THE DISTRIBUTION OF THE VALUES OF ADDITIVE ARITHMETICAL FUNCTIONS

BY

P. D. T. A. ELLIOTT and C. RYAVEC

University of Colorado, Boulder, Colo., U.S.A.

Introduction

A real valued number theoretic function is said to be *additive* if for every pair of coprime positive integers a and b, the relation

$$f(ab) = f(a) + f(b)$$

is satisfied. Thus an additive function is determined by its values on the prime-powers. If, in addition, for each prime p

$$f(p) = f(p^2) = \ldots,$$

then $f(m)$ is said to be *strongly additive*. In this paper we shall confine our attention to strongly additive functions.

The paper falls into three sections.

In the first section we consider those strongly additive functions $f(m)$ which, after a suitable translation, possess a limiting distribution. Theorems 1 and 2 provide a characterization of such functions, essentially in terms of their values on the primes.

A classic result of Erdös and Wintner states that an additive function $f(m)$ has a limiting distribution if and only if the two series

$$\sum_p \frac{f'(p)}{p} \tag{*}$$

and

$$\sum_p \frac{(f'(p))^2}{p}$$

converge.(1) These two conditions are quite restrictive, however, so it is desirable to study

(1) See Notation.

a wider class of additive functions. It is natural to begin with those functions for which *only* the series (*) does not converge.

If this is the case, then $f(m)$ cannot possess a limiting distribution. An old result of Turán, however, says, in effect, that "if a strongly additive function has a finite variance, then the values of the function do not differ from the mean very often". (We give a precise statement of this result in Lemma 4.) Thus we might expect that if the frequencies $\nu_n(m\colon f(m) < z)$ are suitably translated; i.e., by an amount

$$\sum_{p\leqslant n} \frac{f'(p)}{p}, \quad (n = 1, 2, \ldots),$$

then the resulting frequencies

$$\nu_n\left(m\colon f(m) - \sum_{p\leqslant n} \frac{f'(p)}{p} < z\right)$$

will have a limiting distribution. That this is indeed the case was first stated by Erdös (Theorem II, [4]) and proved with the additional hypothesis that $|f(p)|$ is bounded. Erdös claimed that even more is true:

THEOREM III (Erdös). *Let $f(m)$ be additive. Assume that a constant c exists such that if we put $f(m) - c\log m = g(m)$, then $g(m)$ will satisfy*

$$\sum_p \frac{(g'(p))^2}{p} < \infty.$$

Then the frequencies

$$\nu_n\left(m\colon f(m) - c\log m - \sum_{p\leqslant n} \frac{g'(p)}{p} \leqslant z\right)$$

have a limiting distribution. (1)

Erdös also stated that the converse to Theorem III is probably true, although he claimed he could supply a proof only if $f(p) > 0$. All of these cases are included in Theorem 2 of the present paper. We also determine the characteristic function of the limit law, whenever it exists; and necessary and sufficient conditions are given for the continuity of the limit law.

The second section deals with various continuity properties of certain distribution functions associated with additive functions.

In the third, and final section we consider the place of strongly additive functions in

(1) We remark that the above theorem of Erdös does not coincide with his original formulation.

a more general framework. The assumption that $f(m)$ has a limiting distribution is relaxed; and we prove two theorems under weaker hypotheses. In particular (Theorem 4), we prove an analogue of a classical result of Paul Lévy, which has also been considered by Erdös in his paper [4].

Notation

We denote by $|E|$ the cardinality of a typical set E. For each positive integer n, we define the *frequency*

$$\nu_n(m: \dots) = \frac{1}{n} \sum_{\substack{1 \leqslant m \leqslant n \\ \dots}} 1,$$

where the sum counts those integers m for which property ... holds. For an additive function $f(m)$ it will be convenient to define the distribution functions

$$F_n(z) = \nu_n(m: f(m) < z), \quad n = 1, 2, 3, \dots;$$

and we say that $f(m)$ possesses a limiting distribution (or a limit law) if there exists a left continuous function $F(z)$ with the properties $F(-\infty) = 0$, $F(+\infty) = 1$, such that $\lim_{n\to\infty} F_n(z) = F(z)$ for all real points z at which $F(z)$ is continuous.

We shall also use the standard notation $u \ll v$ for functions u and v when there exists a positive constant B so that $|u| \leqslant Bv$, the inequality being uniform over some well-defined region.

We adopt the conventional notation

$$f'(p) = \begin{cases} f(p) & \text{if } |f(p)| \leqslant 1 \\ 1 & \text{otherwise.} \end{cases}$$

Section 1

THEOREM 1. *Let $f(m)$ be a strongly additive function. Then there exist constants $\alpha_1, \alpha_2, \dots$ so that the frequencies*

$$\nu_n(m: f(m) - \alpha_n < z) \tag{1}$$

possess a limiting distribution if and only if $f(m)$ has the form

$$f(m) = c \log m + g(m), \tag{2}$$

where
$$\sum_p \frac{(g'(p))^2}{p} < \infty. \tag{3}$$

In this case we can set

$$\alpha_n = c \log n + \sum_{p \leqslant n} \frac{g'(p)}{p} + \text{constant} + o(1),$$

and apart from the last two terms in this expression, the choice of the numbers α_n *is unique.*

The characteristic function of the limiting distribution, when it exists, will have the form

$$\varphi(t) = \frac{1}{1+itc} \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{k=1}^{\infty} \frac{e^{itg(p^k)}}{p^k}\right) e^{-it(g'(p)/p)},$$

to within a factor $\exp(-it(\text{constant}))$.

The distribution function will be continuous if and only if $\sum_{f(p)\neq 0} (1/p) = \infty$.

We shall deduce this theorem from Lemma 1 and the following

THEOREM 2. *Let* $f(m)$ *be a strongly additive function. Then a necessary and sufficient condition that a constant c and real numbers* $\alpha_1, \alpha_2, \ldots$ *exist so that the frequencies*

$$\nu_n(m\colon f(m) - c \log m - \alpha_n < z), \quad (n = 1, 2, 3 \ldots), \tag{4}$$

possess a limiting distribution is that there exist a (further) constant d, and an additive function $g(m)$, *so that*

$$f(m) = d \log m + g(m), \tag{5}$$

where
$$\sum \frac{(g'(p))^2}{p} < \infty. \tag{6}$$

Moreover, when these conditions are satisfied we can take $c = d$, *and* $\alpha_n = \sum_{p \leqslant n} g'(p)/p$. *The characteristic function of the limiting distribution, when it exists, will have the form*

$$\phi(t) = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{k=1}^{\infty} \frac{e^{itg(p^k)}}{p^k}\right) e^{-it(g'(p)/p)}.$$

The distribution function will be continuous if and only if

$$\sum_{g(p)\neq 0} \frac{1}{p} = \infty.$$

We remark that it would be desirable to find necessary and sufficient conditions so that there exists two sequences $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ such that the frequencies

$$\nu_n(m: \beta_n f(m) - \alpha_n < z), \quad (n = 1, 2, \ldots),$$

possess a limit law. In the above theorems we consider the case $\beta_1 = \beta_2 = \ldots = \text{constant}$.

