INTERTWINING OPERATORS FOR REPRESENTATIONS INDUCED FROM UNIFORM SUBGROUPS

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§ 1. Introduction

Let G be a second countable locally compact group and Γ a discrete uniform subgroup ($\Gamma\diagdown G$ compact). Then the quasi-regular representation $G \times L^2(\Gamma\diagdown G) \to L^2(\Gamma\diagdown G)$ decomposes into a direct sum of irreducibles $\oplus_{i \in I} n_i \pi_i$, each with finite multiplicity. If G is a simply connected nilpotent Lie group, the problem of determining the spectrum $\{\pi_i\} \subseteq G^\wedge$ and multiplicities n_i was first discussed by C. C. Moore [13]; later, L. Richardson [16] and R. Howe [6] independently gave closed formulas for calculating the spectrum and multiplicities in this situation. Recently, Auslander and Brezin [1] and R. Howe [19] have developed inductive proceedures for determining spectra and multiplicities in solvmanifolds.

In this paper we given a construction of intertwining operators between induced representations, reminiscent of a construction of Weil [18], which generalizes Mackey's theorem [8], [10; p. 122–130] on intertwining operators between induced representations of a *finite* group. Recall that if H_1 and H_2 are subgroups with unitary representations π_1 and π_2, and if $\sigma_i = \text{Ind}\,(H_i \uparrow G, \pi_i)$, then for any finite group G Mackey's formula gives the intertwining number for σ_1 and σ_2. In fact, Mackey constructs all intertwining operators explicitly. We employ a different construction and prove that it yields all intertwining operators in the case when G is second countable, H_1 is normal in G and σ_1 irreducible, $H_2\diagdown G$ is compact and π_2 finite dimensional, and $H_1 \cap H_2\diagdown H_1$ is compact (Theorem 3.7 below). This theorem plus some additional work yields the Howe-Richardson results, but it also applies in other situations.

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Our construction is similar to those found in Weil [18], Richardson [16], and Auslander-Brezin [1], though it applies in a more general setting. Our proof that the operators are well defined is different because of the more general context in which we work; it is quite elementary. To show that we have all the intertwining operators we use a simple case of the Mackey subgroup theorem, but in our situation the direct integral can be (and is) written out quite explicitly. We also use the notion of weak containment, especially as it applies to CCR representations. We do not invoke any of the more advanced aspects of the Mackey machine. One main result is the following analog of Mackey's intertwining formula (see 3.7 below).

THEOREM. *Let G be a second countable locally compact group and K, Γ closed subgroups such that* (i) *K is normal in G,* (ii) *$\Gamma\backslash G$ is compact and has finite invariant volume,* (iii) *$K \cap \Gamma\backslash K$ is compact (hence has finite invariant volume),* (iv) *$K\backslash K\Gamma$ is discrete. Let π be a finite dimensional representation of K such that $\sigma = {}_GU^\pi$ is irreducible, let ϱ be any finite dimensional representation of Γ, and let $\tau = {}_GU^\varrho$. Then*

$$\mathrm{Hom}({}_GU^\pi, {}_GU^\varrho) \cong \oplus_{y \in K\backslash G/\Gamma} \mathrm{Hom}_{\Gamma\cap K}(\pi\cdot y\,|\,\Gamma\cap K, \varrho\,|\,\Gamma\cap K).$$

Here $\pi\cdot y(k) = \pi(yky^{-1})$ and Hom (...) is the space of bounded linear intertwining operators. The direct sum is an algebraic direct sum since $\mathrm{Hom}_G(U^\pi, U^\varrho)$ must be finite dimensional if nontrivial, in this context. If π is not finite dimensional, there is still a useful intertwining formula which is a less direct analog of Mackey's formula for finite groups (see Theorem 3.4 below, and the commentary with equations (5) and (6)).

§ 2. Preliminary remarks

Throughout sections 2 and 3, G will be a second countable locally compact group; K, Γ will be closed subgroups such that (i) K is normal in G, (ii) $\Gamma\backslash G$ is compact with a finite G-invariant measure, (iii) $\Gamma \cap K\backslash K$ is compact. *Note:* If Γ is discrete and $\Gamma\backslash G$ compact it is easy to see that G is unimodular, and there is a finite invariant measure on $\Gamma\backslash G$. Also, (i) ... (iii) imply that $\Gamma \cap K\backslash K$ has a finite K-invariant measure. [Pf: First, $\Gamma \cap K\backslash K$ compact $\Rightarrow K\Gamma$ closed. By an elementary theorem [15], both $K\Gamma\backslash G$ and $\Gamma\backslash K\Gamma$ have finite volume. But the map $(\Gamma \cap K)\cdot k \to \Gamma\cdot k$ is a K-equivariant homeomorphism from $\Gamma \cap K\backslash K$ to $\Gamma\backslash K\Gamma$.]

If π is a representation of a subgroup M, its Hilbert space will be denoted by $\mathcal{H}(\pi)$. We let ${}_GU^\pi = \mathrm{Ind}\,(M \uparrow G, \pi)$ be the induced representation. If L is another subgroup, $L \subseteq G$, then $\pi|L$ is π restricted to L; thus, ${}_GU^\pi|L$ is π induced from M to G and then restricted to L. If $x \in G$ and π is a representation of a subgroup M, the *conjugate* π^x or $\pi\cdot x$

$$\begin{tikzcd} & KM \arrow[dr, "r"] & \\ \pi \; M \arrow[ur, "i"] \arrow[dr, "r"'] & & K \\ & M \cap K \arrow[ur, "i"'] & \end{tikzcd}$$

Figure 1

is the representation of $x^{-1}Mx$ modeled in $\mathcal{H}(\pi)$ such that $\pi^x(g)=\pi(xgx^{-1})$ for $g \in x^{-1}Mx$. Usually M will be normal. If π, ϱ are representations of M, then $\operatorname{Hom}_M(\pi, \varrho)$ is the space of bounded linear intertwining operators from $\mathcal{H}(\pi)$ into $\mathcal{H}(\varrho)$; we omit the subscript M when no confusion will result.

Next we establish a few lemmas. The first is a formal property of the induction process.

LEMMA 2.1. *Let K be normal in G, M a closed subgroup such that $KM=G$. If π is a unitary representation of M, then ${}_GU^\pi \,|\, K \cong {}_KU^{\pi|K\cap M}$. That is, the diagram in Figure* 1 *commutes (i=induction, r=restriction).*

Proof. The Hilbert space of ${}_GU^\pi$ is made up of measurable functions $f\colon G \to \mathcal{H}(\pi)$ such that $f(mx)=\pi(m)f(x)$, all $m \in M$, $x \in G$. Such a function is determined by its values on any set meeting each $M\backslash G$ coset at least once, such as K. Moreover, the action of K on f is determined by what happens on $f\,|\,K$, and the spaces $M\backslash G$ and $M \cap K\backslash K$ (which are naturally isomorphic) have the same quasi-invariant measures. Now it is not hard to show that $f \to f\,|\,K$ gives the isomorphism desired. Q.E.D.

The next lemma will be used to make some final reductions in our work. A detailed proof is given in Moore [12].

LEMMA 2.2. *Let Γ be a discrete subgroup such that $\Gamma\backslash G$ is compact, and let π be a finite dimensional representation of G. Let ϱ be an irreducible representation of Γ. Then* $\operatorname{Hom}_G(\pi, {}_GU^\varrho) \cong \operatorname{Hom}_\Gamma(\pi\,|\,\Gamma, \varrho)$.

In fact, the equivalence is given as follows. If $A \in \operatorname{Hom}_\Gamma(\pi\,|\,\Gamma, \varrho)$ define $B\colon \mathcal{H}(\pi) \to \mathcal{H}(U^\varrho)$ by $Bv(g)=A\pi(g)v$; the map $A \to B$ is the desired isomorphism. It maps *onto* because, if π_0 is a subrepresentation of U^ϱ equivalent to π, then π_0 is realized on a space of continuous functions. Given B we get the corresponding A by taking $Av=Bv(e)$.

Next we prove lemmas which show that all representations we deal with are type I. In our applications to nilpotent groups, all representations are known to be type I, and these lemmas are unnecessary. However, they are needed to establish the intertwining theorems in the generality given here.

Lemma 2.3. *Let K be a normal subgroup of G and Γ a closed subgroup of G such that $\Gamma\backslash G$ and $\Gamma\cap K\backslash K$ are compact with finite invariant measures. Let ϱ be a finite dimensional representation of Γ, $\lambda={}_{\Gamma K}U^{\varrho}=\operatorname{Ind}(\Gamma\uparrow K\Gamma,\varrho)$, and consider the restriction $\lambda|K$. Let $S=\{\pi\cdot x\in K^{\wedge}\colon x\in G$ and π any irreducible representation of K occurring in $\lambda|K\}$. Then S is a closed set in the hull-kernel topology of $K^{\wedge}$.*

Proof. By Lemma 2.1, $\lambda|K\cong{}_KU^{\varrho|\Gamma\cap K}$. If $\Gamma\cap K$ is discrete it is well known that the latter is a direct sum of CCR irreducibles [5; section 2], each with finite multiplicity. Actually [17], the result is true for any subgroup $M\subseteq K$ (discrete or not) such that $M\backslash K$ is compact and has finite invariant measure. The spectrum T of this representation, the irreducible CCR representations $\{\pi_i\}\subseteq K^{\wedge}$ occurring in it, form a discrete, closed, Hausdorff subspace in $K^{\wedge}$ [4; Theorem 1.8]. Also, T is invariant under the action of Γ on $K^{\wedge}$, because ${}_{K\Gamma}U^{\varrho\cdot\gamma}\cong{}_{K\Gamma}U^{\varrho}$ for $\gamma\in\Gamma$. Let C be a compact set in G such that $G=\Gamma C$. Let $\{\pi_\alpha\}$ be any net of elements in S which converges to an element $\pi\in K^{\wedge}$. Each π_α is of the form $\zeta_\alpha\cdot x_\alpha$ with $\zeta_\alpha\in T$ and $x\in G$. Because T is Γ-invariant we may choose the ζ_α and x_α so that $x_\alpha\in C$ for all α. By passing to a subnet we may assume that $x_\alpha\to x\in C$. But the map $K^{\wedge}\times G\to K^{\wedge}$ is jointly continuous. [From Fell's description of the topology of $K^{\wedge}$ in terms of positive definite functions on K [4; Theorem 1.5], this is easily seen by examining limits (uniform on compacta) of positive definite functions associated with representations in K.] Thus $\zeta_\alpha=\pi_\alpha\cdot x^{-1}\to\pi\cdot x^{-1}$. Since T is discrete the ζ_α must eventually all be the same element $\zeta\in T$, so that $\pi_\alpha\to\zeta\cdot x\in S$. Q.E.D.

Lemma 2.4. *In the situation of Lemma 2.3 the restriction of ${}_GU^{\varrho}=\operatorname{Ind}(\Gamma\uparrow G,\varrho)$ to K is type I, as are all of its subrepresentations.*

Proof. Let $S=T\cdot G$ as in 2.3. As noted, the elements of T (hence also S) are all CCR representations of K, and S is closed in $K^{\wedge}$. Let $A(K)$ be the group C^* algebra of K; then $I=\operatorname{hull}(S)=\{a\in A(K)\colon\pi(a)=0$, all $\pi\in S\}$ is a closed two sided ideal. The C^* algebra $A(K)/I$, has $A^{\wedge}=\ker(I)=S$. Since all elements of $A^{\wedge}$ are CCR, A is type I. By Mackey's subgroup theorem, the restriction of $\tau={}_GU^{\varrho}$ to K can be written as

$$\tau|K=\int_{K\Gamma\backslash G}^{\oplus}(\lambda|K)\cdot x\,d\mu(x)\quad(\text{where }\lambda={}_{K\Gamma}U^{\varrho}\cong{}_KU^{\varrho|\Gamma\cap K}).$$

For each $x\in G$, $(\lambda|K)\cdot x=(\bigoplus n_i\pi_i)\cdot x$ is a direct sum of irreducibles in S, and so $((\lambda|K)\cdot x)a=0$ for all $a\in I$, $x\in G$. By the direct integral decomposition of $\tau|K$, we get $\tau(a)=0$, all $a\in I$; thus, τ is the liftback under $A(K)\to A(K)/I$ of a representation of the CCR algebra A, and is type I. Q.E.D.

We note one other straightforward fact. If π is a representation of a normal subgroup K such that $\sigma = {}_G U^\pi$ is irreducible, and if $x \in G$, $x \notin K$, then π and $\pi \cdot x$ are inequivalent, so that the stabilizer of π under the action of G on $K^\wedge$ is $\mathrm{Stab}_G(\pi) = K$. This follows from Mackey's theory, but there is a short direct proof. [Since $\sigma = {}_G U^\pi$ is irreducible, so are π and $\pi \cdot x$ as representations of K. If $A \in \mathrm{Hom}_K(\pi, \pi \cdot x)$ then the operator $B\colon \mathcal{H}(\sigma) \to \mathcal{H}(\sigma)$ defined by $Bf(y) = A(f(xy))$ is well defined, for if $k \in K$ we get

$$(Bf)(ky) = A(f(xky)) = A\pi(xkx^{-1})f(xy) = \pi(k)Af(xy).$$

Clearly $B \in \mathrm{Hom}_G(\sigma, \sigma)$ and cannot equal a scalar multiple of the identity operator if $A = 0$ and $x \notin K$. Since σ is irreducible, $A = 0$.]

