phi^*=r$.

PROPOSITION 5.3. *Let A be a JB-algebra of complex type, with state space K. Let $\mathfrak{A}$ be the enveloping C^*-algebra of A, with state space $\mathcal{K}$. If $\varrho\in\partial_e\mathcal{K}$, then $r(\varrho)\in\partial_e K$, and the restriction map r maps F_ϱ bijectively onto $F_{r(\varrho)}$.*

Proof. Consider the GNS representation $\pi_\varrho\colon\mathfrak{A}\to B(H_\varrho)$. Since A generates $\mathfrak{A}$, then $\pi_\varrho|_A$ is irreducible. By Proposition 3.5, $\pi_\varrho|_A$ is dense, and so by Proposition 2.5, $(\pi_\varrho|_A)^*$ maps the normal state space N of $B(H)$ bijectively onto a split face of K. Consider the commutative diagram:

$$\begin{tikzcd} \mathcal{K} \arrow[d, "r"'] & \\ & N \arrow[ul, "\pi_\varrho^*"'] \arrow[dl, "(\pi_\varrho|_A)^*"] \\ K & \end{tikzcd}$$

Since π^* maps N bijectively onto F_ϱ, it follows that r maps F_ϱ bijectively onto a split face of K, and the result follows. □

LEMMA 5.4. *With assumptions as in Proposition 5.3, the restrictions to A of the representations π_ϱ, resp. $\pi_{\Phi^*\varrho}$, are conjugate irreducible representations of A associated with $r(\varrho)$.*

Proof. Let $\xi_\varrho\in H_\varrho$ denote a representing vector for ϱ. Choose an involution j of H_ϱ fixing ξ_ϱ, and define the transpose map $a\mapsto a^t$ on $B(H_\varrho)$ by $a^t=ja^*j$. Then $x\mapsto\pi_\varrho(\Phi(x))^t$ is an irreducible representation of $\mathfrak{A}$, and ξ_ϱ is seen to represent the state $\Phi^*\varrho$ under this representation, which can therefore be identified with $\pi_{\Phi^*\varrho}$.

For $x\in A$, we have $\pi_\varrho(\Phi(x))^t=j\pi_\varrho(x)j$, which proves that the restriction of this representation is conjugate to $\pi_\varrho|_A$.

That $\pi_\varrho|_A$ is associated with $r(\varrho)$ follows from the following, if $x\in A$:

$$((\pi_\varrho|_A)(x)\xi_\varrho|\xi_\varrho)=\langle x,\varrho\rangle=\langle x,r(\varrho)\rangle.$$ □

PROPOSITION 5.5. *With assumptions as in Proposition 5.3 the following are equivalent:*

(i) $\Phi^*\varrho=\varrho$,

(ii) $F_\varrho=\{\varrho\}$,

(iii) $F_\varrho\cap\Phi^*(F_\varrho)\neq\emptyset$.

Moreover, the inverse image $r^{-1}(r(\varrho))$ of $r(\varrho)$ in $\mathcal{K}$ equals the line segment $[\varrho, \Phi^*\varrho]$ which degenerates to a point if the above requirements are fulfilled.

Proof. We prove (i) ⇒ (iii) ⇒ (ii) ⇒ (i). Of these implications, the first is immediate since $\Phi^*(F_\varrho) = F_{\Phi^*\varrho}$.

Assume (iii). Since minimal split faces are either disjoint or equal, we have $\Phi^*\varrho \in F_\varrho$, so ϱ, $\Phi^*\varrho$ are equivalent. From this it follows that π_ϱ, $\pi_{\Phi^*\varrho}$ are unitarily equivalent. Then the same holds, of course, for their restrictions to A. But, by Lemma 5.4, these restrictions are conjugate, so by Proposition 2.4 we have $\dim \pi_\varrho = 1$, or $F_\varrho = \{\varrho\}$.

Assume next that $F_\varrho = \{\varrho\}$, i.e. $\dim \pi_\varrho = 1$. Then we have $\pi_\varrho(x)\xi = \langle x, \varrho\rangle \xi$, and similarly for $\Phi^*\varrho$, thereby proving (since ϱ and $\Phi^*\varrho$ have the same restrictions to A) that $\pi_\varrho|_A$ is unitarily equivalent to $\pi_{\Phi^*\varrho}|_A$. A unitary equivalence of these two representations is also an equivalence of π_ϱ and $\pi_{\Phi^*\varrho}$. Thus $\Phi^*\varrho \in F_\varrho = \{\varrho\}$, so $\Phi^*\varrho = \varrho$.

Finally, assume that $\sigma \in \partial_e K$ and $r(\sigma) = r(\varrho)$. Then $\pi_\sigma|_A$ is an irreducible representation of A associated with $r(\varrho)$, and is therefore either unitarily equivalent or conjugate to $\pi_\varrho|_A$, i.e. unitarily equivalent to either $\pi_\varrho|_A$ or $\pi_{\Phi^*\varrho}|_A$ (Lemma 5.4). Thus π_σ is equivalent to either π_ϱ or $\pi_{\Phi^*\varrho}$, i.e. $\sigma \in F_\varrho$ or $\sigma \in F_{\Phi^*\varrho}$. Since F_ϱ, $F_{\Phi^*\varrho}$ are mapped injectively into K (Prop. 5.3), $\sigma = \varrho$ or $\sigma = \Phi^*\varrho$ follows. Thus $r^{-1}(r(\varrho)) \cap \partial_e \mathcal{K} = \{\varrho, \Phi^*\varrho\}$. Since $r^{-1}(r(\varrho))$ is a closed face of $\mathcal{K}$, the Krein-Milman theorem shows that $r^{-1}(r(\varrho)) = [\varrho, \Phi^*\varrho]$, and the proof is complete. □

Definitions. We divide the pure state space of A in two parts as follows:

$$\partial_{e,0} K = \{\varrho \in \partial_e K \colon F_\varrho = \{\varrho\}\} \tag{5.5}$$

$$\partial_{e,1} K = \partial_e K \setminus \partial_{e,0} K. \tag{5.6}$$

Thus $\partial_{e,0} K$ correspond to one-dimensional representations. $\partial_{e,0} K$ is easily seen to be closed in the facial topology [1; § 6], but we shall not use this fact.

COROLLARY 5.6. *The restriction map $r\colon \mathcal{K} \to K$ maps $\partial_{e,0}\mathcal{K}$ one-to-one onto $\partial_{e,1} K$ and $\partial_{e,1}\mathcal{K}$ two-to-one onto $\partial_{e,1} K$.*

Proof. Since $r(\mathcal{K}) = K$, the Krein-Milman theorem proves that $r(\partial_e \mathcal{K}) \supseteq \partial_e K$. Proposition 5.3 contains the opposite inclusion and proves that $\partial_{e,0}\mathcal{K}$ (resp. $\partial_{e,1}\mathcal{K}$) is mapped into $\partial_{e,0} K$ (resp. $\partial_{e,1} K$). By Proposition 5.5, the proof is completed. □

Our next lemma is essential. It is our only use of the results of § 4, so it is not as innocent as it looks. We repeat our standing hypothesis that A is of complex type.

LEMMA 5.7. *The fixed point set in $\mathfrak{A}$ of Φ is $A + iA$.*

Proof. By Corollary 5.2, $A+iA$ is pointwise fixed by Φ. Let $x\in\mathfrak{A}$ be a fixed point of Φ. Since Φ is a map, we may assume that x is self-adjoint. We shall prove that $x\in A$.