LEMMA 1. *Let $\omega_1, \omega_2, \ldots$ be a sequence of real numbers for which the limit $\lim_{n\to\infty} e^{it\omega_n}$ exists uniformly in some neighborhood of the origin. Then the sequence $\omega_1, \omega_2, \ldots$ itself tends to a limit.*

Proof of Lemma 1. Set

$$\phi(t) = \lim_{n\to\infty} e^{it\omega_n}, \quad |t| \leqslant t_0.$$

Then if $t_1 = 2t$, $|t| \leqslant t_0$, we have

$$\lim_{n\to\infty} e^{it_1\omega_n} = \lim_{n\to\infty} (e^{it\omega_n})^2 = \phi^2(t);$$

and it follows that the limit $\phi(t)$ exists and is continuous for all real t. Moreover, the convergence is uniform on any bounded interval of the real line; and, in particular, $\phi(t)$ is continuous at the origin.

From what we have said it follows that the distribution functions

$$W_n(z) = \begin{cases} 1 & \text{if } z > \omega_n \\ 0 & \text{if } z \leqslant \omega_n \end{cases}$$

converge to a distribution function $W(z)$ (with characteristic function $\phi(t)$) in the usual probabilistic sense. And by a classical result of probability theory we know that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\phi(t)|^2 = A \quad (0 \leqslant A \leqslant 1)$$

is equal to the sum of the squares of the jumps of $W(z)$. Since $|\phi(t)| = 1$ for all t, it is clear that $A = 1$ and that $W(z)$ consists of a single step. It follows that the sequence $\omega_1, \omega_2, \ldots$ converges to the point at which this step occurs.

Proof of Theorem 1. It is clear that the necessity of conditions 2 and 3 follows from the case $c = 0$ of Theorem 2.

We now assume that conditions (2) and (3) are satisfied, so that the first assertion of Theorem 2 holds. The characteristic function of the frequency (4) assumes the form

$$n^{-1} \sum_{m \leqslant n} e^{it(f(m) - c \log m - \zeta_n)}, \quad \zeta_n = \sum_{p \leqslant n} \frac{g'(p)}{p},$$

so that by Theorem 2 there exists a characteristic function $\phi(t)$ with

$$n^{-1} \sum_{m \leqslant n} e^{it(f(m) - c \log m)} = e^{it\zeta_n} \phi(t) + o(1), \quad (n \to \infty).$$

Since the right side is continuous at $t = 0$, by a standard theorem in the theory of probability this is equivalent to the assertion (1) of Theorem 1.

Suppose now that for some further sequence of numbers $\eta_1, \eta_2, \ldots$

$$\nu_n(m: f(m) - \eta_n < z)$$

converges to a limiting distribution with characteristic function $\Psi(t)$. Then

$$e^{it(\alpha_n - \eta_n)} \phi_n(t) \to \Psi(t), \quad \phi_n(t) \to \phi(t), \quad (n \to \infty),$$

the convergence being uniform on any bounded t-interval, so that in some neighborhood of the origin on which $\phi(t)$ does not vanish,

$$\lim_{n \to \infty} e^{it(\alpha_n - \eta_n)}$$

exists uniformly. From Lemma 1 it follows that

$$\eta_n = \alpha_n + \text{constant} + o(1), \quad (n \to \infty),$$

and this completes the proof of Theorem 1 except for the assertion concerning the (possible) continuity of the distribution function. The proof of this assertion is essentially included in the proof of Theorem 4, where the necessary and sufficient condition for continuity is shown to be

$$\text{either} \quad c \neq 0$$

$$\text{or} \quad c = 0, \text{ and } \sum_{g(p) \neq 0} \frac{1}{p} = \infty.$$

By means of the convergence of the series (6), this condition is readily seen to be equivalent to that given in the statement of Theorem 1.

We now prove the necessity of the conditions (5) and (6) of Theorem 2. To do this we need the following lemma.

LEMMA 2. *Let $h(m)$ be an additive function. For each real number t, we define the sum $S(n, t)$ by*

$$S(n, t) = n^{-1} \sum_{m \leqslant n} e^{ith(m)}.$$

Then if there exists a set of positive measure on which $S(n, t)$ does not converge to zero as $n \to \infty$, the function $h(m)$ has the form

$$h(m) = b \log m + u(m), \tag{7}$$

where

$$\sum_p \frac{(u'(p))^2}{p} < \infty, \tag{8}$$

and b is some constant.

Proof of Lemma 2. We recall the definition of a finitely distributed additive function as given by Erdös in [4]. This states that an additive function $h(m)$ is finitely distributed if there exist constants c_1 and c_2 and infinitely many integers n so that there exist $1 \leqslant a_1 < a_2 < \ldots < a_x \leqslant n$ so that $x > c_1 n$ and $|f(a_i) - f(a_j)| < c_2$.

In [6] it was shown that any finitely distributed additive function $h(m)$ must have the form given in (7) and (8). An inspection of the proof given there shows that the hypothesis of Lemma 2 yields the same conclusion.

Proof of Theorem 2 (Necessity). Let the frequency functions (4) possess a limiting distribution whose characteristic function is $\phi(t)$. The characteristic functions, $\phi_n(t)$, of these frequences have the form

$$\phi_n(t) = \int_{-\infty}^{\infty} e^{itz}\, d\nu_n(m: f(m) - c \log m - \alpha_n < z).$$

We adopt the notation of Lemma 2 with $h(m) = f(m) - c \log m$, and we obtain

$$\phi_n(t) = S(n, t)\, e^{it\alpha_n}.$$

Since $\phi_n(t) \to \phi(t)$, then $|S(n, t)| \to |\phi(t)|$. But $\phi(t)$ if a characteristic function, so it is non-zero in some neighborhood of the origin. It follows from Lemma 2 that

$$f(m) - c \log m = b \log m + u(m),$$

and this is a representation of the desired type.