§ 3. Construction of intertwining operators

Throughout this section we consider a system $(G, K, \Gamma; \pi, \varrho)$ of closed subgroups and unitary representations such that

(i) K is normal in G
(ii) $\Gamma \diagdown G$ is compact and has finite invariant measure
(iii) $K \cap \Gamma \diagdown K$ is compact (hence has finite invariant measure)
(iv) $K \diagdown K\Gamma$ is discrete
(v) π is a representation of K such that $\sigma = {}_G U^\pi$ is irreducible
(vi) ϱ is a representation of Γ.

It follows that $K \cap \Gamma \diagdown \Gamma \cong K \diagdown K\Gamma$ is also discrete. Later we will add a finite dimensionality condition on ϱ. We write $\tau = {}_G U^\varrho$.

Suppose that $A \in \mathrm{Hom}(\pi | \Gamma \cap K, \varrho | \Gamma \cap K)$. To justify Mackey's formula (for finite groups) we should try to write down, formally at least, a corresponding element $B = JA \in \mathrm{Hom}_G(\sigma, \tau)$. Recall that σ operates on a space of measurable functions $F\colon G \to \mathcal{H}(\pi)$ such that $F(kx) = \pi(k)(F(x))$ if $k \in K$; likewise for τ with respect to ϱ. A little thought reveals that the following averaging process should produce an intertwining operator from A,

$$(BF)(x) = \textstyle\sum_{\gamma \in \Gamma \cap K \backslash \Gamma} \varrho(\gamma^{-1}) A(F(\gamma x)), \tag{1}$$

the sum taken over any set of coset representatives for $\Gamma \cap K \diagdown \Gamma$. This makes sense as a sum over representatives, because if we replace γ by $\gamma_0 \gamma$ ($\gamma_0 \in \Gamma \cap K$) we get

$$\varrho((\gamma_0\gamma)^{-1}) AF(\gamma_0 \gamma x) = \varrho(\gamma)^{-1}(\varrho(\gamma_0^{-1}) A\pi(\gamma_0)) F(\gamma x) = \varrho(\gamma)^{-1} AF(\gamma x).$$

Next observe that $(BF)(\gamma_1 x) = \varrho(\gamma_1)(BF(x))$ for $\gamma_1 \in \Gamma$ (simply replace γ by $\gamma\gamma_1^{-1}$ in the sum). Clearly, if we can show that the sum converges for almost all x, and determines a bounded operator, then $B \in \mathrm{Hom}(\sigma, \tau)$. We start the proof that B is bounded with two lemmas.

Let $u, v \in \mathcal{H}(\pi)$ and let $\gamma \in \Gamma$. The map $K \to \mathbf{C}$ defined by $k \to \langle A\pi(k)v, \varrho(\gamma)^{-1}A\pi(\gamma k\gamma^{-1})w\rangle$ is constant on $\Gamma \cap K$-cosets in K: if $\gamma_0 \in \Gamma \cap K$ (which is normal in Γ) then

$$\begin{aligned}&\langle A\pi(\gamma_0 k)v, \varrho(\gamma)^{-1}A\pi(\gamma\gamma_0 k\gamma^{-1})w\rangle \\ &\quad = \langle \varrho(\gamma_0)A\pi(k)v, \varrho(\gamma)^{-1}\varrho(\gamma\gamma_0\gamma^{-1})A\pi(\gamma k\gamma^{-1})w\rangle = \langle A\pi(k)v, \varrho(\gamma)^{-1}A\pi(\gamma k\gamma^{-1})w\rangle.\end{aligned}$$

LEMMA 3.1. *Let $(G, K, \Gamma; \pi, \varrho)$ be as above. If $\gamma \notin \Gamma \cap K$, $\gamma \in \Gamma$, then*

$$\int_{\Gamma \cap K \backslash K} \langle A\pi(k)v, \varrho(\gamma)^{-1}A\pi(\gamma k\gamma^{-1})w\rangle\, d\dot{k} = 0$$

for all $v, w \in \mathcal{H}(\pi)$.

Proof. Define T: $\mathcal{H}(\pi) \to \mathcal{H}(\pi)$ by

$$\langle Tv, w\rangle = \int_{\Gamma \cap K \backslash K} \langle A\pi(k)v, \varrho(\gamma)^{-1}A\pi(\gamma k\gamma^{-1})w\rangle\, d\dot{k}.$$

Routine estimates show that T is bounded. If $k_0 \in K$, then

$$\begin{aligned}\langle T\pi(k_0)v, w\rangle &= \int_{\Gamma \cap K \backslash K} \langle A\pi(kk_0)v, \varrho(\gamma)^{-1}A\pi(\gamma k\gamma^{-1})w\rangle\, d\dot{k} \\ &= \int_{\Gamma \cap K \backslash K} \langle A\pi(k)v, \varrho(\gamma)^{-1}A\pi(\gamma kk_0^{-1}\gamma^{-1})w\rangle\, d\dot{k} \\ &= \int_{\Gamma \cap K \backslash K} \langle A\pi(k)v, \varrho(\gamma)^{-1}A\pi(\gamma k\gamma^{-1})\pi^{\gamma}(k_0^{-1})w\rangle\, d\dot{k} \\ &= \langle Tv, \pi^{\gamma}(k_0^{-1})w\rangle = \langle \pi^{\gamma}(k_0)Tv, w\rangle.\end{aligned}$$

Hence $T \in \mathrm{Hom}\,(\pi, \pi \cdot \gamma)$. Now apply the remarks about $\mathrm{Stab}_G(\pi)$ at the end of section 2 to conclude that $T = 0$ if $\gamma \in \Gamma \sim (\Gamma \cap K)$. Q.E.D.

LEMMA 3.2. *Let F: $G \to \mathcal{H}(\pi)$ be continuous, satisfying $F(kx) = \pi(k)F(x)$ for all $k \in K$, $x \in G$, such that $\|F(x)\| = \|F(Kx)\|$ has compact support in $K \backslash G$. For each $x \in G$ let $S_x = \{\gamma \in \Gamma: \gamma x \in \mathrm{supp}\,(F)\}$. Then there is an integer n_F independent of $x \in G$ such that S_x is the union of at most n cosets of $\Gamma \cap K$.*

Proof. As noted, $K \cap \Gamma \backslash \Gamma \cong K \backslash K\Gamma$ is discrete. Clearly S_x is a union of $K \cap \Gamma$-cosets. Choose compacta $C_1, C_2 \subseteq G$ such that $\mathrm{supp}(F) \subseteq K \cdot C_1$ and $G = \Gamma \cdot C_2$. Then $\Gamma \cap K \backslash KC_1C_2^{-1}$

is compact since $K \cap \Gamma \diagdown K$ is compact, and hence meets the discrete set $\Gamma \cap K \diagdown \Gamma$ at finitely many points. Let n be the number of points. To show that this n works, consider any $x \in G$ and let $x = \gamma x_2$ $(\gamma \in \Gamma, x_2 \in C_2)$. Then $S_x \cdot \gamma \cdot x_2 = S_x \cdot x \subseteq \operatorname{supp}(F) \subseteq K \cdot C_1$, so that $S_x \gamma \subseteq K C_1 C_2^{-1}$. Of course we also have $(\Gamma \cap K) S_x \cdot \gamma \subseteq \Gamma$. Hence $\Gamma \cap K \diagdown S_x \gamma \subseteq (\Gamma \cap K \diagdown \Gamma) \cap (\Gamma \cap K \diagdown K C_1 C_2^{-1})$ has at most n elements. Thus $\Gamma \cap K \diagdown S_x \gamma$ and $\Gamma \cap K \diagdown S_x$ have cardinality at most n. Q.E.D.

Among other things, this observation shows that the sum over $\Gamma \cap K \diagdown \Gamma$ defining BF in (1) has at most n_F nonzero entries for each $x \in G$, and so is well defined.

THEOREM 3.3. *Let $(G, K, \Gamma; \pi, \varrho)$ be as above and let $A \in \operatorname{Hom}(\pi | \Gamma \cap K, \varrho | \Gamma \cap K)$. For all continuous functions $F: G \to \mathcal{H}(\pi)$ satisfying $F(kx) = \pi(k) F(x)$ all $k \in K$, $x \in G$, such that $\|F(x)\|$ has compact support in $K \diagdown G$, define $BF(x)$ as in (1). Then $BF \in \mathcal{H}(U^\varrho)$ and B extends uniquely to a bounded linear operator $B: \mathcal{H}(U^\pi) \to \mathcal{H}(U^\varrho)$. In fact, $B \in \operatorname{Hom}(U^\pi, U^\varrho)$ and the map $J: A \to B$ is injective.*

Proof. Write $\sigma = {}_G U^\pi$ and $\tau = {}_G U^\varrho$. Since B is defined on a dense set, the uniqueness of any bounded extension is clear. From what has already been said, only the boundedness of B and the fact: $B = 0 \Rightarrow A = 0$ require proof. First we compute $\|BF\|$ (afterwards we will justify our use of Fubini in interchanging integrals over $\Gamma \cap K \diagdown G$ with sums over $\Gamma \cap K \diagdown \Gamma$).

$$\|BF\|^2 = \int_{\Gamma \backslash G} \sum_{\gamma_1 \in \Gamma \cap K \backslash \Gamma} \sum_{\gamma_2 \in \Gamma \cap K \backslash \Gamma} \langle \varrho(\gamma_1)^{-1} A F(\gamma_1 x), \varrho(\gamma_2)^{-1} A F(\gamma_2 x) \rangle \, d\dot{x}$$

$$= \int_{\Gamma \backslash G} \sum_{\gamma_1, \gamma_3 \in \Gamma \cap K \backslash \Gamma} \langle \varrho(\gamma_1)^{-1} A F(\gamma_1 x), \varrho(\gamma_1^{-1} \gamma_3^{-1}) A F(\gamma_3 \gamma_1 x) \rangle \, d\dot{x}$$

$$= \int_{\Gamma \cap K \backslash G} \sum_{\gamma_3 \in \Gamma \cap K \backslash \Gamma} \langle A F(x), \varrho(\gamma_3)^{-1} A F(\gamma_3 x) \rangle d\dot{x}$$

Fix γ_3; then

$$\int_{\Gamma \cap K \backslash G} \langle A F(x), \varrho(\gamma_3)^{-1} A F(\gamma_3 x) \rangle \, d\dot{x}$$

$$= \int_{K \backslash G} \int_{\Gamma \cap K \backslash K} \langle A F(kx), \varrho(\gamma_3)^{-1} A F(\gamma_3 k \gamma_3^{-1} \cdot \gamma_3 x) \rangle \, d\dot{k} \, d\dot{x}$$

$$= \int_{K \backslash G} \int_{\Gamma \cap K \backslash K} \langle A \pi(k) F(x), \varrho(\gamma_3)^{-1} A \pi(\gamma_3 k \gamma_3^{-1}) F(\gamma_3 x) \rangle \, d\dot{k} \, d\dot{x} = 0$$

unless $\gamma_3 \in \Gamma \cap K$, by Lemma 3.1. Hence by Fubini,

$$\|BF\|^2 = \int_{\Gamma\cap K\backslash G} \langle AF(x), AF(x)\rangle\, d\dot{x} \leqslant \|A\|^2 \int_{K\backslash G}\int_{\Gamma\cap K\backslash K} \|F(kx)\|^2\, d\dot{k}\, d\dot{x}$$

$$= \|A\|^2 \|F\|^2 \cdot \text{volume}\ (\Gamma \cap K \diagdown K) \tag{2}$$

Thus $\|BF\| < +\infty$ and B is bounded. If A is nonzero then for some $F \in \mathcal{H}(\sigma)$, $AF(x)$ is nonzero on a set of positive measure and so, by (2), $\|BF\| \neq 0$.