Since $\mathfrak{A}$ is generated by A, x is a limit of sums of terms of the form $y=a_1 \dots a_m + ib_1 \dots b_n$, where $a_i, b_j \in A$. Since $x=x^*=\Phi(x)$, x is also the limit of sums of terms of the form $\frac{1}{4}(y+y^*+\Phi(y)+\Phi(y)^*)=\frac{1}{2}(a_1 \dots a_m + a_m \dots a_1)$. But A is reversible in any representation (Theorem 4.6), so the latter element belongs to A. Therefore $x\in A$, and the proof is complete. □

LEMMA 5.8. *The fixed point set $\mathcal{K}_0$ in $\mathcal{K}$ of Φ^* is mapped bijectively onto K by the restriction map r.*

Proof. That $r(\mathcal{K}_0)=K$ is obvious since $r(\mathcal{K})=K$, $r\circ\Phi^*=r$ and $\frac{1}{2}(\varrho+\Phi^*\varrho)\in\mathcal{K}_0$ for $\varrho\in\mathcal{K}$.

To prove that r is injective on $\mathcal{K}_0$, let $\varrho, \sigma\in\mathcal{K}_0$ and assume that $r(\varrho)=r(\sigma)$. If $x\in\mathfrak{A}_{sa}$, then $\frac{1}{2}(x+\Phi(x))$ is a self-adjoint fixed point of Φ, so by Lemma 5.7, $\frac{1}{2}(x+\Phi(x))\in A$. Thus

$$\langle x,\varrho\rangle = \langle x, \tfrac{1}{2}(\varrho+\Phi^*\varrho)\rangle = \langle \tfrac{1}{2}(x+\Phi x),\varrho\rangle = \langle \tfrac{1}{2}(x+\Phi x),\sigma\rangle = \langle x, \tfrac{1}{2}(\sigma+\Phi^*\sigma\rangle = \langle x,\sigma\rangle,$$

so $\varrho=\sigma$, and the proof is complete. □

LEMMA 5.9. *Let F be a closed split face of $\mathcal{K}$. If r is one-to-one on $\partial_e F$, r is one-to-one on F.*

Proof. Since Φ^* is an affine automorphism of $\mathcal{K}$, $G=\Phi^*(F)$ is a closed split face of $\mathcal{K}$. Let $\varrho\in\partial_e(F\cap G)$. Then $\varrho\in\partial_e F$ and $\Phi^*\varrho\in\partial_e F$. By assumption, $\varrho=\Phi^*\varrho$. Using the Krein-Milman theorem, we conclude that $F\cap G$ is pointwise fixed by Φ^*.

Now the convex hull co $(F\cup G)$ is a direct convex sum of the three split faces $F\cap G$, $F'\cap G$, $F\cap G'$ (see [1; Prop. II. 6.6]). Also, Φ^* maps $F'\cap G$ onto $F\cap G'$ and vice versa.

Assume now that $\varrho,\sigma\in F$ and $r(\varrho)=r(\sigma)$. Using Lemma 5.8, we find

$$\tfrac{1}{2}(\varrho+\Phi^*\varrho)=\tfrac{1}{2}(\sigma+\Phi^*\sigma), \tag{5.7}$$

since both are fixed points of Φ^*. Write

$$\varrho=\mu\varrho_1+(1-\mu)\varrho_2$$

$$\sigma=\lambda\sigma_1+(1-\lambda)\sigma_2$$

where $0\leqslant\mu\leqslant 1$, $0\leqslant\lambda\leqslant 1$, $\varrho_1, \sigma_1\in F\cap G$, $\varrho_2, \sigma_2\in F\cap G'$.

The relation (5.7) can be written

$$\mu\varrho_1+\tfrac{1}{2}(1-\mu)\varrho_2+\tfrac{1}{2}(1-\mu)\Phi^*\varrho_2=\lambda\sigma_1+\tfrac{1}{2}(1-\lambda)\sigma_2+\tfrac{1}{2}(1-\lambda)\Phi^*\sigma_2.$$

Using affine independence of the split faces $F\cap G$, $F'\cap G$, and $F\cap G'$, we conclude $\mu\varrho_1=\lambda\sigma_1$ and $(1-\mu)\varrho_2=(1-\lambda)\sigma_2$, or $\varrho=\sigma$. We are through. □

THEOREM 5.10. *Let A be a JB-algebra of complex type, and $\mathfrak{A}$ its enveloping C^*-algebra. Let K resp. $\mathcal{K}$, be their state spaces, and r: $\mathcal{K}\to K$ the restriction map. Then A is isomorphic to the self-adjoint part of a C^*-algebra iff there exists a closed split face F of $\mathcal{K}$ such that r maps $\partial_e F$ bijectively onto $\partial_e K$.*

Proof. Assume the existence of such a split face F. Let $J=F_0$ be the annihilator in $\mathfrak{A}$ of F. Then F is affinely homeomorphic to the state space of the C^*-algebra $\mathfrak{A}/J$. By Lemma 5.9 and the Krein-Milman theorem, r is an affine homeomorphism of F onto K. Thus A is isomorphic to the self-adjoint part of the C^*-algebra $\mathfrak{A}/J$.

Conversely, assume that φ: $A\to\mathcal{B}_{sa}$ is a Jordan isomorphism, with $\mathfrak{A}$ a C^*-algebra. Let $\bar\varphi$: $\mathfrak{A}\to\mathcal{B}$ be its extension as a *-homomorphism. Then, with $J=\ker\bar\varphi$, $\mathcal{B}$ can be identified with $\mathfrak{A}/J$, and φ corresponds to the composition of the inclusion $A\to\mathfrak{A}$ and the canonical map $\mathfrak{A}\to\mathfrak{A}/J$. We obtain the following commutative diagram of algebras and homomorphisms, together with its dual diagram of state spaces and affine mappings:

$$\begin{tikzcd}
\mathfrak{A} \arrow[dr] & \\
 & \mathfrak{A}/J \\
A \arrow[uu, hook] \arrow[ur, "\cong" description] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{K} \arrow[dd] & \\
 & F \arrow[ul, hook'] \arrow[dl, "r|_F"] \\
K &
\end{tikzcd}$$

Here we have identified the state space of $\mathfrak{A}/J$ with the closed split face F of $\mathcal{K}$, where $F=J^{\perp}$. This F satisfies the conditions of the theorem. □

Now that we have established the potential usefulness of the enveloping C^*-algebra of a JB-algebra of complex type, we will show how actually to compute it. Inspection of the proof below will reveal that it is the existence of the *-anti-automorphism Φ, established in Corollary 5.2, that characterizes $\mathfrak{A}$.

PROPOSITION 5.11. *Let A be a JB-algebra of complex type. If π: $A\to B(H)_{sa}$ is a faithful representation, the enveloping C^*-algebra $\mathfrak{A}$ is (isomorphic to) the C^*-algebra on $H\oplus H$ generated by $(\pi\oplus\pi^t)(A)$. (π^t is defined relative to some involution on H.)*

Proof. The C^*-algebra $\mathfrak{C}$ generated by $(\pi\oplus\pi^t)(A)$ is contained in $B(H)\oplus B(H)$ (acting on $H\oplus H$). Define the *-anti-automorphism φ on $B(H)\oplus B(H)$ by

$$\varphi(a\oplus b)=b^t\oplus a^t,\quad a, b\in B(H).$$

Then $\varphi\circ(\pi\oplus\pi^t)=\pi\oplus\pi^t$. It follows that $\mathfrak{C}$ is φ-invariant.

