THE (φ, k) RECTIFIABLE SUBSETS OF A HOMOGENEOUS SPACE

BY

JOHN E. BROTHERS

Indiana University, Bloomington, Ind., U.S.A. (1)

1. Introduction

In recent years much of the progress which has been made in geometric measure theory has depended on knowledge of the geometric structure of subsets of n dimensional Euclidean space $\mathbf{R}^n$ relative to some measure such as the k dimensional Hausdorff measure. For example, the proof in [8] of the existence of solutions for the least area problem (Plateau's problem) and the minimal surface problem depends essentially on this structure theory.

Central to the structure theory is the characterization of rectifiable subsets in terms of their projection properties. Such results were obtained first by Besicovitch in [1] for one dimensional Hausdorff measure in the plane, then by Federer in [3] for general measures in $\mathbf{R}^n$. Our goal in the present paper is to give global generalizations of these theorems to measures in a manifold X with a transitive group of diffeomorphisms G.

In order to make the transition from $\mathbf{R}^n$ to X it is necessary to restate the projection properties of a subset A of $\mathbf{R}^n$ without reference to projections. We do this by replacing orthogonal projections of A into $\mathbf{R}^k$ with intersections $A \cap g(P)$, where g is an isometry of $\mathbf{R}^n$ and P is a fixed $n-k$ dimensional plane. For example, the statement "$p(A)$ has Lebesgue measure zero for almost all orthogonal projections p: $\mathbf{R}^n \to \mathbf{R}^k$" is equivalent to "$A \cap g(P)$ is empty for almost all isometries g". Thus in studying subsets of X we are led to consider intersections $A \cap g(B)$, where $g \in G$ and B is a fixed $n-k$ dimensional smooth submanifold of X. The main general results are in § 5; they include as a special case a new characterization of rectifiable subsets of $\mathbf{R}^n$. The proof of the key lemma 3.7 reduces to a new proof of the corresponding lemma [3, 7.3] or [7, 3.3.4] for the case where B is a plane.

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In § 6 we define a generalized integralgeometric measure with respect to B for the case where X is a Riemannian manifold of constant curvature with isometry group G. The integralgeometric formulas of [2] are used to verify that the relationship between this measure and k dimensional Hausdorff measure is the same as that obtained by Federer in [3] for the classical integralgeometric measure in $\mathbf{R}^n$.

2. Preliminaries

The purpose of this section is to fix basic notation and terminology; more details may be found in references such as [7] and [9].

2.1. *Notation.* Throughout this paper X will be an n dimensional separable Riemannian manifold of class ∞. G will be a separable, $m+n$ dimensional Lie group of transformations of X which acts transitively on X. Set $e=\mathbf{1}_X$, the *identity map* of X.

One denotes by L_g and R_g the *left* and *right translations* of G by $g\in G$. Let Ψ be a left invariant Haar measure on G.

Let f: $G\times G\to G$ be such that $f(a, b)=ab^{-1}$.

Fix an origin $o\in X$ and define

$$\pi: G\to X \quad \text{by} \quad \pi(g)=g(o).$$

If $S\subset X$, let $S'=\pi^{-1}(S)$. The isotropy subgroup $I=\{o\}'$ is a closed, m dimensional Lie subgroup of G, and hence has the relative topology.

Let B be a proper $n-k$ dimensional submanifold of class 1 of X, $0<k<n$.

Let φ be a non-negative measure on X such that closed sets are φ measurable.

2.2. *Tangent space.* If M is an l dimensional manifold of class 1 and $x\in M$, then $\mathbf{T}_x(M)$ is the l dimensional real vector space of *tangent vectors* of M at x.

2.3. *Exterior Algebra.* For each finite dimensional vector space V and $l=0, 1, \ldots,$ dim V, $\Lambda_l(V)$ is the associated space of l *vectors* (contravariant skewsymmetric tensors of rank l). Furthermore,

$$\Lambda_*(V)=\bigoplus_{l=0}^{\dim V}\Lambda_l(V)$$

is the corresponding exterior algebra, with exterior multiplication $\wedge$.

Each inner product on V, with the corresponding norm $|\ \ |$, induces an inner product on $\Lambda_*(V)$ with norm also denoted by $|\ \ |$. An orthonormal basis for $\Lambda_*(V)$ is obtained by exterior multiplication from an orthonormal basis for V.

2.4. *Differential.* Suppose M and N are manifolds of class 1 and f: $M\to N$. If $x\in M$, $y=f(x)$ and f is differentiable at x, the differential of f at x is a linear transformation

$$f_{\#}(x)\colon \mathbf{T}_x(M) \to \mathbf{T}_y(N);$$

$f_{\#}(x)$ can be extended to a unique algebra homomorphism

$$f_{\#}(x)\colon \Lambda_*[\mathbf{T}_x(M)] \to \Lambda_*[\mathbf{T}_y(N)].$$

If M and N are Riemannian manifolds and $r = \inf\{\dim M, \dim N\}$, then the *Jacobian* of f at x is

$$Jf(x) = \sup\{|f_{\#}(x)(v)|\colon v \in \Lambda_r[\mathbf{T}_x(M)],\ |v| = 1\}.$$

2.5. *Definition.* $\mathbf{R}^n$ is the *n dimensional Euclidean space* consisting of all sequences $x = (x^1, \ldots, x^n)$ of real numbers, with the metric

$$x \cdot y = \sum_{i=1}^{n} x^i y^i \quad \text{for} \quad x, y \in \mathbf{R}^n.$$

$e_1, \ldots, e_n$ are the *standard orthonormal basis vectors* of $\mathbf{R}^n$. If M is a linear subspace of $\mathbf{R}^n$, then $M^{\perp}$ is the *orthogonal complement* of M.

For $x \in \mathbf{R}^n$ one identifies $\mathbf{T}_x(\mathbf{R}^n)$ with $\mathbf{R}^n$.

$\alpha(n)$ is the volume of the unit ball $\mathbf{R}^n \cap \{x\colon |x| < 1\}$.

2.6. *Definition.* H^l is the *l dimensional Hausdorff measure.* If S is a subset of a metric space Z, then $H^l(S)$ equals the limit, as $r \to 0+$, of the infimum of the sums

$$\sum 2^{-l}\alpha(l)\,(\operatorname{diam} u)^l, \quad u \in U,$$

corresponding to all countable coverings U of S such that $(\operatorname{diam} u) < r$ for $u \in U$.

It follows that if $H^0(S) < \infty$, then $H^0(S)$ is the cardinal number of S.

2.7. *Suslin sets.* The family of Suslin (analytic) subsets of X contains the Borel subsets of X and has the following properties [7, § 2.2]:

Each Suslin set is φ measurable.

If F is a countable, nonempty family of Suslin sets, then $\bigcup F$ and $\bigcap F$ are Suslin sets.

If Y is a manifold and $f\colon X \to Y$ is continuous, then $f(S)$ and $f^{-1}(T)$ are Suslin sets whenever S and T are Suslin subsets of X and of Y, respectively.

2.8. *Definition.* If μ measures Y and $A \subset Y$, then $\mu \llcorner A$ is the measure on Y defined by the formula

$$\mu \llcorner A(S) = \mu(A \cap S) \quad \text{for } S \subset Y.$$

If $f\colon Y \to Z$, then $f_{\#}(\mu)$ is the measure on Z defined by the formula

$$f_{\#}(\mu)(S) = \mu[f^{-1}(S)] \quad \text{for } S \subset Z.$$

2.9. *Definition.* $R \subset X$ is *k rectifiable* if there exists a Lipschitzian function mapping some bounded subset of $\mathbf{R}^k$ onto R.

$R \subset X$ is *countably k rectifiable* if R is the union of a countable family of k rectifiable sets.

$E \subset X$ is *countably* (φ, k) *rectifiable* if there exists a countably k rectifiable set R with $\varphi(E \sim R) = 0$.

$E \subset X$ is (φ, k) *rectifiable* if E is countably (φ, k) rectifiable and $\varphi(E) < \infty$.

$E \subset X$ is *purely* (φ, k) *unrectifiable* if E contains no k rectifiable set R with $\varphi(R) > 0$.

2.10. *Notation.* Let Y be a metric space and $y \in Y$. For $r > 0$ we denote

$$K_{y,r} = K^Y_{y,r} = Y \cap \{x : \operatorname{dist}(x, y) < r\}.$$

If $r > 0$, $s > 0$ and $Y \subset X$, then

$$\mathbf{X}(y, r, Y, s) = X \cap \{x : \operatorname{dist}(x, Y) < s \operatorname{dist}(x, y)\} \cap K_{y,r}.$$

If $A \subset X$, then $S_{A,1}$ is the set of $(a, b) \in G \times B'$ such that for some $\delta > 0$,

$$\lim_{s \to 0^+} \sup_{0 < r < \delta} \varphi(A \cap \mathbf{X}[\pi(a), r, ab^{-1}(B), s])\, r^{-k} s^{-k} = 0;$$

$S_{A,2}$ is the set of $(a, b) \in G \times B'$ such that for all $\delta > 0$,

$$\limsup_{s \to 0^+} \sup_{0 < r < \delta} \varphi(A \cap \mathbf{X}[\pi(a), r, ab^{-1}(B), s])\, r^{-k} s^{-k} = \infty;$$

$$S_{A,3} = G \times B' \cap \{(a, b) : \pi(a) \in \operatorname{cl}[A \cap ab^{-1}(B) \sim \{\pi(a)\}]\}.$$

3. The local structure of a set

Let $O(n)$ denote the orthogonal group of linear isometries of $\mathbf{R}^n$. For each $g \in O(n)$ and $i = 1, \ldots, n$ let g_i be the ith column of the matrix of g with respect to the standard basis of $\mathbf{R}^n$. For $g, g' \in O(n)$ one defines

$$\operatorname{dist}(g, g') = \Big(\sum_{i=1}^{n} |g_i - g'_i|^2\Big)^{\frac{1}{2}};$$

the resulting metric on $O(n)$ is bi-invariant. Set $\mathbf{1}_{\mathbf{R}^n} = \mathbf{1}$, $l = \frac{1}{2}n(n-1)$ and

$$S^{n-1} = \mathbf{R}^n \cap \{x : |x| = 1\}.$$

If $S \subset SO(n)$ and $A \subset \mathbf{R}^n$, denote

$$S(\mathrm{A}) = \bigcup \{g(A) : g \in S\}.$$

3.1. LEMMA. *If* $0<s<1$ *and* $x\in S^{n-1}$, *then*

$$K_{1,s}\{x\}=K_{x,s/\sqrt{2}}\cap S^{n-1}.$$

Proof. We can assume $x=e_n$; set $K=K_{1,s}$. If $g\in O(n)$ and $g(e_n)=e_n$, then $gKg^{-1}=K$ and $gK(e_n)=K(e_n)$. Thus, we need only show that

$$S=K_{e_n,s/\sqrt{2}}\cap S^{n-1}\cap\{x:x^1=\ldots=x^{n-2}=0\},$$

where

$$S=K(e_n)\cap\{x:x^1=\ldots=x^{n-2}=0\}.$$

But from $g(e_n)\in S$ we infer that the matrix of g has the form

$$\begin{Bmatrix} M & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{Bmatrix},$$

where M is an orthogonal matrix of order $n-2$; in fact, we can assume that M is the identity. Then

$$2\operatorname{dist}(g(e_n), e_n)^2=\operatorname{dist}(g, 1)^2,$$

whence follows our assertion about S.

3.2. LEMMA. *Let* L *and* M *be* k *dimensional linear subspaces of* $\mathbf{R}^n$ *and fix* $0<s<\frac{1}{4}$. *If*

$$M\subset\mathbf{X}(0,\infty,L,s)\cup\{0\},$$

then

(i) $L\subset\mathbf{X}(0,\infty,M,2s)\cup\{0\}$,

(ii) $\mathbf{X}(0,\infty,M,s)\subset\mathbf{X}(0,\infty,L,2s)$,

(iii) $M^\perp\subset\mathbf{X}(0,\infty,L^\perp,\sqrt{s})\cup\{0\}$.

