SOLUTION IN BANACH ALGEBRAS OF DIFFERENTIAL EQUATIONS WITH IRREGULAR SINGULAR POINT

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1. Introduction

We have to do with linear first-order differential equations $W'(z) = F(z)\,W(z)$, where z is a complex variable, and F and W are functions taking values in an arbitrary non-commutative Banach algebra $\mathfrak{A}$ with identity E. In [4], E. Hille has discussed the existence and nature of analytic solutions when F is holomorphic, near a regular point of F, and near a regular singular point, and has indicated how the theory will go when the equation has an irregular singular point at infinity of rank 1. The methods are adapted from the classical theory in which $\mathfrak{A}$ is the complex field $\mathfrak{C}$.

The present paper adds to the discussion with an investigation, for the cases $p \geqslant 1$, of the equation

$$z\frac{d}{dz}W(z) = (z^p P_0 + z^{p-1} P_1 + \dots + z P_{p-1} + P_p)\,W(z), \tag{1.1}$$

a general form of first-order differential equation having an irregular singular point of rank p at infinity. Here $P_0, P_1, \dots, P_p$ are given elements of $\mathfrak{A}$, and an analytic and algebraically regular solution W is sought which takes its values $W(z)$ in $\mathfrak{A}$. The analogous equation in which $W(z)$ is a column matrix and the P's are square matrices, over $\mathfrak{C}$, was discussed in detail by G. D. Birkhoff in [1]. He assumed P_0 to be a matrix with distinct characteristic roots, and found solutions by writing $W(z)$ as a sum of Laplace integrals in the manner of Poincaré, using these to obtain asymptotic expansions for the solutions, valid for z tending to infinity in appropriate sectors of the plane, determined by the characteristic roots of P_0. The same procedure is adopted here, under an analogous though lighter restriction on P_0: we find a solution $W(z)$ valid when z lies in appropriate sectors, corresponding to a pole $\varkappa$ of $R(\lambda, P_0)$, the resolvent of P_0, whose residue idempotent has the property of being minimal. No assumptions are made upon the nature of the spectral set of P_0 complementary to $\varkappa$, except that it shall not intervene too awkwardly between $\varkappa$ and ∞. We also find an asymptotic expansion for the solution. The discussion leans heavily on the papers of Birkhoff and Hille. (For completeness, however, sufficient of this background material is included here for this paper to be read without prior acquaintance with these two.) A crucial part of the argument concerns the resolvent of a $p \times p$ matrix $\boldsymbol{C}$ whose elements are in $\mathfrak{A}$: by a detailed analysis we are able to specify the spectrum and the resolvent of $\boldsymbol{C}$ precisely, and thereby clarify some points which are obscure or incompletely covered in Birkhoff's paper. On the other hand, we do not examine here the question of the algebraic regularity of the solution $W(z)$, or the number of solutions.

The paper describes work done at Yale University and at the Mathematical Institute, Oxford. I am greatly indebted to Professor Hille, who suggested this investigation, and who in lectures and conversations introduced me to this subject and its literature. It is a great pleasure to thank him for his help. I must also thank Professor G. Temple for his interest and encouragement. The work was supported in part by the United States Army Research Office (Durham) under grant number DA ARO (D) 31–124–G 179, and by the United States Educational Foundation in Australia, under the Fulbright Act.

2. Reduction of the differential equation

We use capital Roman letters for the elements of $\mathfrak{A}$. The norm is written $\|\cdot\|$. Terms such as 'derivative' and 'holomorphic' for functions on $\mathfrak{C}$ to $\mathfrak{A}$ have the meanings given in [5], Chapter III, § 2; contour integrals are defined as Riemann-Stieltjes limits (see [5], Sections 3.3, 3.11). A prime usually denotes differentiation. As a superfix, T denotes matrix transposition.

We observe first that the substitution in (1.1) of

$$W(z) = Y(z) \exp\left(\frac{\alpha_0}{p} z^p + \frac{\alpha_1}{p-1} z^{p-1} + \dots + \alpha_{p-1} z\right) z^{\alpha_p} \tag{2.1}$$

($\alpha_0, \alpha_1, \dots, \alpha_p$ being constant scalars) leads to the equivalent equation

$$z Y'(z) = \{z^p (P_0 - \alpha_0 E) + z^{p-1}(P_1 - \alpha_1 E) + \dots + (P_p - \alpha_p E)\}\, Y(z). \tag{2.2}$$

The verification is immediate. The observation is in general false if the α's are non-scalars in $\mathfrak{A}$, because the algebra is non-commutative. The transformation was used by Birkhoff.

Given (1.1), we shall assume that a preliminary transformation (2.1) has been made, with scalars which will be determined presently, and we write

$$\bar{P}_j = P_j - \alpha_j E \quad (j = 0, 1, \dots, p). \tag{2.3}$$

We attempt to solve (2.2) by Poincaré's method as extended by Birkhoff; that is, we suppose that there exists a solution

$$Y(z) = \int_c e^{sz^p}(z^{p-1}V_1(s) + z^{p-2}V_2(s) + \dots + zV_{p-1}(s) + V_p(s))\,ds \tag{2.4}$$

in the form of a sum of Laplace integrals, for an appropriate choice of the contour c in the s-plane and functions $V_1, V_2, \dots, V_p$ taking values in $\mathfrak{A}$. In this and the following three sections we consider the determination of the V's, and we return to the choice of the contour c in § 6.

Formal substitution of (2.4) in (2.2) followed by a rearrangement of terms gives

$$\int_c e^{sz^p}\left(\sum_{k=1}^{2p} z^{2p-k}U_k(s)\right)ds = O, \tag{2.5}$$

where
$$U_k(s) = \begin{cases} (psE - \bar{P}_0)V_k(s) - \sum\limits_{j=1}^{k-1} \bar{P}_j V_{k-j}(s) & (1 \leqslant k \leqslant p), \\ (2p-k)V_{k-p}(s) - \sum\limits_{j=k-p}^{p} \bar{P}_j V_{k-j}(s) & (p+1 \leqslant k \leqslant 2p). \end{cases} \tag{2.6}$$

Write the sum in (2.5) in the form

$$\sum_{k=1}^{2p} z^{2p-k}U_k = \sum_{k=p+1}^{2p} z^{2p-k}U_k + z^p \sum_{k=1}^{p} z^{p-k}U_k = S_1 + z^p S_2, \tag{2.7}$$

and integrate by parts in (2.5), using

$$\frac{d}{ds}(e^{sz^p}) = z^p e^{sz^p}.$$

We get
$$\int_c e^{sz^p} S_1\,ds + [S_2 e^{sz^p}]_c - \int_c e^{sz^p}\frac{d}{ds}S_2\,ds = O. \tag{2.8}$$