Consider the extension $(\pi\oplus\pi^t)^-\colon \mathfrak{A}\to\mathcal{C}$ of $\pi\oplus\pi^t$. Since A generates $\mathfrak{A}$ as a C^*-algebra, this extension maps $\mathfrak{A}$ onto $\mathcal{C}$. To complete the proof, we shall show that $(\pi\oplus\pi^t)^-$ is actually injective.

Now $\varphi\circ(\pi\oplus\pi^t)^-\circ\Phi$ is a *-homomorphism $\mathfrak{A}\to\mathcal{C}$ extending $\pi\oplus\pi^t$, and hence it equals $(\pi\oplus\pi^t)^-$. Thus $\varphi\circ(\pi\oplus\pi^t)^-=(\pi\oplus\pi^t)^-\circ\Phi$. It follows that the kernel J of $(\pi\oplus\pi^t)^-$ is a Φ-invariant, closed two-sided ideal of $\mathfrak{A}$. We shall finish by proving that such an ideal J, satisfying $J\cap A=\{0\}$, is trivial.

Letting $F=J^\perp$ be its annihilator in $\mathcal{K}$, we see that F is a closed Φ^*-invariant split face of $\mathcal{K}$, and $r(F)=(J\cap A)^\perp=\{0\}^\perp=K$.

Choose any $\varrho\in\partial_e\mathcal{K}$. By Proposition 5.5, $r^{-1}(r(\varrho))=[\varrho,\Phi^*(\varrho)]$. Since $r(F)=K$, it follows that $F\cap[\varrho,\Phi^*(\varrho)]\neq\emptyset$. Because F is a Φ^*-invariant face, $\varrho\in F$. By the Krein-Milman theorem, then $F=\mathcal{K}$, and so $J=F_\perp=\{0\}$, and the proof is complete. $\square$

§ 6. The normal state space of $B(H)$

Let H be a complex Hilbert space, and denote by N the normal state space of $B(H)$. It is well known [33; Th. 1.15.3] that N can be identified with the set of positive trace class operators of trace 1 on H in such a way that the trace class operator a corresponding to a given $\varrho\in N$ satisfies:

$$\langle x,\varrho\rangle=\operatorname{Tr}(ax),\quad x\in B(H). \tag{6.1}$$

It is also well known (and easily verified) that $\varrho\in N$ is pure iff a is a one-dimensional projection, and this is equivalent to ϱ being a vector state, i.e. of the form $\varrho=\omega_\xi$ where $\omega_\xi(x)=(x\xi\,|\,\xi)$ for some unit vector $\xi\in H$.

If dim $H=2$ then $B(H)_{sa}\cong M_2(\mathbf{C})_{sa}$ is a spin factor of dimension 4, and by the general theory of [6; § 3] N must be a Euclidean ball of dimension 3. We will now make this explicit: A (necessarily normal) state ϱ on $M_2(\mathbf{C})$ is given by a positive matrix of trace 1, i.e. of the form

$$a=\frac{1}{2}\begin{bmatrix}1+\alpha_1 & \alpha_2+i\alpha_3\\ \alpha_2-i\alpha_3 & 1-\alpha_1\end{bmatrix} \tag{6.2}$$

where $\det(a)=\frac{1}{4}(1-\alpha_1^2-\alpha_2^2-\alpha_3^2)\geqslant 0$. Now $\varphi\colon(\alpha_1,\alpha_2,\alpha_3)\to\varrho$ is the desired affine isomorphism of the standard 3-ball $\mathbf{E}^3=\{(\alpha_1,\alpha_2,\alpha_3)\colon\sum\alpha_i^2\leqslant 1\}$ onto the state space of $M_2(\mathbf{C})$. Henceforth we will identify the state space of $M_2(\mathbf{C})$ and the standard 3-ball $\mathbf{E}^3$ via this isomorphism. (Also we identify $M_2(\mathbf{C})$ with the linear transformations on $\mathbf{C}^2$ in the usual way.)

The affine automorphisms of N correspond to the unital order automorphisms of $B(H)$, these are precisely the Jordan automorphisms, which in turn can be either *-automorphisms or *-anti-automorphisms [25]. We will now explain how one can use the concept of orientation for the balls $B(\varrho, \sigma) \subseteq N$ $(\varrho, \sigma \in \partial_e N)$ to separate the two cases.

Definitions. A parametric 3-*ball* is an affine isomorphism φ of $\mathbf{E}^3$ onto a convex set (in some linear space), the range $B = \varphi(\mathbf{E}^3)$ is said to be a 3-*ball*, and φ is said to be a *parametrization* of B. If φ and ψ are two parametric 3-balls such that $\varphi(\mathbf{E}^3) = \psi(\mathbf{E}^3)$, then $\psi^{-1} \circ \varphi$ is an orthogonal transformation of $\mathbf{E}^3$ and we write $\varphi \sim \psi \bmod \mathrm{O}(3)$; if in addition $\det(\psi^{-1} \circ \varphi) = +1$, then we write $\varphi \sim \psi \bmod \mathrm{SO}(3)$. The parametrizations of any given 3-ball B fall in two equivalence classes modulo SO(3), each of which is called an *orientation* of B. An affine isomorphism $\psi: B_1 \to B_2$ between two 3-balls oriented by parametrizations φ_1 and φ_2 respectively, is said to be *orientation preserving* if $\det(\varphi_2^{-1} \circ \psi \circ \varphi_1) = +1$, and *orientation reversing* if $\det(\varphi_2^{-1} \circ \psi \circ \varphi_1) = -1$. Unless otherwise stated, we will assume that the state space of $M_2(\mathbf{C})$ is oriented by the natural parametrization $\varphi: (\alpha_1, \alpha_2, \alpha_3) \to \varrho$ (cf. (6.2)).

Recall from [11] that a map $\varphi: \mathfrak{A} \to \mathfrak{B}$ between C^*-algebras is called *2-positive* if $\varphi \otimes 1_2$ is a positive map from $\mathfrak{A} \otimes M_2(\mathbf{C})$ into $\mathfrak{A} \otimes M_2(\mathbf{C})$ (1_2 is the identity map on $M_2(\mathbf{C})$). By [43] φ is called *2-copositive* if $\varphi \otimes t_2$ is positive (t_2 is the transpose map on $M_2(\mathbf{C})$).

LEMMA 6.1. *A unital order automorphism Φ of $M_2(\mathbf{C})$ is a *-automorphism (*-anti-automorphism) iff it is 2-positive (2-copositive), and in this case Φ^* is an orientation preserving (orientation reversing) affine automorphism of the state space of $M_2(\mathbf{C})$.*

Proof. Clearly *-automorphisms are 2-positive [11], and by [12] a 2-positive Jordan homomorphism is a *-homomorphism. The statement on *-anti-automorphisms and 2-copositivity follows by composing φ with the transpose map.