Proof. Let P and $P^\perp$ be the orthogonal projections of $\mathbf{R}^n$ on L and on $L^\perp$, respectively. Then $(\operatorname{kernel} P)\cap M=\{0\}$; let $P'=(P|M)^{-1}$. Since P' is linear, a Lipschitz constant for P' is

$$\sup\{|P'(v)|:v\in L\cap S^{n-1}\};$$

it is easy to see that this is less than $(1-s^2)^{-1}<2$. Considering $0\neq y\in L$, we set $x=P'(y)$ and conclude that

$$\operatorname{dist}(y, M)\leqslant|x-y|=\operatorname{dist}(x, L)<2s|y|.$$

Next suppose $x\in\mathbf{X}(0,\infty,M,s)$ and $y\in M$ is such that $|x-y|=\operatorname{dist}(x, M)$. Then

$$\operatorname{dist}(x, L)\leqslant|y-x|+\operatorname{dist}(y, L)<s|x|+s|y|\leqslant 2s|x|.$$

Finally, fix $0 \neq x \in M^{\perp}$ and set $y = P'[P(x)]$. Then

$$(1-s^2)|y|^2 < |P(y)|^2 = |P(x)|^2 \leqslant |P^{\perp}(x)|\,|P^{\perp}(y)| < |P^{\perp}(x)|\,s\,|y|;$$

consequently, $|P^{\perp}(y)| < s|P^{\perp}(x)|$ and we conclude that

$$|P(x)|^2 < s|x|^2.$$

3.3. Lemma. *Suppose* $0 \in B \subset \mathbf{R}^n$. *If* $0 < s < 1$, *then*

$$K_{1,s}(B) \subset \mathbf{X}(0, \infty, B, s) \cup \{0\}.$$

Proof. For each $0 \neq x \in B$ and $g \in K_{1,s}$ we infer from 3.1 that

$$\operatorname{dist}(g(x), B) \leqslant |g(x) - x| < s|x|.$$

3.4. Lemma. *Suppose* $0 \in B \subset \mathbf{R}^n$. *There exist* $r_1 > 0$ *and* $0 < s_1 < 1$ *such that if* $0 < s < s_1$, *then*

$$\mathbf{X}(0, r_1, B, s/6) \subset K_{1,s}(B).$$

Proof. For each $0 \neq w \in \mathbf{T}_0(B)$ define

$$J_w = \mathbf{T}_0(B)^{\perp} + \mathbf{R}w.$$

Let Π be the orthogonal projection of $\mathbf{R}^n$ on $\mathbf{T}_0(B)$. Choose $\delta < 1$ so that $B \cap K_{0,\delta}$ is connected and closed relative to $K_{0,\delta}$, $\Pi | B \cap K_{0,\delta}$ is univalent, and $(\Pi | B \cap K_{0,\delta})^{-1}$ has Lipschitz constant 0.975^{-1}.

The remainder of the proof is divided into three parts; from Part 3 we have for $0 < s < s_1$

$$\mathbf{X}(0, r_1, B, s/6) \subset \bigcup \{\mathbf{X}(0, r_1, Y_w, s/6) : w \in \mathbf{T}_0(B)\} \subset K_{1,s}(B).$$

Part 1. *There exist* $0 < r_0 < \delta$ *and* $0 < r_1 < r_0/4$ *such that the following are true:*

(i) *For* $0 \neq w \in \mathbf{T}_0(B)$

$$Y_w = J_w \cap B \cap K_{0,\delta}$$

is the connected image of a curve in J_w, *and whenever* $0 < r < 2r_1$, $rS^{n-1} \cap Y_w$ *consists of two points.*

(ii) *If* $0 \neq y \in B \cap K_{0,r_1}$, *then the orthogonal projection* v *of* y *on* $\mathbf{T}_y(B)$ *is not zero; define*

$$P_y = \mathbf{R}y + [(\mathbf{R}v)^{\perp} \cap \mathbf{T}_y(B)].$$

(iii) *If* $0 \neq y \in B \cap K_{0,r_1}$, *then*

$$B \cap K_{y,r_1} \subset \mathbf{X}(y, \infty, P_y, 2^{-4}) \cup \{y\},$$

$$\mathbf{T}_0(B) \subset \mathbf{X}(0, \infty\, P_y, 2^{-4}) \cup \{0\}.$$

Proof. We can choose δ so that $F = (\Pi | B \cap K_{0,\delta})^{-1}$ is uniformly differentiable on the open set $U = \Pi(B \cap K_{0,\delta})$, hence there exist $0 < r_0 < \delta$ and $0 < r_1 < r_0/4$ such that for $y \in B \cap K_{0,r_0}$,

$$\mathbf{T}_y(B) \subset \mathbf{X}(0, \infty, \mathbf{T}_0(B), 2^{-7}) \cup \{0\}, \tag{*}$$

and for $x \in U$,

$$B \cap K_{F(x), 2r_1} \subset F(K_{x, 2r_1}) \subset \mathbf{X}(F(x), \infty, F(x) + \mathbf{T}_{F(x)}(B), 2^{-7}) \cup \{F(x)\}.$$

Thus, if $y \in B \cap K_{0,2r_1}$, then

$$0 \in B \cap K_{y,2r_1} \subset \mathbf{X}(y, \infty, y + \mathbf{T}_y(B), 2^{-7}) \cup \{y\}, \tag{**}$$

whence

$$\mathbf{R}y \subset \mathbf{X}(0, \infty, \mathbf{T}_y(B), 2^{-7}) \cup \{0\}. \tag{***}$$

Consider $0 \neq w \in \mathbf{T}_0(B)$. Since $\Pi(J_w) = \mathbf{R}w$, $\gamma = F | \mathbf{R}w \cap U$ is a curve in J_w and

$$\gamma(\mathbf{R}w \cap U) = Y_w$$

is connected. Now fix $0 \neq y \in Y_w \cap K_{0,2r_1}$.

$$\mathbf{R}\gamma'[\Pi(y)] = J_w \cap \mathbf{T}_y(B),$$

hence

$$J_w^{\perp} + \mathbf{T}_y(B)^{\perp} = [J_w \cap \mathbf{T}_y(B)]^{\perp}.$$

We therefore infer from (***) that

$$\{0\} = \mathbf{R}y \cap [J_w^{\perp} + \mathbf{T}_y(B)^{\perp}] = \mathbf{R}y \cap \mathbf{R}\gamma'[\Pi(y)]^{\perp}$$

and

$$\{0\} = \mathbf{T}_y(|y| S^{n-1}) \cap \mathbf{R}\gamma'[\Pi(y)],$$

and the last assertion of (i) follows from this.

Consider $0 \neq y \in B \cap K_{0,r_1}$. From (***) we infer that the orthogonal projection of $\mathbf{R}^n$ on $\mathbf{T}_y(B)$ maps y to $v \neq 0$. Suppose $w = u + \alpha y$, where $u \in (\mathbf{R}v)^{\perp} \cap \mathbf{T}_y(B)$. Then by (***)

$$\operatorname{dist}(w, \mathbf{T}_y(B))^2 = |\alpha y - \alpha v|^2 = \operatorname{dist}(\alpha y, \mathbf{T}_y(B))^2 < 2^{-7} |\alpha y|^2 < 2^{-7} |w|^2,$$

and we use (*) and 3.2 (ii) to conclude that

$$P_y \subset \mathbf{X}(0, \infty, \mathbf{T}_y(B), 2^{-7}) \cup \{0\} \subset \mathbf{X}(0, \infty, \mathbf{T}_0(B), 2^{-6}) \cup \{0\},$$

whence by 3.2 (i)

$$\mathbf{T}_0(B) \subset \mathbf{X}(0, \infty, P_y, 2^{-5}) \cup \{0\}.$$

Similarly, we infer from (**) that since $y + P_y = P_y$,

$$B \cap K_{y, r_1} \subset \mathbf{X}(y, \infty, P_y, 2^{-4}) \cup \{y\},$$

which completes the proof of (iii).

PART 2. *If* $0 < s < \frac{1}{8}$, $0 < r < r_1$, $0 \neq w \in \mathbf{T}_0(B)$ *and*

$$\{y_0, y_1\} = Y_w \cap rS^{n-1},$$

then
$$Y_w \cap \{x: r - rs < |x| < r + rs\} \subset K_{y_0, 4rs} \cup K_{y_1, 4rs}$$

Proof. Choose $0 < \alpha < \pi/2$ and, for each $i = 0, 1$, $0 \leq \theta_i < \pi/2$ such that

$$\sin \alpha = 4s, \ \cos \theta_i = |\Pi(y_i)|/r.$$

One verifies that $\sin \theta_i < \sqrt{5}/10$. Define

$$\beta_i = \sup \{\theta_i - \alpha, 0\},$$

$$R = J_w \cap \{x: r - rs < |x| < r + rs\},$$

and

$$\begin{aligned} L^i &= J_w \cap \{x: |x| \sin \beta_i \leq |x - \Pi(x)| \leq |x| \sin (\theta_i + \alpha)\} \\ &= J_w \cap \{x: |x| \cos \beta_i \geq |\Pi(x)| \geq |x| \cos (\theta_i + \alpha)\} \\ &= J_w \cap \{x: |\Pi(x)| \tan \beta_i \leq |x - \Pi(x)| \leq |\Pi(x)| \tan (\theta_i + \alpha)\}. \end{aligned}$$

Clearly,

$$\Pi(L^i \cap R) = \mathbf{R}w \cap \{z: (r + rs) \cos \beta_i \geq |z| \geq (r - rs) \cos (\theta_i + \alpha)\}.$$

We can assume $|w| = 1$; set

$$K^i = \operatorname{cl} (K_{y_i, 4rs} \cap J_w).$$

We will next show that $K^i \subset L^i$ for each $i = 0, 1$. Fix a 2-plane $Q \subset J_w$ containing y_i and w and fix ϱ such that $M = \mathbf{T}_0(B)^\perp + \varrho w$ intersets K^i;

$$M \cap K^i = M \cap \{x: |x - \xi| \leq \sigma\}$$

with $\xi \in K^i \cap Q \cap M$ and $\sigma \geq 0$. Inasmuch as $K^i \cap Q \subset L^i$ and $|\Pi(\xi)| = \varrho$, we have

$$\varrho \tan \beta_i + \sigma \leq |\xi - \varrho w| \leq \varrho \tan (\theta_i + \alpha) - \sigma.$$

Thus if $y \in M \cap K^i$, then the triangle inequality implies that

$$\varrho \tan \beta_i \leqslant |y - \varrho w| \leqslant \varrho \tan (\theta_i + \alpha),$$

whence $y \in L^i$.

Finally, suppose there exists $x \in Y_w \cap R$ such that

$$|x - y_0| \geqslant 4rs, \quad |x - y_1| \geqslant 4rs.$$

Since Y_w is connected and $\varrho S^{n-1} \cap Y_w$ consists of two points for $0 < \varrho < 2r_1$, there must exist $x_0 \in Y_w \cap R$ and y_i such that $|x_0 - y_i| = 4rs$. Then $x_0 \in L^i$ and

$$3.9rs \leqslant |\Pi(x_0) - \Pi(y_i)| \leqslant (r + rs) \cos \beta_i - (r - rs) \cos (\theta_i + \alpha) = \delta_i.$$

If $\theta_i \leqslant \alpha$, then it is easy to verify that $\delta_i < 2rs$, which is a contradiction. On the other hand, if $\theta_i > \alpha$, then

$$\delta_i < 2rs(4 \sin \theta_i + 1) < 3.8rs,$$

which completes the proof.