In order that (2.4) satisfy (2.2), it suffices then to choose the V's so as to satisfy

$$S_1 = \frac{d}{ds}S_2$$

identically in z, i.e. to satisfy

$$U_{k+p}=\frac{dU_k}{ds}\ (k=1,2,\ldots,p), \tag{2.9}$$

and to choose the contour so that the sum of the integrated terms in (2.8) vanishes and the integrals converge. Thus we are led, by the substitution of (2.6) in (2.9), to the system of equations

$$(\boldsymbol{A}-ps\boldsymbol{E})\,\boldsymbol{v}'(s)=\boldsymbol{B}\boldsymbol{v}(s), \tag{2.10}$$

where $\boldsymbol{v}(s)$ is the column matrix $(V_1(s), V_2(s), \ldots, V_p(s))^T$, $\boldsymbol{A}$ and $\boldsymbol{B}$ are the triangular matrices

$$\boldsymbol{A}=\begin{pmatrix} \overline{P}_0 & & & \\ \overline{P}_1 & \overline{P}_0 & & \cdot \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ \overline{P}_{p-1} & & \overline{P}_1 & \overline{P}_0 \end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix} E+\overline{P}_p & \overline{P}_{p-1} & \overline{P}_{p-2} & & \overline{P}_1 \\ & 2E+\overline{P}_p & \overline{P}_{p-1} & \cdot & \\ & & \cdot & \cdot & \overline{P}_{p-2} \\ & & & \cdot & \overline{P}_{p+1} \\ & & & & pE+\overline{P}_p \end{pmatrix}, \tag{2.11}$$

and $\boldsymbol{E}$ is the identity matrix. Write $\boldsymbol{R}(\lambda, \boldsymbol{A})=(\lambda\boldsymbol{E}-\boldsymbol{A})^{-1}$, the resolvent matrix of $\boldsymbol{A}$; (2.10) is

$$\boldsymbol{v}'(s)=-\boldsymbol{R}(ps,\boldsymbol{A})\,\boldsymbol{B}\boldsymbol{v}(s). \tag{2.12}$$

Let $\mathfrak{B}_p, \equiv \mathfrak{B}_p(\mathfrak{A})$, denote the Banach space of vectors composed of p components belonging to $\mathfrak{A}$, with $\mathfrak{C}$ for scalar field, and norm

$$\boldsymbol{|}\boldsymbol{v}\boldsymbol{|}=\|V_1\|+\|V_2\|+\ldots+\|V_p\|. \tag{2.13}$$

If $\mathfrak{A}$ has finite dimension n, the dimension of $\mathfrak{B}_p$ is np; if $\mathfrak{A}$ is infinite-dimensional, so is $\mathfrak{B}_p$. To find a solution $\boldsymbol{v}$ of (2.12), we consider the analogous equation

$$\boldsymbol{V}'(s)=-\boldsymbol{R}(ps,\boldsymbol{A})\,\boldsymbol{B}\boldsymbol{V}(s) \tag{2.14}$$

in $\mathfrak{M}_p, \equiv \mathfrak{M}_p(\mathfrak{A})$, the algebra of $p\times p$ matrices with elements in $\mathfrak{A}$. As norm for $\mathfrak{M}_p$ we may take the maximum of row sums: if $\boldsymbol{X}\in\mathfrak{M}_p$ and the (i,j) element of $\boldsymbol{X}$ is X_{ij},

$$\boldsymbol{|}\boldsymbol{X}\boldsymbol{|}=\max_i\sum_{j=1}^{p}\|X_{ij}\|. \tag{2.15}$$

$\mathfrak{M}_p$ is then a Banach algebra over $\mathfrak{C}$, with $\mathfrak{A}$ as a left and right operator domain.

If $\boldsymbol{V}$ is a solution of (2.14), each of its columns is separately a solution of (2.12); conversely, any p solutions $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ of (2.12) can be put together to form a solution $\boldsymbol{V}$ of (2.14). If the $\boldsymbol{v}$'s are linearly independent over $\mathfrak{C}$, it is not necessarily true that $\boldsymbol{V}$ is regular in $\mathfrak{M}_p$: for example, the idempotent matrix $\boldsymbol{J}$ ((3.8), below) satisfies

$\boldsymbol{J}(\boldsymbol{E}-\boldsymbol{J})=\boldsymbol{O}$ and is therefore singular, but it is a triangular matrix, and its columns are therefore linearly independent. However, it is true that if $\boldsymbol{V}$ is regular, its columns are linearly independent. For suppose that $\lambda_1 \boldsymbol{v}_1+\ldots+\lambda_p \boldsymbol{v}_p=\boldsymbol{o}$ is a non-trivial linear relation among its columns. Then if $\boldsymbol{\Lambda}$ is the matrix each of whose columns is $(\lambda_1 E, \ldots, \lambda_p E)^T$, we have $\boldsymbol{V\Lambda}=\boldsymbol{O}$, and $\boldsymbol{V}$ is singular.

A regular solution of (2.14) therefore provides p linearly independent solutions of (2.12), while any non-zero solution of (2.14) provides at least one non-zero solution of (2.12). We therefore consider equation (2.14). Its solutions depend upon the singularities of $\boldsymbol{R}(ps, \boldsymbol{A})\,\boldsymbol{B}$.

In the following discussion we assume $p>1$. The case $p=1$ is trivially exceptional, and may be dealt with similarly.

3. The resolvent of $\boldsymbol{A}$

To find $\boldsymbol{R}(\lambda, \boldsymbol{A})$ explicitly, assume it to be a lower triangular matrix $\boldsymbol{X}$, and equate corresponding elements in the identity $\boldsymbol{X}(\lambda \boldsymbol{E}-\boldsymbol{A})=\boldsymbol{E}$. The elements of $\boldsymbol{X}$ are easily determined recursively, and the matrix is found to be a two-sided inverse. In this way we obtain

$$\boldsymbol{R}(\lambda, \boldsymbol{A})=\begin{pmatrix} R & & & & \\ S_1 & R & & & \\ S_2 & S_1 & R & & \\ \vdots & \ddots & \ddots & \ddots & \\ S_{p-1} & \cdots & S_2 & S_1 & R \end{pmatrix}, \tag{3.1}$$

where $R=R(\lambda, \overline{P}_0)=(\lambda E-\overline{P}_0)^{-1}$, the resolvent of $\overline{P}_0$ in $\mathfrak{A}$,

$$S_1=R\overline{P}_1R,$$
$$S_2=R\overline{P}_2R+R\overline{P}_1R\overline{P}_1R,$$

and generally
$$S_r=\sum_{(r)} R\overline{P}_{i_1}R\overline{P}_{i_2}R\ldots R\overline{P}_{i_h}R, \tag{3.2}$$

where '(r)' beneath the summation sign means that the sum is taken over all ordered partitions $(i_1, i_2, \ldots, i_h)$ of r:

$$i_1+i_2+\ldots+i_h=r, \quad i_1\geqslant 1, i_2\geqslant 1, \ldots, i_h\geqslant 1.$$

These formulae show that the singularities of $\boldsymbol{R}(ps, \boldsymbol{A})$ occur precisely at the singularities of $R(ps, \overline{P}_0)$. In general, a simple pole of the latter will produce a pole of the pth order in the former.