The last statement of the lemma can be proved by direct computation. (If Φ is of the form $\Phi(x) = u^*xu$ with u unitary, one finds $\det(\varphi^{-1} \circ \Phi^* \circ \varphi) = 1$; and if Φ is of the form $\Phi(x) = u^*x^tu$, one finds $\det(\varphi^{-1} \circ \Phi^* \circ \varphi) = -1$.) But it is easier to proceed otherwise: The topological group $\mathcal{G}$ of affine automorphisms of the state space of $M_2(\mathbf{C})$ is isomorphic to O(3) with the orientation-preserving maps corresponding to the unit-component SO(3) and the orientation-reversing maps corresponding to the other component. On the other hand we can identify $\mathcal{G}$ with the group of unital order automorphisms of $M_2(\mathbf{C})$ via $\Phi \to \Phi^*$, and we note that the latter group admits a partition into the closed sets of *-automorphisms and of *-anti-automorphisms, respectively. Hence these two sets correspond to the components of $\mathcal{G}$. Since the set of *-automorphisms contains the identity, it must correspond to the unit component of $\mathcal{G}$. This completes the proof. □

Consider now an arbitrary two-dimensional projection p of H. Let $u: \mathbf{C}^2 \to H$ be a (complex linear) isometry onto pH, and define $\psi: B(H) \to M_2(\mathbf{C})$ by

$$\psi(x) = u^*xu. \tag{6.3}$$

Note that ψ is a *-isomorphism of $pB(H)p \subseteq B(H)$ onto $M_2(\mathbf{C})$, but not a *-homomorphism of all of $B(H)$ unless dim $H=2$. (However, it follows from the trivial half of the Neumark-Stinespring theorem that ψ is 2-positive. This observation will be useful later.)

The dual map ψ^* is an affine isomorphism from the state space of $M_2(\mathbf{C})$ onto the set B_p of all $\varrho \in N$ with support projection p (which can be identified with the state space of $pB(H)p$), and this set is equal to the face $B(\omega_\xi, \omega_\eta)$ for an arbitrary pair of unit vectors ξ, η spanning pH. (See the proof of [6; Th. 3.11]. The relevant facts can also be found in [17] and [31].) We give B_p the orientation determined by this parametrization ψ^*. It follows from Lemma 6.1 that this orientation is independent of the choice of the isometry u. We call it the *standard orientation* for B_p. Unless otherwise stated, we will assume that each of these 3-balls is given the standard orientation.

PROPOSITION 6.2. *Let H_i be a complex Hilbert space and N_i the normal state space of $B(H_i)$ for $i=1, 2$; also let Φ: $B(H_1) \to B(H_2)$ be a Jordan isomorphism. If Φ is a *-isomorphism (*-anti-isomorphism), then Φ maps each 3-ball $B_q \subseteq N_2$ orientation preservingly (orientation reversingly) onto the 3-ball $B_q \subseteq N_1$ where $p = \Phi^{-1}(q)$.*

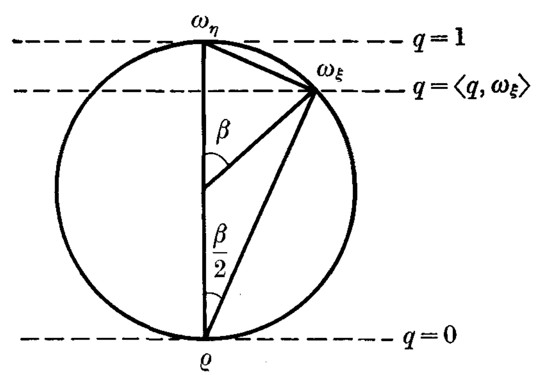
Proof. Note Φ maps $pB(H_1)p$ onto $\Phi(p)B(H_2)\Phi(p) = qB(H_2)q$; the proposition now follows from Lemma 6.1. □

By Proposition 6.2 a conjugation of H will implement a change of orientation for all 3-balls $B(\varrho, \sigma) \subseteq N$ $(\varrho, \sigma \in \partial_e N)$.

The next proposition gives further information about the interplay between the (flat, complex) geometry of H and the (curved, real) geometry of $\partial_e N$.

PROPOSITION 6.3. *Let ξ, η be distinct unit vectors in H such that $(\xi|\eta) \in \mathbf{R}^+$, let α be the corresponding angle (i.e. $\cos \alpha = (\xi|\eta)$), and let $\beta \in [0, \pi]$ be the length in radians of the great circle arc spanned by the corresponding vector states $\omega_\xi, \omega_\eta \in B(\omega_\xi, \omega_\eta)$. Then $\beta = 2\alpha$. In particular, ω_ξ and ω_η are antipodal iff $\xi \perp \eta$.*

Proof. Let q be the projection of H onto the one-dimensional subspace $[\eta]$, and let $B = B(\omega_\xi, \omega_\eta)$. Now choose $\varrho \in B$ antipodal to ω_η, i.e. with $\frac{1}{2}(\varrho + \omega_\eta)$ the center of B. By (6.1), (6.2) the normalized trace of $M_2(\mathbf{C})$ is the center of $\mathbf{E}^3$. Hence $\frac{1}{2} = \langle q, \frac{1}{2}(\varrho + \omega_\eta)\rangle$, so $\langle q, \varrho\rangle = 0$. The picture below shows a plane cross-section of B with level lines for the affine function q. (Note that if $\varrho = \omega_\xi$ then $\beta = \pi$, and the picture is simplified.)



Using elementary geometry, we find

$$\cos^2\frac{\beta}{2} = \langle q, \omega_\xi\rangle = (q\xi|\xi) = ((\xi|\eta)\eta|\xi) = |(\xi|\eta)|^2 = \cos^2\alpha. \qquad \square$$

§ 7. Orientability

In this paragraph we will define and investigate a notion of orientation for sets of 3-balls contained in a given compact convex set K (in some locally convex Hausdorff space).

Definitions. A *facial* 3-*ball* of K is a 3-ball which is a face of K. By $\mathcal{B}(K)$ we denote the topological space of all affine isomorphisms φ from $\mathbf{E}^3$ onto a face of K equipped with the topology of pointwise convergence. (By finite dimensionality and compactness of $\mathbf{E}^3$, this coincides with the topology of uniform convergence.)

We will now identify parametrizations modulo O(3) and SO(3) as explained in § 6. This means that we pass from $\mathcal{B}(K)$ to $\mathcal{B}(K)/\mathrm{O}(3)$ and $\mathcal{B}(K)/\mathrm{SO}(3)$, where the groups O(3) and SO(3) act continuously (from the right) on $\mathcal{B}(K)$ through the formula:

$$(\varphi\circ g)(\theta) = \varphi(g(\theta)) \tag{7.1}$$

with $\varphi\in\mathcal{B}(K)$, $\theta\in\mathbf{E}^3$ and $g\in\mathrm{O}(3)$ or $g\in\mathrm{SO}(3)$.

We call $\mathcal{B}(K)/\mathrm{O}(3)$ *the space of all facial* 3-*balls of* K. Set-theoretically, it can be identified with the set of all facial 3-balls of K. Topologically, two such balls are close to each other if they can be parametrized such that pairs of points corresponding to a common $\theta\in\mathbf{E}^3$ are close to each other in the given topology of K.

We call $\mathcal{B}(K)/\mathrm{SO}(3)$ *the space of all oriented facial* 3-*balls of* K. To specify an element of this space, one must give a facial ball of K together with an orientation.

Clearly, we have canonical maps

$$\mathcal{B}(K) \to \mathcal{B}(K)/\mathrm{SO}(3) \to \mathcal{B}(K)/\mathrm{O}(3), \tag{7.2}$$

where the last map is two-to-one.