PART 3. *There exists $0 < s_1 < \frac{1}{8}$ such that if $0 \neq w \in \mathbf{T}_0(B)$, then for $0 < s < s_1$,*

$$\mathbf{X}(0, r_1, Y_w, s/4) \subset K_{1, s\sqrt{2}}(Y_w).$$

Proof. For $0 < s < 1$ let $t(s) = \sin \theta$, where $s = 2 \sin \theta/2$. Recalling that

$$\lim_{s \to 0^+} \frac{s}{t(s)} = 1,$$

we choose $0 < s_1 < \frac{1}{8}$ so that for $0 < s < s_1$,

$$s < \tfrac{3}{2} t(s).$$

Fix $0 < s < s_1$ and $0 < r < r_1$, and set

$$\{y_0, y_1\} = Y_w \cap rS^{n-1}.$$

Consider
$$x_0 \in J_w \cap \{x\colon \text{dist}(x, P_{y_i}) < rs/4\}.$$

From Part 1 (ii) and (iii) and 3.2 (iii) we infer that $x_0 = \alpha y_i + v$, $v \in \mathbf{T}_0(B)^{\perp}$, and

$$|v|^2 = \text{dist}(x_0, P_{y_i})^2 + \text{dist}(v, P_{y_i}^{\perp})^2 < (rs/4)^2 + 2^{-4}|v|^2,$$

whence
$$\text{dist}(x_0, \mathbf{R}y_i) \leqslant |x_0 - \alpha y_i| = |v| < rs/3.$$

Suppose $|y| = r$ and $|y - y_i| \geqslant rs$ for $i = 0, 1$. From Part 2 and Part 1 (iii) we infer that

$$Y_w \cap \{x\colon r - rs < |x| < r + rs\} \subset Y_w \cap (K_{y_0, 4rs} \cup K_{y_1, 4rs})$$

$$\subset J_w \cap [\{x\colon \operatorname{dist}(x, P_{y_0}) < rs/4\} \cup \{x\colon \operatorname{dist}(x, P_{y_1}) < rs/4\}]$$

$$\subset \mathbf{R}^n \cap [\{x\colon \operatorname{dist}(x, \mathbf{R}y_0) < rs/3\} \cup \{x\colon \operatorname{dist}(x, \mathbf{R}y_1) < rs/3\}].$$

Consequently, if $\operatorname{dist}(y, Y_w) < rs$, then for some i

$$\operatorname{dist}(y, Y_w) > \operatorname{dist}(y, \mathbf{R}y_i) - rs/3 \geqslant rt(s) - rs/3 \geqslant rs/3;$$

therefore,

$$\mathbf{X}(0, \infty, Y_w, s/3) \cap rS^{n-1} \subset (K_{y_0, rs} \cup K_{y_1, rs}) \cap rS^{n-1}.$$

Finally, we use 3.1 to conclude that

$$\mathbf{X}(0, r_1, Y_w, s/3) = \bigcup_{0<r<r_1} [\mathbf{X}(0, \infty, Y_w, s/3) \cap rS^{n-1}]$$

$$\subset \bigcup_{0<r<r_1} [K_{1, s\sqrt{2}}(Y_w \cap rS^{n-1})] \subset K_{1, s\sqrt{2}}(Y_w).$$

3.5. LEMMA. *Suppose $0 \in B \subset \mathbf{R}^n$. There exists $r_1 > 0$ such that if $0 \neq y \in B \cap K_{0, r_1}$, then*

$$I_y = O(n) \cap \{g\colon g(y) \in B\}$$

is a closed $l-k$ dimensional submanifold of class 1 of $O(n)$. Further, there exist positive numbers s_3, C_1, C_2 such that if $0 \neq y \in B \cap K_{0, r_1}$ and $0 < s < s_3$, then

$$C_1 s^{l-k} \leqslant H^{l-k}(I_y \cap K_{g, s}) \leqslant C_2 s^{l-k}$$

whenever $g \in I_y$.

Proof. Choose r_0 and r_1 as in 3.4 Part 1. By Part 1 (ii) we then have $y \notin \mathbf{T}_y(B)^{\perp}$ for $y \in B_0 = B \cap K_{0, r_1} \sim \{0\}$, whence

$$\dim [\mathbf{T}_y(B) \cap \mathbf{T}_y(|y|\, S^{n-1})] = n - k - 1, \qquad (*)$$

and it follows that the map

$$F_0\colon O(n) \times B_0 \to \mathbf{R}^n, \; F_0(g, y) = g(y),$$

has rank n at each point of $O(n) \times B_0$. Further, the map

$$F\colon O(n) \times B_0 \to O(n) \times \mathbf{R}^n, \; F(g, y) = (g, g(y)),$$

imbeds $O(n) \times B_0$ as a proper submanifold Φ of $O(n) \times \mathbf{R}^n$. Defining

$$p\colon O(n) \times \mathbf{R}^n \to \mathbf{R}^n, \; p(g, x) = x,$$

$$q\colon O(n) \times \mathbf{R}^n \to O(n), \; q(g, x) = g,$$

we observe that $p(\Phi)=K_{0,r_1}\sim\{0\}$ and, since $p\circ F=F_0$, whenever $0<|x|<r_1$,

$$\Phi_x=p^{-1}\{x\}\cap\Phi$$

is an $l-k$ dimensional submanifold of Φ which is properly imbedded by q in $O(n)$. One verifies that if $y\in B_0$, then

$$q(\Phi_y)^{-1}=I_y;$$

I_y is closed because $B_0\cup\{0\}$ is closed relative to K_{0,r_1}.

The remainder of the proof is divided into three parts. If $n-k>1$, the estimate for $H^{l-k}(I_y\cap K_{g,s})$ follows from Parts 1 and 3; in case $n-k=1$ one uses Part 1 and 3.4 Part 1 (i).

Let $\Pi\colon \mathbf{R}^n\to\mathbf{T}_0(B)$ be the orthogonal projection and set $S_0=\mathbf{T}_0(B)\cap S^{n-1}$.

PART 1. *There exist $0<s_1<1$ and positive numbers c, c_0, $c_i<1$ such that for $Y\in B_0$, $g\in I_y$, $0<s<s_1$,*

$$c_i s^\lambda H^{n-k-1}[(|y|^{-1}B)\cap K_{g(y/|y|),c_i s}\cap S^{n-1}]\leqslant cH^{l-k}(I_y\cap K_{g,s})$$
$$\leqslant c_0 s^\lambda H^{n-k-1}[(|y|^{-1}B)\cap K_{g(y/|y|),s}\cap S^{n-1}],$$

where $\lambda=\frac{1}{2}(n-1)(n-2)$.

Proof. Whenever $0\neq x\in\mathbf{R}^n$ define

$$\pi_x\colon O(n)\to S^{n-1},\ \pi_x(g)=g(x/|x|).$$

Then for $y\in B_0$, $\pi_y^{-1}[\pi_y(I_y)]=I_y$ and

$$\pi_y(I_y)=(|y|^{-1}B)\cap S^{n-1}.$$

Fix x. From the existence of a cross-section for π_x in a neighborhood of $\pi_x(\mathbf{1})$ we infer the existence of $0<s_1<1$ and $c_0>0$, $c_i>0$ such that if $0<s<s_1$, then

$$H^\lambda[\pi_x^{-1}\{z\}\cap K_{\mathbf{1},s}]\geqslant c_i s^\lambda\quad\text{for}\quad z\in K_{\pi_x(\mathbf{1}),c_i s}\cap S^{n-1}$$

and
$$H^\lambda[\pi_x^{-1}\{z\}\cap K_{\mathbf{1},s}]\leqslant c_0 s^\lambda\quad\text{for}\quad z\in\pi_x(K_{\mathbf{1},s}).$$

Moreover, since $\pi_{g(x)}=\pi_x\circ R_g$ for $g\in O(n)$, since L_g preserves the fibers of π_x, and since L_g and R_g are isometries of $O(n)$, we conclude that whenever $0\neq x\in\mathbf{R}^n$,

$$H^\lambda[\pi_x^{-1}\{z\}\cap K_{g,s}]\geqslant c_i s^\lambda\quad\text{for}\quad z\in K_{\pi_x(g),c_i s}\cap S^{n-1}$$

and
$$H^\lambda[\pi_x^{-1}\{z\}\cap K_{g,s}]\leqslant c_0 s^\lambda\quad\text{for}\quad z\in\pi_x(K_{g,s}).$$

Let us consider $\alpha \in H = \Lambda_{n-k-1}[\mathbf{T}_1(\pi_x^{-1}\{\pi_x(1)\})^\perp]$, $|\alpha| = 1$. Since the metric on $O(n)$ is bi-invariant,

$$H \cap \{\beta\colon |\beta| = 1\} = \{(\text{ad } g)_\#(1)(\alpha)\colon g(x) = x\},$$

and we conclude from this that $c = |\pi_{x\#}(1)(\alpha)| > 0$ and is independent of choice of α. As before we infer that

$$|\pi_{x\#}(g)(\alpha)| = c$$

for $g \in O(n)$ and $\alpha \in \Lambda_{n-k-1}[\mathbf{T}_g(\pi_x^{-1}\{\pi_x(g)\})^\perp]$, $|\alpha| = 1$.

Finally, consider $y \in B_0$ and $0 < s < s_1$. We have $J(\pi_y | I_y) = c$, hence application of the coarea formula [7, 3.2.12] or [5, 3.1] yields

$$cH^{l-k}(I_y \cap K_{g,s}) = \int_{\pi_y(I_y)} H^\lambda[\pi_y^{-1}\{z\} \cap K_{g,s}]\, dH^{n-k-1}z.$$

The desired inequalities now follow from 3.1 and our estimates for the integrand.

PART 2. *Suppose $n-k>1$. Fix $0<r<r_1$, set*

$$\beta = \Pi[(r^{-1}B) \cap S^{n-1}],$$

and define

$$\varrho\colon \beta \to S_0, \quad \varrho(x) = x/|x|.$$

The following are true:

(i) *β is a compact $n-k-1$ dimensional submanifold of $\mathbf{T}_0(B)$ of class 1.*

(ii) *ϱ is one-to-one onto S_0.*

(iii) $2^{n-k-1} \geq J\varrho \geq 2^{-2(n-k-1)}$.

(iv) *There exists $0 < s_2 < s_1$ such that for $y \in (r^{-1}B) \cap S^{n-1}$,*

$$K_{\varrho\circ\Pi(y),s/16} \cap S_0 \subset \varrho \circ \Pi[K_{y,s} \cap (r^{-1}B) \cap S^{n-1}] \subset K_{\varrho\circ\Pi(y),2s} \cap S_0.$$

Proof. That (i) is true follows from (*) and the fact that 2 is a Lipschitz constant for $(\Pi | B_0)^{-1}$. Moreover, the three starred formulas in the proof of 3.4 Part 1 together with 3.2 (ii) imply that for $y \in B_0$

$$B_0 \cup \mathbf{R}y \cup \mathbf{T}_y(B) \subset \mathbf{X}(0, \infty, \mathbf{T}_0(B), \tfrac{1}{8}) \cup \{0\}. \qquad (**)$$

One uses this to verify that

$$\beta \subset \mathbf{T}_0(B) \cap \{x\colon \tfrac{1}{2} < |x| < 1\}.$$

Suppose $\varrho(x_1)=\varrho(x_2)$, $|x_1|<|x_2|$. Let λ: $[0, 1]\to\mathbf{T}_0(B)$ parametrize the line segment from x_1 to x_2, and

$$\gamma=(\Pi\,|\,r^{-1}B_0)^{-1}\circ\lambda\colon [0,1]\to\mathbf{T}_0(B)^{\perp}+\mathbf{R}x_1.$$

Application of the mean value theorem to $|\gamma|^2$ yields the existence of $0<t_0<1$ such that $\gamma(t_0)\cdot\gamma'(t_0)=0$. We have by (**)

$$\gamma'(t_0)\in\mathbf{X}(0, \infty, \mathbf{R}x_1, \tfrac{1}{8}),$$

whence by 3.2 (iii)

$$\gamma(t_0)\in\mathbf{X}(0, \infty, \mathbf{T}_0(B)^{\perp}, \tfrac{1}{2}),$$

which contradicts (**). Thus ϱ is one-to-one.