To justify our use of Fubini, we show that

$$\int_{\Gamma\backslash G} \sum_{\gamma_1,\gamma_2\in\Gamma\cap K\backslash\Gamma} |\langle \varrho(\gamma_1)^{-1} AF(\gamma_1 x), \varrho(\gamma_2)^{-1} AF(\gamma_2 x)\rangle|\, d\dot{x}$$

is finite. By 3.2, this expression is dominated by

$$\int_{\Gamma\backslash G} \sum_{\gamma_1,\gamma_2\in\Gamma\cap K\backslash\Gamma} \|A\|^2 \|F(\gamma_1 x)\| \|F(\gamma_2 x)\|\, d\dot{x}$$

$$\leqslant \|A\|^2 \int_{\Gamma\backslash G} \left(\sum_{\gamma_1\in\Gamma\cap K\backslash\Gamma} \|F(\gamma_1 x)\|\right)\left(\sum_{\gamma_2\in\Gamma\cap K\backslash\Gamma} \|F(\gamma_2 x)\|\right) d\dot{x}$$

$$\leqslant \|A\|^2 n_F^2 \int_{\Gamma\backslash G} \|F\|_\infty^2\, d\dot{x} < +\infty. \qquad \text{QED}$$

We note two facts about this construction. Let $\Gamma_0 = \Gamma \cap K$. First, if $y \in G$ then equation (1) determines a map

$$\mathrm{J} : \mathrm{Hom}_{\Gamma_0}(\pi \cdot y \,|\, \Gamma_0, \varrho \,|\, \Gamma_0) \to \mathrm{Hom}_G(U^{\pi\cdot y}, U^{\varrho}). \tag{3}$$

But $U^\pi \cong U^{\pi\cdot y}$ under the isometry I^y: $\mathcal{H}(U^\pi) \to \mathcal{H}(U^{\pi\cdot y})$ given by $I^y F(x) = F(yx)$. This induces an isomorphism Φ^y: Hom $(U^{\pi\cdot y}, U^\varrho) \to$ Hom (U^π, U^ϱ) if we take $(\Phi^y T) F = T(I^y F)$. Thus, for each $y \in G$, we may regard the map (3) as carrying Hom $(\pi\cdot y \,|\, \Gamma_0, \varrho \,|\, \Gamma_0)$ into Hom (U^π, U^ϱ) by replacing J with $\mathbf{J} = \Phi^y \circ J$. Now if $k \in K$, $\gamma \in \Gamma$, routine calculations show that R: $A \to \varrho(\gamma)^{-1} A \pi(k)^{-1}$ is an isomorphism between Hom $(\pi\cdot y \,|\, \Gamma_0, \varrho \,|\, \Gamma_0)$ and Hom$(\pi \cdot ky\gamma \,|\, \Gamma_0, \varrho \,|\, \Gamma_0)$. Further calculations show that the maps in Figure 2 commute. Thus the J-image of Hom $(\pi\cdot y \,|\, \Gamma_0, \varrho \,|\, \Gamma_0)$ in Hom (U^π, U^ϱ) depends only upon the $K \diagdown G/\Gamma = G/K\Gamma$ double coset to which y belongs. In this manner we obtain a linear map

$$\mathbf{J}\colon \oplus_{y\in K\backslash G/\Gamma} \mathrm{Hom}\ (\pi\cdot y \,|\, \Gamma \cap K, \varrho \,|\, \Gamma \cap K) \to \mathrm{Hom}\ (U^\pi, U^\varrho). \tag{4}$$

We do not yet know if $\mathbf{J}$ is injective or surjective, though it is injective when restricted to each subspace in the direct sum. We shall examine these questions in a slightly different context. Second, if $B_i = \mathbf{J} A_i$ $(i = 1,2)$ and if A_1 and $\varrho(\gamma) A_2$ have orthogonal ranges for all $\gamma \in \Gamma_0 \diagdown \Gamma$, then B_1 and B_2 have orthogonal ranges (as a computation like that in Theorem 3.3 easily shows).

Given the system $(G, K, \Gamma; \pi, \varrho)$, Mackey's formula has as its most direct (formal)

$$\begin{tikzcd}
\mathrm{Hom}(U^{\pi\cdot y}, U^{\varrho}) \arrow[dr, "\Phi^{y}"] & & \mathrm{Hom}(U^{\pi\cdot ky\gamma}, U^{\varrho}) \arrow[dl, "\Phi^{ky\gamma}"'] \\
& \mathrm{Hom}(U^{\pi}, U^{\varrho}) & \\
\mathrm{Hom}(\pi\cdot y\,|\,\Gamma_0, \varrho\,|\,\Gamma_0) \arrow[uu, "J"] \arrow[ur, "\mathbf{J}"'] \arrow[rr, "R"'] & & \mathrm{Hom}(\pi\cdot ky\gamma\,|\,\Gamma_0, \varrho\,|\,\Gamma_0) \arrow[uu, "J"'] \arrow[ul, "\mathbf{J}"]
\end{tikzcd}$$

Figure 2. Here $\Gamma_0 = \Gamma \cap K$.

generalization the statement

$$\mathrm{Hom}_G(U^{\pi}, U^{\varrho}) \cong \oplus_{y\in K\backslash G/\Gamma}\, \mathrm{Hom}_{K\cap\Gamma}(\pi\cdot y\,|\,\Gamma\cap K, \varrho\,|\,\Gamma\cap K). \tag{5}$$

However, we may form the related system $(G, K, H; \pi, \lambda)$, where $H = K\Gamma$ and $\lambda = {}_HU^{\varrho}$, which also satisfies properties (i) ... (vi) as noted below. For this system Mackey's formula says (formally) that

$$\mathrm{Hom}_G(U^{\pi}, U^{\lambda}) \cong \oplus_{y\in K\backslash G/H}\, \mathrm{Hom}_K(\pi\cdot y, \lambda\,|\,K). \tag{6}$$

But it is obvious that $\mathrm{Hom}_G(U^{\pi}, U^{\varrho}) \cong \mathrm{Hom}_G(U^{\pi}, U^{\lambda})$, and by an earlier lemma we know that $\lambda\,|\,K \cong {}_KU^{\varrho|\Gamma\cap K}$, so that for all y we have

$$\mathrm{Hom}_K(\pi\cdot y, \lambda\,|\,K) \cong \mathrm{Hom}_K(\pi\cdot y, {}_KU^{\varrho|\Gamma\cap K}).$$

If π is finite dimensonal then formulas (5) and (6) are equivalent by C. Moore's formula (Lemma 2.2), because

$$\mathrm{Hom}_K(\pi\cdot y, {}_KU^{\varrho|\Gamma\cap K}) \cong \mathrm{Hom}_{\Gamma\cap K}(\pi\cdot y\,|\,\Gamma\cap K, \varrho\,|\,\Gamma\cap K),$$

so either formula could be taken as the generalization of Mackey's formula in this case. If $\dim\pi = +\infty$ formula (5) breaks down, but formula (6) remains valid, so it should be regarded as the correct generalized intertwining formula. In our later applications we will use both formulas, though we lean most heavily on (6).

Let $H = K\Gamma$; compactness of $K\cap\Gamma\diagdown K$ insures that $K\Gamma$ is closed. Obviously $H\diagdown G$ is compact, and in fact has an invariant measure. [Lift the G-invariant measure on $\Gamma\diagdown G$ over to $H\diagdown G$ under the continuous, onto, G-equivariant map $\Gamma x \to Hx$.] Furthermore, $K\diagdown KH = K\diagdown K\Gamma$ is discrete. Thus if $\lambda = {}_HU^{\varrho}$ we may carry out the preceeding construction using the system $(G, K, H; \pi, \lambda)$ in place of $(G, K, \Gamma; \pi, \varrho)$; this transition amounts to assuming that $\Gamma \supseteq K$ in that discussion. Now for each $y \in G$, J maps $\mathrm{Hom}_K(\pi\cdot y, \lambda|K)$ into $\mathrm{Hom}(U^{\pi\cdot y}, U^\lambda)$ and $\mathbf{J}$ is a linear map,

$$\mathbf{J}\colon \oplus_{y\in G/H} \mathrm{Hom}_K(\pi\cdot y, \lambda|K) \to \mathrm{Hom}_G({}_GU^\pi, {}_GU^\lambda). \tag{7}$$

This map is injective. In fact, if y and z are in different $G/H = K\diagdown G/H$ cosets, then $\pi\cdot(y\gamma^{-1})$ and $\pi\cdot z$ are inequivalent in $K^\wedge$ for all $\gamma\in\Gamma$. Hence, if $A_1 \in \mathrm{Hom}_K(\pi\cdot y, \lambda|K)$ and $A_2 \in \mathrm{Hom}_K(\pi\cdot z, \lambda|K)$, then $\varrho(\gamma)A_1$ and A_2 have ranges in $\mathcal{H}(\lambda)$ which correspond to distinct irreducibles and are therefore orthongonal, which makes (7) an injection.

THEOREM 3.4. *Let $(G, K, \Gamma, \pi, \varrho)$ be as above and suppose that ϱ is finite dimensional. Then*

$$\begin{aligned}\mathrm{Hom}(U^\pi, U^\varrho) &\cong \mathrm{Hom}(U^\pi, U^\lambda)\\ &\cong \oplus_{y\in K\diagdown G/H} \mathrm{Hom}_K(\pi\cdot y, \lambda|K)\\ &\cong \oplus_{y\in K\diagdown G/H} \mathrm{Hom}_K(\pi\cdot y, {}_KU^{\varrho|\Gamma\cap K}),\end{aligned}$$

where $\lambda = {}_{K\Gamma}U^\varrho$.

Proof. By 2.1, $\lambda|K \cong {}_KU^{\varrho|\Gamma\cap K}$. By induction in stages, ${}_GU^\lambda \cong {}_GU^\varrho$, so it suffices to show that the map $\mathbf{J}$ in (7) is an isomorphism. We already know it is injective, so our whole problem is to show it surjective: that our construction produces all possible intertwining operators. Here we need $\lambda|K$ type I.

Let $\sigma = {}_GU^\pi$ and $\tau = {}_GU^\lambda$. We need explicit descriptions of the direct integral decompositions of $\sigma|K$ and $\tau|K$, whose existence is guaranteed by the Mackey subgroup theorem [9],

$$\sigma|K \cong \int_{K\diagdown G}^{\oplus} \pi\cdot z\, dm(\dot z), \qquad \tau|K \cong \int_{H\diagdown G}^{\oplus} (\lambda|K)\cdot x\, d\mu(\dot x). \tag{8}$$

We want to compare these decompositions.

Let $C\subseteq G$ be a fixed measurable transveral for $H\diagdown G$. If μ is the invariant measure on $H\diagdown G$ we may regard μ as a measure back on C, and identify the spaces $(H\diagdown G, \mu)$ and (C, μ). Likewise, we identify $(K\diagdown H, \nu)$ and (D, ν) indiscriminately, where D is a (discrete, countable) transversal for $K\diagdown H$ and ν the counting measure.

We begin with the decomposition of $\tau|K = U^\lambda|K$. Using the transversal C we may identify $F\in\mathcal{H}(U^\lambda)$ with a field of vectors $F^\sim = \{F_x^\sim : x\in C\}$ defined by taking $F_x^\sim(h) = F(hx)$, all $h\in H$. Then $F_x^\sim \in \mathcal{K}_x$, where we set $\mathcal{K}_x = \mathcal{H}(\lambda)$ for all $x\in C$, and we may identify

$$\mathcal{H}(U^\lambda) \cong L^2(H\backslash G, \mu; \mathcal{H}(\lambda)) = L^2(C, \mu; \mathcal{H}(\lambda)) = \int_C^{\oplus} \mathcal{K}_x\, d\mu(x). \tag{9}$$

In this concrete setting, $U^\lambda | K$ decomposes into the concrete representations $(\lambda | K) \cdot x$, all modeled on $\mathcal{H}(\lambda)$,

$$\tau | K = \int_C^\oplus (\lambda | K) \cdot x \, d\mu(x). \tag{10}$$

The verifications are obvious, following Mackey's proof of the subgroup theorem.

We want to compare $\tau | K$ with $U^{\pi \cdot y} | K$ for various $y \in G$. The decomposition (8), corresponding to the scheme $K \uparrow G \downarrow K$, is not convenient; it will be easier to compare decompositions over the same base space. To this end we set $\pi'(y) = {}_H U^{\pi \cdot y} = \mathrm{Ind}\,(K \uparrow H, \pi \cdot y)$. Then we decompose ${}_G U^{\pi \cdot y} | K$ over $H \diagdown G = C$ as follows. First decompose

$$\mathcal{H}(U^{\pi \cdot y}) \cong \int_C^\oplus \mathcal{K}'_x \, d\mu(x) \quad (\text{where } \mathcal{K}'_x = \mathcal{H}({}_H U^{\pi \cdot y}), \text{ all } x \in C) \tag{11}$$

by identifying a vector $F \in \mathcal{H}(U^{\pi \cdot y})$ with the field of vectors $F^\sim = \{F_x^\sim : x \in C\}$, where $F_x^\sim(h) = F(hx) \in \mathcal{H}({}_H U^{\pi \cdot y})$. It is easily verified that changing the representative of F affects the equivalence class of $F_x^\sim$ only on a null set in C, that

$$\int_{K \diagdown G} \|F(g)\|^2 \, dm(g) = \int_C \|F_x^\sim\|^2 \, d\mu(x)$$

if Haar measure m on $K \diagdown G$ is suitably normalized, and that the vectors $\{F^\sim : F \in \mathcal{H}(U^{\pi \cdot y})\}$ span a dense set in $\int_C^\oplus \mathcal{K}'_x \, d\mu(x)$, so that (11) holds. The corresponding decomposition of $U^{\pi \cdot y} | K$ is

$$U^{\pi \cdot y} | K \cong \int_C^\oplus ({}_H U^{\pi \cdot y}) \cdot x \, d\mu(x) = \int_C^\oplus \pi'(y) \cdot x \, d\mu(x). \tag{12}$$

Applying the construction of Theorem 3.3 to the system $(H, K, H; \pi \cdot y, \lambda)$, we define a map J': $\mathrm{Hom}_K(\pi \cdot y, \lambda | K) \to \mathrm{Hom}_K(\pi'(y), \lambda)$. If $F \in \mathcal{H}(\pi'(y))$ is continuous on H with supp $\|F\|$ compact modulo K, and if $A \in \mathrm{Hom}_K(\pi \cdot y, \lambda | K)$, then (1) gives

$$(J'A)\, F(h) = \textstyle\sum_{\gamma \in D = K \diagdown H} \lambda(\gamma)^{-1} A F(\gamma h) \quad \text{all} \quad h \in H.$$

We next consider the constant operator field $B_x = J'A$: $\mathcal{H}(\pi'(y)) \to \mathcal{H}(\lambda)$, all $x \in C$, which induces a bounded operator $B = \int_C^\oplus B_x \, d\mu(x)$ from

$$\mathcal{H}(U^{\pi \cdot y}) = \int_C^\oplus \mathcal{K}'_x \, d\mu(x) \quad \text{into} \quad \mathcal{H}(U^\lambda) = \int_C^\oplus \mathcal{K}_x \, d\mu(x).$$

LEMMA 3.5. *Let $y \in G$ and let $A \in \mathrm{Hom}_K(\pi \cdot y, \lambda | K)$. If we decompose $\mathcal{H}(U^{\pi \cdot y})$ and $\mathcal{H}(U^\lambda)$ as in* (9) *and* (11), *then the operator $JA \in \mathrm{Hom}(U^{\pi \cdot y}, U^\lambda)$ is decomposable*

$$JA = \int_C^\oplus (JA)_x \, d\mu(x) \tag{13}$$

where $(JA)_x = J'A$: $\mathcal{K}'_x \to \mathcal{K}_x$ for all $x \in C$ (the constant operator field above).