For the proof of the sufficiency of the conditions (5) and (6) of Theorem 2 we require a further preliminary result.

LEMMA 3. *Let $h(m)$ be a strongly additive function for which the series $\sum_p (h^2(p)/p) < \infty$. Then the frequencies*

$$\nu_n\left(m: h(m) - \sum_{p \leqslant n} \frac{h(p)}{p} < z\right), \quad (n = 1, 2, \ldots),$$

converge to a limit law with characteristic function

$$\Psi(t)=\prod_p\left(1-\frac{1-e^{ith(p)}}{p}\right)e^{-it(h(p)/p)}.$$

Moreover, if the series $\sum_{h(p)\neq 0}(1/p)$ *converges, then the limit law is discrete; but if it diverges, then the limit law is continuous (either singular or absolutely continuous).*

Proof of Lemma 3. This lemma can be found as Theorem 4.4 of Kubilius' monograph [5].

Proof of Theorem 2 (Sufficiency). It will clearly suffice to prove that if an additive function $g(n)$ satisfies

$$\sum_p \frac{(g'(p))^2}{p}<\infty,$$

and if constants $\alpha_1, \alpha_2, \ldots$ are defined by

$$\alpha_n=\sum_{p\leqslant n}\frac{g'(p)}{p},$$

then the frequencies $\nu_n(m\colon g(m)-\alpha_n<z)$ have a limiting distribution. Accordingly, we define a strongly additive function $\gamma(m)$ by

$$\gamma(p)=\begin{cases} g'(p) & p>2\\ 0 & p=2.\end{cases}$$

Hence, we can apply Lemma 3 to $\gamma(m)$ to deduce that

$$\nu_n\left(m\colon \gamma(m)-\alpha_n+\frac{g'(2)}{2}<z\right)\to\Gamma(z), \tag{9}$$

as $n\to\infty$. The characteristic function of $\Gamma(z)$ has the form

$$\Psi_1(t)=\prod_p\left(1-\frac{1-e^{it\gamma(p)}}{p}\right)e^{-it(\gamma(p)/p)}.$$

We next define a *multiplicative* function $h(m)$ by

$$h(p^j)=e^{itg(p^j)}-e^{it\gamma(p)}\sum_{k=0}^{j-1}h(p^k),\quad j\geqslant 1,$$
$$h(1)=1.$$

It follows from the definition that

$$h(p) = e^{itg(p)} - e^{it\gamma(p)}.$$

We now show that for each value of t the series

$$\sum_{d=1}^{\infty} \frac{h(d)}{d} \tag{10}$$

is uniformly absolutely convergent. To this end we note that the following inequalities hold:

$$\begin{aligned} &|h(p^j)| \leqslant 2^j, \quad p \geqslant 3, \\ &|h(2^j)| \leqslant 2, \quad j = 1, 2, \dots. \end{aligned} \tag{11}$$

These are readily verified by induction noting that

$$h(2^j) = e^{itg(2^j)} - e^{itg(2^{j-1})}.$$

It follows that

$$\sum_{j=0}^{\infty} \frac{h(p^j)}{p^j} \left(1 + e^{it\gamma(p)} \sum_{l=1}^{\infty} p^{-l}\right) = \sum_{j=0}^{\infty} p^{-j} \left(h(p^j) + e^{it\gamma(p)} \sum_{k=0}^{j-1} h(p^k)\right) = \sum_{j=0}^{\infty} \frac{e^{itg(p^j)}}{p^j},$$

where the change in the order of summation is justified since the series involved are absolutely convergent. Hence,

$$\sum_{j=0}^{\infty} \frac{h(p^j)}{p^j} = \left(1 + \frac{e^{it\gamma(p)}}{p-1}\right)^{-1} \sum_{j=0}^{\infty} p^{-j} e^{itg(p^j)}.$$

Since $h(m)$ is multiplicative, for each positive integer n,

$$\sum_{d=1}^{\infty} \frac{|h(d)|}{d} \leqslant \prod_{p \leqslant n} \left(1 + \sum_{p^j \leqslant n} \frac{|h(p^j)|}{p^j}\right).$$

It will therefore be sufficient to prove that the double series

$$\sum_{p} \sum_{j=1}^{\infty} \frac{|h(p^j)|}{p^j}$$

converges. But for primes $p > 2$, we have

$$\sum_{j=1}^{\infty} \frac{|h(p^j)|}{p^j} \leqslant \left|\frac{e^{itg(p)} - e^{it\gamma(p)}}{p}\right| + \sum_{j=2}^{\infty} \left(\frac{2}{p}\right)^j = \left|\frac{e^{itg(p)} - e^{it\gamma(p)}}{p}\right| + \frac{4}{p(p-2)},$$

and

$$\sum_{j=1}^{\infty} \frac{|h(2^j)|}{2^j} \leqslant \frac{|e^{itg(2)} - 1|}{2} + 1.$$

Therefore,

$$\sum_{p}\sum_{j=1}^{\infty}\frac{|h(p^j)|}{p^j}\leqslant\frac{|e^{itg(2)}-1|}{2}+1+\sum_{\substack{p>2\\|g(p)|>1}}\frac{|e^{itg(p)}-e^{it\gamma(p)}|}{p}+\sum_{p>2}\frac{4}{p(p-2)}$$

$$\leqslant 2+2\sum_{|g(p)|\geqslant 1}\frac{1}{p}+\sum_{p>2}\frac{4}{p(p-2)}.$$

Hence, the series (10) is uniformly absolutely convergent. Moreover, it is convenient to note at this point that the same series is uniformly bounded for all real t.

We now note that the relation

$$e^{itg(m)}=\sum_{d|m}h(d)\,e^{it\gamma(m/d)}$$

holds, since it is true for prime powers and both sides of the equation are multiplicative. Thus, if $\phi_n(t)$ denote the characteristic functions

$$\phi_n(t)=\frac{1}{n}\sum_{m\leqslant n}e^{itg(m)},$$

then

$$\phi_n(t)=\frac{1}{n}\sum_{m\leqslant n}\sum_{d|m}h(d)\,e^{it\gamma(m/d)}=\frac{1}{n}\sum_{d\leqslant n}h(d)\sum_{m\leqslant(n/d)}e^{it\gamma(m)}. \tag{12}$$

We shall now show that the frequencies

$$\nu_n\left(m: g(m)-\sum_{p\leqslant n}\frac{g'(p)}{p}<z\right)$$

possess a limit law by showing that the corresponding characteristic functions

$$\phi_n(t)\,e^{-it\alpha_n}=\frac{1}{n}\sum_{m\leqslant n}e^{itg(m)-it\alpha_n}$$

converge. To do this we shall use the representation (12) and the convergence of the frequencies (9).