Suppose $y\in B_0$. One verifies with the help of (**) that

$$\mathbf{R}y+\mathbf{T}_y(B\cap|y|S^{n-1})\subset\mathbf{X}(0, \infty, \mathbf{T}_0(B), \tfrac{1}{4}).$$

Furthermore, if $x\in\mathbf{X}(0, \infty, \mathbf{T}_0(B), \frac{1}{4})$, then $|\Pi(x)|>\frac{1}{2}|x|$, and therefore

$$\sigma_y=(\Pi\,|\,[\mathbf{R}y+\mathbf{T}_y(B\cap|y|S^{n-1})])^{-1}$$

has Lipschitz constant 2.

Considering now the assertion (iii) we see that the upper bound for $J\varrho$ follows from the fact that 2 is a Lipschitz constant for ϱ. Fix $y\in(r^{-1}B)\cap S^{n-1}$ and choose $v\in\mathbf{T}_y[(r^{-1}B)\cap S^{n-1}]$, $|v|=1$. Then

$$1=|v\wedge y|=|\sigma_{ry}[\Pi(v)\wedge\Pi(y)]|\leqslant 4|\Pi(v)\wedge\Pi(y)|,$$

and we conclude that for $x=\Pi(y)$ and $w=\Pi(v)/|\Pi(v)|$,

$$|\varrho_{\#}(x)(w)|\geqslant\tfrac{1}{4}.$$

Thus $\varrho_{\#}(x)$ is one-to-one and, since β is compact, ϱ is a diffeomorphism of β onto S_0; this completes the proof of (ii). It follows that $\|\varrho_{\#}^{-1}\|\leqslant 4$, hence 4 is a Lipschitz constant for ϱ^{-1} with respect to the Riemannian metric on S_0. This implies that

$$J\varrho(x)\geqslant\|\varrho_{\#}^{-1}[\varrho(x)]\|^{-(n-k-1)}\geqslant 2^{-2(n-k-1)}.$$

Finally, we observe that for $0<s<1$,

$$K_{x,s/2}\cap\beta\subset\Pi[K_{y,s}\cap(r^{-1}B)\cap S^{n-1}]\subset K_{x,s}\cap\beta$$

and

$$\varrho(K_{x,s}\cap\beta)\subset K_{\varrho(x),2s}.$$

Furthermore,

$$\varrho(K_{x,s}\cap\beta)\supset K^{S_0}_{\varrho(x),s/4}=K_{\varrho(x),\tau}\cap S_0$$

where $\tau = 2 \sin s/8$. Thus (iv) will hold if we choose $0 < s_2 < s_1$ so that for $0 < s < s_2$,

$$2 \sin s/8 \geqslant s/8.$$

PART 3. *Suppose $n-k>1$. There exists $0<s_3<s_2$ such that if $y \in B_0$ and $0<s<s_3$, then*

$$\alpha(n-k-1)2^{4-5(n-k)}s^{n-k-1} \leqslant H^{n-k-1}[(|y|^{-1}B) \cap K_{y/|y|,s} \cap S^{n-1}] \leqslant \alpha(n-k-1)2^{4(n-k)-3}s^{n-k-1}.$$

Proof. Let $|y| = r$. We infer from Part 2 that whenever S is a Borel subset of $(r^{-1}B) \cap S^{n-1}$,

$$H^{n-k-1}[\varrho \circ \Pi(S)] = \int_{\Pi(S)} J\varrho \, dH^{n-k-1}$$

and

$$2^{-3(n-k-1)}H^{n-k-1}(S) \leqslant 2^{-2(n-k-1)}H^{n-k-1}[\Pi(S)] \leqslant H^{n-k-1}[\varrho \circ \Pi(S)] \leqslant 2^{n-k-1}H^{n-k-1}(S).$$

If $z \in S_0$ and $0<s<1$, set

$$\theta_z(s) = \alpha(n-k-1)^{-1}s^{-(n-k-1)}H^{n-k-1}(S_0 \cap K_{z,s}).$$

Observing that $\theta_z(s)$ is independent of z and

$$\lim_{s \to 0^+} \theta_z(s) = 1$$

we choose $0 < s_3 < s_2/2$ so that if $0 < s < 2s_3$, then

$$\tfrac{1}{2} < \theta_z(s) < 2.$$

Finally, Part 2 (iv) allows us to conclude that for $0 < s < s_3$,

$$2^{-3(n-k-1)}H^{n-k-1}[(r^{-1}B) \cap S^{n-1} \cap K_{y/r,s}] \leqslant \alpha(n-k-1)2^{n-k}s^{n-k-1}$$

and
$$2^{n-k-1}H^{n-k-1}[(r^{-1}B) \cap S^{n-1} \cap K_{y/r,s}] \geqslant \alpha(n-k-1)2^{3-4(n-k)}s^{n-k-1}.$$

3.6. LEMMA. *If ψ measures $O(n)$, $\psi(T)=0$ and T is H^l measurable, then for H^l almost all $g \in T$, $\limsup_{s \to 0^+} \psi(K_{g,s})s^{-l}$ equals either 0 or ∞.*

Proof. The ratios

$$H^l(K_{g,s})/[\alpha(l)s^l]$$

corresponding to $g \in O(n)$ and $s>0$ are independent of g and approach 1 as $s \to 0^+$. Consequently, our assertion follows from application of [7, 2.9.17] with $\varphi = H^l$.

3.7. LEMMA. *Suppose $0 \in B \subset X = \mathbf{R}^n$. Let A be a Suslin subset of $\mathbf{R}^n$. Then H^l almost every $g \in O(n)$ satisfies one of the following conditions:*

(i) *For some* $\delta > 0$,

$$\lim_{s\to 0^+} \sup_{0<r<\delta} \varphi[A \cap K_{g,s}(B) \cap K_{0,r} \sim \{0\}]\, r^{-k} s^{-k} = 0.$$

(ii) *For all* $\delta > 0$,

$$\limsup_{s\to 0^+} \sup_{0<r<\delta} \varphi[A \cap K_{g,s}(B) \cap K_{0,r} \sim \{0\}]\, r^{-k} s^{-k} = \infty.$$

(iii) $0 \in \mathrm{cl}[A \cap g(B) \sim \{0\}]$.

Proof. We readopt the notation introduced in the first paragraph of the proof of 3.5. We can assume $A \subset K_{0,r_1} \sim \{0\}$. Let φ' be the measure on Φ such that for $S \subset \Phi$

$$\varphi'(S) = \int_A^* H^{l-k}(\Phi_x \cap S)\, d\varphi x,$$

where "$\int^*$" means "upper integral". For each positive integer ν we consider the measure ψ_ν over $O(n)$ defined by the formula

$$\psi_\nu(T) = \sup_{0<r<1/\nu} \varphi'[p^{-1}(K_{0,r}) \cap q^{-1}(T) \cap \Phi]\, r^{-k}.$$

Letting

$$P_\nu = O(n) \cap \{g : \lim_{s\to 0^+} \psi_\nu(K_{g,s})\, s^{-l} = 0\},$$

$$Q_\nu = O(n) \cap \{g : \limsup_{s\to 0^+} \psi_\nu(K_{g,s})\, s^{-l} = \infty\},$$

$$R_\nu = q[p^{-1}(A \cap K_{0,1/\nu}) \cap \Phi],$$

we note that R_ν is H^l measurable and

$$\psi_\nu[O(n) \sim R_\nu] = 0,$$

and infer from 3.6 that

$$H^l[O(n) \sim (P_\nu \cup Q_\nu \cup R_\nu)] = 0.$$

Observing that $\psi_\nu \geq \psi_{\nu+1}$, $P_\nu \subset P_{\nu+1}$, $Q_\nu \supset Q_{\nu+1}$, we also let

$$P = \bigcup_{\nu=1}^{\infty} P_\nu,\quad Q = \bigcap_{\nu=1}^{\infty} Q_\nu,\quad R = \bigcap_{\nu=1}^{\infty} R_\nu$$

and verify that

$$\bigcap_{\nu=1}^{\infty} (P_\nu \cup Q_\nu \cup R_\nu) \subset P \cup Q \cup R,$$

hence

$$H^l[O(n) \sim (P \cup Q \cup R)] = 0.$$

Next consider $H^{l-k} \llcorner \Phi_x$, $0 < |x| < r_1$, and $0 < s < s_3/2$. Choosing $g_0 \in O(n)$ such that $y = g_0^{-1}(x) \in B$, one verifies that

$$q(\Phi_x) = g_0[q(\Phi_y)] = g_0(I_y^{-1}).$$

Since the metric on $O(n)$ is bi-invariant, this implies that for $(h, x) \in \Phi_x$

$$H^{l-k}[\Phi_x \cap q^{-1}(K_{h,s})] = H^{l-k}(I_y \cap K_{h^{-1}g_0, s}).$$

With the help of 3.5 we conclude that if $g \in O(n)$, then

$$H^{l-k}[\Phi_x \cap q^{-1}(K_{g,s})] \leqslant C_2(2s)^{l-k};$$

furthermore,

$$p[q^{-1}(K_{g,s}) \cap \Phi] = K_{g,s}(B) \sim \{0\},$$

hence if $x \in K_{g,s/2}(B)$, then

$$C_1(s/2)^{l-k} \leqslant H^{l-k}[\Phi_x \cap q^{-1}(K_{g,s})].$$

We now have for $r > 0$

$$\varphi'[p^{-1}(K_{0,r}) \cap q^{-1}(K_{g,s}) \cap \Phi] = \int^*_{A \cap K_{g,s}(B) \cap K_{0,r}} H^{l-k}[\Phi_x \cap q^{-1}(K_{g,s})]\, d\varphi x,$$

hence

$$C_1(s/2)^{l-k}\varphi[A \cap K_{g,s/2}(B) \cap K_{0,r}] \leqslant \varphi'[p^{-1}(K_{0,r}) \cap q^{-1}(K_{g,s}) \cap \Phi]$$
$$\leqslant C_2(2s)^{l-k}\varphi[A \cap K_{g,s}(B) \cap K_{0,r}];$$

therefore, $g \in P$ if and only if g satisfies condition (i) and $g \in Q$ if and only if g satisfies condition (ii). We complete the proof by observing that $g \in R$ if and only if for each ν,

$$A \cap g(B) \cap K_{0,1/\nu} \neq \varnothing.$$

3.8. LEMMA. *Let Y be a Riemannian manifold and $F: X \to Y$ a diffeomorphism.*

For some $\delta > 0$,

$$\lim_{s \to 0^+} \sup_{0<r<\delta} \varphi[A \cap \mathbf{X}(x, r, B, s)]\, r^{-k} s^{-k} = 0$$

if and only if for some $\delta > 0$,

$$\lim_{s \to 0^+} \sup_{0<r<\delta} F_{\#}(\varphi)\,[F(A) \cap \mathbf{X}(F(x), r, F(B), s)]\, r^{-k} s^{-k} = 0.$$

For all $\delta > 0$,

$$\limsup_{s \to 0^+} \sup_{0<r<\delta} \varphi[A \cap \mathbf{X}(x, r, B, s)]\, r^{-k} s^{-k} = \infty$$

if and only if for all $\delta > 0$,

$$\limsup_{s \to 0^+} \sup_{0<r<\delta} F_{\#}(\varphi)\,[F(A) \cap \mathbf{X}(F(x), r, F(B), s)]\, r^{-k} s^{-k} = \infty.$$

Proof. Fix $x \in X$ and choose $\varrho > 0$ so that $F|K_{x,\varrho}$ and $(F|K_{x,\varrho})^{-1}$ have Lipschitz constant $M \geq 1$. Then for $0 < s < 1$ and $0 < r < \varrho/2$,

$$\mathbf{X}[F(x), rM^{-2}, F(B), sM^{-4}] \subset F(\mathbf{X}[x, rM^{-1}, B, sM^{-2}]) \subset \mathbf{X}[F(x), r, F(B), s].$$

3.9. THEOREM. *Assume that G is a group of isometries of X with $m = \frac{1}{2}n(n-1)$. If A is a Suslin subset of X and $(a, b) \in G \times B'$, then for H^m almost every $g \in I$ there exists $i \in \{1, 2, 3\}$ such that*

$$(a, bg) \in S_{A,i}.$$

Proof. We can assume that $a = b = e$. Let

$$\exp\colon K_{0,\sigma} \to X,$$

where $K_{0,\sigma} \subset \mathbf{T}_o(X)$, be the normal coordinates of X at o defined with respect to the Riemannian connection of X (see [9]). Recalling that $\exp(K_{0,\sigma}) = K_{o,\sigma}$ we set

$$B_0 = \exp^{-1}(B \cap K_{o,\sigma}), \quad A_0 = \exp^{-1}(A \cap K_{o,\sigma}),$$

$$\varphi_0 = \exp_{\#}^{-1}(\varphi \llcorner K_{o,\sigma}).$$

From 3.8 and the identity

$$g \circ \exp = \exp \circ g_{\#}(o) \quad \text{for } g \in I$$

we infer that it suffices to show that for H^m almost all $g \in I$ one of the following holds with $h = g_{\#}(o)$:

(i) For some $\delta > 0$,

$$\lim_{s \to 0^+} \sup_{0<r<\delta} \varphi_0[A_0 \cap \mathbf{X}(0, r, h(B_0), s)]\, r^{-k} s^{-k} = 0.$$

(ii) For all $\delta > 0$,

$$\limsup_{s \to 0^+} \sup_{0<r<\delta} \varphi_0[A_0 \cap \mathbf{X}(0, r, h(B_0), s)]\, r^{-k} s^{-k} = \infty.$$

(iii) $0 \in \operatorname{cl}[A_0 \cap h(B_0) \sim \{0\}]$.