Proof. It suffices to show that $JA = B$ on a dense set of vectors $F \in \mathcal{H}(U^{\pi\cdot y})$. Consider F which are continuous, such that supp $\|F\|$ is compact modulo K. Then for any H-coset Hg_0, $Hg_0 \cap \text{supp } \|F\| = \text{supp } \|F|Hg_0\|$ is compact modulo K. [Indeed, writing supp $\|F\| = XK$ for some compact set $X \subseteq G$, we have $Hg_0 \cap XK = (Hg_0 \cap X)\cdot K$ since K is normal.] Under our map (11), F corresponds to a field $F^{\sim} = \{F_x^{\sim} : x \in C\}$ with $F_x^{\sim}(h) = F(hx) \in \mathcal{H}({}_HU^{\pi\cdot y})$ a continuous function on H such that supp $\|F_x^{\sim}\|$ is compact modulo K. Thus the pointwise formula (1) applies both to $(JA)F$ and to $(J'A)F_x^{\sim}$. For $x \in C$ we get

$$(JA)_x F_x^{\sim}(h) = ((J'A)F_x^{\sim})(h) = \textstyle\sum_{\gamma \in D = K\backslash H} \lambda(\gamma)^{-1} A F_x^{\sim}(\gamma h);$$

BF corresponds to this field of vectors in $\int_C^{\oplus} \mathcal{K}_x d\mu(x)$. On the other hand, if $Q = (JA)F$ and if $g = hx$ for $h \in H$, $x \in C$, then

$$Q(hx) = ((JA)F)(hx) = \textstyle\sum_{\gamma \in D = K\backslash H} \lambda(\gamma)^{-1} AF(\gamma hx).$$

But Q corresponds to $Q^{\sim} = \{Q_x^{\sim}\}$ in $\int_C^{\oplus} \mathcal{K}_x d\mu(x)$, where $Q_x^{\sim}(h) = Q(hx)$. For μ-a.e. $x \in C$ we get

$$Q_x^{\sim}(h) = \textstyle\sum_{\gamma \in K\backslash H} \lambda(\gamma)^{-1} AF(\gamma hx).$$

Clearly then, $(JA)F = BF$ as required. Q.E.D.

Let $\mathcal{H}_1$ be the subspace of $\mathcal{H}(U^\lambda)$ spanned by the ranges of all the "standard" intertwining operators $\mathbf{J}A \in \text{Hom}(U^\pi, U^\lambda)$ obtained from $A \in \text{Hom}_K(\pi\cdot y, \lambda|K)$ under the map (7). For each $y \in G$, $\mathbf{J}A = \Phi^y(JA) = JA \circ I^y$ (recall Figure 2). Clearly $\mathbf{J}A$ and JA have the same range in $\mathcal{H}(U^\lambda)$, so we may determine $\mathcal{H}_1$ by examining the operators $JA \in \text{Hom}(U^{\pi\cdot y}, U^\lambda)$ for each $y \in G$ separately. From our decomposition (13) it follows that $\mathcal{H}_1$ has the form

$$\mathcal{H}_1 = \int_C^{\oplus} (\mathcal{K}_x)_1 \, d\mu(x)$$

where $(\mathcal{K}_x)_1 = \mathcal{K}_1 =$ closed span of $\{J'A(\mathcal{H}({}_HU^{\pi\cdot y})) : y \in G,\ A \in \text{Hom}_K(\pi\cdot y, \lambda|K)\} \subseteq \mathcal{H}(\lambda)$. All fibers are the same. Let $\mathcal{K}_2 = \mathcal{K}_1^{\perp}$ in $\mathcal{H}(\lambda)$, and set $(\mathcal{K}_x)_2 = \mathcal{K}_2$ for all $x \in C$; then $\mathcal{H}_2 = \mathcal{H}(U^\lambda) \ominus \mathcal{H}_1$ decomposes

$$\mathcal{H}_2 = \int_C^{\oplus} (\mathcal{K}_x)_2 \, d\mu(x), \quad (\mathcal{K}_x)_2 = \mathcal{K}_2 \quad \text{for all } x. \tag{14}$$

The subspaces $\mathcal{H}_1$, $\mathcal{H}_2$ are $U^\lambda(G)$-invariant. The following important observation follows from the definition of $\mathcal{K}_1$.

There cannot be a $y \in G$ and $A \neq 0$ in $\text{Hom}_K(\pi\cdot y, \lambda|K)$ such that range $(J'A) \subseteq \mathcal{K}_2$. (15)

Otherwise JA would intertwine $\mathcal{H}(U^{\pi\cdot y})$ and $\mathcal{H}_2$.

Now we are ready to prove $\mathbf{J}$ surjective. First, suppose $T \in \operatorname{Hom}(U^\pi, U^\lambda)$ has range $(T) \subseteq \mathcal{H}_1$. We show directly that T is in the range of $\mathbf{J}$ in (7). The idea of the proof is simple, though somewhat obscured by the notation needed to write it all out.

LEMMA 3.6. *Let $(G, K, \Gamma; \pi, \varrho)$ be as in Theorem 3.4; let $H = K\Gamma$ and $\lambda = {}_H U^\varrho$. Then the range of $\mathbf{J}$, the space of linear operators*

$$\mathfrak{X} = \mathbf{J}(\oplus_{y \in K\backslash G/H} \operatorname{Hom}_K(\pi \cdot y, \lambda | K)) \subseteq \operatorname{Hom}(U^\pi, U^\lambda),$$

is finite dimensional. If $\mathcal{H}_1$ is defined as above and if $T \in \operatorname{Hom}(U^\pi, U^\lambda)$ has range in $\mathcal{H}_1$, then $T \in \mathfrak{X}$.

Proof. Notice that U^π has finite (or zero) multiplicity in $U^\lambda \cong U^\varrho$; since $\mathbf{J}$ is injective, $\dim \mathfrak{X} < +\infty$. Now let $\mathfrak{X}^y = \operatorname{span}\{\operatorname{range}(\mathbf{J}A): A \in \operatorname{Hom}_K(\pi \cdot y, \lambda | K)\}$. These $U^\lambda(G)$-invariant subspaces are identical for y in the same G/H coset (see Figure 2); they are orthogonal otherwise, as we have remarked in connection with (7). Hence $\mathcal{H}_1 = \oplus\{\mathfrak{X}^y: y \in G/H\}$. Moreover, $\lambda | K \cong_K U^{\varrho | \Gamma \cap K}$ contains at most a finite number of direct sum copies of $\pi \cdot y$, so that $\operatorname{Hom}_K(\pi \cdot y, \lambda | K) = \mathbf{C}E'_1 \oplus \ldots \oplus \mathbf{C}E'_m$ where the E'_j are isometries onto pairwise orthogonal $\lambda(K)$-invariant subspaces in $\mathcal{H}(\lambda)$ on which $\lambda | K$ acts like $\pi \cdot y$. Let $E_j = \mathbf{J}(E'_j)$; write range $(E_j) = \mathfrak{X}^y_j \subseteq \mathfrak{X}^y$. We assert that the E_j are isometries into $\mathfrak{X}^y$ with pairwise orthogonal ranges such that $\mathfrak{X}^y = \mathfrak{X}^y_1 \oplus \ldots \oplus \mathfrak{X}^y_m$. In fact, if $A = c_1 E'_1 + \ldots + c_m E'_m \in \operatorname{Hom}_K(\pi \cdot y, \lambda | K)$, then by (2) (replacing $(G, K, \Gamma; \pi, \varrho)$ with $(G, K, H; \pi, \lambda)$ in that calculation) we get

$$\begin{aligned} \|\mathbf{J}A(F)\|^2 &= \int_{K\backslash G} \|AF(x)\|^2 \, dm(x) = \int_{K\backslash G} \|\textstyle\sum_j c_j E'_j(F(x))\|^2 \, dm(x) \\ &= (\textstyle\sum_j |c_j|^2) \cdot \displaystyle\int_{K\backslash G} \|F(x)\|^2 \, dm(x) = \|A\|^2 \cdot \|F\|^2 \end{aligned} \tag{16}$$

for all $F \in \mathcal{H}(U^\pi)$. Similarly,

$$(\mathbf{J}A_1(F), \mathbf{J}A_2(F)) = \int_{K\backslash G} (A_1 F(x), A_2 F(x)) \, dm(x),$$

so that $E_j = \mathbf{J}(E'_j)$ are isometries with orthogonal ranges. The inclusion $(\mathfrak{X}^y \subseteq \ldots)$ follows because $\operatorname{Hom}_K(\pi \cdot y, \lambda | K) = \oplus \mathbf{C}E_j$.

Thus, $\mathcal{H}_1 = \oplus_{\alpha \in I} \mathfrak{X}_\alpha$ where $\{\mathfrak{X}_\alpha\} = \{\mathfrak{X}^y_j: y \in G/H, 0 \leqslant j \leqslant m(y)\}$. Let $P_\alpha: \mathcal{H}_1 \to \mathfrak{X}_\alpha$ be the projection. Suppose that $T \in \operatorname{Hom}(U^\pi, U^\lambda)$ has range in $\mathcal{H}_1$. Then $P_\alpha T$ also intertwines, and if $P_\alpha T \neq 0$, then it and the "standard" isometry $E_\alpha = \mathbf{J}(E'_\alpha)$ both map $\mathcal{H}(U^\pi)$ to $\mathfrak{X}_\alpha$. By irreducibility, $P_\alpha T = c(\alpha) E_\alpha$ for $c(\alpha) \in \mathbf{C}$, so that $T = \sum_\alpha P_\alpha T = \sum c(\alpha) E_\alpha$ is in $\mathfrak{X}$. Q.E.D.

Let P_1, P_2 be the projections onto $\mathcal{H}_1$, $\mathcal{H}_2$ in $\mathcal{H}(U^\lambda) = \mathcal{H}_1 \oplus \mathcal{H}_2$. If $T \in \operatorname{Hom}(U^\pi, U^\lambda)$ has range $(T) \nsubseteq \mathcal{H}_1$, then $P_2 T$ is a nonzero intertwining operator with range in $\mathcal{H}_2$. Since

U^π is irreducible, there must be an intertwining isometry T with range in $\mathcal{H}_2$. We complete the proof that $\mathbf{J}$ is surjective by showing that (15) is violated in this situation.

Now T intertwines $\sigma = U^\pi$ with τ_2, the subrepresentation of $\tau = U^\lambda$ on the subspace $\mathcal{H}_2$, so we may identify σ as a subrepresentation of τ_2. Moreover, $\tau | K$ is type I on K, by 2.4, hence so are $\tau_2 | K$ and $\sigma | K$. Let $\lambda_2 = \lambda | K$ restricted to the K-invariant subspace $\mathcal{K}_2$ defined in (14). The decomposition $\lambda | K = \oplus_{i \in I} n_i \pi_i$ induces a decomposition $\lambda_2 = \oplus m_i \pi_i$, $0 \leqslant m_i \leqslant n_i$, so that $\operatorname{sp}(\lambda_2) = \{\pi_i : m_i \neq 0\} \subseteq \operatorname{sp}(\lambda | K) = \{\pi_i\}$; both spectra are discrete sets in $K^\wedge$ consisting of CCR representations. Thus $S_2 = \bigcup \{\operatorname{sp}(\lambda_2) \cdot g : g \in G\}$ is a closed set of CCR representations in $K^\wedge$, by exactly the same reasoning as in 2.3 and 2.4.

Consider the direct integral decomposition corresponding to the scheme $K \uparrow G \downarrow K$,

$$\sigma | K = \int_{K \backslash G}^{\oplus} \pi \cdot z \, d\dot{z}.$$

As noted at the end of section 2, the $\pi \cdot z$ are inequivalent in $K^\wedge$ since $\operatorname{Stab}_G(\pi) = K$. Let $R = \pi \cdot G$, the orbit in $K^\wedge$. Now R and S_2 are G-invariant in $K^\wedge$. If $R \cap S_2 \neq \emptyset$ we obtain a contradiction because there would be a $\pi_i \in \operatorname{sp}(\lambda_2)$ and $x, z \in G$ such that $\pi \cdot z \cong \pi_i \cdot z$, or $\pi \cdot z x^{-1} \cong \pi_i \leqslant \lambda_2 \leqslant \lambda | K$; i.e. there would be a nontrivial operator $A \in \operatorname{Hom}(\pi \cdot z x^{-1}, \lambda | K)$ with range in $\mathcal{K}_2$, violating (15).