Let $\varkappa$ be an isolated point of Sp (P_0), the spectrum of P_0, and σ the complement of $\varkappa$ in Sp (P_0), assumed non-empty. Now Sp $(\bar{P}_0) = $ Sp $(P_0) - \alpha_0$. Choose

$$\alpha_0 = \varkappa; \tag{3.3}$$

then (1) if Γ is an oriented envelope of 0, e.g. a sufficiently small circle about 0 containing no other points of the spectrum, the integral

$$J = J_\varkappa = \frac{1}{2\pi i}\int_\Gamma R(\lambda, \bar{P}_0)\, d\lambda \tag{3.4}$$

defines a proper idempotent J in $\mathfrak{A}$: $J^2 = J, J \neq E, O$; J commutes with P_0. The functions $JR(\lambda, \bar{P}_0)$ and $(E-J)R(\lambda, \bar{P}_0)$ have holomorphic extensions in the complements of $\{0\}$ and $\sigma - \varkappa$ respectively; and for $|\lambda| > 0$,

$$JR(\lambda, \bar{P}_0) = \frac{J}{\lambda} + \sum_{n=1}^{\infty} \frac{(J\bar{P}_0)^n}{\lambda^{n+1}}. \tag{3.5}$$

Assume that $JP_0 = \varkappa J$. Then

$$R(\lambda, \bar{P}_0) = JR(\lambda, \bar{P}_0) + (E-J)R(\lambda, \bar{P}_0) = \frac{J}{\lambda} + H(\lambda), \tag{3.6}$$

where $H(\lambda)$ is holomorphic away from $\sigma - \varkappa$: that is, $R(\lambda, \bar{P}_0)$ has a simple pole at the origin with residue J, and the spectrum of $\boldsymbol{A}$ consists of a pole at the origin of order $\leqslant p$, together with $\sigma - \varkappa$.

In the rest of the paper the discussion refers to a fixed simple pole $\varkappa$, and the suffix in $J_\varkappa$ and other dependent expressions will be omitted. The only assumption made upon σ is that implied in the existence of the sector Σ, in § 6 below. If there are several such poles, each gives rise to solutions of (1.1) valid for z in appropriate sectors. We do not attempt to discuss solutions which may be determined by more complicated singularities.

The same spectral resolution can be applied to $\boldsymbol{A}$ in the algebra $\mathfrak{M}_p$. Thus the integral

$$\boldsymbol{J} = \boldsymbol{J}_\varkappa = \frac{1}{2\pi i}\int_\Gamma \boldsymbol{R}(\lambda, \boldsymbol{A})\, d\lambda = \text{residue of } \boldsymbol{R}(\lambda, \boldsymbol{A}) \text{ at } 0, \tag{3.7}$$

defines a lower triangular idempotent matrix

(1) [5], Section 5.6.

$$\boldsymbol{J}=\begin{pmatrix} J & & & & & \\ K_1 & J & & & & \\ K_2 & K_1 & J & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ K_{p-1} & & & K_2 & K_1 & J \end{pmatrix}; \tag{3.8}$$

and clearly, for $r=1,2,\ldots,p-1$,

$$K_r=\frac{1}{2\pi i}\int_\Gamma S_r\,d\lambda=\text{residue of } S_r \text{ at } 0. \tag{3.9}$$

It will be convenient on occasions to write $S_0=R$, $K_0=J$.

LEMMA 1. *Let the residue idempotent J at the simple pole 0 of $R(\lambda,\overline{P}_0)$ be minimal, that is, let $J\mathfrak{A}J$ be a division algebra isomorphic with the complex field.*(1) *Then by successive choice of $\alpha_1,\alpha_2,\ldots,\alpha_{p-1}$ we can ensure that $\boldsymbol{R}(\lambda,\boldsymbol{A})$ has a simple pole at the origin.*

The proof is by induction on the diagonals of $\boldsymbol{R}$. The leading-diagonal elements R have simple poles already. Again,

$$S_1=\left(\frac{J}{\lambda}+\ldots\right)\overline{P}_1\left(\frac{J}{\lambda}+\ldots\right)=\frac{J\overline{P}_1J}{\lambda^2}+\frac{K_1}{\lambda}+\ldots.$$

Since, by assumption, to every $A\in\mathfrak{A}$ there corresponds a scalar a such that $JAJ=aJ$, we can define numbers $\pi_0,\pi_1,\ldots,\pi_p$ by

$$JP_jJ=\pi_jJ\quad(j=0,1,\ldots,p). \tag{3.10}$$

Choose
$$\alpha_1=\pi_1. \tag{3.11}$$

Then $J\overline{P}_1J=(\pi_1-\alpha_1)J=O$, and S_1 has at most a simple pole at $\lambda=0$.

Assume that for some r less than $p-1$, the poles at 0 of $R,S_1,\ldots,S_r$ have, by choice of $\alpha_1,\ldots,\alpha_r$, been reduced to orders $\leqslant 1$; we show that we can do the same for S_{r+1} by choosing α_{r+1} appropriately. Now

$$\begin{aligned} S_{r+1}&=\sum_{(r+1)}R\overline{P}_{i_1}R\ldots R\overline{P}_{i_h}R\\ &=R\overline{P}_1S_r+R\overline{P}_2S_{r-1}+\ldots+R\overline{P}_rS_1+R\overline{P}_{r+1}R\\ &=\left(\frac{J}{\lambda}+\ldots\right)\overline{P}_1\left(\frac{K_r}{\lambda}+\ldots\right)+\ldots+\left(\frac{J}{\lambda}+\ldots\right)\overline{P}_r\left(\frac{K_1}{\lambda}+\ldots\right)\\ &\qquad+\left(\frac{J}{\lambda}+\ldots\right)\overline{P}_{r+1}\left(\frac{J}{\lambda}+\ldots\right)=\frac{G_{r+1}}{\lambda^2}+\frac{K_{r+1}}{\lambda}+\ldots,\quad\text{say.}\end{aligned}$$

(1) [7], p. 45.

It will be sufficient therefore if α_{r+1} is chosen so that the expression

$$G_{r+1} = J\overline{P}_1 K_r + J\overline{P}_2 K_{r-1} + \dots + J\overline{P}_r K_1 + J\overline{P}_{r+1} J \tag{3.12}$$

is zero. To show that this choice is possible, we use the set of identities

$$K_t J + K_{t-1} K_1 + \dots + K_1 K_{t-1} + J K_t = K_t \quad (t = 0, 1, 2, \dots, p-1) \tag{3.13}$$

(obtained by comparing corresponding elements in the identity $\boldsymbol{J}^2 = \boldsymbol{J}$). We have

$$G_{r+1} = \sum_{j=0}^{r} J\overline{P}_{j+1} \sum_{i=0}^{r-j} K_{r-j-i} K_i = \sum_{i=0}^{r} J \left(\sum_{k=i}^{r} \overline{P}_{r-k+1} K_{k-i} \right) K_i. \tag{3.14}$$

Let the numbers θ_{j+1} $(j = 1, 2, \dots, p-1)$ be those defined under the basic assumption by

$$J(\overline{P}_1 K_j + \overline{P}_2 K_{j-1} + \dots + \overline{P}_{j-1} K_2 + \overline{P}_j K_1) J = \theta_{j+1} J. \tag{3.15}$$

Then (3.14) becomes

$$G_{r+1} = \theta_{r+1} J + G_r K_1 + G_{r-1} K_2 + \dots + G_2 K_{r-1} + G_1 K_r + J(P_{r+1} - \alpha_{r+1} E) J.$$

The inductive hypothesis implies that $G_1 = G_2 = \dots = G_r = O$, so $G_{r+1} = (\theta_{r+1} + \pi_{r+1} - \alpha_{r+1}) J$, and choice of α_{r+1} so that $G_{r+1} = O$ is therefore possible. Then S_{r+1} has at most a simple pole at $\lambda = 0$; the result follows by induction. The α's are determined by

$$\alpha_j = \pi_j + \theta_j \quad (j = 1, 2, \dots, p-1). \tag{3.16}$$

More specifically, we define $\theta_1 = 0$, and then determine $\alpha_1, \theta_2, \alpha_2, \theta_3, \dots, \alpha_{p-1}$ successively by using (3.16) and (3.15) alternately, so that $G_1 = \dots = G_{p-1} = O$.