Using the differentiability of the adjoint representation of I we infer that the representation of I in the orthogonal group $\mathbf{O}$ of $\mathbf{T}_o(X)$ which corresponds $g_{\#}(o)$ with g is a differentiable isomorphism of I onto an open subgroup of $\mathbf{O}$. Thus, proving our assertion is equivalent to showing that for H^m almost all $h \in \mathbf{O}$ one of the conditions (i), (ii), (iii) is satisfied. But this follows from 3.3, 3.4 and 3.7.

3.10. THEOREM. *Assume that X has a linear connection which is invariant under the action of G, that I acts transitively on the space γ_{n-k} of $n-k$ dimensional linear subspaces of $\mathbf{T}_o(X)$, and that B is a totally geodesic submanifold of X. If A is a Suslin subset of X and $(a, b) \in G \times B'$, then for H^m almost every $g \in I$ there exists $i \in \{1, 2, 3\}$ such that*

$$(a, bg) \in S_{A, i}.$$

Proof. We can assume that $a = b = e$. Let

$$\exp: K_{0,\sigma} \to X,$$

where $K_{0,\sigma} \subset \mathbf{T}_o(X)$, be the normal coordinates of X at o (see [9]). Set $U = \exp(K_{0,\sigma})$, $A_0 = \exp^{-1}(A \cap U)$, $\varphi_0 = \exp_{\#}^{-1}(\varphi \llcorner U)$, $\lambda_0 = \mathbf{T}_o(B)$.

Inasmuch as each $g \in I$ leaves the connection invariant, we have

$$\exp[g_{\#}(o)(\lambda_0) \cap K_{0,\sigma}] = g(B) \cap U.$$

We use this together with 3.8 to infer that it suffices to show that for H^m almost all $g \in I$ one of the following holds with $\lambda = g_{\#}(o)(\lambda_0)$:

(i) For some $\delta > 0$,

$$\lim_{s \to 0^+} \sup_{0<r<\delta} \varphi_0[A_0 \cap \mathbf{X}(0, r, \lambda, s)]\, r^{-k} s^{-k} = 0.$$

(ii) For all $\delta > 0$,

$$\limsup_{s \to 0^+} \sup_{0<r<\delta} \varphi_0[A_0 \cap \mathbf{X}(0, r, \lambda, s)]\, r^{-k} s^{-k} = \infty.$$

(iii) $0 \in \mathrm{cl}[A_0 \cap \lambda \sim \{0\}]$.

Using the differentiability of the adjoint representation of I we infer that the representation of I in the group of nonsingular endomorphisms of $\mathbf{T}_o(X)$ which corresponds $g_{\#}(o)$ with g is of class ∞. Thus the map of I onto γ_{n-k} which carries g onto $g_{\#}(o)(\lambda_0)$ is a fibre map. We conclude that proving our assertion is equivalent to showing that for almost all $\lambda \in \gamma_{n-k}$ one of the conditions (i), (ii), (iii) is satisfied. But this follows from application of 3.9 with $X = \mathbf{R}^n$ and $B = \lambda_0$, or from [7, 3.3.4].

3.11. *Remark.* If B is an $n-k$ dimensional linear subspace of $\mathbf{R}^n$, then the proofs of 3.4 and 3.5 can be greatly simplified.

We first show that if $0 < s < 1$, then

$$K_{1,s}(B) = \mathbf{X}(0, \infty, B, \tau) \cup \{0\},$$

$$\tau = s[\tfrac{1}{2}(1-s^2/8)]^{\frac{1}{2}}.$$

In fact, if $2\sin\theta/2 = s/\sqrt{2}$, then $\tau = \sin\theta$, and from 3.1 we infer that if $x \in B \cap S^{n-1}$, then

$$K_{1,s}(\mathbf{R}x) \sim \{0\} = \bigcup_{r>0} \{y : \operatorname{dist}(y, \mathbf{R}x) < \tau r\} \cap rS^{n-1} = \mathbf{X}(0, \infty, \mathbf{R}x, \tau).$$

Furthermore, if $\Pi: \mathbf{R}^n \to B$ is the orthogonal projection, then for $y \in \mathbf{X}(0, \infty, B, \tau)$, we have

$$y \in \mathbf{X}(0, \infty, \mathbf{R}[\Pi(y)], \tau),$$

hence

$$\mathbf{X}(0, \infty, B, \tau) = \bigcup \{\mathbf{X}(0, \infty, \mathbf{R}x, \tau) : x \in B \cap S^{n-1}\} = K_{1,s}(B) \sim \{0\}.$$

Next we consider the estimate in 3.5. Fix $0 \neq y \in B$; we can assume $|y| = 1$. Further, if $g \in O(n)$ and $g(y) \in B$, then $I_{g(y)} = I_y g^{-1}$ and thus we need only consider I_y from now on. Cover I_y by open subsets $U_1, \ldots, U_\alpha$ of $O(n)$ on each of which is defined a coordinate map

$$f_i : U_i \to \mathbf{R}^l$$

such that f_i and f_i^{-1} have the Lipschitz constant M, and such that

$$f_i(I_y \cap U_i) \subset \mathbf{R}^l \cap \{x : x^1 = \ldots = x^k = 0\} = P.$$

Let σ be the Lebesgue number of the covering $U_1, \ldots, U_\alpha$ and consider $g \in I_y$, $0 < s < \sigma$. Choosing $U_\beta \supset K_{g,s}$ we set $z = f_\beta(g)$ and verify that

$$\begin{aligned}\alpha(l-k) M^{2(k-l)} s^{l-k} = M^{k-l} H^{l-k}(P \cap K_{z,s/M}) &\leqslant H^{l-k}(I_y \cap K_{g,s}) \\ &\leqslant M^{l-k} H^{l-k}(P \cap K_{z,sM}) = \alpha(l-k) M^{2(l-k)} s^{l-k}.\end{aligned}$$

4. Purely (φ, k) unrectifiable sets

Assign a left invariant Riemannian metric to G. Let $\sigma_e: N_e \to G$ be a cross-section such that $\sigma_e(o) = e$. For $a \in G$ let $N_a = aN_e$ and $\sigma_a = L_a \circ \sigma_e \circ a^{-1}$. Define $\chi_a: N_a \times I \to G$ by the formula $\chi_a(x, g) = \sigma_a(x) g$.

Assume φ to be Borel regular.

4.1. LEMMA. *If $A \subset X$ and $\delta > 0$, then the function mapping (a, b) onto*

$$\limsup_{s \to 0^+} \sup_{0<r<\delta} \varphi(A \cap \mathbf{X}[\pi(a), r, ab^{-1}(B), s])\, r^{-k} s^{-k}$$

is a Borel function on $G \times G$.

The proof is analogous to that of [7, 3.3.4].

4.2. LEMMA. *If A is a Suslin subset of X, then $S_{A,3}$ is a Suslin subset of $G \times G$.*

Proof. For each positive integer i the set S_i of $(a_0, b_0, a, b) \in G \times B' \times A' \times B'$ such that

$$f(a_0, b_0) = f(a, b) \quad \text{and} \quad 0 < \operatorname{dist}[\pi(a_0), \pi(a)] < i^{-1}$$

is a Suslin set. Consequently,

$$S_{A,3} = \bigcap_{i=1}^{\infty} \{(a_0, b_0) : (a_0, b_0, a, b) \in S_i\}$$

is a Suslin set.

4.3. LEMMA. *Assume that φ is finite. There exists a Borel regular measure φ' on G such that:*

(i) *If $a \in G$, then $\varphi' \llcorner N'_a = \chi_{a\#}[\varphi \llcorner N_a \times H^m]$.*

(ii) *If φ and $H^k \llcorner A$ have the same null sets and $H^k(A) < \infty$, then φ' and $H^{k+m} \llcorner A'$ have the same null sets.*

Proof. Let $\mathcal{G}$ be the principal fibre bundle with bundle space G, base space X and structure group I. Orient I and denote by $\vec{I}$ the unit positively oriented m-vectorfield on I. We can clearly assume φ to have compact support. Keeping [2, 3.8] in mind we define $L_{\mathcal{G}}$ as in [2, 3.3] and set

$$\varphi' = \|L_{\mathcal{G}}(\varphi)\|,$$

where φ is regarded as a 0-current. From [2, 3.3 and 3.1] we infer that for $a \in G$,

$$L_{\mathcal{G}}(\varphi) \llcorner N'_a = \chi_{a\#}(\varphi \llcorner N_a \times I) = \chi_{a\#}(0, \vec{I})\, \chi_{a\#}(\varphi \llcorner N_a \times H^m).$$

Further, $\chi_{a\#}(0, \vec{I})$ is the restriction to N'_a of the left invariant m-vectorfield on G which agrees with $\vec{I}$ on I; consequently,

$$|\chi_{a\#}(0, \vec{I})| = 1,$$

which implies (*i*).

Suppose φ and $H^k \llcorner A$ have the same null sets, $H^k(A) < \infty$. Then Fubini's theorem implies that $\varphi \times H^m$ and $H^k \llcorner A \times H^m$ have the same null sets. Furthermore, we infer from [7, 2.10.27 and 2.10.45] or [4, 3.2 and 4.1] that $H^k \llcorner A \times H^m$ and $H^{k+m} \llcorner A \times I$ have the same null sets, hence conclude from (i) that (ii) is true.

4.4. LEMMA. *Suppose A is a purely (φ, k) unrectifiable Suslin subset of X such that $\varphi(A) < \infty$ and $\varphi(W) = 0$ whenever $W \subset A$ and $H^k(W) = 0$. If Y is a separable Riemannian manifold of class 1 and dimension j, then $A \times Y$ is a purely $(\varphi \times H^j, k+j)$ unrectifiable subset of $X \times Y$.*

Proof. Let R be a $k+j$ rectifiable Suslin subset of $A \times Y$. We infer from Fubini's theorem that we can conclude that $\varphi \times H^j(R) = 0$ if we can show that $\varphi(S) = 0$ where

$$S = A \cap \{x: H^j[R \cap \{x\} \times Y] > 0\}.$$

But from [7, 3.2.21] or [6] it follows that S is countably (H^k, k) rectifiable, hence (φ, k) rectifiable by our hypothesis.