LEMMA 7.1. *The mappings in formula* (7.1) *are continuous and open, the three occurring spaces are Hausdorff, and* ψ: $\mathcal{B}(K)/\mathrm{SO}(3) \to \mathcal{B}(K)/\mathrm{O}(3)$ *is a locally trivial* $\mathbf{Z}_2$*-bundle.*

Proof. The first two statements follow easily from the definitions.

Consider the mapping of $\mathcal{B}(K)/\mathrm{SO}(3)$ which interchanges the two elements of each fibre for ψ. Clearly, it is induced by any element of O(3) with determinant -1, and is therefore continuous. This mapping gives the desired $\mathbf{Z}_2$-action on $\mathcal{B}(K)/\mathrm{SO}(3)$.

To prove local triviality we consider a facial ball of K for which we choose a parametrization φ. We shall be through if we can find a neighbourhood V of φ in $\mathcal{B}(K)$ without any pair of elements parametrizing the same ball with opposite orientations.

Assume that K sits in a locally convex Hausdorff space X. (In the applications of interest to us, K will be the state space of a JB-algebra A of complex type and $X = A^*$.) Let $A(\mathbf{E}^3, X)$ be the topological vector space of all affine mappings of $\mathbf{E}^3$ into X equipped with the topology of pointwise convergence. Then $\mathcal{B}(K) \subseteq A(\mathbf{E}^3, X)$, and the topology of $\mathcal{B}(K)$ is that inherited from $A(\mathbf{E}^3, X)$. Observing that the injective mappings form an open subset of $A(\mathbf{E}^3, X)$, we choose a convex neighbourhood U of φ in $A(\mathbf{E}^3, X)$ which consists of injective mappings only. Let $V = U \cap \mathcal{B}(K)$.

Now we assume for contradiction that $\varphi_1, \varphi_2 \in V$ parametrize the same ball with opposite orientations. For $t \in [0, 1]$, define γ_t: $\mathbf{E}^3 \to \mathbf{E}^3$ by

$$\gamma_t = \varphi_1^{-1} \circ (t\varphi_1 + (1-t)\varphi_2).$$

By the properties of U, each γ_t is injective. Hence $t \to \det(\gamma_t)$ is a continuous function from $[0, 1]$ into $\mathbf{R} \setminus \{0\}$. But $\det(\gamma_0) = -1$ and $\det(\gamma_1) = 1$, which gives a contradiction. □

In principle, the discussion above is meaningful for an arbitrary compact convex K, but it is mainly of interest when K is the state space of a JB-algebra of complex type. In this case, the facial 3-balls are precisely the sets $B(\varrho, \sigma)$ where ϱ, σ are distinct pure states which are equivalent (not separated by a split face, cf. § 2).

Definitions. The state space K of a JB-algebra of complex type is said to be *orientable* if the $\mathbf{Z}_2$-bundle $\mathcal{B}(K)/\mathrm{SO}(3) \to \mathcal{B}(K)/\mathrm{O}(3)$ is trivial. A continuous cross-section of this bundle is called a *global orientation*, or simply an *orientation*, of K.

Note that a cross-section of this $\mathbf{Z}_2$-bundle can be specified by choosing an orientation for each of the facial 3-balls in K. The requirement of continuity is a precise version of the statement that "the orientation is never suddenly reversed by passage from one ball to a neighbouring one".

Let K be the state space of a JB-algebra A of complex type and let $\pi: A \to B(H)_{sa}$ be any irreducible concrete representation associated with a given pure state $\varrho \in K$. For each facial 3-ball in F_ϱ we consider the orientation carried over by π^* from the standard orientation of the corresponding facial 3-ball in the normal state space of $B(H)$ (cf. Proposition 2.2). We say this orientation is *induced by* π. By the results of § 6, this orientation is given by the parametrization Θ^* where $\Theta: A \to M_2(\mathbf{C})$ is defined by

$$\Theta(x) = u^*\pi(x)u, \tag{7.3}$$

for an appropriate isometry $u: \mathbf{C}^2 \to H$.

LEMMA 7.2. *Let $\mathfrak{A}$ be a unital C^*-algebra with state space K and let ψ be an affine isomorphism of the state space of $M_2(\mathbf{C})$ onto a facial 3-ball B of K equipped with the orientation induced by the customary GNS-representation $\pi_\varrho: \mathfrak{A} \to B(H_\varrho)$ for some pure state ϱ such that $B \subseteq F_\varrho$. Then ψ is orientation preserving (orientation reversing) iff $\psi = \Psi^*$ where Ψ is a 2-positive (2-copositive) unital linear map from $\mathfrak{A}$ onto $M_2(\mathbf{C})$.*

Proof. Let Θ be defined as in (7.3) for all $x \in \mathfrak{A}$. Then $\beta = \Theta^*$ is an orientation preserving affine map from the state space of $M_2(\mathbf{C})$ onto B. Now $\beta^{-1} \circ \psi$ is an affine map from the state space of $M_2(\varphi)$ onto itself. By Lemma 6.2, $\beta^{-1} \circ \psi$ is the dual of a unital order automorphism Φ of $M_2(\mathbf{C})$ which is 2-positive (2-copositive) iff $\beta^{-1} \circ \psi$ is orientation preserving (orientation reversing), i.e. iff ψ is orientation preserving (orientation reversing) with respect to the given orientation of B.

Since Θ is 2-positive, the composed map $\Psi = \Phi \circ \Theta$ will be 2-positive (2-copositive) iff Φ is 2-positive (2-copositive), i.e. iff ψ is orientation preserving (orientation reversing). Since $\psi = \beta \circ \Phi^* = \Theta^* \circ \Phi^* = \Psi^*$, the proof is complete. □

THEOREM 7.3. *The state space K of a unital C^*-algebra $\mathfrak{A}$ is orientable. Specifically, there is a (unique) global orientation of K such that for each facial 3-ball B in K and each pure state ϱ with $B \subseteq F_\varrho$, then the orientation of B is that induced by the GNS-representation π_ϱ: $\mathfrak{A} \to B(H_\varrho)$.*

Proof. For each pure state ϱ we give each facial 3-ball in F_ϱ the orientation induced by the GNS-representation π_ϱ. Note that this determines a well defined orientation for each facial 3-ball in K; for if ϱ and σ are two pure states with $F_\varrho = F_\sigma$, then the GNS-representations π_ϱ and π_σ are unitarily equivalent, hence they induce the same orientations (cf. Proposition 6.3).

This choice of orientations gives a cross-section of the $\mathbf{Z}_2$-bundle $\mathcal{B}(K)/\mathrm{SO}(3)\to\mathcal{B}(K)/\mathrm{O}(3)$; it remains to prove that it is continuous.

Let $\mathcal{X}$ and $\mathcal{Y}$ be the sets of orientation preserving, respectively orientation reversing, affine isomorphisms of the state space of $M_2(\mathbf{C})$ onto a facial 3-ball of K. We will regard the elements of $\mathcal{X}$ and $\mathcal{Y}$ as parametric facial 3-balls in K by identifying the state space of $M_2(\mathbf{C})$ with $\mathbf{E}^3$.