From the convergence of the frequencies (9) we see that

$$\sum_{m\leqslant n}e^{it\gamma(m)}=ne^{-it(g'(2)/2+\alpha_n)}\,\Psi_1(t)+o(n).$$

Now choose $\varepsilon>0$ and choose D so that

$$\sum_{d>D}\frac{|h(d)|}{d}<\varepsilon.$$

We divide the sum (12) into two parts according to whether $d \leqslant D$, or $d > D$. We estimate the first of these two sums by

$$e^{-it(g'(2)/2)}\,\Psi_1(t)\sum_{d\leqslant D}\frac{h(d)}{d}\,e^{it\alpha_{n/d}}+\frac{1}{n}\sum_{d\leqslant D}h(d)\,o\left(\frac{n}{d}\right),$$

and for all sufficiently large n,

$$\frac{1}{n}\sum_{d\leqslant D}\left|h(d)\,o\left(\frac{n}{d}\right)\right|<\varepsilon.$$

If $d \leqslant D$, an application of the Cauchy–Schwarz inequality shows that

$$\left|\alpha_n-\alpha_{n/d}\right|=\left|\sum_{(n/d)<p\leqslant n}\frac{g'(p)}{p}\right|\leqslant\left(\sum_{(n/d)<p\leqslant n}\frac{1}{p}\right)^2\left(\sum_{(n/d)<p\leqslant n}\frac{(g'(p))^2}{p}\right)=o(1)$$

(uniformly for $1 \leqslant d \leqslant D$) as $n \to \infty$, since the series (6) converges. Hence, for sufficiently large n,

$$\frac{1}{d}\sum_{d\leqslant D}h(d)\sum_{m\leqslant(n/d)}e^{it\gamma(m)}=e^{it(-(g'(2)/2)+\alpha_n)}\,\Psi(t)\sum_{d=1}^{\infty}\frac{h(d)}{d}+2\theta\varepsilon,$$

where $|\theta| \leqslant 1$.

The second of the two sums (corresponding to $D \leqslant d \leqslant n$) is estimated by

$$\left|\frac{1}{n}\sum_{D<d\leqslant n}h(d)\sum_{m\leqslant(n/d)}e^{it\gamma(m)}\right|=\left|\sum_{D<d\leqslant n}\frac{h(d)}{d}\left(\frac{n}{d}\right)^{-1}\sum_{m\leqslant(n/d)}e^{it\gamma(m)}\right|\leqslant\sum_{D<d}\frac{|h(d)|}{d}<\varepsilon.$$

We deduce that

$$\int_{-\infty}^{\infty}e^{itz}d\nu_n(m\colon g(m)-\alpha_n<z)\to e^{-it(g'(2)/2)}\,\Psi_1(t)\sum_{d=1}^{\infty}\frac{h(d)}{d}$$

as $n \to \infty$.

Since the right hand side is continuous at $t = 0$, then the frequencies

$$\nu_n\left(m\colon g(m)-\sum_{p\leqslant n}\frac{g'(p)}{p}<z\right)=\nu_n\left(m\colon f(m)-c\log m-\sum_{p\leqslant n}\frac{g'(p)}{p}<z\right)$$

possess a limit law. A straightforward calculation yields the characteristic function of this limit law:

$$\phi(t)=\prod_p\left(1-\frac{1}{p}\right)\left(1+\sum_{j=1}^{\infty}\frac{e^{itg(p^j)}}{p^j}\right)e^{-it(g'(p)/p)}.$$

In order to consider the (possible) continuity of the limit law we recall that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|\phi(t)|^2dt$$

equals the sum of the squares of the jumps of the distribution; and, hence, the distribution is continuous if and only if the previous limit is zero. We show that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|\phi(t)|^2\,dt=0 \tag{13}$$

if and only if

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|\Psi_1(t)|^2\,dt=0, \tag{14}$$

were $\phi(t)$ and $\Psi_1(t)$ are connected by the relation

$$\phi(t)=e^{it(g'(2)/2)}\,\Psi_1(t)\sum_{d=1}^{\infty}\frac{h(d)}{d}.$$

Since the infinite series (10) is uniformly bounded for all t, it follows that the truth of (14) implies that of (13). We now prove the opposite implication.

Choose a large prime q. From the inequalities (11), we see that there is a positive constant λ_1 depending at most on q so that

$$\left|\prod_{p\geqslant q}\left(\sum_{j=0}^{\infty}\frac{h(p^j)}{p^j}\right)\right|\geqslant\lambda_1.$$

For the odd primes $p<q$ we employ the estimate

$$\left|\prod_{3\leqslant p<q}\left(\sum_{j=0}^{\infty}\frac{h(p^j)}{p^j}\right)\right|=\left|\prod_{3\leqslant p<q}\left(1+\frac{e^{it\gamma(p)}}{p-1}\right)^{-1}\sum_{j=0}^{\infty}\frac{e^{itg(p^j)}}{p^j}\right|$$
$$\geqslant\prod_{3\leqslant p<q}\left(1+\frac{1}{p-1}\right)^{-1}\left(1-\frac{1}{p}\left(1-\frac{1}{p}\right)^{-1}\right)\geqslant\lambda_2,$$

where λ_2 is a positive constant.

We conclude that

$$\lambda_1\lambda_2|\Psi_1(t)|\leqslant 2\,|\phi(t)|\,\Big|\sum_{j=0}^{\infty}2^{-j}\,e^{itg(2^j)}\Big|^{-1},$$

whenever the right hand side is defined. We may suppose that $g(2)\neq 0$; for if not, then

$$\Big|\sum_{j=0}^{\infty}2^{-j}\,e^{itg(2^j)}\Big|\leqslant 1,$$

and (14) would follow immediately from (13).