4.5. LEMMA. *Let μ be a measure on $\mathbf{R}^{m+n} \times \mathbf{R}^{n-k}$ such that closed subsets of $\mathbf{R}^{m+n} \times \mathbf{R}^{n-k}$ are μ measurable, and F: $\mathbf{R}^{m+n} \times \mathbf{R}^{n-k} \to \mathbf{R}^{m+n}$ be the projection. If W is a purely $(\mu, m+n)$ unrectifiable subset of $\mathbf{R}^{m+n} \times \mathbf{R}^{n-k}$ and for each $w \in W$ there exists $\delta > 0$ such that*

$$\lim_{s \to 0+} \sup_{0<r<\delta} \mu[W \cap \mathbf{X}(w, r, F^{-1}\{F(w)\}, s)] r^{-m-n} s^{-m-n} = 0,$$

then $\mu(W) = 0$.

This is [7, 3.3.8].

4.6. LEMMA. *If A is a purely (φ, k) unrectifiable Suslin subset of X such that $\varphi(A) < \infty$ and $\varphi(W) = 0$ whenever $W \subset A$ and $H^k(W) = 0$, then*

$$\varphi' \times H^{m+n-k}(A' \times B' \cap S_{A,1}) = 0.$$

Proof. Since B is separable, it is sufficient to show that

$$\varphi' \times H^{m+n-k}(S_{A,1} \cap A' \times B' \cap N'_e) = 0. \tag{*}$$

Let $B_0 = \sigma_e(B \cap N_e)$. We shall show that

$$\varphi' \times H^{n-k}(S_{A,1} \cap A' \times B_0) = 0.$$

Inasmuch as $\varphi' \times H^{n-k}$ and $(R_h \times R_h)_\#(\varphi' \times H^{n-k})$ have the same null sets for $h \in H$, it will follow that

$$\varphi' \times H^{n-k}(S_{A,1} \cap A' \times B_0 h) = 0,$$

and 4.1, Fubini's theorem and [7, 2.10.45] or [4, 4.1] will imply (*).

If $g \in f(G \times B_0)$, then it is clear that

$$F_g = G \times B_0 \cap f^{-1}\{g\}$$

is a proper $n-k$ dimensional submanifold of $G \times B_0$ which is mapped diffeomorphically by $\pi \circ p$ onto $g(B \cap N_e)$, where p: $G \times G \to G$ is the projection $p(a, b) = a$. Let C be the set of $\zeta \in A' \times B_0$ such that for some $\delta > 0$, the limit, as $s \to 0^+$, of the numbers

$$\sup_{0<r<\delta} \varphi' \times H^{n-k}[A' \times B_0 \cap \mathbf{X}_{G\times B_0}(\zeta, r, F_{f(\zeta)}, s)] r^{-m-n} s^{-m-n}$$

is zero. ($\mathbf{X}_{G\times B_0}$ indicates use of the metric in $G \times B_0$.)

The remainder of the proof is divided into two parts.

PART 1. $A' \times B_0 \cap S_{A,1} \subset C$.

Proof. Fix $(a_0, b_0) \in A' \times B_0 \cap S_{A,1}$; set $g_0 = a_0 b_0^{-1}$. Choose coordinates ψ for X in a neighborhood $N \subset N_e$ of $\pi(b_0)$ having compact closure in N_e so that $\psi[\pi(b_0)] = 0$ and

$$\psi(N \cap B) = \psi(N) \cap \beta,$$

where β is the linear subspace of $\mathbf{R}^n$ spanned by $e_{k+1}, \ldots, e_n$. Choose coordinates ψ_0 for I in a neighborhood M_I of e having compact closure so that $\psi_0(e) = 0$. Then

$$\Phi: M \times B_0 \cap N' \to \mathbf{R}^n \times \mathbf{R}^m \times \beta,$$

where $M = g_0[\chi_e(N \times M_I)]$ and

$$\Phi = (\psi \times \psi_0) \circ \chi_e^{-1} \circ L_{g_0}^{-1} \times \psi \circ \pi,$$

is a coordinate system at (a_0, b_0) with

$$\Phi(a_0, b_0) = (0, 0, 0) = 0;$$

setting
$$\alpha = \{(w, 0, w) : w \in \beta\}$$

we have
$$\Phi(M \times B_0 \cap N' \cap F_{g_0}) = \Phi(M \times B_0 \cap N') \cap \alpha.$$

Defining $A_0 = \psi[g_0^{-1}(A) \cap N]$ and $\varphi_0 = \psi_\#[g_{0\#}^{-1}(\varphi) \llcorner N]$, we infer from 3.8 that for some $\delta > 0$,

$$\lim_{s \to 0^+} \sup_{0<r<\delta} \varphi_0[A_0 \cap \mathbf{X}(0, r, \beta, s)] r^{-k} s^{-k} = 0.$$

Further, by 4.3

$$\Phi_\#(\varphi' \times H^{n-k} \llcorner M \times B_0 \cap N') = \varphi_0 \times \psi_{0\#}(H^m \llcorner M_I) \times (\psi \circ \pi)_\#(H^{n-k} \llcorner B_0 \cap N'),$$

hence we infer from application of 3.8 to Φ that $(a_0, b_0) \in C$ if

$$\lim_{s \to 0^+} \sup_{0<r<\delta} \varphi_0 \times H^m \times H^{n-k}[A_0 \times \mathbf{R}^m \times \beta \cap \mathbf{X}(0, r, \alpha, s)] r^{-m-n} s^{-m-n} \qquad (^{**})$$

is equal to zero.

Let P: $\mathbf{R}^n \times \mathbf{R}^m \times \beta \to \mathbf{R}^n$ be the projection. Fix $0 < s < \frac{1}{2}$. Using the orthonormal basis

$$\{2^{-\frac{1}{2}}(e_i, 0, e_i): i = k+1, \dots, n\} \cup \{2^{-\frac{1}{2}}(e_i, 0, -e_i): i = k+1, \dots, n\} \cup \{(e_i, 0, 0): i = 1, \dots, k\}$$

of $\mathbf{R}^n \times \{0\} \times \beta$, we verify that

$$\mathbf{X}(0, \infty, \alpha, s) = \mathbf{R}^n \times \mathbf{R}^m \times \beta \cap \{(u, v, w): \sum_{i=1}^{k} (u \cdot e_i)^2$$

$$< (1-s^2)^{-1}[s^2 \sum_{i=k+1}^{n} (u \cdot e_i)^2 + (s^2-1)|v|^2 + (s^2|w|^2 - \tfrac{1}{2} \sum_{i=k+1}^{n} [(u-w) \cdot e_i]^2)]\}.$$

For each $w_0 \in \beta$ define E_{w_0} on β by the formula

$$E_{w_0}(w) = s^2|w|^2 - \tfrac{1}{2}|w_0 - w|^2.$$

Observing that

$$\sup E_{w_0} = E_{w_0}[(1-2s^2)^{-1}w_0] = s^2(1-2s^2)^{-1}|w_0|^2,$$

we conclude that

$$P[\mathbf{X}(0, \infty, \alpha, s)] = \mathbf{R}^n \cap \{u_0 + w_0 : w_0 \in \beta, u_0 \in \beta^\perp, w \in \beta, v \in \mathbf{R}^m,$$

$$|u_0|^2 < (1-s^2)^{-1}[s^2|w_0|^2 + (s^2-1)|v|^2 + E_{w_0}(w)]\}$$

$$\subset \mathbf{R}^n \cap \{u : \sum_{i=1}^{k} (u \cdot e_i)^2 < 2s^2(1-2s^2)^{-1}|u|^2\} \subset \mathbf{X}(0, \infty, \beta, 2s).$$

Next we fix $r > 0$ and verify that

$$\mathbf{X}(0, r, \alpha, s) \subset \mathbf{R}^n \times \mathbf{R}^m \times \beta \cap \{(u, v, w): \sum_{i=1}^{k} (u \cdot e_i)^2 + \tfrac{1}{2} \sum_{i=k+1}^{n} [(u-w) \cdot e_i]^2 + |v|^2 < (rs)^2\}$$

$$\subset \mathbf{R}^n \times \mathbf{R}^m \times \beta \cap \{(u, v, w): \tfrac{1}{4}|u-w|^2 + |v|^2 < (rs)^2\},$$

whence for $u \in \mathbf{R}^n$ we have

$$\mathbf{X}(0, r, \alpha, s) \cap P^{-1}\{u\} \subset \{u\} \times (\mathbf{R}^m \cap K_{0,rs}) \times (\beta \cap K_{u,2rs}).$$

Finally, we apply Fubini's theorem to conclude that (**) is not greater than

$$\lim_{s \to 0+} \sup_{0<r<\delta} 2^n \alpha(m) \alpha(n-k) \varphi_0[A_0 \cap \mathbf{X}(0, r, \beta, 2s)] r^{-k} (2s)^{-k} = 0.$$

PART 2. $\varphi' \times H^{n-k}(C) = 0$.

Proof. Let U be an open subset of $G \times B_0$ in which there are coordinates

$$\chi: U \to \mathbf{R}^{m+n} \times \mathbf{R}^{n-k}$$

such that for each $g \in f(U)$ there exists $w \in \mathbf{R}^{m+n}$ with

$$\chi(F_g \cap U) = \chi(U) \cap F^{-1}\{F(w)\},$$

where F: $\mathbf{R}^{m+n} \times \mathbf{R}^{n-k} \to \mathbf{R}^{m+n}$ is the projection.

By 4.3 and 4.4, C is a purely $(\varphi' \times H^{n-k},\ m+n)$ unrectifiable subset of $G \times B_0$, hence application of 3.8 to χ and 4.5 to $\mu = \chi_\#(\varphi' \times H^{n-k} \llcorner U)$ enables us to conclude that

$$\varphi' \times H^{n-k}(C \cap U) = 0.$$

4.7. LEMMA. *If A is a Suslin subset of X and $\varphi(A) < \infty$, then*

$$\Psi[f(A' \times \beta' \cap S_{A,2})] = 0.$$

Proof. Since B is separable and

$$R_h \times R_h(S_{A,2}) = S_{A,2} \quad \text{for } h \in I,$$

it is sufficient to show that

$$\Psi[f(A' \times B_0 \cap S_{A,2})] = 0,$$

where $B_0 = \sigma_e(B \cap N_e)$.

Defining $F_g = G \times B_0 \cap f^{-1}\{g\}$ as in 4.6, we also denote by D the set of $\zeta \in A' \times B_0$ such that for all $\delta > 0$,

$$\limsup_{t \to 0+} \varphi' \times H^{n-k}[A' \times B_0 \cap \{\xi\colon \operatorname{dist}(\xi, F_{f(\zeta)}) < t\} \cap K_{\zeta,\delta}]\, t^{-m-n} = \infty.$$

The remainder of the proof is divided into two steps.

PART 1. $A' \times B_0 \cap S_{A,2} \subset D$.

Proof. Fix $(a_0, b_0) \in A' \times B_0 \cap S_{A,2}$. Proceeding as in the first two paragraphs of the proof of 4.6 Part 1 we conclude that for all $\delta > 0$,

$$\limsup_{s \to 0+} \sup_{0 < r < \delta} \varphi_0[A_0 \cap \mathbf{X}(0, r, \beta, s)]\, r^{-k} s^{-k} = \infty.$$

Further, we see as in 3.8 that $(a_0, b_0) \in D$ if for all $\delta > 0$,

$$\limsup_{t \to 0+} \varphi_0 \times H^m \times H^{n-k}[A_0 \times \mathbf{R}^m \times \beta \cap \{z\colon \operatorname{dist}(z, \alpha) < t\} \cap K_{0,\delta}]\, t^{-m-n} \tag{*}$$

is equal to infinity.