By Lemma 7.2 each $\psi\in\mathcal{X}$ is the dual of a 2-positive linear map of $\mathfrak{A}$ onto $M_2(\mathbf{C})$ and each $\psi\in\mathcal{Y}$ is the dual of a 2-copositive linear map of $\mathfrak{A}$ onto $M_2(\mathbf{C})$. The sets of 2-positive and 2-copositive maps from $\mathfrak{A}$ to $M_2(\mathbf{C})$ are seen to be closed in the pointwise topology; hence $\mathcal{X}$ and $\mathcal{Y}$ are closed subsets of $\mathcal{X}\cup\mathcal{Y}=\mathcal{B}(K)$. Clearly, there can not be any pair (φ,ψ) with $\varphi\in\mathcal{X}$, $\psi\in\mathcal{Y}$ and $\varphi\sim\psi$ mod SO(3). Hence the direct images of $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{B}(K)/\mathrm{SO}(3)$ are disjoint closed subsets, and each of these sets will determine a continuous cross-section of $\mathcal{B}(K)/\mathrm{SO}(3)\to\mathcal{B}(K)/\mathrm{O}(3)$. The former of the two is the cross-section we are interested in, so the proof is complete. $\square$

The orientation defined in Theorem 7.3 will be called the *standard orientation* for the state space K of the given C^*-algebra $\mathfrak{A}$.

We will now show that there exist JB-algebras of complex type which are not orientable. By Theorem 7.3, such algebras can not be isomorphic to the self-adjoint part of a C^*-algebra.

PROPOSITION 7.4. *Let $T\subseteq\mathbf{C}$ be the unit circle and let $A\subseteq C(T, M_2(\mathbf{C})_{sa})$ consist of all f such that*

$$f(-\lambda)=f(\lambda)^t,\quad \textit{all } \lambda\in T. \tag{7.4}$$

Then A is a JB-algebra of complex type with non-orientable state space.

Proof. Clearly A is a JB-algebra with the algebraic operations and the norm inherited from the C^*-algebra $\mathfrak{A}=C(T, M_2(\mathbf{C}))$.

Consider a pure state ϱ of A. By the Krein-Milman theorem we can extend ϱ to a pure state $\bar{\varrho}$ of $\mathfrak{A}$. It is well known that $\bar{\varrho}$ must be of the form $f\mapsto\langle f(\lambda),\theta\rangle$, where $\lambda\in T$ and θ is a pure state of $M_2(\mathbf{C})$. From this it follows that the evaluation map $f\mapsto f(\lambda)$, which is a dense concrete representation of A, is associated with ϱ. Therefore A is of complex type.

As in § 6 we identify $\mathbf{E}^3$ with the state space of $M_2(\mathbf{C})$. Then we define parametric facial 3-balls α_t of K by the formula

$$\langle f,\alpha_t(\theta)\rangle=\langle f(e^{t\pi i}),\theta\rangle,$$

where $f \in A$, $\theta \in \mathbf{E}^3$ and $t \in [0, 1]$. Now $(\alpha_t)_{0 \leqslant t \leqslant 1}$ is a continuous path in $\mathcal{B}(K)$ such that α_0 and α_1 parametrize the same facial ball. If K is orientable, then α_0 and α_1 should belong to the same orientation. But $\alpha_1^{-1} \circ \alpha_2$ is the dual of the transpose map on $M_2(\mathbf{C})$; and by Lemma 6.1, it must be orientation reversing. This contradiction completes the proof. $\square$

Remark. It can be shown that there exists a (discontinuous) affine isomorphism of the state space of the JB-algebra A in Proposition 7.4 onto the state space of a C^*-algebra. One such C^*-algebra consists of all $f \in C(T, M_2(\mathbf{C})_{sa})$ satisfying $f(-\lambda) = f(\lambda)$ for all $\lambda \in T$. Indeed, the state spaces of both of these algebras can be identified with the same set of $M_2(\mathbf{C})_{sa}$-valued measures on T. We do not know if the state space of an arbitrary JB-algebra A of complex type is affinely isomorphic to the state space of a C^*-algebra; nor do we know if A^{**} is isomorphic to the self-adjoint part of a von Neumann algebra.

§ 8. The main theorem

In this chapter we prove the converse of Theorem 7.3; thus, orientability of the state space is a sufficient as well as a necessary condition for a JB-algebra of complex type to be isomorphic to the self-adjoint part of a C^*-algebra.

To make use of the assumed orientability, we shall need to construct convergent nets of facial balls. We have simple examples showing that, even if K is the state space of a C^*-algebra, the mapping $(\varrho, \sigma) \mapsto B(\varrho, \sigma)$ need not be continuous from its domain in $\partial_e K \times \partial_e K$ (pairs of distinct but equivalent extreme points) to $\mathcal{B}(K)/\mathrm{O}(3)$. To overcome this obstacle, we shall use the existence of a continuous map $\varrho \mapsto \varrho_a$, mapping extreme points to equivalent extreme points. Then we show that $\varrho \mapsto B(\varrho, \varrho_a)$ is continuous.

Definition. If ϱ is a state of the JB-algebra A and $a \in A$ with $\langle a^2, \varrho \rangle \neq 0$, the transformed state ϱ_a is defined by

$$\langle x, \varrho_a \rangle = \langle a^2, \varrho \rangle^{-1} \langle \{axa\}, \varrho \rangle. \tag{8.1}$$

(Recall that $\{axa\}$ is the Jordan triple product $\{axa\} = 2a \circ (a \circ x) - a^2 \circ x$, which reduces to the ordinary product axa when A is concretely represented on a Hilbert space.)

Suppose now that A is of complex type, ϱ is pure, let $\pi_\varrho \colon A \to B(H_\varrho)$ be a dense concrete representation associated with ϱ, and let $\xi_\varrho \in H_\varrho$ be the vector representing ϱ. Then ϱ_a is pure, with the representing vector

$$\xi_{\varrho_a} = \langle a^2, \varrho \rangle^{-\frac{1}{2}} \pi_\varrho(a) \xi_\varrho. \tag{8.2}$$

Thus, ϱ_a is equivalent to ϱ (cf. Proposition 3.6).

The facial ball $B(\varrho, \varrho_a)$ is defined iff ϱ and ϱ_a are distinct. This is equivalent to $\xi_\varrho, \xi_{\varrho_a}$ being linearly independent, or $|(\xi_\varrho|\xi_{\varrho_a})|<1$, since both are unit vectors. Thus, the domain of the mapping $\varrho \mapsto B(\varrho, \varrho_a)$ is the weak *-open subset of $\partial_e K$ consisting of those $\varrho \in \partial_e K$ for which

$$\langle a, \varrho\rangle^2 < \langle a^2, \varrho\rangle. \tag{8.3}$$

LEMMA 8.1. *Let $\mathfrak{A}$ be a C^*-algebra, and $\mathcal{K}$ its state space. Then, if $a \in \mathfrak{A}_{sa}$, the mapping $\varrho \mapsto B(\varrho, \varrho_a)$ is continuous from its domain in $\partial_e \mathcal{K}$ to the space $\mathcal{B}(\mathcal{K})/\mathrm{O}(3)$ of facial balls.*

Proof. For each $\varrho \in \partial_e \mathcal{K}$ satisfying (8.3), we define an isometry u_ϱ: $\mathbf{C}^2 \to H_\varrho$ mapping e_1 to ξ_ϱ and with ξ_{ϱ_a} in its image. Applying the Gram-Schmidt process to $\xi_\varrho, \xi_{\varrho_a}$, we find that this can be done by mapping e_2 to

$$\eta_\varrho = (\langle a^2, \varrho\rangle - \langle a, \varrho\rangle^2)^{-\frac{1}{2}}(\pi_\varrho(a)\xi_\varrho - \langle a, \varrho\rangle \xi_\varrho).$$

We define the map α_ϱ: $\mathbf{E}^3 \to B(\varrho, \varrho_a)$ by identifying $\mathbf{E}^3$ with the state space of $M_2(\mathbf{C})_{sa}$ and setting

$$\langle x, \alpha_\varrho(\theta)\rangle = \langle u_\varrho^* \pi_\varrho(x) u_\varrho, \theta\rangle, \tag{8.4}$$

where $x \in \mathfrak{A}$, $\theta \in \mathbf{E}^3$. We will now complete the proof by demonstrating that $\varrho \mapsto \alpha_\varrho$ is a continuous map from its domain in $\partial_e \mathcal{K}$ to $\mathcal{B}(\mathcal{K})$.