Let ε satisfy

$$0<\varepsilon<\frac{\pi}{4|g(2)|},$$

and define the set E by

$$E = \left\{ t: \left| t - \frac{\pi(2l+1)}{g(2)} \right| \leqslant \varepsilon;\ l = 0, \pm 1, \pm 2, \ldots \right\}.$$

If $t \notin E$, then

$$\operatorname{Re}\{1 + e^{itg(2)}\} = 2\cos^2\left(\frac{tg(2)}{2}\right) \geqslant 2\cos^2\left(\frac{\pi}{2}(2l+1) + \frac{\varepsilon g(2)}{2}\right) = 2\sin^2\left(\frac{\varepsilon g(2)}{2}\right) = \lambda_3 > 0.$$

Hence
$$\operatorname{Re}\left\{\sum_{j=0}^{\infty} 2^{-j} e^{itg(2^j)}\right\} \geqslant \tfrac{1}{2}\operatorname{Re}\{1 + e^{itg(2)}\} + \tfrac{1}{2} - \sum_{j=2}^{\infty} 2^{-j} \geqslant \lambda_3,$$

Therefore

$$\frac{1}{T}\int_0^T |\Psi_1(t)|^2 dt = \frac{1}{T}\int_{t\in E\cap[0,T]} |\Psi_1(t)|^2 dt + \frac{1}{T}\int_{t\in[0,T]-E} |\Psi_1(t)|^2 dt$$
$$\leqslant T^{-1}\mu([0,T]\cap E) + 4(\lambda_1\lambda_2\lambda_3)^{-2} T^{-1}\int_0^T |\phi(t)|^2 dt.$$

Since
$$\mu([0,T]\cap E) \leqslant \frac{2T|g(2)|\varepsilon}{\pi},$$

we have
$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T |\Psi_1(t)|^2 dt \leqslant \frac{2|g(2)|\varepsilon}{\pi}.$$

As $\varepsilon > 0$ can be taken arbitrarily small, (14) holds. It follows from Lemma 3 that the limiting distribution of the frequencies (4) are continuous if and only if $\sum_{g(p)\neq 0} 1/p$ diverges. This finishes the proof of Theorem 2.

Section 2

We see from Lemma 2 that an additive function is finitely distributed if $S(n, t)$ fails to converge to zero on a set of positive measure. It is natural, therefore, to inquire what properties of $f(m)$ correspond to the converse proposition; namely, that $S(n, t) \to 0$ in measure as $n \to \infty$. We consider this question in Theorem 3. Moreover, as a corollary to this theorem, we give a necessary condition that real numbers $\beta_1, \beta_2, \ldots$ exist so that the frequences $\nu_n(m: \beta_n f(m) - \alpha_n < z)$ possess a limiting distribution.

THEOREM 3. *The following three conditions are equivalent:*

(i) *For any $\delta > 0$, $F_n(z+\delta) - F_n(z) \to 0$ uniformly for all real z as $n \to \infty$.*

(ii)
$$\int_{-T}^{T} |S(n,t)|\, dt \to 0$$

for all real T as $n \to \infty$.

(iii)
$$\limsup_{T\to\infty}\limsup_{n\to\infty}\int_{-T}^{T}|S(n,t)|\,dt=0.$$

Proof. ((i) ⇒ (ii))

We employ the representation (see [1]),

$$\frac{1}{n^2}\sum_{\substack{k_1,\,k_2\leqslant n\\ |f(k_1)-f(k_2)|\leqslant 1}}(1-|f(k_1)-f(k_2)|)=\int_{-\infty}^{\infty}|S(n,t)|^2\left(\frac{\sin \pi t}{\pi t}\right)^2 dt. \tag{15}$$

From the hypothesis (i) with $\delta=1$, we see that

$$\frac{1}{n}\sum_{\substack{k_1=1\\ z\leqslant f(k_1)<z+1}}^{n}1\to 0\quad (n\to\infty)$$

uniformly in z. Applying the last result with $z=f(1), f(2), \dots, f(n)$ in turn, we obtain

$$\frac{1}{n^2}\sum_{\substack{k_1,\,k_2\leqslant n\\ |f(k_1)-f(k_2)|\leqslant 1}}1\to 0\quad (n\to\infty). \tag{16}$$

It follows from (15) and (16) that for all real T,

$$\lim_{n\to\infty}\int_{-T}^{T}|S(n,t)|\,dt=0,$$

which is (ii).

((ii)) ⇒ (iii)). The proof is immediate.

((iii) ⇒ (i)).

We again appeal to an integral representation:

$$\frac{1}{n}\sum_{\substack{k\leqslant n\\ |f(k)-y|<1}}(1-|f(k)-y|)=\int_{-\infty}^{\infty}S(n,t)\,e^{-iyt}\left(\frac{\sin \pi t}{\pi t}\right)^2 dt. \tag{17}$$

From the inequalities

$$\left|\int_{-\infty}^{\infty}S(n,t)\,e^{-iyt}\left(\frac{\sin \pi t}{\pi t}\right)^2 dt\right|\leqslant\int_{-\infty}^{\infty}|S(n,t)|\left(\frac{\sin \pi t}{\pi t}\right)^2 dt\leqslant\int_{-T}^{T}|S(n,t)|^2\,dt+\frac{2}{\pi^2 T},$$

we deduce that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}S(n,t)\,e^{-iyt}\left(\frac{\sin \pi t}{\pi t}\right)^2 dt\leqslant\limsup_{T\to\infty}\limsup_{n\to\infty}\int_{-T}^{T}|S(n,t)|\,dt=0,$$

by hypothesis (iii). Therefore, for each y, the left hand side of (17) tends to zero as $n\to\infty$. Noting that

$$\frac{1}{2n}\sum_{\substack{k\leqslant n\\ |f(k)-y|<\frac{1}{2}}} 1 \leqslant \sum_{\substack{k\leqslant n\\ |f(k)-y|<1}} (1-|f(k)-y|)\to 0$$

as $n\to\infty$, we may apply this result with $y=z,\ z+1, \ldots, z+[\delta]+1$ in turn to obtain

$$\frac{1}{n}\sum_{\substack{k\leqslant n\\ z\leqslant f(k)<z+\delta}} 1 \leqslant 2\sum_{y=0}^{[\delta]+1}\frac{1}{n}\sum_{\substack{k\leqslant n\\ |f(k)-y|<\frac{1}{2}}} 1 \to 0;$$

that is

$$\lim_{n\to\infty}(F_n(z+\delta)-F_n(z))=0,$$

which is condition (i). This establishes Theorem 3.

COROLLARY. *Let $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ be two real sequences such that the frequences $\nu_n(m\colon \beta_n f(m)-\alpha_n<z)$ have a limiting distribution function $F(z)$. Then, either, $\lim_{n\to\infty}\beta_n=0$, or we may choose β_n identically equal to* 1.

Proof. Assume that $\limsup_{n\to\infty}|\beta_n|=c_1>0$. Then there is a subsequence $\beta_{n_1}, \beta_{n_2}, \ldots$ for which either $\beta_{n_j}\geqslant c_1/2$ or $\beta_{n_j}\leqslant -c_1/2$. We assume first that the former case holds.