Let $P\colon \mathbf{R}^n \times \mathbf{R}^m \times \beta \to \mathbf{R}^n$ be the projection. Fix $0 < s < 1$ and $r > 0$. Suppose $u \in \mathbf{R}^n$, $\operatorname{dist}(u, \beta) = |u - w|$, $w \in \beta$. Then

$$\operatorname{dist}[(u, 0, w), \alpha] = |u - w|,$$

hence

$$\mathbf{X}(0, r, \beta, s) \subset \mathbf{R}^n \cap \{x\colon \operatorname{dist}(x, \beta) < rs\} = P[\mathbf{R}^n \times \mathbf{R}^m \times \beta \cap \{z\colon \operatorname{dist}(z, \alpha) < rs\}].$$

Furthermore, if $u \in \mathbf{X}(0, r, \beta, s/2)$ and $\operatorname{dist}[z, (u, 0, w)] < rs/2$, then $\operatorname{dist}(z, \alpha) < rs$. Thus, we can use Fubini's theorem to conclude that (*) is not less than

$$\limsup_{(r,s)\to(0,0)} 2^{-m-n}\alpha(m+n-k)\,\varphi_0[A_0 \cap \mathbf{X}(0, r, \beta, s/2)]\,(rs/2)^{-k} = \infty.$$

PART 2. $\Psi[f(D)] = 0$.

Proof. Let U be an open subset of $G \times B_0$ such that there are coordinates $\chi\colon U \to \mathbf{R}^{m+n} \times \mathbf{R}^{n-k}$ and $\chi_0\colon f(U) \to \mathbf{R}^{m+n}$ with

$$\chi_0 \circ f \,|\, U = F \circ \chi,$$

where $F\colon \mathbf{R}^{m+n} \times \mathbf{R}^{n-k} \to \mathbf{R}^{m+n}$ is the projection. Also choose U so that

$$\varphi' \times H^{n-k}(A' \times B_0 \cap U) < \infty.$$

Letting $A_1 = \chi(A' \times B_0 \cap U)$ and $\mu = \chi_\#(\varphi' \times H^{n-k} \llcorner U)$, we denote by D_1 the set of $(x, y) \in A_1$ such that for all $\delta > 0$,

$$\limsup_{t\to 0^+} \mu[A_1 \cap F^{-1}(K_{x,t}) \cap K_{(x,y),\delta}]\, t^{-m-n} = \infty.$$

We see as in 3.8 that $\chi(D \cap U) = D_1$, hence $\Psi[f(D \cap U)] = 0$ if $H^{m+n}[F(D_1)] = 0$.

Let ν be the measure on $\mathbf{R}^{m+n}$ such that

$$\nu(S) = \mu[A_1 \cap F^{-1}(S)]$$

for $S \subset \mathbf{R}^{m+n}$. It is clear that closed subsets of $\mathbf{R}^{m+n}$ are ν measurable. Hence the standard theorem on differentiation with respect to Lebesgue measure assures us that

$$\limsup_{t\to 0^+} \frac{\nu(K_{x,t})}{H^{m+n}(K_{x,t})} < \infty$$

for H^{m+n} almost all $x \in \mathbf{R}^{m+n}$. On the other hand, if $x \in F(D_1)$, then

$$\limsup_{t\to 0^+} \nu(K_{x,t})\, t^{-m-n} = \infty,$$

and we conclude that

$$H^{m+n}[F(D_1)] = 0.$$

4.8. THEOREM. *Suppose one of the following two conditions is satisfied:*

(i) *G is the group of isometries of X,* $\dim G = \frac{1}{2}n(n+1)$.

(ii) *X has a linear connection which is invariant under the action of G, I acts transitively on the space of $n-k$ dimensional linear subspaces of $\mathbf{T}_o(X)$, and B is totally geodesic.*

Let A be a purely (φ, k) unrectifiable Suslin subset of X such that $\varphi(A) < \infty$ and $\varphi(W) = 0$ whenever $W \subset A$ and $H^k(W) = 0$. Then

$$\varphi' \times H^{m+n-k}(A' \times B' \cap S_{A,1}) = 0,$$

$$\varphi' \times H^{m+n-k}[A' \times B' \sim (S_{A,2} \cup S_{A,3})] = 0,$$

$$\Psi[f(A' \times B' \cap S_{A,2})] = 0.$$

Proof. The first assertion is the conclusion of 4.6.

From 3.9 or 3.10 we infer that if $(a, b) \in A' \times B'$, then

$$H^m[\{a\} \times bI \sim (S_{A,1} \cup S_{A,2} \cup S_{A,3})] = 0.$$

Assured by 4.1 and 4.2 that $S_{A,1}$, $S_{A,2}$, $S_{A,3}$ are Suslin sets, we conclude from application of the coarea formula [7, 3.2.12] or [5, 3.1] to $\pi | B'$ that

$$H^{m+n-k}[\{a\} \times B' \sim (S_{A,1} \cup S_{A,2} \cup S_{A,3})] = 0.$$

Thus by Fubini's theorem

$$\varphi' \times H^{m+n-k}[A' \times B' \sim (S_{A,1} \cup S_{A,2} \cup S_{A,3})] = 0,$$

and this implies the second assertion.

The third assertion is the conclusion of 4.7.

5. The structure theorems

For this section we shall assume that either condition (i) or condition (ii) of 4.8 is satisfied. Let G have a left invariant metric.

5.1. THEOREM. *If $E \subset X$ with $H^k(E) < \infty$, then there exists a countably k rectifiable Borel subset R of X such that $E \sim R$ is purely (H^k, k) unrectifiable and*

$$(E \sim R) \cap g(B) = \varnothing$$

for Ψ almost all $g \in G$. Furthermore, if $H^k(E \cap R) > 0$ and E is H^k measurable, then

$$H = G \cap \{g\colon E \cap R \cap g(B) \neq \varnothing\}$$

is Ψ measurable and $\Psi(H) > 0$.

Proof. Since H^k is Borel regular, we can assume E to be a Borel set. Proceeding as in [7, 3.2.14] we obtain a countably k rectifiable Borel subset R of X such that $A = E \sim R$ is purely (H^k, k) unrectifiable. R is constructed by maximizing the finite measure $H^k \llcorner E$ on the class of countably k rectifiable Borel subsets of X.

Suppose S is a Suslin subset of $A' \times B'$ such that

$$(H^k \llcorner A)' \times H^{m+n-k}(S) = 0.$$

Using 4.3 and Fubini's theorem we infer that

$$H^{k+m} \times H^{m+n-k}(S) = 0,$$

hence conclude from [4, 4.1] or [7, 2.10.45] that

$$H^{2m+n}(S) = 0.$$

Applying this to 4.8 and using 4.1 and 4.2 we have

$$H^{2m+n}(A' \times B' \cap S_{A,1}) = 0,$$

$$H^{2m+n}[A' \times B' \sim (S_{A,2} \cup S_{A,3})] = 0.$$

Consequently, since $(a, b) \in S_{A,i}$ implies $(ah, bh) \in S_{A,i}$ for $h \in I$, we can apply the Eilenberg inequality [4, 3.2] or [7, 2.10.27] to conclude that

$$\Psi[f(A' \times B' \cap S_{A,1})] = 0,$$

$$\Psi[f(A' \times B' \sim [S_{A,2} \cup S_{A,3}])] = 0,$$

$$\Psi[f(A' \times B' \cap S_{A,2})] = 0.$$

Therefore, if we show that

$$\Psi[f(A' \times B' \cap S_{A,3})] = 0,$$

it will follow that $\Psi[f(A' \times B')] = 0$, which is equivalent to $A \cap g(B) = \emptyset$ for Ψ almost all $g \in G$.

Fix $g_0 \in G$ and let $\Sigma\colon U \to G \times G$ be a cross-section for f in a neighborhood U of g_0. Let $\sigma\colon V \to G$ be a cross-section for π in a neighborhood V of o such that $\sigma(o) = e$. Define the diffeomorphism

$$\chi\colon U \times V \times I \to W \subset G \times G$$

by the formula $\chi(g, x, h) = \Sigma(g)(\sigma(x)h, \sigma(x)h)$. Let I_0 be a compact neighborhood of e in I; set $c = H^m(I_0)$. Choose compact neighborhoods $U_0 \subset U$ of g_0 and $V_0 \subset V$ of o and let M be a Lipschitz constant for

$$(\chi \mid W_0)^{-1}, \; W_0 = \chi(U_0 \times V_0 \times I_0).$$

Choose U_0, V_0, I_0 so that also $H^{k+m} \times H^{m+n-k}(A' \times B' \cap W_0) < \infty$. Suppose $(a, b) = \chi(ab^{-1}, x, h_0) \in W_0$. Then if

$$\Phi(a, b) = \{(ah, bh) : h \in I\},$$

it follows that
$$\Phi(a, b) \cap W_0 = \chi(\{ab^{-1}\} \times \{x\} \times h_0^{-1} I_0)$$
and
$$H^m[\Phi(a, b) \cap W_0] \geq M^{-m} c.$$

We use the Eilenberg inequality, 4.3 and [7, 2.10.45] to obtain c_0 such that

$$\int^* H^m(A' \times B' \cap W_0 \cap f^{-1}\{g\})\, dH^{m+n} g \leq c_0 H^{k+m} \times H^{m+n-k}(A' \times B' \cap W_0) < \infty.$$

Consider $(a, b) \in A' \times B' \cap S_{A,3} \cap (\text{interior } W_0)$; set $g = ab^{-1}$. There is an infinite subset S of $A' \times B' \cap f^{-1}\{g\} \cap W_0$ such that $\pi \circ p \mid S$ is one-to-one, where $p: G \times G \to G$ is the projection on the first factor. Thus $\Phi(w) \cap \Phi(z) = \emptyset$ for $w, z \in S$, $w \neq z$, and $H^m(A' \times B' \cap W_0 \cap f^{-1}\{g\}) = \infty$; therefore, $H^{m+n}[f(A' \times B' \cap S_{A,3}) \cap \text{interior } U_0] = 0$.

Regarding verification of the last assertion of our theorem we infer from [7, 3.2.29] that we can assume R to be a proper k dimensional submanifold of X of class 1. We use the Fubini theorem, 4.3 and the Eilenberg inequality to infer that

$$H^{2m+n}(C) > 0, \quad C = (E \cap R)' \times B'.$$

Application of the coarea formula [5, 3.1] or [7, 3.2.11] yields

$$\int_C J(f \mid R' \times B')\, dH^{2m+n} = \int H^m[C \cap f^{-1}\{g\}]\, dH^{m+n} g.$$

Set $H = \{g\colon H^m[C \cap f^{-1}\{g\}] > 0\}$. C is H^{2m+n} measurable, hence H is H^{m+n} measurable and it is sufficient to show that $J(f \mid R' \times B')(z) \neq 0$ for some $z \in C$.

We can assume that $(e, e) \in C$ and

$$\mathbf{T}_o(R) + \mathbf{T}_o(B) = \mathbf{T}_o(X).$$

Choose orthonormal vectors $u_1, \ldots, u_{k+m}$ in $\mathbf{T}_e(R')$ and orthonormal vectors $v_1, \ldots, v_{n-k}$ in $\mathbf{T}_e(B')$ such that $v_1, \ldots, v_{n-k}, u_{k+1}, \ldots, u_{k+m}$ is an orthonormal basis of $\mathbf{T}_e(B')$ and $u_{k+1}, \ldots, u_{k+m}$ is an orthonormal basis of $\mathbf{T}_e(I)$. Using [2, 4.1] one proceeds as in the proof of [2, 4.2] to verify that

$$J(f \mid R' \times B')(e, e) = 2^{m/2} \left| \left[\bigwedge_{i=1}^{k+m} u_i\right] \wedge \left[\bigwedge_{i=1}^{n-k} v_i\right] \right| > 0.$$

5.2. *Definition.* If $x \in X$, the *k dimensional upper density* of φ at x is

$$\Theta^{*k}(\varphi, x) = \limsup_{r \to 0+} \alpha(k)^{-1} r^{-k} \varphi(K_{x,r}).$$

5.3. THEOREM. *Suppose $W \subset X$, $\varphi(W) < \infty$, $\varphi(S) = 0$ whenever $S \subset W$ and $H^k(S) = 0$, and*

$$\Theta^{*k}(\varphi \llcorner W, x) > 0 \quad \text{for } \varphi \text{ almost all } x \in W.$$

Then there exists a countably (φ, k) rectifiable and φ measurable set Q such that $W \sim Q$ is purely (φ, k) unrectifiable and

$$(W \sim Q) \cap g(B) = \varnothing$$

for Ψ almost all $g \in G$. Furthermore, if $\varphi(W \cap Q) > 0$ and W is a Borel set, then there exists a Ψ measurable subset H of G such that $\Psi(H) > 0$ and

$$W \cap Q \cap g(B) \neq \varnothing$$

whenever $g \in H$.