It is clear that the minimality of J is crucial to the proof of the lemma. Next we show that, with this restriction, the simple pole is the only case to be considered.

LEMMA 2. *If the residue idempotent J of $\overline{P}_0$ is minimal, $R(\lambda, \overline{P}_0)$ has at most a simple pole at the origin.*

Proof. By the properties of J,

$$J(P_0 - \varkappa E) = J^2 (P_0 - \varkappa E) = J(P_0 - \varkappa E) J = (\pi_0 - \varkappa) J. \tag{3.17}$$

If $\pi_0 = \varkappa$, then we have (3.6), and 0 is a simple pole. Suppose $\pi_0 \neq \varkappa$. From (3.5) we get

$$J R(\lambda, \overline{P}_0) = \frac{1}{\lambda - \pi_0 + \varkappa} J,$$

and (3.6) shows that the singularity of $R(\lambda, \overline{P}_0)$ at 0 is in fact removable.

Henceforth we assume that $R(\lambda, \bar{P}_0)$ has a simple pole at $\lambda=0$ with minimal residue idempotent there, and that the α's have been chosen so that $\boldsymbol{R}(\lambda, \boldsymbol{A})$ has a simple pole at $\lambda=0$. We remark that in general another simple pole $\lambda=\mu$ of $R(\lambda, P_0)$ would determine a different choice of the α's.

Equation (2.14) now has a regular singularity at $s=0$, and can be written

$$s\boldsymbol{V}'(s) = -s\boldsymbol{R}(ps, \boldsymbol{A})\boldsymbol{B}\boldsymbol{V}(s) = \left(\sum_{m=0}^{\infty} s^m \boldsymbol{C}_m\right)\boldsymbol{V}(s), \tag{3.18}$$

with

$$\boldsymbol{C} = \boldsymbol{C}_0 = -p^{-1}\boldsymbol{J}\boldsymbol{B}. \tag{3.19}$$

We can suppose that the series converges in norm for $|s| \leqslant \varrho$, for some $\varrho > 0$.

4. Solution of (3.18)

Define (1) the *commutator* $\mathfrak{T}_{\boldsymbol{A}}$ of an element $\boldsymbol{A}$ of $\mathfrak{M}_p$ to be the bounded linear transformation of $\mathfrak{E}(\mathfrak{M}_p)$ given by

$$\mathfrak{T}_{\boldsymbol{A}}[\boldsymbol{X}] = \boldsymbol{A}\boldsymbol{X} - \boldsymbol{X}\boldsymbol{A} \quad (\boldsymbol{X} \in \mathfrak{M}_p). \tag{4.1}$$

The solving of (3.18) proceeds as follows. If we attempt to make a trial solution by expressing $\boldsymbol{V}(s)$ as a power series in s, three distinct cases present themselves.

Case A. *No positive integer belongs to* $\mathrm{Sp}(\mathfrak{T}_{\boldsymbol{C}})$. In this case the formal substitution of

$$\boldsymbol{V}(s) = \sum_{n=0}^{\infty} \boldsymbol{A}_n s^{\boldsymbol{C}+n\boldsymbol{E}}, \quad \boldsymbol{A}_0 = \boldsymbol{E} \tag{4.2}$$

in (3.18) leads to a set of equations

$$(n\,\mathfrak{E} - \mathfrak{T}_{\boldsymbol{C}})[\boldsymbol{A}_n] = \sum_{k=1}^{n} \boldsymbol{C}_k \boldsymbol{A}_{n-k} \quad (n=1, 2, \ldots), \tag{4.3}$$

from which the coefficients $\boldsymbol{A}_n$ can be determined successively, and with these values, the series in (4.2) converges absolutely for $0<|s|<\varrho$ and is an actual solution of (3.18), *fundamental* in the sense that $\boldsymbol{V}(s)$ has an inverse in $\mathfrak{M}_p$ when s is in the punctured disc.

Case B. *Some positive integers belong to* $\mathrm{Sp}(\mathfrak{T}_{\boldsymbol{C}})$, *but they are all poles of the resolvent operator* $\mathfrak{R}(\lambda, \mathfrak{T}_{\boldsymbol{C}})$. In this case we make a formal substitution of the form

(1) For explication of the following remarks, see the discussion in Hille [4], of which they are an abridgement. We denote elements of $\mathfrak{E}(\mathfrak{M}_p)$, the algebra of bounded linear transformations on $\mathfrak{M}_p$, by Fraktur capital letters, writing $\mathfrak{E}$ for the identity operator.

$$\boldsymbol{W}(s,\eta)=\sum_{n=0}^{\infty}\boldsymbol{A}_n(\eta)\,s^{\boldsymbol{C}+(n+\eta)\boldsymbol{E}},\quad \boldsymbol{A}_0(\eta)=\eta^N\boldsymbol{E} \tag{4.4}$$

for $\boldsymbol{V}(s)$, where η is a small scalar parameter, and N is the sum of the orders of the poles of $\mathfrak{R}(\lambda,\mathfrak{T}_{\boldsymbol{C}})$ which occur at the positive integers. This leads to equations like (4.3), from which the coefficients can be determined successively, and $\lim_{\eta\to 0}\boldsymbol{W}(s,\eta)$ is then a solution of (3.18) for $0<|s|<\varrho$, which may, however, be identically zero. To obtain a fundamental non-zero solution it may be necessary to form

$$\lim_{\eta\to 0}\frac{\partial^N}{\partial\eta^N}\boldsymbol{W}(s,\eta);$$

this fundamental solution in general contains logarithmic terms, up to $(\log s)^N$.

Case C. Sp $(\mathfrak{T}_{\boldsymbol{C}})$ *contains positive integers which are not poles of* $\mathfrak{R}(\lambda,\mathfrak{T}_{\boldsymbol{C}})$. This case appears to be somewhat intractible.

It is clear from these results that the nature of $\mathrm{Sp}(\mathfrak{T}_{\boldsymbol{C}})$ must be clarified before we attempt to solve (3.18). Here we are helped by the following result.

LEMMA 3.(1)

(i) $\mathrm{Sp}\,(\mathfrak{T}_{\boldsymbol{C}})\subseteq\{\alpha-\beta:\ \alpha,\beta\in\mathrm{Sp}\,(\boldsymbol{C})\}$.

(ii) *Suppose that* γ, *belonging to* $\mathrm{Sp}\,(\mathfrak{T}_{\boldsymbol{C}})$, *can be written as the difference of poles* α_i, β_i *of* $\boldsymbol{R}(\lambda,\boldsymbol{C})$ *(say of orders* μ_i,ν_i *respectively) in only a finite number of ways. Then* γ *is a pole of* $\mathfrak{R}(\lambda,\mathfrak{T}_{\boldsymbol{C}})$, *of order* $\leqslant\max_i(\mu_i+\nu_i-1)$.