On the other hand, we have already noted in (9), (10), and (12) that

$$\tau_2 | K \cong \int_C^{\oplus} \lambda_2 \cdot x \, d\mu(x).$$

Let $I_2 = \operatorname{hull}(S_2)$ in the group C^* algebra $A(K)$; thus $S_2 = \ker(I_2)$ since S_2 is closed in $K^\wedge$. Let $\{a_n : n = 1, 2, \ldots\}$ be a norm dense set in I_2; clearly $\pi'(a_n) = 0$ for all $\pi' \in S_2$. Since $\operatorname{sp}(\lambda_2) \cdot x \mathrel{\hat{\subseteq}} S_2$, $(\lambda_2 \cdot x) a_n = 0$ for all n and $x \in C$. Thus $(\tau_2 | K) a_n = 0$ all n, which implies that $(\sigma | K) a_n = 0$ for all n. But then

$$(\sigma | K) \, a_n = \int_X^{\oplus} (\pi \cdot z) \, a_n \, dm(z) = 0$$

where X is a measurable transversal for $K \diagdown G$ and $m =$ Haar measure on $K \diagdown G$ identified with a measure on X. For each n there is a null set $N_n \subseteq X$ such that $(\pi \cdot z) a_n = 0$ all $z \in X \sim N_n$ which means that $(\pi \cdot z) a_n = 0$ all n and all $z \in X \sim N$ where $N = \bigcup_{n=1}^{\infty} N_n$. By norm continuity of representations, we get $(\pi \cdot z) a = 0$ all $a \in A(K)$, $z \in X \sim N$. This means precisely that $\{\pi \cdot z : z \in X \sim N\}$ is weakly contained in S_2 (in $K^\wedge$). Since S_2 is hullkernel closed, we must have $\{\pi \cdot z : z \in X \sim N\} \subseteq S_2$. Hence $S_2 \cap R$ is nonempty. This completes the proof of Theorem 3.4. Q.E.D.

By applying Lemma 2.2 we get the analog of Mackey's intertwining formula for finite groups, if π is finite dimensional.

COROLLARY 3.7. *If in the situation of Theorem* 3.4 *the representation* π *is finite dimensional, then*

$$\mathrm{Hom}(U^\pi, U^\varrho) \cong \oplus_{y\in K\backslash G/\Gamma} \mathrm{Hom}_{\Gamma\cap K}(\pi\cdot y \,|\, \Gamma\cap K, \varrho\,|\,\Gamma\cap K). \tag{17}$$

§ 4. Algebraic preliminaries on nilpotent Lie groups

In discussing a simply connected nilpotent Lie group N we use the rational structure corresponding to a given uniform discrete subgroup Γ. The subgroups H such that $H\cap\Gamma\diagdown H$ is compact turn out to be the rational subgroups of N. We use the following basic facts, see [2; Appendix] for further details. Define $\mathfrak{n}_{\mathbf{Q}} = \mathbf{Q}$-span $\{\log(\Gamma)\}$. This is a Lie algebra over $\mathbf{Q}$ by the Campbell-Hausdorff formula and $N_{\mathbf{Q}} = \exp(\mathfrak{n}_{\mathbf{Q}})$ is a dense subgroup in N. A closed connected subgroup $H\subseteq N$ is *rational* if $H = (H\cap N_{\mathbf{Q}})^-$; equivalently, $H\cap\Gamma\diagdown H$ is compact. A real linear functional $f\in\mathfrak{n}^*$ is *rational* ($f\in\mathfrak{n}^*_{\mathbf{Q}}$) if $f(\mathfrak{n}_{\mathbf{Q}})\subseteq\mathbf{Q}$, or equivalently $f(\log\Gamma)\subseteq\mathbf{Q}$. If H is a closed connected subgroup and $\Gamma' = H\cap\Gamma$ is uniform in H, then the rational points $H_{\mathbf{Q}}$ it determines coincide with $H\cap N_{\mathbf{Q}}$.

In the following, a number of our previous notational conventions are inoperative. We shall use general facts about Kirillov theory; see [7] or [14] for an account. Kirillov theory associates to each irreducible $\sigma\in N^\wedge$ the $\mathrm{Ad}'(N)$-orbit O of some $f\in\mathfrak{n}^*$; $\sigma=\sigma^f$ is obtained by inducing $f^\sim(\exp X) = e^{2\pi i\langle f, X\rangle}$ from any maximal subordinate subgroup M_f up to N. If O contains a rational element $f\in\mathfrak{n}^*_{\mathbf{Q}}$, then there are rational maximal subordinate subalgebras $\mathfrak{m}_f$, and furthermore the normalizer $\{X\in\mathfrak{n}: [X,\mathfrak{l}]\subseteq\mathfrak{l}\}$ of any rational subalgebra $\mathfrak{l}\subseteq\mathfrak{n}$ is rational and properly contains $\mathfrak{l}$ unless $\mathfrak{l}=\mathfrak{n}$; see [2; Appendix]. Thus we get rational subalgebras $\mathfrak{m}_f = \mathfrak{n}_1\subseteq\mathfrak{n}_2\subseteq\ldots\subseteq\mathfrak{n}_k=\mathfrak{n}$ such that $\mathfrak{n}_j$ is a proper ideal in $\mathfrak{n}_{j+1}$, $\mathfrak{n}_j\lhd\mathfrak{n}_{j+1}$.

If $\phi\in\mathfrak{n}^*$ is a Lie algebra homomorphism of $\mathfrak{n}$, then $\phi\cdot n = \mathrm{Ad}'(n)\phi$ is equal to ϕ for all $n\in N$; thus the orbits $(f+\phi)\cdot N$ and $f\cdot N$ are equal for any $f\in\mathfrak{n}^*$.

LEMMA 4.1. *Suppose* $f, f'\in\mathfrak{n}^*$ *are both rational and lie in the same* $\mathrm{Ad}'(N)$-*orbit* O. *Then there is an* $x\in N_{\mathbf{Q}}$ *such that* $f' = f\cdot x$.

Proof. Let $\mathfrak{m}_f$ be a rational maximal subordinate subalgebra for f. From the proof of the theorem in [14], Part II, Ch. 1, Sec. 3 it follows that one can choose rational elements $\{X_1, \ldots, X_d\}\subseteq\mathfrak{n}_{\mathbf{Q}}$ ($d = n - \dim\mathfrak{m}_f$) spanning a real subspace transverse to $\mathfrak{m}_f$, and a basis of rational elements $\{e_1, \ldots, e_n\}\subseteq\mathfrak{n}^*$ with the following properties. Let $\mathbf{t} = (t_1, \ldots, t_d)\in\mathbf{R}^d$ and write

$$g(\mathbf{t}) = \mathrm{Ad}'(\exp(t_1X_1)\cdot\ldots\cdot\exp(t_dX_d))f = \textstyle\sum_{j=1}^n Q_j(\mathbf{t})\,e\,.$$

Then

(i) $\mathbf{t} \to g(\mathbf{t})$ is a homeomorphism of $\mathbf{R}^d$ onto O.

(ii) The Q_j are polynomials with rational coefficients.

(iii) There exist indices $1 \leq j_1 < \ldots < j_d \leq n$ such that $Q_{j_k}(\mathbf{t}) = t_k +$ (polynomial in $t_1, \ldots, t_{k-1}$), and for $j < j_k$, $Q_j(\mathbf{t})$ is a polynomial in $t_1, \ldots, t_{k-1}$.

Now pick $\mathbf{t}$ such that $g(\mathbf{t}) = f'$. If t_k is the first irrational component of $\mathbf{t}$, then $Q_{j_k}(\mathbf{t}) e_{j_k}$ is not a rational function, contradicting rationality of f. Hence $\mathbf{t} \in \mathbf{Q}^d$ and the Campbell-Hausdorff formula shows that $\exp(t_1 X_1) \cdot \ldots \cdot \exp(t_d X_d) = \exp(X) = x$ for some $X \in \mathfrak{n}_{\mathbf{Q}}$. Then $f \cdot x = f'$. Q.E.D.

COROLLARY 4.2. *Suppose $f \in \mathfrak{n}^*$ is rational and $\mathfrak{n}_0$ is a rational ideal in $\mathfrak{n}$ which contains a maximal subordinate subalgebra $\mathfrak{m}$ for f. If $y \in N$ is such that $f \cdot y$ is rational when restricted to $\mathfrak{n}_0$, there exists an $x \in N_{\mathbf{Q}}$ such that $f \cdot y = f \cdot x$ on $\mathfrak{n}_0$.*

Proof. Let $f_1, f_2 \in \mathfrak{n}^*$ and assume that $\mathfrak{n}_0$ contains a maximal subordinate subalgebra $\mathfrak{m}$ for f_1. Then

$$\text{If } f_1 \text{ and } f_2 \text{ agree on } \mathfrak{n}_0, \text{ they are in the same } \mathrm{Ad}'(N)\text{-orbit in } \mathfrak{n}^*. \tag{18}$$

In fact, $\mathfrak{m}$ is clearly subordinate to f_2. The characters $f_j^\sim(n) = \exp[2\pi i\langle f_j, \log n\rangle]$ agree on M, so $\mathrm{Ind}(M \uparrow N, f_1^\sim) \cong \mathrm{Ind}(M \uparrow N, f_2^\sim)$ is irreducible. By [7; Theorem 5.2], $\mathfrak{m}$ must be maximal subordinate to f_2. Moreover, since f_1, f_2 give the same irreducible representation of N, they are in the same $\mathrm{Ad}'(N)$-orbit. Now let $f' = f \cdot y \mid \mathfrak{n}_0$. Clearly $\mathfrak{m}_1 = \mathrm{Ad}(y)\mathfrak{m}$ lies in $\mathfrak{n}_0$ since $\mathfrak{n}_0 \lhd \mathfrak{n}$, and is maximal subordinate to $f \cdot y$ on $\mathfrak{n}$. Let l be any rational extension of f' to $\mathfrak{n}$ (exists since $\mathfrak{n}_0$ is rational). Since $f \cdot y$ and l agree on $\mathfrak{n}_0$ they lie in a single $\mathrm{Ad}'(N)$-orbit and so do f, l. By 4.1, there is an $x \in N_{\mathbf{Q}}$ such that $f \cdot x = l$, so $f \cdot x = f \cdot y$ on $\mathfrak{n}_0$. Q.E.D.

We will deal with the decomposition of ${}_N U^\varrho$ where ϱ is an arbitrary one dimensional representation of Γ. We need a few facts which are trivial, or reduce to the preceeding lemmas, when $\varrho \equiv 1$. Given ϱ we cannot in general find an element $f \in \mathfrak{n}^*$ such that $\varrho(\exp X) = \exp[2\pi i\langle f, X\rangle]$ for $X \in \log \Gamma$ and f is a Lie algebra homomorphism. We say that $f \in \mathfrak{n}^*$ is *ϱ-rational* if $f \in \phi + \mathfrak{n}_{\mathbf{Q}}^*$, where $\phi \in \mathfrak{n}^*$ is a homomorphism such that $e^{2\pi i\phi} \mid \Gamma = \varrho$ on a subgroup of finite index in Γ ($\Leftrightarrow e^{2\pi i\phi}/\varrho$ is a root of unity, all $\gamma \in \Gamma$, since Γ is finitely generated [11]. We thank Prof. Wolf Beigelbock for the present definition in place of our original, more cumbersome version. Although ϕ is not uniquely determined, if it exists, any other ψ differs by an element in $\mathfrak{n}_{\mathbf{Q}}^*$ because

$$\exp[2\pi i\langle \phi - \psi, X\rangle] = \frac{\exp[2\pi i\langle \phi, X\rangle]}{\exp[2\pi i\langle \psi, X\rangle]} \cdot \frac{\varrho(\exp X)}{\varrho(\exp X)} \quad \text{is a root of unity,}$$

all $X \in \log \Gamma$. Thus, all ϱ-rational elements may be associated with the same homomorphism ϕ; they comprise the set $\phi + \mathfrak{n}_{\mathbf{Q}}^* \subseteq \mathfrak{n}^*$. Obviously if $\varrho \equiv 1$ the ϱ-rational elements reduce to $\mathfrak{n}_{\mathbf{Q}}^*$.

LEMMA 4.3. *Let Γ be a discrete uniform subgroup in N, ϱ a one-dimensional representation of Γ, and let $f \in \mathfrak{n}^*$ be ϱ-rational. Let $\mathfrak{n}_0$ be a rational ideal containing a maximal subordinate subalgebra $\mathfrak{m}$ for f and set $\Gamma_0 = \Gamma \cap N_0$, $\varrho_0 = \varrho \,|\, \Gamma_0$. If there is a $y \in N$ such that $f \cdot y$ is ϱ_0-rational on $\mathfrak{n}_0$, then there is an $x \in N_{\mathbf{Q}}$ such that $f \cdot y = f \cdot x$ on $\mathfrak{n}_0$.*

Proof. Let f' stand for $f \,|\, \mathfrak{n}_0$, all $f \in \mathfrak{n}^*$. Let ϕ be a homomorphism of $\mathfrak{n}$ such that the ϱ-rational elements are $\phi + \mathfrak{n}_{\mathbf{Q}}^*$; then the ϱ_0-rational elements on $\mathfrak{n}_0$ are $\phi' + (\mathfrak{n}_0)_{\mathbf{Q}}^*$. Now ϱ-rationality of $f \Rightarrow f - \phi$ rational; ϱ_0-rationality of $f' \cdot y = (f \cdot y)' \Rightarrow f' \cdot y - \phi' = (f - \phi)' \cdot y$ is rational on $\mathfrak{n}_0$. Clearly $\mathfrak{m}$ is maximal subordinate to $f - \phi$ as well as to f. By Lemma 4.2, there is an $x \in N_{\mathbf{Q}}$ such that $f \cdot x - \phi = (f - \phi) \cdot x$ equals $f \cdot y - \phi = (f - \phi) \cdot y$ on $\mathfrak{n}_0$, so that $f \cdot x = f \cdot y$ on $\mathfrak{n}_0$. Q.E.D.

Notice that if f is ϱ-rational in $\mathfrak{n}^*$, there exist rational maximal subordinate subalgebras for f since f and $f - \phi$ (rational!) have the same subordinate subalgebras.