Since $F(z)$ is a distribution function, there exist real numbers z and δ so that $F(z+\delta)-F(z)=c_2>0$. For this choice of z and δ, we have for all sufficiently large n

$$\nu_n(m\colon \beta_n f(m)-\alpha_n<z+\delta)-\nu_n(m\colon \beta_n f(m)-\alpha_n<z)\geqslant c_2/2.$$

Consequently, for all sufficiently large n in the subsequence $n_1, n_2, \ldots$ we have

$$F_n\left(\frac{z+\alpha_n}{\beta_n}+\frac{2\delta}{c_1}\right)-F_n\left(\frac{z+\alpha_n}{\beta_n}\right)\geqslant\frac{c^2}{2}.$$

Thus, with z replaced by $(z+\alpha_n)\beta_n^{-1}$ $(n=n_j)$ we see that condition (i) of Theorem 3 fails. It follows that condition (ii) also fails, in which case $f(m)$ has the form (7) and (8) by Lemma 2. Hence, we may choose the $\beta_n=1$ by Theorem 1.

Similarly, if there is a subsequence $\beta_{n_1}, \beta_{n_2}, \ldots$ so that $\beta_{n_j}\leqslant -(c_1/2)$, we can deduce the same conclusion. Q.e.d.

Section 3

Suppose that the distribution functions $F_n(z)$ converge to a continuous distribution $F(z)$. Then for each z

$$\lim_{\delta\to 0+}(F(z+\delta)-F(z))=0. \tag{18}$$

A theorem of P. Lévy [5] states that in the present circumstances, this can occur if and only if the series $\sum_{f(p)\neq 0}1/p$ diverges. Even when the functions $F_n(z)$ do not converge, a meaning can be given to this result, provided that we replace the condition (18) by

$$\limsup_{\delta\to 0+}\ \limsup_{n\to\infty}\ (F_n(z+\delta)-F_n(z))=0, \tag{19}$$

uniformly in z.

Note that a distribution function is continuous if and only if it is uniformly continuous, so that the last condition is a natural generalization of (18).

The sufficiency of Lévy's condition was proved by Erdös (Theorem IV [4]) subject to $|f(p)|\leqslant B$, and formulated in the following different manner:

THEOREM IV (*Erdös*). *Let $f(m)$ be a (strongly) additive function such that* (19) *holds. Then to every $\varepsilon>0$, there exists a $\delta>0$ such that if $a_1<a_2<\ldots<a_x\leqslant n$ is a sequence of integers with $|f(a_i)-f(a_j)|<\delta$, then $x<\varepsilon n$ for n sufficiently large.*(1)

Accordingly, we state the following theorem.

THEOREM 4. *The following two conditions are equivalent:*

$$\limsup_{\delta\to 0+}\ \limsup_{n\to\infty}\ (F_n(z+\delta)-F_n(z))=0,$$

uniformly in z, and

$$\sum_{f(p)\neq 0}\frac{1}{p}=\infty. \tag{20}$$

The content of this theorem was succintly stated by Erdös ([4], p. 17) in the following surrealistic manner: "If $\sum_{f(p)\neq 0}1/p=\infty$, the distribution function tries to be continuous whether it exists or not."

Proof of Theorem (Necessity). We assume to the contrary that

$$\sum_{f(p)\neq 0}\frac{1}{p}<\infty.$$

Let $\mathfrak{p}$ denote the set of primes for which $f(p)\neq 0$. Then a simple application of the sieve of Eratosthenes shows that the number of integers in the interval $1\leqslant m\leqslant n$ which are not divisible by any prime in $\mathfrak{p}$ is equal to

$$(1+o(1))\,n\prod_{p\in\mathfrak{p}}\left(1-\frac{1}{p}\right).$$

On each of these integers $f(m)=0$. Taking $z=-\delta/2$, we see that

$$\limsup_{\delta\to 0+}\ \limsup_{n\to\infty}\ (F_n(\delta/2)-F_n(-\delta/2))\geqslant\prod_{p\in\mathfrak{p}}\left(1-\frac{1}{p}\right)>0,$$

which contradicts the hypothesis.

(1) We note the obvious misprint in the statement of this theorem on p. 2 of [4] in which the rôles of ε and δ have somehow been reversed.

We now prove the sufficiency. Hence, we assume that (20) holds. Suppose that (19) fails. Then there exists a decreasing sequence $\delta_1 \geqslant \delta_2 \geqslant \ldots > 0$ and a sequence $z_1, z_2, \ldots$ (where $z_k = z_k(\delta_k)$) such that

$$\limsup_{n\to\infty} \; (F_n(z_k+\delta_k) - F_n(z_k)) \geqslant \gamma > 0.$$

Thus, we obtain a sequence of integers $n_1 < n_2 < \ldots$ so that for n_l sufficiently large,

$$F_{n_l}(z_k+\delta_k) - F_{n_l}(z_k) \geqslant \frac{\gamma}{2}.$$

It follows that the intervals $1 \leqslant m \leqslant n_l$ contain at least $(\gamma/2)n_l$ integers a_i on which

$$|f(a_i) - f(a_j)| < \delta_k \leqslant \delta_1, \tag{21}$$

and so $f(m)$ is finitely distributed. Therefore, $f(m)$ has the form

$$f(m) = c \log m + g(m), \tag{22}$$

where
$$\sum_p \frac{(g'(p))^2}{p} < \infty.$$

If $c=0$, then $f(m)=g(m)$; and it follows from Theorem 2 that the frequencies

$$\nu_n\left(m: f(m) - \sum_{p\leqslant n} \frac{f'(p)}{p} < z\right)$$

have a *continuous* limiting distribution since $\sum_{f(p)\neq 0} 1/p = \infty$. Therefore, (21) cannot hold, and we may suppose that $c \neq 0$ hence forth.

For convenience we set

$$\phi_n(t) = \frac{1}{n}\sum_{m\leqslant n} e^{itf(m)}.$$

It follows from Theorem 2 and from the fact that $f(m)$ has the form (22), that there is a characteristic function $\Psi(t)$ so that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m\leqslant n} e^{itg(m)} \cdot \exp\left(-it\sum_{p\leqslant n}\frac{g'(p)}{p}\right) = \Psi(t).$$

Integrating by parts, using the fact that for any fixed ε, $0<\varepsilon<1$,

$$\sum_{p\leqslant m}\frac{g'(p)}{p} = (1+o(1))\sum_{p\leqslant n}\frac{g'(p)}{p}$$

uniformly for $\varepsilon n \leqslant m < n$ as $n\to\infty$, we obtain

$$\phi_n(t) = (1+o(1))\,\frac{n^{itc}}{1+itc}\,\Psi(t)\exp\left(it\sum_{p\leqslant n}\frac{g'(p)}{p}\right),$$

uniformly for any compact interval of t values.