We show in the next section that the only singularities of $\boldsymbol{R}(\lambda,\boldsymbol{C})$ are simple poles at $0,\ -p^{-1},\ -2p^{-1},\ldots,\ -1$. It then follows from Lemma 3 that 1 is the only positive integer which could be in $\mathrm{Sp}\,(\mathfrak{T}_{\boldsymbol{C}})$ and that it would then occur as a simple pole. This confines the discussion to Cases A and B, with the necessity of at most one differentiation in the latter case, and thus represents a considerable simplification.

5. Spectrum and resolvent of C

Write
$$\boldsymbol{H}=\boldsymbol{JBJ}=-p\boldsymbol{CJ}. \tag{5.1}$$

Then, since $\boldsymbol{J}$ is idempotent, we have

$$-p\boldsymbol{C}^2=\boldsymbol{HC},\quad \boldsymbol{H}^2=-p\boldsymbol{CH}. \tag{5.2}$$

(1) Foguel [3]; quoted in [4].

LEMMA 4. $\operatorname{Sp}(\boldsymbol{C}) = -p^{-1}\operatorname{Sp}(\boldsymbol{H})$.

Proof. By using (5.1) we can easily verify the identities

$$p^2\lambda(\lambda\boldsymbol{E}-\boldsymbol{C}) = (p\lambda\boldsymbol{E}-\boldsymbol{H}-p\boldsymbol{C})(p\lambda\boldsymbol{E}+\boldsymbol{H}), \tag{5.3}$$

$$(p\lambda\boldsymbol{E}+\boldsymbol{H}+p\boldsymbol{C})(\lambda\boldsymbol{E}-\boldsymbol{C}) = \lambda(p\lambda\boldsymbol{E}+\boldsymbol{H}), \tag{5.4}$$

$$\frac{p\lambda\boldsymbol{E}+\boldsymbol{H}+p\boldsymbol{C}}{p\lambda} = \left(\frac{p\lambda\boldsymbol{E}-\boldsymbol{H}-p\boldsymbol{C}}{p\lambda}\right)^{-1}. \tag{5.5}$$

Suppose $\lambda \neq 0$. If $-p\lambda \notin \operatorname{Sp}(\boldsymbol{H})$, the right-hand side of (5.3) is a regular element of $\mathfrak{M}_p$ and so $\lambda \notin \operatorname{Sp}(\boldsymbol{C})$. Conversely if $\lambda \notin \operatorname{Sp}(\boldsymbol{C})$, (5.4) and (5.5) show that $-p\lambda \notin \operatorname{Sp}(\boldsymbol{H})$. Thus the lemma is established, except for the role of the point $\lambda = 0$.

Again, (5.2) can be written

$$(\boldsymbol{H}+p\boldsymbol{C})\,\boldsymbol{C} = \boldsymbol{O}, \quad (\boldsymbol{H}+p\boldsymbol{C})\,\boldsymbol{H} = \boldsymbol{O}. \tag{5.6}$$

If $\boldsymbol{H}+p\boldsymbol{C} = \boldsymbol{O}$, then $(p\lambda\boldsymbol{E}+\boldsymbol{H})^{-1} = p^{-1}(\lambda\boldsymbol{E}-\boldsymbol{C})^{-1}$, and the lemma follows immediately. If $\boldsymbol{H}+p\boldsymbol{C} \neq \boldsymbol{O}$, then $\boldsymbol{C}$ and $\boldsymbol{H}$ are singular elements, so that 0 belongs to the spectrum of both. The lemma now follows in this case also.

We get at the spectrum of $\boldsymbol{C}$ through $\boldsymbol{H}$, which is a more amenable matrix. To do this, it is convenient for the purposes of exposition to introduce new elements $\overline{P}_{p+1}, \overline{P}_{p+2}, \ldots, \overline{P}_{2p-1}$ in $\mathfrak{A}$ which we take to be of the form

$$\overline{P}_{p+1} = -\alpha_{p+1}J,\ \overline{P}_{p+2} = -\alpha_{p+2}J, \ldots,\ \overline{P}_{2p-1} = -\alpha_{2p-1}J;$$

the scalars $\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{2p-1}$ are to be fixed presently. Let $\boldsymbol{A}^*$ be the $2p \times 2p$ matrix got by enlarging $\boldsymbol{A}$, using the new $\overline{P}$'s:

$$\boldsymbol{A}^* = \begin{pmatrix} \overline{P}_0 & & & \\ \overline{P}_1 & \overline{P}_0 & & \\ \vdots & \ddots & \ddots & \\ \overline{P}_{2p-1} & \cdots & \overline{P}_1 & \overline{P}_0 \end{pmatrix}.$$

This has a resolvent $\boldsymbol{R}^*(\lambda, \boldsymbol{A}^*)$ of the same form as (3.1), with elements S_r defined by (3.2) for $r = 1, 2, \ldots, 2p-1$, those for $r = 1, 2, \ldots, p-1$ being the same as before. The residue idempotent $\boldsymbol{J}^*$ has the same form as (3.8), $K_p, \ldots, K_{2p-1}$ being the residues of $S_p, \ldots, S_{2p-1}$, so that (3.9) holds for $r = 0, 1, \ldots, 2p-1$, and (3.13) for $t = 0, 1, \ldots, 2p-1$. But $S_p, \ldots, S_{2p-1}$ do not necessarily have simple poles at the origin, so we use the process in the proof of Lemma 1 to reduce their poles to orders $\leqslant 1$ by choosing $\alpha_p, \ldots, \alpha_{2p-1}$ appropriately. That is, we define G_{r+1} for $r = 0, 1, 2, \ldots, 2p-2$ by (3.12), and θ_{j+1} for $j = 1, 2, \ldots, 2p-1$ by (3.15), and define the α's by

$$\alpha_p = \pi_p + \theta_p, \quad \alpha_j = \theta_j \quad (j = p+1, \ldots, 2p-1). \tag{5.7}$$

That is, having previously found $\alpha_1, \theta_2, \alpha_2, \theta_3, \ldots, \alpha_{p-1}$ in succession, we continue in the same way to find $\theta_p, \alpha_p, \theta_{p+1}, \alpha_{p+1}, \ldots, \alpha_{2p-1}$, using (5.7) in place of (3.16), so that

$$G_j = O \quad (j = 1, 2, \ldots, 2p-1). \tag{5.8}$$

$\boldsymbol{R}^*(\lambda, \boldsymbol{A}^*)$ now has a simple pole at $\lambda = 0$.