Finally, we show that ϱ-rational functionals always exist. That is, we need to show that there is a Lie homomorphism $\phi \in \mathfrak{n}^*$ such that $e^{2\pi i \phi} = \varrho$ on a subgroup of finite index in Γ, since Γ is finitely generated [11]. Then $\phi + \mathfrak{n}_{\mathbf{Q}}^*$ gives all the ϱ-rational elements. To construct ϕ, consider the commutator subgroup $N' = [N, N]$. As in [11; pp. 7–9], $\Gamma \cap N' = \Gamma'$ is uniform in N', so the quotient homomorphism $p: N \to N/N'$ carries Γ to a discrete uniform subgroup in the vector group N/N'; $p(\Gamma)$ is an additive lattice. Choose $\{x_1, ..., x_m\} \subseteq \Gamma$ so $p(x_j)$ form a $\mathbf{Z}$-basis for $p(\Gamma)$, take $X_j = \log(x_j)$, and take $s_j \in \mathbf{R}$ so that $\varrho(x_j) = e^{2\pi i s_j}$. Clearly, $\mathbf{R}$-span $\{X_1, ..., X_m\}$ is a subspace in $\mathfrak{n}$ transverse to $[\mathfrak{n}, \mathfrak{n}]$. Define $f \in \mathfrak{n}^*$ so $f \equiv 0$ on $[\mathfrak{n}, \mathfrak{n}]$ and $\langle f, X_j \rangle = s_j$. Obviously f is a Lie algebra homomorphism and $f^\sim = e^{2\pi i f}$ is a homomorphism of N. Let $\Gamma_2 = [\Gamma, \Gamma]$; then $\varrho \equiv 1$ on Γ_2 and $[N, N]/\Gamma_2$ is compact, so that Γ_2 has finite index in $\Gamma \cap [N, N]$. Let Γ_1 be the subgroup of N generated by $\Gamma_2 \cup \{x_1, ..., x_m\}$. Now $\varrho(x_j) = f^\sim(x_j)$ and $\varrho \equiv f^\sim \equiv 1$ on Γ_2, so $\varrho \equiv f^\sim$ on Γ_1. But Γ_1 has finite index in Γ, so we may take $\phi = f$. [For some finite set F, $\Gamma \cap [N, N] = F \cdot \Gamma_2$. But $p(\Gamma_1) = p(\Gamma)$, so $x \in \Gamma \Rightarrow$ there are $y \in \Gamma_1$, $n \in \Gamma \cap [N, N]$ such that $x = ny$. In turn, $n \in F \cdot \Gamma_2$, so that $x \in F \cdot \Gamma_2 \cdot \Gamma_1 = F \cdot \Gamma_1$.]

§ 5. The multiplicity formula for nilmanifolds

Now that we have cleared away the algebraic background, we are ready to apply Theorem 3.4 to get the multiplicity formula for nilmanifolds $\Gamma \backslash N$. Our proof is rather different from those of Howe and Richardson. With 3.4 in hand there is no need to carry out inductions on dim N, or to separate out the case of one-dimensional center. Modulo certain rationality questions based on the work of section 4, the multiplicity formula emerges via straightforward calculations.

THEOREM 5.1. *Let ϱ be a one-dimensional representation of discrete uniform subgroup Γ in a simply connected nilpotent Lie group N, and let $\tau = {}_N U^{\varrho}$. Suppose that σ is an irreducible representation occurring in τ, $\sigma \leqslant \tau$, corresponding to the orbit O in $\mathfrak{n}^*$. Then*

$$O \text{ contains } \varrho\text{-rational elements.} \tag{19}$$

Let f be any ϱ-rational element in $\mathfrak{n}^*$, $\mathfrak{m}$ any rational maximal subordinate subalgebra (they exist), and $\sigma = \operatorname{Ind}(M \uparrow N, f^{\sim})$ where $f^{\sim}(\exp X) = e^{2\pi i \langle f, X\rangle}$ on M. Then $\operatorname{Hom}_N(\sigma, \tau)$ *is isomorphic to*

$$\oplus^*_{x \in M\backslash N/\Gamma} \operatorname{Hom}_{x^{-1}Mx}(f^{\sim} \cdot x \,|\, \Gamma \cap x^{-1}Mx, \varrho \,|\, \Gamma \cap x^{-1}Mx), \tag{20}$$

where (*) stands for sum over the rational double cosets (those which meet $N_{\mathbf{Q}}$), and the x are rational coset representatives.

Notes. Statement (19) generalizes Moore's observation [12, Cor. 2], which applies when $\varrho \equiv 1$ and Γ is a lattice subgroup ($\log \Gamma$ an additive lattice in $\mathfrak{n}$). Formula (20) is essentially Richardson's formula [16], except that in his work the maximal subordinate subalgebras M had to be chosen to satisfy certain technical requirements in addition to rationality (without which the formula becomes meaningless on the right). These technical requirements are sometimes troublesome to verify; their absence in (20) leads to slightly easier calculations. Howe's procedure [6] differs from ours in the preliminary necessary condition (19). In his proceedure one sorts out the orbits O which contain elements f which (i) are rational on $[\mathfrak{n}, \mathfrak{n}]$, (ii) admit rational maximal subordinate subalgebras such that $e^{2\pi i f} = \varrho$ on $\Gamma \cap M$. For these orbits there is a multiplicity formula similar to (20) intertwining operators are not discussed in [6]. Yet another approach to multiplicity formulas is given in [2].

Proof of (19). As in section 4, there is a group homomorphism $\phi^{\sim} = e^{2\pi i \phi}$ of N which agrees with ϱ on a subgroup Γ_1 of finite index in Γ. Then $\varrho' = \varrho/\phi^{\sim}$ is identically 1 on Γ_1, so ϱ' very nearly reduces to the trivial character on Γ. If ζ is any one-dimensional representation of N, we note that

$$U^{(\zeta|H)\otimes\pi} \cong \zeta \otimes U^{\pi} \text{ for any representation } \pi \text{ on any subgroup } H.$$

[Taking $\mathcal{H}((\zeta\,|\,H)\otimes\pi) = \mathcal{H}(\pi)$ and $\mathcal{H}(\zeta \otimes U^{\pi}) = \mathcal{H}(U^{\pi})$, the isomorphism is effected by $T\colon \mathcal{H}(U^{\pi}) \to \mathcal{H}(U^{(\zeta|H)\otimes\pi})$ where $Tf(x) = \zeta(x) f(x)$.] Let $f \in \mathfrak{n}^*$, let M be any maximal subordinate subgroup, and write $f^{\sim}$ for $e^{2\pi i f}$ restricted to M. Let $\zeta = e^{-2\pi i \phi}$, the conjugate of $\phi^{\sim}$. Then

$$U^{f^{\sim}} \leqslant U^{\varrho} \Leftrightarrow U^{\zeta|M \otimes f^{\sim}} \cong \zeta \otimes U^{f^{\sim}} \leqslant \zeta \otimes U^{\varrho} \cong U^{\zeta|\Gamma \otimes \varrho}.$$

But

$$\zeta \,|\, M \otimes f^{\sim} = e^{2\pi i (f-\phi)} \quad \text{on } M,$$

$$\zeta \,|\, \Gamma \otimes \varrho = \varrho / \phi^{\sim} = \varrho' \quad \text{on } \Gamma,$$

and $f \in \mathfrak{n}^*$ is ϱ-rational $\Leftrightarrow f - \phi$ is rational. Thus, proof of (19) reduces to showing that, for $h \in \mathfrak{n}^*$,

$$T^{h^{\sim}} \leqslant U^{\varrho'} \Rightarrow \text{the orbit } O(h) \text{ meets } \mathfrak{n}^*_{\mathbf{Q}}. \tag{21}$$

If $\varrho' \equiv 1$ this is well known [13; Cor. 2]. To prove (21) we exploit the fact that $\varrho' \equiv 1$ on the subgroup Γ_1 of finite index in Γ. Suppose that $\sigma = \operatorname{Ind}(M \uparrow N, h^{\sim})$ appears in $\tau = {}_N U^{\varrho'}$. If we induce $\varrho_1' = \varrho' \,|\, \Gamma_1 = 1 \,|\, \Gamma_1$ to a representation τ_1 on N, then τ is a subrepresentation of τ_1 and so $\sigma \leqslant \tau \leqslant \tau_1$. By Moore's observation, $O(h)$ contains points rational with respect to the rational structure determined by Γ_1. Since $[\Gamma : \Gamma_1] < +\infty$, these groups determine the same rational structure $N_{\mathbf{Q}}$, so that $O(h)$ meets $\mathfrak{n}^*_{\mathbf{Q}}$. Q.E.D.

Before we prove (20) we need a refinement of the formula in Theorem 3.4, showing how rationality considerations affect that formula.

LEMMA 5.2. *Let N, Γ, ϱ be as in 5.1. Let f be a ϱ-rational element in $\mathfrak{n}^*$, $\mathfrak{m}$ a rational maximal subordinate subalgebra for f, N_0 a rational normal subgroup containing $M = \exp(\mathfrak{m})$, and set $\Gamma_0 = \Gamma \cap N_0$, $\varrho_0 = \varrho \,|\, \Gamma_0$. Let $f^{\sim} = e^{2\pi i f}$ on M, $\sigma_0 = \operatorname{Ind}(M \uparrow N_0, f^{\sim})$, $\sigma = \operatorname{Ind}(M \uparrow N, f^{\sim})$, $\tau_0 = \operatorname{Ind}(\Gamma_0 \uparrow N_0, \varrho_0)$, and $\tau = \operatorname{Ind}(\Gamma \uparrow N, \varrho)$. Then in the intertwining formula of 3.4, applied to the system $(N, N_0, \Gamma; \sigma_0, \varrho)$,*

$$\operatorname{Hom}_N(\sigma, \tau) \cong \oplus_{y \in N_0 \backslash N / \Gamma} \operatorname{Hom}_{N_0}(\sigma_0 \cdot y, \tau_0),$$

only rational double cosets (those meeting $N_{\mathbf{Q}}$) can contribute nonzero multiplicity.

Proof. Let $f' = f \,|\, \mathfrak{n}_0$ (ϱ_0-rational in $\mathfrak{n}_0^*$). For any $y \in N$, $\sigma_0 \cdot y$ on N_0 is induced from $f^{\sim} \cdot y$ on $y^{-1} M y$; thus $\sigma_0 \cdot y$ is associated with the N_0-orbit $O = (f' \cdot y) \cdot N_0$. If $\sigma_0 \cdot y \leqslant \tau_0$, it follows that there are ϱ_0 hyphen-rational points in O; there is an $x \in N_0$ such that $f' \cdot yx$ is ϱ_0-rational on $\mathfrak{n}_0$. Apply Lemma 4.3; we know f is ϱ-rational and $f \cdot yx \,|\, \mathfrak{n}_0 = f' \cdot yx$ is ϱ_0-rational. Hence there is a $z \in N_{\mathbf{Q}}$ such that $f \cdot yx = f \cdot z$ on $\mathfrak{n}_0$. Hence $\sigma_0 \cdot yx = \sigma_0 \cdot z$. Since $\operatorname{Ind}(N_0 \uparrow N, \sigma_0)$ is irreducible, σ_0 is stabilized by N_0 under action of N (remarks following 2.4), so that $N_0 yx = N_0 z$ and $N_0 y \Gamma = N_0 (yxy^{-1}) y \Gamma = N_0 yx \Gamma = N_0 z \Gamma$. Q.E.D.

Proof of (20). Let $\mathfrak{m}=\mathfrak{n}_1\subseteq\ldots\subseteq\mathfrak{n}_k=\mathfrak{n}$ be the successive normalizers $\mathfrak{n}_{j+1}=\{X\in\mathfrak{n}: [X,\mathfrak{n}_j]\subseteq\mathfrak{n}_j\}$, each a rational subalgebra with $\mathfrak{n}_j\lhd\mathfrak{n}_{j+1}$. We write $N_j=\exp(\mathfrak{n}_j)$, $\Gamma_j=\Gamma\cap N_j$, $\sigma_j=\text{Ind}\,(M\uparrow N_j,\pi)$ where π is the one-dimensional representation $e^{2\pi i f}$ on M. For $x\in N$ we write

$$N_j(x)=x^{-1}N_jx,\qquad \sigma_j(x)=\text{Ind}\,(M\uparrow N_j,\pi)\cdot x,$$
$$\Gamma_j(x)=\Gamma\cap x^{-1}N_jx,\qquad \tau_j(x)=\text{Ind}\,(\Gamma_j(x)\uparrow N_j(x),\varrho_j(x)),$$
$$\varrho_j(x)=\varrho\,|\,\Gamma_j(x).$$

If ζ is a representation on $H\subseteq N$ we define $\zeta\cdot x(h')=\zeta(xh'x^{-1})$ on $H'=x^{-1}Hx$; thus $\sigma_j(x)=\sigma_j\cdot x$ is a representation on $N_j(x)$. We proceed by downward induction on $j=k,k-1,\ldots,1$ to prove that

$$\text{Hom}_N(\sigma,\tau)\cong\oplus^*_{x\in N_j\backslash N/\Gamma}\,\text{Hom}_{N_j(x)}(\sigma_j\cdot x,\tau_j(x)),\tag{22}$$

where the sum is taken only over the rational double cosets, choosing rational representatives x. The result is obvious, saying Hom $(\sigma,\tau)\cong$ Hom (σ,τ), if $j=k$. Assuming the result true for indices $k,\ldots,j+1$ we wish to prove it true for index j. Because the general case is a bit confusing notationally, we first give the simpler case $j=k-1$. Since $N_{k-1}\lhd N$, Theorem 3.4 directly yields

$$\text{Hom}(\sigma,\tau)\cong\oplus_{y\in N_{k-1}\backslash N/\Gamma}\,\text{Hom}_{N_{k-1}}(\sigma_{k-1}\cdot y,\tau_{k-1})\tag{23}$$

(without any rationality conditions on double cosets or their representatives). By Lemma 5.2, only rational double cosets contribute to this sum. Furthermore, for any $y\in N$, $N_{k-1}=N_{k-1}(y)$ since $N_{k-1}\lhd N$, and $\Gamma_{k-1}=\Gamma_{k-1}(y)$; thus, $\varrho_{k-1}=\varrho_{k-1}(y)$ and $\tau_{k-1}=\tau_{k-1}(y)$. Hence we may rewrite (23) as

$$\text{Hom}(\sigma,\tau)\cong\oplus^*_{y\in N_{k-1}\backslash N/\Gamma}\,\text{Hom}_{N_{k-1}(y)}(\sigma_{k-1}\cdot y,\tau_{k-1}(y)),\tag{24}$$

which is the desired formula for $j=k-1$.