Employing the representation

$$\int_{-\infty}^{\infty}\phi_n\left(\frac{t}{2\delta_k}\right)e\left(-\frac{tz_k}{2\delta_k}\right)\left(\frac{\sin\pi t}{\pi t}\right)^2 dt = \frac{1}{n}\sum_{m\leqslant n}\left(1-\left|\frac{f(m)-z_k}{2\delta_k}\right|\right) \qquad \left|\frac{f(m)-z_k}{2\delta_k}\right|\leqslant 1$$

for $n=n_1, n_2, \dots,$ we see that the proportion of a_i in the interval $[1, n]$ which satisfy (21) does not exceed

$$2\int_{-\infty}^{\infty}\phi_n\left(\frac{t}{2\delta_k}\right)e\left(-\frac{tz_k}{2\delta_k}\right)\left(\frac{\sin\pi t}{\pi t}\right)^2 dt.$$

Set $T=\delta_k^{-2}$. We have the estimates

$$2\int_{|t|\geqslant T\delta_k}\left|\phi_n\left(\frac{t}{2\delta_k}\right)e\left(-\frac{tz_k}{2\delta_k}\right)\left(\frac{\sin\pi t}{\pi t}\right)^2 dt\right| \leqslant 4\int_{T\delta_k}^{\infty}\frac{dt}{(\pi t)^2}\leqslant\frac{4\delta_k}{\pi^2},$$

and (for all sufficiently large n taken from the sequence $n_1, n_2, \dots$)

$$\left|2\int_{|t|\leqslant T\delta_k}\left(\frac{t}{2\delta_k}\right)e\left(-\frac{tz_k}{2\delta_k}\right)\left(\frac{\sin\pi t}{\pi t}\right)^2 dt\right| = \left|2\delta_k\int_{-T}^{T}\phi_n(u)\,e\left(-\frac{uz_k}{2}\right)\left(\frac{\sin\pi u\,\delta_k}{\pi u\delta_k}\right)^2 du\right.$$

$$\leqslant 2\,\delta_k(1+o(1))\int_{-T}^{T}\frac{du}{1+|u|}\leqslant 10\,\delta_k\,|\log\delta_k|.$$

Combining these last two inequalities we see that

$$0<\gamma\leqslant\limsup_{n\to\infty}\,(F_n(z_k+\delta_k)-F_n(z_k))\leqslant\frac{4\delta_k}{\pi^2}+10\,\delta_k\,|\log\delta_k|,$$

which cannot hold if δ_k is sufficiently small. This completes the proof of the sufficiency when $c\neq 0$, and the proof of Theorem 4.

It is clear that the assertion concerning the continuity of the limiting distribution in Theorem 1 can be proved in exactly the above manner.

Suppose that we again weaken the requirement that the distribution functions $F_n(z)$ have a limiting distribution, and assume only that

$$\limsup_{z\to\infty}\,\limsup_{n\to\infty}\,(F_n(z)-F_n(-z))=1. \tag{23}$$

This condition is certainly true when the $F_n(z)$ have a limiting distribution; and it is natural to ask whether there exist necessary and sufficient conditions, corresponding to the Erdös-

Wintner criteria (16) and (17) when $F_n(z) \to F(z)$, for (14) to hold. The following theorem establishes such conditions.

THEOREM 5. *The following two propositions are equivalent:*

PROPOSITION 1. $$\limsup_{z\to\infty} \limsup_{n\to\infty} \nu_n(m: |f(m)| < z) = 1.$$

PROPOSITION 2. (i) $$\liminf_{n\to\infty} \left| \sum_{p\leqslant n} \frac{f'(p)}{p} \right| < \infty$$

(ii) $$\sum_p \frac{(f'(p))^2}{p} < \infty.$$

We shall need several times the well-known

LEMMA 4 (Turán). *Let $h(m)$ be a strongly additive function. For each positive integer m set*

$$E(n) = \sum_{p\leqslant n} \frac{h(p)}{p}$$

and
$$D^2(n) = \sum_{p\leqslant n} \frac{h^2(p)}{p}.$$

Then the following inequality is satisfied for some positive constant C

$$\sum_{m\leqslant n} |h(m) - E(n)|^2 \leqslant Cn\, D^2(n).$$

Proof of Lemma. This result was first proved for real-valued functions by Turán. It was generalized by Kubilius to complex valued additive functions (see Lemma 3.1 of [5]).

Proof of Theorem (Prop. 1 ⇒ Prop. 2). We deduce from Proposition 1 that there exists a z_0 so that

$$\limsup_{n\to\infty} \nu_n(m: |f(m)| < z_0) \geqslant \tfrac{1}{2}.$$

Therefore, $f(m)$ is finitely distributed; and we can put

$$f(m) = c \log m + g(m), \tag{24}$$

where
$$\sum_p \frac{(g'(p))^2}{p} < \infty. \tag{25}$$

Let $\mathfrak{P} = \{p: |g(p)| > 1\}$, and set

$$\alpha = \prod_{p\in\mathfrak{P}} \left(1 - \frac{1}{p}\right).$$

Hence, either $\mathcal{D}$ is empty or $0<\alpha<1$. For the remainder of the proof we shall assume that $0<\alpha<1$, as the former assumption involves no additional difficulties.

We now show that $c=0$. It will then follow from (24) and (25) that (ii) holds.

Choose $\varepsilon>0$ so small that $\alpha+5\varepsilon<1$. Then by Proposition 1, there exists a z_1 so that

$$\liminf_{n\to\infty} \nu_n(m: |f(m)|\geq z_1)<\varepsilon.$$

We may therefore choose a sequence $n_1<n_2<\ldots$ so that

$$\nu_{n_j}(m: |f(m)>z_1)\leq 2\varepsilon \quad (j=1, 2, \ldots). \tag{26}$$

Let Σ' indicate a summation over integers not divisible by any prime in $\mathcal{D}$. (Note that the number of such positive integers not exceeding n_j is $(1+o(1))(1-\alpha)n_j$.)