We shall need

LEMMA 5. *For* $r, t = 0, 1, \ldots, 2p-2$,

$$\begin{aligned} &K_{t+1}(\bar{P}_1 K_r + \bar{P}_2 K_{r-1} + \ldots + \bar{P}_r K_1 + \bar{P}_{r+1} J) \\ &\quad = (K_t \bar{P}_1 + K_{t-1} \bar{P}_2 + \ldots + K_1 \bar{P}_t + J\bar{P}_{t+1}) K_{r+1}. \end{aligned} \tag{5.9}$$

Proof. From the definition (3.2) of the S's we have, for $j = 1, 2, \ldots, 2p-2$,

$$\begin{aligned} S_{j+1} &= S_j \bar{P}_1 R + S_{j-1} \bar{P}_2 R + \ldots + S_1 \bar{P}_j R + R\bar{P}_{j+1} R \\ &= R\bar{P}_1 S_j + R\bar{P}_2 S_{j-1} + \ldots + R\bar{P}_j S_1 + R\bar{P}_{j+1} R. \end{aligned}$$

Therefore

$$\begin{aligned} &S_{t+1}(\bar{P}_1 S_r + \bar{P}_2 S_{r-1} + \ldots + \bar{P}_r S_1 + \bar{P}_{r+1} J) \\ &\quad = (S_t \bar{P}_1 R + S_{t-1} \bar{P}_2 R + \ldots + S_1 \bar{P}_t R + R\bar{P}_{t+1} R)(\bar{P}_1 S_r + \ldots + \bar{P}_{r+1} J) \\ &\quad = (S_t \bar{P}_1 + S_{t-1} \bar{P}_2 + \ldots + S_1 \bar{P}_t + R\bar{P}_{t+1}) S_{r+1}. \end{aligned} \tag{5.10}$$

In the first and last expressions in these equations write each S_j as its power series $K_j \lambda^{-1} + \ldots$ in λ, and equate coefficients of λ^{-2}: (5.9) follows.

We are now in a position to evaluate the matrix product $\boldsymbol{H} = \boldsymbol{JBJ}$. Write

$$\boldsymbol{B} = \boldsymbol{P} + \Omega E, \quad \boldsymbol{P} = \begin{pmatrix} \bar{P}_p & \bar{P}_{p-1} & \cdot & \bar{P}_1 \\ & \bar{P}_p & \cdot & \\ & & \cdot & \bar{P}_{p-1} \\ & & & \bar{P}_p \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \cdot & \\ & & & p \end{pmatrix}. \tag{5.11}$$

The product $\boldsymbol{J}\Omega\boldsymbol{J}$ is a lower triangular matrix; we show that $\boldsymbol{JPJ}$ is also, with zeros in the leading diagonal, and hence that $\boldsymbol{H}$ is lower triangular. The (i, j) element of $\boldsymbol{JPJ}$ is

$$\begin{aligned} (\boldsymbol{JPJ})_{i,j} &= (i\text{th row of } \boldsymbol{J})\, \boldsymbol{P}\, (j\text{th column of } \boldsymbol{J}) \\ &= \sum_{\gamma=0}^{p-j} \left(\sum_{\beta=1}^{\min(i, j+\gamma)} K_{i-\beta} \bar{P}_{p-j-\gamma+\beta} \right) K_\gamma. \end{aligned} \tag{5.12}$$

Consider the ith row. Suppose $j \geqslant i$: the inside sum runs from 1 to i, and the double sum

$$= \sum_{\beta=1}^{i} K_{i-\beta}\left(\sum_{\gamma=0}^{p-j+\beta-1} \bar{P}_{p-j+\beta-\gamma} K_\gamma - \sum_{\gamma=p-j+1}^{p-j+\beta-1} \bar{P}_{p-j+\beta-\gamma} K_\gamma\right).$$

The second inside sum is void when $\beta = 1$. Thus

$$(\boldsymbol{JPJ})_{i,j} = \sum_{\beta=1}^{i-1} K_{i-\beta} \sum_{\gamma=0}^{p-j+\beta-1} \bar{P}_{p-j+\beta-\gamma} K_\gamma + J \sum_{\gamma=0}^{p-j+i-1} \bar{P}_{p-j+i-\gamma} K_\gamma - \sum_{\beta=2}^{i} K_{i-\beta} \sum_{\gamma=p-j+1}^{p-j+\beta-1} \bar{P}_{p-j+\beta-\gamma} K_\gamma.$$

The second term on the right is G_{p-j+i}, by (3.12), while to the first we can apply Lemma 5. Thus the (i,j) element of $\boldsymbol{JPJ}$ equals

$$\sum_{\beta=1}^{i-1}\left(\sum_{\delta=0}^{i-\beta-1} K_\delta \bar{P}_{i-\beta-\delta}\right) K_{p-j+\beta} + G_{p-j+i} - \sum_{\alpha=2}^{i} K_{i-\alpha} \sum_{\gamma=p-j+1}^{p-j+\alpha-1} \bar{P}_{p-j+\alpha-\gamma} K_\gamma.$$

The two double sums cancel each other, and $G_{p-j+i} = O$. Thus $(\boldsymbol{JPJ})_{i,j} = O$ for $j \geqslant i$.

Suppose $j < i$. Then we write

$$(\boldsymbol{JPJ})_{i,j} = \sum_{\gamma=0}^{i-j-1} \sum_{\beta=1}^{j+\gamma} + \sum_{\gamma=i-j}^{p-j} \sum_{\beta=1}^{i} = \sum\nolimits_1 + \sum\nolimits_2, \quad \text{say.}$$

Now $$\sum\nolimits_2 = \sum_{\beta=1}^{i} K_{i-\beta}\left(\sum_{\gamma=0}^{p-j+\beta-1} - \sum_{\gamma=0}^{i-j-1} - \sum_{\gamma=p-j+1}^{p-j+\beta-1}\right) \bar{P}_{p-j+\beta-\gamma} K_\gamma = T_1 - T_2 - T_3, \quad \text{say.}$$

Here we deal with the first double sum by using Lemma 5 and (3.12) as before: T_1 equals G_{p-j+i} plus a double sum which cancels with T_3. Thus $T_1 - T_3 = O$, and

$$\begin{aligned}(\boldsymbol{JPJ})_{i,j} &= \sum\nolimits_1 - T_2 \\ &= \sum_{\gamma=0}^{i-j-1}\left(\sum_{\beta=1}^{j+\gamma} - \sum_{\beta=1}^{i}\right) K_{i-\beta} \bar{P}_{p-j+\beta-\gamma} K_\gamma \\ &= -\sum_{\gamma=0}^{i-j-1} \sum_{\delta=1}^{i-j-\gamma} K_{i-j-\gamma-\delta} \bar{P}_{p+\delta} K_\gamma \quad (\beta = j+\gamma+\delta) \\ &= \sum_{\delta=1}^{i-j} \alpha_{p+\delta} \sum_{\eta=0}^{i-j-\delta} K_\eta K_{i-j-\delta-\eta}.\end{aligned}$$

From (5.7) and (3.13) we deduce:

$$(\boldsymbol{JPJ})_{i,j} = O \ (j \geqslant i), \quad \sum_{\delta=1}^{i-j} \theta_{p+\delta} K_{i-j-\delta} \quad (j < i). \tag{5.13}$$

Write
$$\Theta = \begin{pmatrix} 0 & & & & & \\ \theta_{p+1} & 0 & & & & \\ \theta_{p+2} & \theta_{p+1} & 0 & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & & \cdot & \cdot & \cdot & \\ \theta_{2p-1} & & \cdot & \theta_{p+2} & \theta_{p+1} & 0 \end{pmatrix}, \quad \Phi = \Theta + \Omega. \tag{5.14}$$

It can be verified by direct evaluation that

$$\boldsymbol{JPJ} = \boldsymbol{J}\Theta\boldsymbol{J}, \tag{5.15}$$

so that, by (5.11),

$$\boldsymbol{H} = \boldsymbol{JBJ} = \boldsymbol{J}(\Theta + \Omega)\boldsymbol{J} = \boldsymbol{J}\Phi\boldsymbol{J}. \tag{5.16}$$