In general we must grapple with notation, though the concept is simple. Assume the result true for indices $k,\ldots,j+1$ so that

$$\text{Hom}(\sigma,\tau)\cong\oplus^*_{x\in N_{j+1}\backslash N/\Gamma}\,\text{Hom}_{N_{j+1}(x)}(\sigma_{j+1}\cdot x,\tau_{j+1}(x)).\tag{25}$$

Consider a typical term on the right; we take x rational, so $N_{j+1}(x)=x^{-1}N_{j+1}x$ is a rational subgroup. Theorem 3.4 says that

$$\text{Hom}_{N_{j+1}(x)}(\sigma_{j+1}\cdot x,\tau_{j+1}(x))=\oplus^*_{y\in N_j(x)\backslash N_{j+1}(x)/\Gamma_{j+1}(x)}\,\text{Hom}_{N_j(x)}((\sigma_j\cdot x)\cdot y,\tau_j(x)).\tag{26}$$

Note: $(\sigma_j \cdot x) \cdot y = \sigma_j \cdot xy$. Again, 5.2 insures that only rational double cosets (with rational representatives y) can contribute to the sum. For any $y \in N_{j+1}(x)$ we have $N_j(x) = N_j(xy)$ because $N_j \lhd N_{j+1}$; thus, $\Gamma_j(x) = \Gamma_j(xy)$, and $\varrho_j(x) = \varrho_j(xy)$, $\tau_j(x) = \tau_j(xy)$. Thus (26) becomes

$$\mathrm{Hom}_{N_{j+1}(x)}(\ldots) \cong \oplus^*_{y \in N_j(x) \backslash N_{j+1}(x)/\Gamma_{j+1}(x)} \mathrm{Hom}_{N_j(xy)}(\sigma_j \cdot xy, \tau_j(xy)) \tag{27}$$

for all rational $x \in N_{\mathbf{Q}}$. We now combine (25) and (27). Let $S \subseteq N_{\mathbf{Q}}$ be a set of representatives for the rational double cosets in $N_{j+1} \backslash N/\Gamma$. We note that $N_{\mathbf{Q}} = \bigcup_{x \in S} (N_{j+1})_{\mathbf{Q}} x\Gamma$. [Inclusion $(\supseteq)$ is obvious. But if $z \in N_{\mathbf{Q}}$, then $N_{j+1} z\Gamma$ is a rational double coset, equal to $N_{j+1} x\Gamma$ for some $x \in S$; thus $z = nx\gamma$ and $n = z\gamma^{-1}x^{-1} \in N_{\mathbf{Q}} \cap N_{j+1} = (N_{j+1})_{\mathbf{Q}}$.] For each $x \in S$ let $S_x \subseteq (N_{j+1})_{\mathbf{Q}}$ be representatives for the rational double cosets in $N_j(x) \backslash N_{j+1}(x)/\Gamma_{j+1}(x)$. Since $\Gamma_{j+1}(x) = \Gamma \cap N_{j+1}(x)$ and $N_{j+1}(x)$ is rational, we again have $(N_{j+1}(x))_{\mathbf{Q}} = \bigcup \{(N_j(x))_{\mathbf{Q}} y \Gamma_{j+1}(x) \colon y \in S_x\}$. Thus,

$$\begin{aligned} N_{\mathbf{Q}} = \bigcup_{x \in S} (N_{j+1})_{\mathbf{Q}} x\Gamma &= \bigcup_{x \in S} x \cdot (x^{-1} N_{j+1} x)_{\mathbf{Q}} \Gamma \\ &= \bigcup_{x \in S} \bigcup_{y \in S_x} x(x^{-1} N_j x)_{\mathbf{Q}} y \Gamma_{j+1}(x) \Gamma = \bigcup_{x \in S,\, y \in S_x} (N_j)_{\mathbf{Q}} xy\Gamma. \end{aligned}$$

These unions are disjoint, so the set $\{xy \colon x \in S, y \in S_x\}$ is a set of representatives for the rational double cosets in $N_j \backslash N/\Gamma$. Therefore, combining (25) and (27), we get

$$\mathrm{Hom}_N(\sigma, \tau) \cong \oplus^*_{x \in N_j \backslash N/\Gamma} \mathrm{Hom}_{N_j(x)}(\sigma_j \cdot x, \tau_j(x))$$

as required. Continuing the induction to $j = 1$ we get

$$\mathrm{Hom}(\sigma, \tau) \cong \oplus^*_{x \in M \backslash N/\Gamma} \mathrm{Hom}_{x^{-1}Mx}(f^{\sim} \cdot x, \tau_1(x)).$$

By Lemma 2.2,

$$\mathrm{Hom}_{x^{-1}Mx}(f^{\sim} \cdot x, \tau_1(x)) \cong \mathrm{Hom}_{x^{-1}Mx \cap \Gamma}(f^{\sim} \cdot x \,|\, \Gamma \cap x^{-1}Mx, \varrho \,|\, \Gamma \cap x^{-1}Mx).$$

Combining these remarks we get formula (20). Q.E.D.

The isomorphism in formula (20) can be written down as follows (see section 6 for details). If

$$A \in \mathrm{Hom}_{\Gamma \cap x^{-1}Mx}(f^{\sim} \cdot x \,|\, \Gamma \cap x^{-1}Mx, \varrho \,|\, \Gamma \cap x^{-1}Mx),$$

we define $B \in \mathrm{Hom}\,(\sigma, \tau)$ by letting

$$BF(y) = \sum_{\gamma \in \Gamma \cap x^{-1}Mx \backslash \Gamma} \varrho(\gamma)^{-1} A F(x\gamma y) \tag{28}$$

for continuous F with compact support modulo M. Of course A is a scalar here. In particular, if $\varrho \equiv 1$ these formulas tell us how to set up orthogonal subspaces of functions in $\mathcal{H}(U^{\varrho}) = L^2(\Gamma \backslash N)$, one for each $x \in (\Gamma \backslash N/M)^*$, which span the σ-primary subspace in L^2. This formula was first obtained in [16].

§ 6. Computational details for (28)

We must unravel the isomorphisms used to obtain (20) to get the concrete isometries $\mathbf{I}^x$: $\mathrm{Hom}(f^\sim\cdot x\,|\,\Gamma\cap x^{-1}Mx,\ \varrho\,|\,\Gamma\cap x^{-1}Mx)\to\mathrm{Hom}(\sigma,\tau)$ we want. The details are the same regardless of which $x\in(\Gamma\diagdown N/M)^*$ we consider, so we shall construct the isometry $\mathbf{I}$:$\mathrm{Hom}(f^\sim\,|\,\Gamma\cap M,\ \varrho\,|\,\Gamma\cap M)\to\mathrm{Hom}(\sigma,\tau)$ and then indicate the simple changes which occur when we replace $f^\sim\to f^\sim\cdot x$ and $M\to x^{-1}Mx$.

Define N_j, Γ_j, σ_j, ϱ_j, τ_j as above (setting $x=e$); then let

$$H_j=\Gamma_jN_{j-1},\qquad \lambda_j=\mathrm{Ind}\,(\Gamma_j\uparrow H_j,\varrho_j),\quad \text{for } 2\leqslant j\leqslant k,$$

and

$$\tilde\sigma_j=\mathrm{Ind}\,(N_{j-1}\uparrow N_j,\sigma_{j-1}),\qquad \tilde\tau_j=\mathrm{Ind}\,(H_j\uparrow N_j,\lambda_j).$$

These objects are related as shown in Figure 3, where rising arrows indicate induction, and falling arrows restriction. Obviously $\sigma_j\cong\sigma_j^\sim$ and $\tau_j\cong\tau_j^\sim$, by induction in stages. We obtain $\mathbf{I}$ by writing out explicitly the isomorphisms and "lift operations" (indicated by arrows) in the following scheme. Here, by definition, $\mathrm{Hom}(\sigma_1\,|\,\Gamma_1,\varrho_1)\equiv\mathrm{Hom}(f^\sim\,|\,\Gamma\cap M,\ \varrho\,|\,\Gamma\cap M)$; the first $\cong$ follows from Lemma, 2.2.

$$\begin{aligned}
&\mathrm{Hom}\,(\sigma_1\,|\,\Gamma_1,\varrho_1)\cong\mathrm{Hom}\,(\sigma_1,\tau_1)\cong\mathrm{Hom}\,(\sigma_1,\lambda_2\,|\,N_1)\to\\
&\to\mathrm{Hom}\,(\tilde\sigma_2,\tilde\tau_2)\cong\mathrm{Hom}\,(\sigma_2,\tau_2)\cong\mathrm{Hom}\,(\sigma_2,\lambda_3\,|\,N_2)\to\\
&\qquad\qquad\cdots\\
&\qquad\qquad\ldots\cong\mathrm{Hom}\,(\sigma_{k-2},\lambda_{k-1}\,|\,N_{k-2})\to\\
&\to\mathrm{Hom}\,(\tilde\sigma_{k-1},\tilde\tau_{k-1})\cong\mathrm{Hom}\,(\sigma_{k-1},\tau_{k-1})\cong\mathrm{Hom}\,(\sigma_{k-1},\lambda_k\,|\,N_{k-1})\to\\
&\to\mathrm{Hom}\,(\tilde\sigma_k,\tilde\tau_k)\cong\mathrm{Hom}\,(\sigma_k,\tau_k)\equiv\mathrm{Hom}\,(\sigma,\tau).
\end{aligned}\tag{29}$$

Fortunately, things combine to give a simple formula at the end (and each time we reach $\mathrm{Hom}\,(\sigma_j,\tau_j)$). Within $\mathcal{H}(\sigma_j)$ let $\mathcal{H}(\sigma_j)_0$ be the continuous functions with compact support modulo $N_1=M$; likewise, let $\mathcal{H}(\tau_j)_0$ be the continuous functions in $\mathcal{H}(\tau_j)$. The isomorphisms $\sigma_j\cong\tilde\sigma_j$, $\tau_j\cong\tilde\tau_j$ are given by isometries W_j, T_j defined by specifying their actions on $\mathcal{H}(\sigma_j)_0$, $\mathcal{H}(\tilde\tau_j)_0$. If $f\in\mathcal{H}(\sigma_j)_0$,

$$[W_jf(n)](n_{j-1})=f(n_{j-1}n)\quad n_{j-1}\in N_{j-1},\ n\in N_j\tag{30}$$

defines a vector $W_jf(n)\in\mathcal{H}(\sigma_{j-1})_0$ which varies continuously in $n\in N_j$ and has compact support modulo N_{j-1}, so that $W_jf\in\mathcal{H}(\tilde\sigma_j)_0$. If $f\in\mathcal{H}(\tau_j)_0$,

$$[T_jf(n)](h_j)=f(h_jn)\quad h_j\in H_j,\ n\in N_j\tag{31}$$

defines a vector $T_jf(n)\in\mathcal{H}(\lambda_j)_0$ (continuous functions on H_j varying like ϱ along Γ_j-cosets)

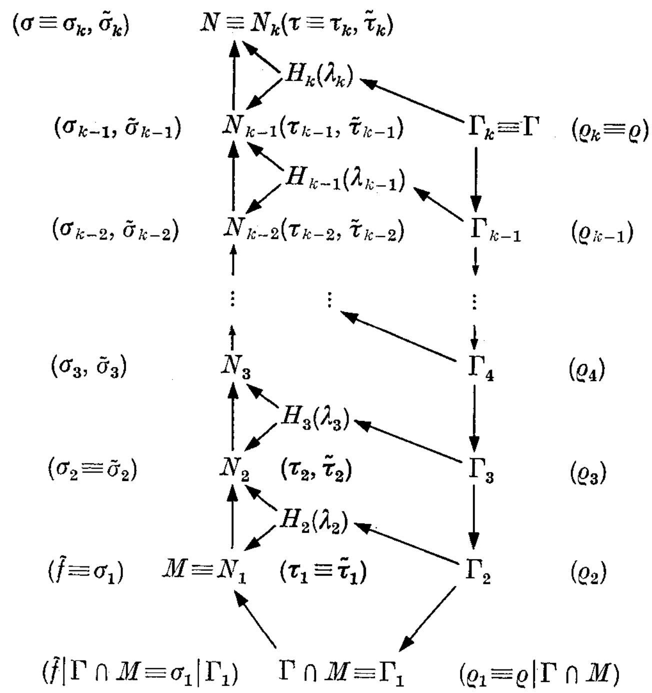

Figure 3.

which varies continuously with $n \in N_j$, so that $T_j f \in \mathcal{H}(\tilde{\tau}_j)_0$. The first isomorphism in each row of (29) maps A to $T_j^{-1} A W_j$.