Applying Lemma 4 to the strongly additive function $h(m)$ defined by $h(p)=g'(p)$, we have

$$\sum_{m\leq n_j}|g(m)-E(n_j)|^2=\sum_{m\leq n_j}{}'|h(m)-E(n_j)|^2\leq \sum_{m\leq n_j}|h(m)-E(n_j)|^2\ll n_j D^2(n_j), \tag{27}$$

where
$$D^2(n_j)\ll \sum_p \frac{(g'(p))^2}{p}\ll 1, \quad (j=1,2,\ldots).$$

Next, by an application of the Cauchy–Schwarz inequality, we see that

$$E^2(n_j)\leq \sum_{p\leq n_j}\frac{(g'(p))^2}{p}\sum_{p\leq n_j}\frac{1}{p}\ll \log\log n_j. \tag{28}$$

Combining inequalities (27) and (28) yields

$$\sum_{m\leq n_j}{}' g^2(m)\leq 2\sum_{m\leq n_j}{}'|g(m)-E(n_j)|^2+2\sum_{m\leq n_j}|E(n_j)|^2\ll n_j\log\log n_j. \tag{29}$$

We now suppose $c\neq 0$ and obtain a contradiction.

By (26), for all but $2\varepsilon n_j$ of the integers $m\leq n_j$, we have

$$|g(m)|=|-c\log m+f(m)|\geq |c|\log m-z_1.$$

Hence, for all but $3\varepsilon n_j$ of these integers, with n_j sufficiently large,

$$|g(m)|\geq \frac{|c|}{2}\log m.$$

Now, the number of $m\leq n_j$ which are not divisible by any prime in $\mathcal{D}$, and for which $|g(m)|\geq(|c|/2)\log m$, exceeds

$$\Big(\sum_{m\leq n_j}{}' 1\Big)-3\varepsilon n_j=(1+o(1))(1-\alpha)n_j-3\varepsilon n_j\geq(1-\alpha-4\varepsilon)n_j>\varepsilon n_j,$$

where these inequalities hold for all sufficiently large n_j. Thus, for all such n_j,

$$\sum_{m\leqslant n_j}' g^2(m) \geqslant \left(\frac{|c|}{2}\right)^2 \sum_{m\leqslant \varepsilon n_j} \log^2 m \gg n_j \log^3 n_j,$$

and the last inequality contradicts (29) when n_j is sufficiently large. Therefore, $c=0$, and (ii) holds.

To show that (i) holds, we restate (27) in the form

$$\sum_{m\leqslant n_j}' |f(m)-E(n_j)|^2 \ll n_j,$$

since now $f(m)=g(m)$. In the above sum there are at least $(1-\alpha-3\varepsilon)n_j$ integers $m\leqslant n_j$ for which $|f(m)|\leqslant z_1$; and so

$$(1-\alpha-3\,\varepsilon)\,n_j E^2(n_j) \leqslant 2 \sum_{m\leqslant n_j}' |E(n_j)-f(m)|^2 + 2 \sum_{\substack{m\leqslant n_j\\ |f(m)|\leqslant z_1}}' |f(m)|^2 \ll n_j.$$

Hence, $E^2(n_j)\ll 1$, which proves (i).

(Prop. 2 $\Rightarrow$ Prop. 1)

Let $\mathcal{D}=\{p_i\colon |f(p_i)|>1\}$. It follows from (ii) that

$$\sum_{p_i\in\mathcal{D}} \frac{1}{p_i} < \infty.$$

In terms of the set $\mathcal{D}$ we define a strongly additive function $h(m)$ by

$$h(p)=\begin{cases} f(p) & \text{if } p\notin\mathcal{D}\\ 1 & \text{otherwise.}\end{cases}$$

With this definition of $h(p)$, the condition that $\Sigma_p(h^2(p)/p)<\infty$ is precisely the condition (ii). It then follows from Lemma 3 that if $\varepsilon>0$, there exists a z so that for all sufficiently large n,

$$\nu_n\left(m\colon \left|h(m)-\sum_{p\leqslant n}\frac{h(p)}{p}\right|<z\right)\geqslant 1-\varepsilon.$$

The condition (ii) guarantees the existence of a positive number B and a sequence $n_1<n_2<\dots$ of integers so that $|A(n_j)|\leqslant B$, where we have put

$$A(n)=\sum_{p\leqslant n}\frac{h(p)}{p}.$$

We deduce that for all sufficiently large n_j,

$$\nu_{n_j}(m\colon |h(m)|<z+B)\leqslant 1-\varepsilon.$$

Let r be a positive integer. It is clear that for each positive integer n,

$$\nu_n(m\colon |f(m)| < z + B + \sum_{i=1}^{r} (1 + |f(p_i)|)) \geqslant \nu_n(m\colon |h(m)| < z + B) - \nu_n(m\colon p_i \mid m \text{ for some } i > r),$$

and the last frequency does not exceed

$$\frac{1}{n} \sum_{i=r+1}^{\infty} \left[\frac{n}{p_i}\right] \leqslant \sum_{i=r+1}^{\infty} \frac{1}{p_i}.$$

Using this last inequality with $n = n_1, n_2, \ldots$ in turn, we obtain

$$\lim_{z\to\infty} \limsup_{n\to\infty} \nu_n(|f(m)| < z + B + \sum_{i=1}^{r} (1 + |f(p_i)|)) \geqslant 1 - \varepsilon - \sum_{i=r+1}^{\infty} \frac{1}{p_i}.$$

Since $\varepsilon > 0$ and the positive integer r are otherwise arbitrary, we deduce that

$$\lim_{z\to\infty} \limsup_{n\to\infty} \nu_n(|f(m)| < z) = 1,$$

and this completes the proof of Theorem 5.

References

[1]. Davenport, H., *Analytic methods for Diophantine equations and inequalities*. Ann Arbor Publishers, 1962.

[2]. Erdös, P., On the density of some sequences of numbers. III. *J. London Math. Soc.*, 13 (1938), 119–127.

[3]. —— Additive arithmetical functions and statistical independence. *Amer. J. Math.*, 61 (1939), 713–721.

[4]. —— On the distribution function of additive functions. *Ann. of Math.*, 47 (1946), 1–20.

[5]. Kubilius, J., *Probabilistic methods in the theory of numbers*. Translations of mathematical monographs, No. 11, Providence, 1964.

[6]. Ryavec, C., A characterization of finitely distributed additive functions. *J. Number Theory* (to appear).

[7]. Turán, P., On a theorem of Hardy and Ramanujan. *J. London Math. Soc.*, 9 (1934), 274–276.

[8]. —— Über einige Verallgemeinerungen eines Satzes von Hardy und Ramanujan. *J. London Math. Soc.*, 11 (1936), 125–133.

Received June 7, 1970.