The (i, j) element of $\boldsymbol{H}$ is

$$H_{i,j} = \sum_{\delta=1}^{i-j} \theta_{p+\delta} K_{i-j-\delta} = j K_{i-j} + \sum_{\beta=1}^{i-j} \beta K_{i-j-\beta} K_\beta \quad (i \geqslant j;\ \text{empty sums are zero}). \tag{5.17}$$

LEMMA 6.
$$\operatorname{Sp}(\boldsymbol{C}) \subseteq \left\{0, -\frac{1}{p}, -\frac{2}{p}, \ldots, -1\right\}.$$

The singularities at $-p^{-1}, \ldots, -1$ *are simple poles at most, and* 0 *is a pole of order exactly* 1.(1)

Proof. $\boldsymbol{H}$ is now known to be a lower triangular matrix, and its leading diagonal is $(J, 2J, 3J, \ldots, pJ)$. The resolvant $\boldsymbol{R}(\lambda, \boldsymbol{H})$ can be formulated by the method used to derive (3.1), although in this case the formula is more complicated since elements of a given diagonal are not necessarily equal. Let $L_m = R(\lambda, mJ)$; we find

$$[\boldsymbol{R}(\lambda, \boldsymbol{H})]_{r,r} = L_r,$$

while for $r > s$,

$$[\boldsymbol{R}(\lambda, \boldsymbol{H})]_{r,s} = \sum_{(r,s)} L_r H_{r,i_1} L_{i_1} H_{i_1,i_2} L_{i_2} H_{i_2,i_3} \ldots H_{i_{l-1},s} L_s, \tag{5.18}$$

where '(r, s)' beneath the summation sign means that the sum is taken over all ordered sets $(i_1, i_2, \ldots, i_{l-1})$ of integers for which

$$r > i_1 > i_2 > \ldots > i_{l-1} > s.$$

Since J is a proper idempotent,

$$L_m = R(\lambda, mJ) = \frac{E - J}{\lambda} + \frac{J}{\lambda - m} \quad (m \neq 0), \tag{5.19}$$

(1) The lemma can be strengthened: see the last paragraph of this section.

and the spectrum of mJ consists of simple poles at 0 and m. It follows from (5.18) that $\mathrm{Sp}(\boldsymbol{H})$ consists of simple poles (at most) at $\lambda = 1, 2, \ldots, p$, together with a pole of order $\leqslant p$ at $\lambda = 0$.

Now by (5.4) and (5.5),

$$\boldsymbol{R}(\lambda, \boldsymbol{C}) = -\lambda^{-1}\boldsymbol{R}(-p\lambda, \boldsymbol{H})(p\lambda\boldsymbol{E} + \boldsymbol{H} + p\boldsymbol{C}). \tag{5.20}$$

From this the lemma follows, except for the order of the pole at 0. It remains to show that this is exactly 1. We do this by first obtaining the Laurent series for $\boldsymbol{R}(\lambda, \boldsymbol{H})$ about the origin. We state and prove two lemmas on the way.

LEMMA 7. *Let Δ be a lower triangular scalar matrix, of order p. Then there exists a unique matrix Z, of the same kind, such that*

$$\boldsymbol{J}\Delta\boldsymbol{J}Z\boldsymbol{J} = \boldsymbol{J} = \boldsymbol{J}Z\boldsymbol{J}\Delta\boldsymbol{J}, \tag{5.21}$$

if, and only if, none of the elements in the leading diagonal of Δ vanishes.

Proof. Write $\boldsymbol{M} = \boldsymbol{J}\Delta\boldsymbol{J}Z\boldsymbol{J}$, $\boldsymbol{N} = \Delta\boldsymbol{J}Z\boldsymbol{J}$, and let $\delta_{i,j}$ $\zeta_{i,j}$, $\boldsymbol{M}_{i,j}$ and $\boldsymbol{N}_{i,j}$ denote the (i,j) elements of $\Delta, Z, \boldsymbol{M}$ and $\boldsymbol{N}$ respectively. We assume $\zeta_{i,j} = 0$ for $i < j$, and determine the diagonals of Z inductively, starting with the leading one. For the kth subdiagonal, $0 \leqslant k \leqslant p - s$, $s = 1, 2, \ldots, p$,

$$\boldsymbol{M}_{s+k,s} = \sum_{\alpha=s}^{s+k} \boldsymbol{K}_{s+k-\alpha}\boldsymbol{N}_{\alpha,s}. \tag{5.22}$$

It can be verified that $\boldsymbol{M}_{s+k,s}$ contains only those elements of Z which are in the leading diagonal and k subsequent diagonals. For $\boldsymbol{M} = \boldsymbol{J}$ it is necessary that $\zeta_{j,j}\delta_{j,j} = 1$ $(j = 1, 2, \ldots, p)$. Suppose, for $j = 0, 1, \ldots, k-1$, that $\zeta_{s+j,s}$ $(s = 1, 2, \ldots, p-j)$ have been chosen so that

$$\boldsymbol{M}_{s+j,s} = \sum_{\alpha=s}^{s+j} \boldsymbol{K}_{s+j-\alpha}\boldsymbol{N}_{\alpha,s} = \boldsymbol{K}_j \quad (j = 0, 1, \ldots, k-1). \tag{5.23}$$

Then, for $s = 1, 2, \ldots, p-k$,

$$\boldsymbol{M}_{s+k,s} = \sum_{\alpha=s}^{s+k} \sum_{\beta=0}^{s+k-\alpha} \boldsymbol{K}_\beta \boldsymbol{K}_{s+k-\alpha-\beta}\boldsymbol{N}_{\alpha,s} \quad \text{(by (3.13))}$$

$$= \sum_{\beta=0}^{k} \boldsymbol{K}_\beta \sum_{\alpha=s}^{s+k-\beta} \boldsymbol{K}_{s+k-\alpha-\beta}\boldsymbol{N}_{\alpha,s} = \boldsymbol{J}\boldsymbol{M}_{s+k,s} + \sum_{\beta=1}^{k} \boldsymbol{K}_\beta \boldsymbol{K}_{k-\beta},$$

i.e.

$$\boldsymbol{M}_{s+k,s} = \boldsymbol{J}\boldsymbol{M}_{s+k,s} + \boldsymbol{K}_k - \boldsymbol{J}\boldsymbol{K}_k. \tag{5.24}$$

Similarly, by using instead the factorization $M = J\Delta JZ \cdot J$, we find

$$M_{s+k,s} = M_{s+k,s} J + K_k - K_k J. \tag{5.25}$$

Premultiply (5.25) by J and add to (5.24); we get

$$M_{s+k,s} = K_k + JM_{s+k,s} J - JK_k J. \tag{5.26}$$

The only element from the kth subdiagonal of Z which occurs in $JM_{s+k,s}J$ is $\zeta_{s+k,s}$, and its coefficient there is $\delta_{s+k,s+k} J$. Thus (5.26) shows that, by the minimal property of J and under the proviso in the statement of the lemma, we can choose $\zeta_{s+k,s}$ so that $M_{s+k,s} = K_k$. It is easily seen that $M_{s,s} = J$, $M_{s+1,s} = K_1$; the first equation of (5.21) follows by induction. It is clear the Z is determined uniquely.

Similarly, there is a unique matrix Z' for which $JZ'J\Delta J = J$. It follows that $JZ'J = JZJ$, and so $Z' = Z$. The lemma is proved.