The second isomorphism is obtained as in Lemma 2.1 via the isometry R_j: $\mathcal{H}(\tau_j) \to \mathcal{H}(\lambda_{j+1} | N_j) = \mathcal{H}(\lambda_{j+1})$,

$$R_j \phi(\gamma_{j+1} n_j) = \varrho(\gamma_{j+1}) \phi(n_j) \quad \text{all } h_{j+1} = \gamma_{j+1} n_j \in H_{j+1}$$

for $\phi \in \mathcal{H}(\tau_j)_0$. Thus, an intertwining operator $A_j \in \operatorname{Hom}(\sigma_j, \tau_j)$ is identified with $A_j^\vee = R_j A_j \in \operatorname{Hom}(\tau_j, \lambda_{j+1} | N_j)$

$$(A_j^\vee f)(\gamma_{j+1} n_j) = \varrho(\gamma_{j+1})(A_j f)(n_j) \quad \text{all } h_{j+1} = \gamma_{j+1} n_j \in H_{j+1}$$

if $f \in \mathcal{H}(\sigma_j)_0$.

The process of lifting $A \in \operatorname{Hom}(\sigma_1 | \Gamma_1, \varrho_1)$ to an operator in $\operatorname{Hom}(\sigma, \tau)$ begins by using the construction in C. Moore's Lemma 2.2 to get $A_1 \in \operatorname{Hom}(\sigma_1, \tau_1)$: if $v \in \mathcal{H}(\sigma_1)(\equiv \mathbf{C})$,

$$(A_1 v)(n_1) = A \sigma_1(n_1) v \quad \text{all } n_1 \in N_1.$$

Inductively, given $A_{j-1} \in \operatorname{Hom}(\sigma_{j-1}, \tau_{j-1})$ we lift it to $A_j \in \operatorname{Hom}(\sigma_j, \tau_j)$ by first transforming it to $A_{j-1}^\vee = R_{j-1} A_{j-1} \in \operatorname{Hom}(\sigma_{j-1}, \lambda_j | N_{j-1})$. Next, lift $A_{j-1}^\vee$ to $\tilde{A}_j \in \operatorname{Hom}(\tilde{\sigma}_j, \tilde{\tau}_j)$, applying the standard construction of Theorem 3.3 to the system $(N_j, N_{j-1}, \Gamma_j; \sigma_{j-1}, \lambda_j)$ If $f \in \mathcal{H}(\tilde{\sigma}_j)_0$, the pointwise formula (1) is valid, and takes the form

$$(\tilde{A}_j f)(n_j) = \sum_{\gamma \in \Gamma_{j-1} \backslash \Gamma_j} \lambda_j(\gamma^{-1}) A^\vee_{j-1}(f(\gamma n_j)) \quad \text{all} \quad n_j \in N_j.$$

Finally, transform $\tilde{A}_j$ to $A_j = T_j^{-1} \tilde{A}_j W_j \in \operatorname{Hom}(\sigma_j, \tau_j)$ to complete a full step in the process. We assert that $A_j f$ is given by the following pointwise formula for $f \in \mathcal{H}(\sigma_j)_0$, $1 \leqslant j \leqslant k$:

$$A_j f(n_j) = \sum_{\gamma \in \Gamma_1 \backslash \Gamma_j} \varrho(\gamma^{-1}) A(f(\gamma n_j)) \quad \text{all} \quad n_j \in N_j. \tag{32}$$

If U is a compact set in N_j, only a finite number of terms in the right hand sum are nonzero on U, so the function $B_j f(n_j)$ defined by the sum is continuous on N_j, and hence lies in $\mathcal{H}(\tau_j)_0$ if $f \in \mathcal{H}(\sigma_j)_0$. Trivially, $A_1 f(n) = B_1 f(n)$, all $n \in N_1$. Assuming the formula true for index $j-1$ we show it valid for j by showing that $T_j(B_j f) = T_j(A_j f)$ in $\mathcal{H}(\tilde{\tau}_j)$. Now, $T_j(B_j f)$ maps $N_j \to \mathcal{H}(\lambda_j)_0$: for $n \in N_j$,

$$\begin{aligned}[][T_j(B_j f)(n)](h_j) &= B_j f(h_j n) = B_j f(\gamma_j n_{j-1} n) \\ &= \sum_{\gamma \in \Gamma_1 \backslash \Gamma_j} \varrho(\gamma^{-1}) A(f(\gamma\gamma_j n_{j-1} n)) = \varrho(\gamma_j) \sum_{\gamma \in \Gamma_1 \backslash \Gamma_j} \varrho(\gamma^{-1}) A(f(\gamma n_{j-1} n))\end{aligned}$$

if $h_j = \gamma_j n_{j-1} \in H_j$. On the other hand, $A_j f = T_j^{-1} \tilde{A}_j W_j f$, so $T_j(A_j f) = \tilde{A}_j(W_j f)$. Since $f \in \mathcal{H}(\sigma_j)_0$ it follows that $W_j f \in \mathcal{H}(\tilde{\sigma}_j)_0$, so the pointwise formula for lifting $A^\vee_{j-1}$ to $\tilde{A}_j$ applies:

$$\tilde{A}_j(W_j f)(n) = \sum_{\gamma \in \Gamma_{j-1} \backslash \Gamma_j} \lambda_j(\gamma^{-1}) A^\vee_{j-1}(W_j f(\gamma n)) \quad \text{all} \quad n \in N_j$$

(finitely many nonzero terms). Since $\tilde{A}_j(W_j f) \in \mathcal{H}(\tilde{\tau}_j)$, this is a function on H_j for each $n \in N_j$; if $h_j = \gamma_j n_{j-1}$,

$$\begin{aligned}[][\tilde{A}_j(W_j f)(n)](h_j) &= \sum_\gamma [\lambda_j(\gamma^{-1})(A^\vee_{j-1}(W_j f(\gamma n)))](h_j) \\ &= \sum_\gamma [A^\vee_{j-1}(W_j f(\gamma n))](h_j \gamma^{-1}) \\ &= \sum_\gamma [A^\vee_{j-1}(W_j f(\gamma n))](\gamma_j \gamma^{-1} \cdot \gamma n_{j-1} \gamma^{-1}) \\ &= \sum_\gamma \varrho(\gamma_j) \varrho(\gamma^{-1}) [A_{j-1}(W_j f(\gamma n))](\gamma n_{j-1} \gamma^{-1}).\end{aligned}$$

Since $\gamma n \in N_j$, $\phi = W_j f\ (\gamma n)$ lies in $\mathcal{H}(\sigma_{j-1})_0$, so the induction hypothesis applies,

$$\begin{aligned} &= \varrho(\gamma_j) \sum_\gamma \varrho(\gamma^{-1}) \sum_{\gamma' \in \Gamma_1 \backslash \Gamma_{j-1}} \varrho(\gamma')^{-1} A([W_j f(\gamma n)](\gamma' \gamma n_{j-1} \gamma^{-1})) \\ &= \gamma(\gamma_j) \sum_{\gamma, \gamma'} \varrho(\gamma'\gamma)^{-1} A(f(\gamma' \gamma n_{j-1} \gamma^{-1} \cdot \gamma n)) = \varrho(\gamma_j) \sum_{\gamma \in \Gamma_1 \backslash \Gamma_j} \varrho(\gamma^{-1}) A(f(\gamma n_{j-1} n)),\end{aligned}$$

as required. When $j = k$ we get $\mathrm{I}(A) = A_k = B$,

$$Bf(n) = \sum_{\gamma \in \Gamma_1 \backslash \Gamma} \varrho(\gamma^{-1}) Af(\gamma n) \quad \text{for} \quad f \in \mathcal{H}(\sigma)_0.$$

If $x \in (\Gamma \backslash N/M)^*$, this discussion applies verbaitm if we substitute $\bar{f} \to \bar{f} \cdot x$, $M \to x^{-1} M x$; we get intertwining operators $A_k^x \in \operatorname{Hom}(\operatorname{Ind}(x^{-1} M x \uparrow N, \bar{f} \cdot x), \tau)$. These are to be identified with operators $\mathbf{I}^x(A) \in \mathrm{Hom}\,(\sigma, \tau)$ using the natural isomorphism $f(n) \to f(xn)$ between Ind (...) and σ. Thus we get

$$\mathbf{I}^x(A)\, f(n) = \textstyle\sum_{\gamma \in \Gamma_1 \backslash \Gamma} \varrho(\gamma^{-1})\, A f(x\gamma n) \quad \text{for} \quad f \in \mathcal{H}(\sigma)_0.$$

§ 7. Other examples

As another application, using Theorem 3.7 together with Mackey's theory of induced representations, let G be the group of Euclidean motions of the plane, and let Γ be the subgroup generated by translations by integers and rotations by multiples of $\pi/2$. We can regard G as the semidirect product of $K = \mathbf{C}$ by the unit circle T, where $e^{i\theta} \in T$ acts on $\mathbf{C}$ by multiplication. We identify $K^\wedge$ with $\mathbf{C}$ by letting w correspond to $\pi_w(z) = e^{2\pi i \operatorname{Re}(zw)}$. The action of T on $K^\wedge$ is still multiplication, so the T-orbits in $K^\wedge$ are the circles $|z| = r$, $r \geqslant 0$. If $r > 0$, the stabilizer of a point in the orbit is 1; for $r = 0$ the stabilizer is T. Thus the irreducible representations of G are given by $\sigma_r = \mathrm{Ind}\,(K \uparrow G, \pi_{r+i0})$, $r > 0$, and by $\mu_n(z, e^{i\theta}) = e^{in\theta}$ for $n \in \mathbf{Z}$ (corresponding to $r = 0$).

Now let ϱ be the trivial representation on Γ, and let $\tau = \mathrm{Ind}\,(\Gamma \uparrow G, \varrho)$. The σ_r fit neatly into our theory, and we can apply 3.4 (or 3.7). Since

$${}_K U^{\varrho|\Gamma \cap K} = \mathrm{Ind}\,(\mathbf{Z} + i\mathbf{Z} \uparrow \mathbf{C}, 1) \cong \oplus_{\gamma \in \mathbf{Z} + i\mathbf{Z}}\, \pi_\gamma,$$

and the Γ-orbit of π_w is just $\{\pi_{\pm w}, \pi_{\pm iw}\}$, it's easy to check that σ_r appears in $\tau \Leftrightarrow$ there is a point in $\mathbf{Z} + i\mathbf{Z}$ with norm $r \Leftrightarrow r^2$ is the sum of two integer squares. Then

$$\begin{aligned} \mathrm{mult}\,(\sigma_r) &= \tfrac{1}{4}\,(\text{total number of ways of writing } r^2 \text{ as } m^2 + n^2,\ m, n \in \mathbf{Z}) \\ &= \tfrac{1}{4}\,(\text{number of ``integral points'' on } |z| = r). \end{aligned}$$

The μ_n, being one-dimensional, appear in $\tau \Leftrightarrow$ the function μ_n is in $L^2(\Gamma \diagdown G)$; thus the μ_{4n} $(n \in \mathbf{Z})$ appear, each with multiplicity 1.

This result is not new; see [20; Thm 1], and also [21]. Our method works, however, on the other Euclidean motion groups. For instance, let G be the Euclidean motion group of $\mathbf{R}^3$ and Γ the group generated by integral translations and rotations by $\pi/2$ about the coordinate axes. If K is the subgroup of all translations then $K^\wedge \cong \mathbf{R}^3$, the orbits are 2-spheres about the origin, and the representation σ_r corresponding to the sphere $\|x\| = r$ of radius $r > 0$ occurs with multiplicity

$$\begin{aligned} &\tfrac{1}{8}\,(\text{number of points in } \mathbf{Z}^3 \cap \{x\colon \|x\| = r\}) \\ &\quad = \tfrac{1}{8}\,(\text{number of ways of writing } r^2 = m^2 + n^2 + p^2 \text{ with } m, n, p \in \mathbf{Z}). \end{aligned}$$

The details are straightforward.

§ 8. Remarks on adelic groups

Finally, we make an application to adelic nilpotent groups. The main result is due to Moore [13]; since we need much of his discussion to justify our arguments here, we merely indicate how his final result is obtained by the above methods. Let $N_{\mathbf{Q}}$ be a nilpotent algebraic group over $\mathbf{Q}$ and let $N = N_{\mathbf{A}}$ be its adelized version. Then $N_{\mathbf{Q}} \backslash N_{\mathbf{A}}$ is compact, and we are back in the general situation studied in this paper. All the arguments of the last two sections go through, with one simplification: a one-dimensional representation ϱ of $N_{\mathbf{Q}}$ automatically extends to $N_{\mathbf{A}}$. However, the final formula in Theorem 5.1 simplifies greatly. If two elements l, l' in an orbit $\mathcal{O}$ are ϱ-rational, they are in the same $\mathrm{Ad}'(N_{\mathbf{Q}})$-orbit (because they are conjugate under a rational element of $N_{\mathbf{A}}$). Hence only one $M \backslash N / N_{\mathbf{Q}}$ double coset can have rational elements. Thus $\mathrm{Hom}\,(\sigma, \tau) \cong \mathbf{C}$ if $\mathcal{O} = \mathcal{O}(\sigma)$ has ϱ-rational elements, and is $\cong 0$ otherwise. That is, the multiplicity of σ in τ is 1 or 0 depending on whether $\mathcal{O}(\sigma)$ has ϱ-rational elements or not.

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