From (8.4) we see that we have to prove that $\varrho \mapsto u_\varrho^* \pi_\varrho(x) u_\varrho$ is a continuous map into $M_2(\mathbf{C})$, whenever $x \in \mathfrak{A}$. That is, we shall prove the continuity of the map $\varrho \mapsto (u_\varrho^* \pi_\varrho(x) u_\varrho e_i | e_j)$, where $i=1, 2$ and $j=1, 2$.

We check only the case $i=1$, $j=2$, leaving the others to the reader. Computing, we find

$$(u_\varrho^* \pi_\varrho(x) u_\varrho e_1 | e_2) = (\pi_\varrho(x)\xi_\varrho | \eta_\varrho) = (\langle a^2, \varrho\rangle - \langle a, \varrho\rangle^2)^{-\frac{1}{2}}(\langle ax, \varrho\rangle - \langle a, \varrho\rangle\langle x, \varrho\rangle),$$

which is a w^*-continuous function of ϱ. □

Remark. The analogue of Lemma 8.1 is not generally valid when $\mathfrak{A}$ is replaced by a JB-algebra of complex type. (There are counterexamples.) We will return to this question in a future paper.

Whenever F is a split face of K, we denote by $\mathcal{B}(F)$ the set of parametric facial balls of F, with the topology inherited from $\mathcal{B}(K)$. It is easily seen that the quotient topology of $\mathcal{B}(F)/\mathrm{O}(3)$ is that inherited from $\mathcal{B}(K)/\mathrm{O}(3)$.

LEMMA 8.2. *Assume A is a JB-algebra of complex type, with state space K. If $\varrho \in \partial_e K$, then $\mathcal{B}(F_\varrho)/\mathrm{O}(3)$ is path-connected.*

Proof. Let $\pi: A \to B(H)_{sa}$ be an irreducible representation associated with ϱ. Then the affine isomorphism π^* of the normal state space N of $B(H)$ induces a continuous surjection of $\mathcal{B}(N)/\mathrm{O}(3)$ onto $\mathcal{B}(F_\varrho)/\mathrm{O}(3)$, where N is given the σ-weak topology (i.e. $\sigma(B(H)_*, B(H))$). Therefore we need only prove that $\mathcal{B}(N)/\mathrm{O}(3)$ is path-connected.

It is enough to join two facial balls of N having one σ in common, by a continuous path. We write those balls $B(\sigma, \tau)$ and $B(\sigma, \tau')$, where σ, τ are antipodal in $B(\sigma, \tau)$, and correspondingly for τ'.

Choosing representing vectors ξ_σ, ξ_τ, $\xi_{\tau'}$ in H, we find, using Lemma 6.1, that $\xi_\sigma \perp \xi_\tau$ and $\xi_\sigma \perp \xi_{\tau'}$. We may arrange it so that $(\xi_\tau | \xi_{\tau'})$ is non-negative and real.

We can then find a continuous path (ζ_t) in the unit sphere of H, joining ξ_τ and $\xi_{\tau'}$, of the form

$$\zeta_t = (\cos t)\xi_\tau + (\sin t)\eta$$

where η is a unit vector orthogonal to ξ_τ. To complete the proof, we shall show that $t \mapsto B(\sigma, \omega_{\zeta_t})$ is a continuous path in $B(N)$.

Choose $a \in B(H)_{sa}$ such that $a\xi_\tau = a\xi_{\tau'} = \xi$. Then, since $a\zeta_t \in \mathbf{R}\xi_\sigma$, we have $B(\sigma, \omega_{\zeta_t}) = B((\omega_{\zeta_t})_a, \omega_{\zeta_t})$. By Lemma 8.1, the latter is a continuous function of t. $\square$

If A is of complex type, let $\varrho \in \partial_e K$ and let $\tilde{\varrho} \in \partial_e \mathcal{K}$ extend ϱ. Consider the GNS representation $\pi_{\tilde{\varrho}}$, and the following commutative diagram of affine isomorphisms, where N is the normal state space of $B(H_{\tilde{\varrho}})$:

$$\begin{tikzcd} & F_{\tilde\varrho} \arrow[d, "r"] \\ N \arrow[ur, "\pi_{\tilde\varrho}^*"] \arrow[r, "(\pi_{\tilde\varrho}|_A)^*"'] & F_\varrho \end{tikzcd}$$

By definition, $\pi_{\tilde{\varrho}}^*$ maps the facial balls of N orientation preservingly onto those of $F_{\tilde{\varrho}}$ (with standard orientation). Thus the orientation of the balls of F_ϱ induced by $\pi_{\tilde{\varrho}}|_A$ is the image, under r, of the standard orientation on $F_{\tilde{\varrho}}$. Thus the following lemma states that any given consistent orientation of the facial balls of F_ϱ is induced by some irreducible representation of A associated with ϱ.

LEMMA 8.3. *Let A be a JB-algebra of complex type with state space K. Assume that K is orientable and that an orientation has been chosen. Let $\mathfrak{A}$ be the enveloping C^*-algebra of A and let $\mathcal{K}$ be its state space with standard orientation. If $\varrho \in \partial_e K$, then there exists $\tilde{\varrho} \in \partial_e \mathcal{K}$ extending ϱ such that the restriction map $r: \mathcal{K} \to K$ maps the facial balls of $F_{\tilde{\varrho}}$ orientation preservingly onto those of F_ϱ, while it maps the facial balls of $F_{\Phi^*\tilde{\varrho}}$ orientation reversingly onto those of F_ϱ.*

Proof. Let $\varrho_1 \in \mathcal{K}$ be a pure state extension of ϱ, and consider the commutative diagram

$$\begin{array}{ccc} \mathcal{B}(F_{\varrho_1})/\mathrm{SO}(3) & \longrightarrow & \mathcal{B}(F_{\varrho})/\mathrm{SO}(3) \\ \downarrow & & \downarrow \\ \mathcal{B}(F_{\varrho_1})/\mathrm{O}(3) & \longrightarrow & \mathcal{B}(F_{\varrho})/\mathrm{O}(3) \end{array}$$

where the horizontal arrows are induced by r. Since both the above $\mathbf{Z}_2$-bundles are trivial, and the base space is path-connected (Lemma 8.2), r must either preserve the orientation for all the facial balls in F_{ϱ_1}, or else reverse the orientation for all of them. In the former case we are done with $\bar{\varrho} = \varrho_1$; in the latter we are done with $\bar{\varrho} = \Phi^* \varrho_1$ since the dual of the *-anti-isomorphism Φ will reverse orientations, while $r \circ \Phi^* = r$. □

We are now ready for our main result.