Proof. For each $i = 1, 2, \ldots$ consider the set

$$E_i = W \cap \{x \colon \Theta^{*k}(\varphi \llcorner W, x) > 1/i\}.$$

Using [3, 3.1] we see that $\infty > i\varphi(W) \geqslant H^k(E_i)$, hence we can apply 5.1 to E_i to obtain the countably k rectifiable Borel set R_i. Letting

$$F = W \sim \bigcup_{i=1}^{\infty} E_i, \quad Q = F \cup \bigcup_{i=1}^{\infty} R_i,$$

we conclude that $\varphi(F) = 0$,

$$W \sim Q \subset \bigcup_{i=1}^{\infty} (E_i \sim R_i),$$

and each $E_i \sim R_i$ is purely (φ, k) unrectifiable. We complete the proof by observing that if $\varphi(W \cap Q) > 0$ and W is a Borel set, then each E_i is a Borel set and $H^k(E_i \cap R_i) > 0$ for some i.

6. Integralgeometric measure

For this section we shall assume that G is the group of isometries of X with $\dim G = \frac{1}{2}n(n+1)$.

G is unimodular. Let Ψ be the Haar measure on G having H^n as its π image; Ψ is independent of choice of π.

Define $\beta(n, k) = \binom{n}{k}^{-1} \alpha(n)^{-1} \alpha(k)\, \alpha(n-k)$.

In 6.3 and 6.4 we shall be concerned with a set $\mathcal{E}$ of closed $n-k$ dimensional submanifolds of X such that G acts transitively on $\mathcal{E}$ and such that if $E \in \mathcal{E}$, then $G \cap \{g \colon g(E) = E\}$ is transitive on E. Also assume that $\mathcal{E}$ has a G invariant Haar measure. The members of $\mathcal{E}$ are necessarily of class ∞.

If X is connected, then X is isometric to either an Euclidean space, a sphere, a real projective space or a simply connected hyperbolic space; see [9, p. 308]. For each of these spaces $\mathcal{E}$ can be taken to be the set of closed, totally geodesic submanifolds of X of dimension $n-k$.

In [2, § 7] it is shown that there is a Haar measure on $\mathcal{E}$ such that if R is an (H^k, k) rectifiable subset of X, then

$$H^k(R) = \beta(n, k)^{-1} \int_{\mathcal{E}} H^0(R \cap E)\, d\Phi E.$$

If $\mathcal{E}$ is the space of $n-k$ dimensional planes in $\mathbf{R}^n$, then

$$\Phi = \lambda_\#(\Phi_n \times H^k),$$

where Φ_n is the Haar measure on $O(n)$ such that $\Phi_n[O(n)] = 1$ and

$$\lambda\colon O(n) \times \mathbf{R}^k \to \mathcal{E}$$

is defined by the formula

$$\lambda(g, w) = g(\mathbf{R}^n \cap \{x\colon x^i = w^i,\ i = 1, \ldots, k\}).$$

6.1. Theorem. *Assume $H^{n-k}(B) < \infty$. There exists a Borel regular measure $\mathfrak{J}_B$, the integralgeometric measure corresponding to B, such that if A is a Borel subset of X, then*

$$\mathfrak{J}_B(A) = \beta(n, k)^{-1} H^{n-k}(B)^{-1} \int_G H^0[A \cap g(B)]\, d\Psi g.$$

Furthermore, $\mathfrak{J}_B(W) = 0$ whenever $H^k(W) = 0$.

Proof. By using local cross-sections for π we can find a countably $n-k$ rectifiable Borel subset B^* of G such that $\pi | B^*$ is one-to-one onto B. Then for $g \in G$ and $A \subset X$,

$$H^0[A \cap g(B)] = H^0(A' \times B^* \cap f^{-1}\{g\}).$$

We apply [7, 2.10.10] to f in order to infer the existence of a Borel regular measure ψ on $G \times G$ such that for every Borel set $S \subset G \times G$,

$$\psi(S) = \int H^0(S \cap f^{-1}\{g\})\, d\Psi g.$$

The definition of ψ is analogous to that of H^l in 2.6; given a countable covering U of S and $u \in U$, one replaces

$$2^{-l}\alpha(l)\,(\text{diameter } u)^l \quad \text{with} \quad \Psi[f(u)].$$

Thus define for each Borel set $A \subset X$,

$$\mathfrak{I}_B(A) = \beta(n, k)^{-1} H^{n-k}(B)^{-1} \psi(A' \times B^*),$$

and for each $W \subset X$,

$$\mathfrak{I}_B(W) = \inf \{\mathfrak{I}_B(A)\colon W \subset A,\ A \text{ a Borel set}\}.$$

Suppose $H^k(W) = 0$; we can assume W to be a Borel set. From Fubini's theorem and 4.3 we infer that $H^{k+m}(W') = 0$, whence $H^{m+n}(W' \times B^*) = 0$ by Fubini's theorem and [4, 4.2]. Consequently,

$$\Psi[f(W' \times B^*)] = 0,$$

and we conclude from the definition of ψ that $\psi(W' \times B^*) = 0$.

6.2. THEOREM. *Suppose $H^{n-k}(B) < \infty$, $A \subset X$ and $H^k(A) < \infty$.*

(i) *If A is purely (H^k, k) unrectifiable, then $\mathfrak{I}_B(A) = 0$.*

(ii) *$H^k(A) \geq \mathfrak{I}_B(A)$, with equality if and only if A is (H^k, k) rectifiable.*

Proof. Let $A_0 \supset A$ be a Borel set such that $H^k(A_0) = H^k(A)$ and $\mathfrak{I}_B(A_0) = \mathfrak{I}_B(A)$. Applying 5.1 and 6.1 we obtain a countably k rectifiable Borel subset R of A_0 such that $A_0 \sim R$ is purely (H^k, k) unrectifiable and $\mathfrak{I}_B(A_0 \sim R) = 0$. Thus if A is purely (H^k, k) unrectifiable, we conclude from 6.1 that

$$\mathfrak{I}_B(A) \leq \mathfrak{I}_B(A_0 \sim R) + \mathfrak{I}_B(A \cap R) = 0.$$

For the general case we have by [2, 5.15]

$$\mathfrak{I}_B(A) = \mathfrak{I}_B(R) + \mathfrak{I}_B(A_0 \sim R) = H^k(R) \leq H^k(A);$$

in particular, if $\mathfrak{I}_B(A) = H^k(A)$, then A is (H^k, k) rectifiable. On the other hand, if A is (H^k, k) rectifiable, then we infer from [7, 3.2.29] that we can assume A_0 to be (H^k, k) rectifiable, hence conclude using [2, 5.15] and 6.1 that $\mathfrak{I}_B(A) = H^k(A)$.

6.3. THEOREM. *There exists a Borel regular measure $\mathfrak{I}_{\mathcal{E}}$, the integralgeometric measure corresponding to $\mathcal{E}$, such that if A is a Borel subset of X, then*

$$\mathfrak{I}_{\mathcal{E}}(A) = \beta(n, k)^{-1} \int_{\mathcal{E}} H^0(A \cap E)\, d\Phi E.$$

If $o \in Y \in \mathcal{E}$, then there is an open neighborhood β of o in Y such that

$$\mathfrak{I}_{\mathcal{E}} = \mathfrak{I}_\beta$$

Proof. Proceeding as in [2, § 7] we define

$$K = G \cap \{g \colon g(Y) = Y\},$$

$\dim K = n - k + \lambda$. Choose a left invariant metric on G so that $H^m(I) = 1$ and

$$\pi_\#(a) \,|\, \mathbf{T}_a(G), \quad \pi_\#(z) \,|\, \mathbf{T}_z(K)$$

are orthogonal projections whenever $a \in G$, $z \in K$. Letting $gK \in G/K$ correspond to $g(Y) = \pi(gK)$, we identify $\mathcal{E}$ with G/K.

Φ is the Haar measure on $\mathcal{E}$ such that for each Borel subset S of G,

$$\int H^{n-k+\lambda}(S \cap E)\, d\Phi E = H^\lambda(K \cap I)\, \Psi(S).$$

Assign a Riemannian metric to $\mathcal{E}$ and let $P\colon G \to G/K = \mathcal{E}$ be the projection. By Weil's condition [10, 9] or [7, 2.7.11] K is unimodular, hence there is a differentiable function ϱ on $\mathcal{E}$ such that $(JP^{-1}) = \varrho \circ P$. Inasmuch as $\Psi = H^{m+n}$, application of the coarea formula [5, 3.1] or [7, 3.2.11] to P enables us to conclude that $\Phi = H^\lambda(K \cap I)\varrho H^{m+k-\lambda}$.

Let α be an open neighborhood of the identity in K such that α has compact closure, $\alpha^{-1} = \alpha$, and $\alpha = \pi^{-1}[\pi(\alpha)] \cap K$. Define for $A \subset X$

$$\mathfrak{I}_{\mathcal{E}}(A) = \mathfrak{I}_\beta(A), \quad \beta = \pi(\alpha).$$

Considering a fixed Borel set $A \subset X$, we shall complete the proof by showing that

$$H^{n-k}(\beta) \int H^0(A \cap E)\, d\Phi E = \int H^0[A \cap g(\beta)]\, d\Psi g. \tag{*}$$

First suppose the right integral to be finite. Let η and ζ denote the characteristic functions of β and of A, respectively. Application of the coarea formula to P yields

$$\infty > H^\lambda(K \cap I) \int H^0[A \cap g(\beta)]\, d\Psi g = \int_{G/K} \int_F H^0[A \cap g(\beta)]\, dH^{n-k+\lambda} g\, d\Phi F.$$

Consider $F \in G/K$, $g_F \in F$, and suppose $A \cap g_F(\beta)$ is infinite. Then $A \cap g_F(\beta)$ has a cluster point $y \in g_F(Y)$. Since

$$K \cap \{g \colon y \in g_F g(\beta)\}$$

is open in K, we conclude that

$$H^0[A \cap g(\beta)] < \infty$$

for each $g \in F$, for Φ almost all $F \in G/K$. Fixing such an F and $g_F \in F$ we observe that $W = g_F^{-1}(A) \cap Y$ is countable and use the coarea formula to compute

$$\int_F H^0[A \cap g(\beta)]\, dH^{n-k+\lambda} g = \int_K \sum_W (\zeta \circ g_F)(\eta \circ z^{-1})\, dH^{n-k+\lambda} z$$

$$= \sum_{x \in W} \zeta \circ g_F(x) \int_K \eta \circ z^{-1}(x)\, dH^{n-k+\lambda} z$$

$$= \sum_{x \in W} \zeta \circ g_F(x)\, H^{n-k+\lambda}(\alpha) = H^0[A \cap \pi(F)]\, H^{n-k}(\beta)\, H^\lambda(K \cap I),$$

which implies (*).

Finally, suppose $H^{m+n}(S) > 0$, where

$$S = G \cap \{g : H^0[A \cap g(\beta)] = \infty\}.$$

S is H^{m+n} measurable, hence

$$\Phi\{E: H^0(A \cap E) = \infty\} \geqslant \Phi[P(S)] > 0,$$

and thus both integrals in (*) are infinite.

6.4. THEOREM. *Assume $A \subset X$ with $H^k(A) < \infty$.*

(i) *If A is purely (H^k, k) unrectifiable, then $\mathfrak{J}_\varepsilon(A) = 0$.*

(ii) *$H^k(A) \geqslant \mathfrak{J}_\varepsilon(A)$, with equality if and only if A is (H^k, k) rectifiable.*

Proof. Choose β so that $\mathfrak{J}_\varepsilon = \mathfrak{J}_\beta$ and apply 6.2.

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