THEOREM 8.4. *Let A be a JB-algebra, and K its state space. Then A is isomorphic to the self-adjoint part of a C^*-algebra iff the following two conditions are satisfied:*

(i) *K has the 3-ball property*

(ii) *K is orientable.*

Proof. The necessity of the two conditions follows from Lemma 3.4 and Theorem 7.3.

To prove sufficiency, we assume (i) and (ii) and fix a global orientation of K. We adopt the notation of Lemma 8.3, and we will denote by F the "σ-convex hull" of all $F_{\bar{\varrho}}$ (i.e. the set of states $\sum_{i=1}^{\infty} \lambda_i \sigma_i$ where $\lambda_i \geqslant 0$, $\sum_{i=1}^{\infty} \lambda_i = 1$, and each σ_i belongs to some $F_{\bar{\varrho}}$ where $\varrho \in \partial_e K$). In symbols:

$$F = \sigma\text{-co}\,\Big(\bigcup_{\varrho \in \partial_e K} F_{\bar{\varrho}}\Big). \tag{8.5}$$

By [6; § 5], σ-co $(\partial_e \mathcal{K})$ is a split face of $\mathcal{K}$. Clearly, F is a split face of σ-co $(\partial_e \mathcal{K})$, hence of K, with extreme boundary

$$\partial_e F = \bigcup_{\varrho \in \partial_e \mathcal{K}} \partial_e F_{\bar{\varrho}} = \{\bar{\varrho} : \varrho \in \partial_e K\}. \tag{8.6}$$

By [17] the w^*-closure $\bar{F}$ of F is a split face. We claim that $\partial_e \bar{F} = \partial_e F$; by (8.6) this implies that r is a bijection from $\partial_e \bar{F}$ to $\partial_e K$, and by Theorem 5.11 this will complete the proof.

Since $\partial_e F = F \cap \partial_e \mathcal{K} \subseteq \bar{F} \cap \partial_e \mathcal{K} = \partial_e \bar{F}$, we only have to prove $\partial_e \bar{F} \subseteq \partial_e F$. For contradiction we assume $\sigma \in \overline{\partial_e F}$, $\sigma \notin \partial_e F$. Let $\varrho = r(\sigma)$, and note that, by (8.6), $\sigma \neq \bar{\varrho}$. By Proposition 5.6 it follows that $\sigma = \Phi^* \bar{\varrho}$, and that the GNS-representation $\pi_{\bar{\varrho}}$ is not one-dimensional. Now we choose $a \in A$ satisfying (8.3); which is possible by the density of $\pi_{\bar{\varrho}}(A)$ in $B(H\text{-})_{sa}$. (We only need to have $\xi_{\bar{\varrho}}$, $\pi_{\bar{\varrho}}(a)\xi_{\bar{\varrho}}$ linearly independent.)

By Milman's theorem, $\partial_e F$ is w^*-dense in $\partial_e \bar{F}$. From (8.6) we conclude that there exists a net $\{\varrho_\gamma\}$ in $\partial_e K$ such that $\bar{\varrho}_\gamma \to \sigma = \Phi^*(\bar{\varrho})$ in $\partial_e \mathcal{K}$, and such that (8.3) holds for all ϱ_γ (i.e. with ϱ_γ in place of ϱ). Note that $\varrho_\gamma = r(\bar{\varrho}_\gamma)$ also converges to ϱ in $\partial_e K$.

By Lemma 8.1, the balls $B(\bar{\varrho}_\gamma, (\bar{\varrho}_\gamma)_a)$ converge to $B(\sigma, \sigma_a)$ in the topology of $\mathcal{B}(\mathcal{K})/\mathrm{O}(3)$. This is also true if these balls, with standard orientation, are viewed as members of $\mathcal{B}(\mathcal{K})/\mathrm{SO}(3)$. This follows from the orientability of $\mathcal{K}$, but it can also be seen directly from the proof of Lemma 8.1.

It follows by the definition of the states involved, and by Lemma 8.3, that r maps $B(\bar{\varrho}_\gamma, (\bar{\varrho}_\gamma)_a)$ orientation preservingly onto $B(\varrho_\gamma, (\varrho_\gamma)_a)$ and $B(\sigma, \sigma_a) = B(\Phi^*\bar{\varrho}, (\Phi^*\bar{\varrho})_a)$ orientation reversingly onto $B(\varrho, \varrho_a)$. Hence, since r is continuous, the balls $B(\varrho_\gamma, (\varrho_\gamma)_a)$ will converge to $B(\varrho, \varrho_a)$ in $\mathcal{B}(K)/\mathrm{O}(3)$; but the orientation is reversed in the limit, so $B(\varrho_\gamma, (\varrho_\gamma)_a)$ does not converge to $B(\varrho, \varrho_a)$ in $\mathcal{B}(K)/\mathrm{SO}(3)$. This is a contradiction since K was equipped with a global orientation. $\square$

Let A be a JB-algebra and $A \oplus iA$ its complexification as a linear space. By a C^*-*structure* on $A \oplus iA$ we mean a triple consisting of a product, an involution, and a norm, organizing $A \oplus iA$ to a C^*-algebra with the given JB-algebra A is its self-adjoint part.

COROLLARY 8.5. *If A is a JB-algebra whose state space satisfies the conditions* (i), (ii) *of Theorem* 8.5, *then the C^*-structures on $A \oplus iA$ are in* 1–1 *correspondence with the global orientations of K.*

Proof. To each C^*-structure on $A \oplus iA$ is associated a global orientation (the standard one), as explained in § 7.

Conversely, for each global orientation of K, the proof of Theorem 8.4 yields a w^*-closed split face F of $\mathcal{K}$ such that $\partial_e F$ is mapped bijectively onto $\partial_e K$ by r, and such that the facial balls in F are mapped orientation preservingly onto those of K (in the standard orientation of $\mathcal{K}$ and the given orientation of K). Let J be the annihilator of F in $\mathfrak{A}$. Then the canonical isomorphism for $A \oplus iA$ and $\mathfrak{A}/J$ (see proof of Theorem 5.11) induces a C^*-structure on $A \oplus iA$, which in turn defines the given orientation of K (since r is orientation preserving from the state space F of $\mathfrak{A}/J$ to K).

Finally we consider two C^*-structures on $A \oplus iA$ which determine the same standard orientation of K. Then for each $\varrho \in \partial_e K$, the GNS-representations of $A \oplus iA$ associated with ϱ for each of the two C^*-structures, will be the same. Hence, the two structures must coincide, and we are done. $\square$

Combining Theorem 8.4 with the main result of [6], we obtain the following (see [6] for the definition of concepts not previously used in this paper):

COROLLARY 8.6. *A compact convex set K (in some locally convex Hausdorff space) is affinely homeomorphic to the state space of a C^*-algebra iff:*

(i) *every norm exposed face is projective,*

(ii) *every $a \in A(K)$ admits a decomposition $a = a^+ - a^-$ with $a^+, a^- \in A(K)^+$ and $a^+ \perp a^-$,*

(iii) *the σ-convex hull of $\partial_e K$ is a split face,*

(iv) *$B(\varrho, \sigma)$ is a norm exposed face affinely isomorphic to a 3-dimensional Euclidean ball or to a line segment for each pair ϱ, σ of distinct extreme points,*

(v) *K is orientable.*

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