The lemma is clearly a statement about a subset of regular elements in the algebra $J\mathfrak{M}_p J$. We shall write $Z = \Delta^\circ$, and call this matrix the *reciprocal* of Δ. With this notation, we now establish the Laurent series for $R(\lambda, H)$ about the origin.

LEMMA 8. *$R(\lambda, H)$ has a simple pole at $\lambda = 0$, and for $0 < |\lambda| < \nu = \lim_{n\to\infty} \|F^n\|^{-1/n}$,*

$$R(\lambda, H) = \frac{E - J}{\lambda} - F - \lambda F^2 - \lambda^2 F^3 - \dots, \tag{5.27}$$

where $F = J\Phi^\circ J$, Φ° being the reciprocal of $\Phi = \Theta + \Omega$.

Proof. We may assume from what is known so far that $R(\lambda, H)$ has a representation

$$R(\lambda, H) = \frac{Q^{p-1}}{\lambda^p} + \frac{Q^{p-2}}{\lambda^{p-1}} + \dots + \frac{Q}{\lambda^2} + \frac{U}{\lambda} - F - \lambda F^2 - \lambda^2 F^3 - \dots, \tag{5.28}$$

where U is idempotent, and the series converges for $0 < |\lambda| < \nu$, say. Now

$$R(\lambda, H)(\lambda E - H) = E = (\lambda E - H) R(\lambda, H), \tag{5.29}$$

and since $JH = H = HJ$,

$$R(\lambda, H)(\lambda J - H) = J = (\lambda J - H) R(\lambda, H);$$

therefore

$$\lambda R(\lambda, H)(E - J) = E - J = \lambda (E - J) R(\lambda, H). \tag{5.30}$$

Substitute from (5.28) in (5.29), and compare coefficients of powers of λ: we find

$$Q = UH, \tag{5.31}$$

$$-\boldsymbol{FH} = \boldsymbol{U} - \boldsymbol{E} = -\boldsymbol{HF}, \tag{5.32}$$

$$\boldsymbol{F}^2\boldsymbol{H} = \boldsymbol{F} = \boldsymbol{HF}^2. \tag{5.33}$$

Again, substitute from (5.28) in (5.30): we find

$$\boldsymbol{U}(\boldsymbol{E} - \boldsymbol{J}) = \boldsymbol{E} - \boldsymbol{J} = (\boldsymbol{E} - \boldsymbol{J})\,\boldsymbol{U}, \tag{5.34}$$

$$\boldsymbol{F}(\boldsymbol{E} - \boldsymbol{J}) = \boldsymbol{O} = (\boldsymbol{E} - \boldsymbol{J})\,\boldsymbol{F}. \tag{5.35}$$

We first use (5.34) to show that $\boldsymbol{U} = \boldsymbol{E} - \boldsymbol{J}$. Let $U_{i,j}$ denote the (i, j) element of $\boldsymbol{U}$, so that $U_{i,j} = O$ for $i < j$, and $U_{j,j} = E - J$, by (5.19). Let $1 < k \leqslant p$, and assume that for $j = 1, 2, \ldots, k-1$

$$U_{r,r-j} = -K_j \quad (r = j+1, j+2, \ldots, p); \tag{5.36}$$

we shall deduce the same identities for $j = k$. Comparing the $(r, r-k)$ elements in the first equation of (5.34) we get

$$U_{r,r-k}(E-J) - U_{r,r-k+1}K_1 - \ldots - U_{r,r-1}K_{k-1} - (E-J)\,K_k = -K_k,$$

which, with (5.36) and (3.13), gives

$$U_{r,r-k} = U_{r,r-k}J - K_k + K_k J. \tag{5.37}$$

Similarly, the second equation of (5.34) gives

$$U_{r,r-k} = JU_{r,r-k} - K_k + JK_k. \tag{5.38}$$

In a similar fashion, by comparing the $(r, r-k)$ elements in $\boldsymbol{U}^2 = \boldsymbol{U}$ and using (5.36) and (3.13), we find

$$U_{r,r-k} = U_{r,r-k}J + JU_{r,r-k} + JK_k + K_k J - K_k. \tag{5.39}$$

Add (5.37) to (5.38) and subtract (5.39), to find $U_{r,r-k} = -K_k$, which was to be proved. It can be verified that (5.36) holds for $j = 1$; it therefore holds for $j = 1, 2, \ldots, p-1$; i.e. $\boldsymbol{U} = \boldsymbol{E} - \boldsymbol{J}$.

From (5.31) it follows that $\boldsymbol{Q} = \boldsymbol{O}$, and so $\lambda = 0$ is a simple pole of $\boldsymbol{R}(\lambda, \boldsymbol{H})$. Again, (5.32) gives

$$\boldsymbol{FH} = \boldsymbol{J} = \boldsymbol{HF}; \tag{5.40}$$

and (5.35) gives

$$\boldsymbol{F} = \boldsymbol{JF} = \boldsymbol{FJ} = \boldsymbol{JFJ}. \tag{5.41}$$

It remains to verify that $\boldsymbol{F} = \boldsymbol{J}\boldsymbol{\Phi}^{\circ}\boldsymbol{J}$. Since $\boldsymbol{H} = \boldsymbol{J}\boldsymbol{\Phi}\boldsymbol{J}$, this value for $\boldsymbol{F}$ satisfies (5.40) and so (5.32), and by (5.41), also (5.33). Thus the representation (5.27) with this $\boldsymbol{F}$ satisfies (5.29) for $0<|\lambda|<\nu$; the uniqueness of the representation implies the result.

The Laurent series for the resolvent of $\boldsymbol{C}$ can now be written down. Using (5.20), we get

$$\boldsymbol{R}(\lambda, \boldsymbol{C}) = \frac{\boldsymbol{E}+p\boldsymbol{FC}}{\lambda} - (p\boldsymbol{F})^2\boldsymbol{C} + \lambda(p\boldsymbol{F})^3\boldsymbol{C} - \lambda^2(p\boldsymbol{F})^4\boldsymbol{C} + \dots . \tag{5.42}$$

Since $\boldsymbol{C}$ is singular (cf. (5.6)), the residue is non-zero and $\boldsymbol{R}(\lambda, \boldsymbol{C})$ has a simple pole at $\lambda=0$. Lemma 6 is now fully established.

The conclusions in the last paragraph of § 4 follow. We can write

$$\mathfrak{R}(1+\eta, \mathfrak{T}_{\boldsymbol{C}}) = \frac{\mathfrak{J}}{\eta} - \mathfrak{F} - \eta\mathfrak{F}^2 - \eta^2\mathfrak{F}^3 - \dots \quad (0<|\eta|<\delta) \tag{5.43}$$

for some $\delta>0$, $\mathfrak{J}$ being an idempotent operator. If $\mathfrak{J}=\mathfrak{O}$, we have Case A. Suppose $\mathfrak{J}\neq\mathfrak{O}$. We substitute (4.4) with $N=1$ into (3.18) and equate coefficients of all powers of s except the first, obtaining

$$\begin{aligned} &\boldsymbol{A}_0(\eta) = \eta\boldsymbol{E}, \\ &\boldsymbol{A}_1(\eta) = \mathfrak{R}(1+\eta, \mathfrak{T}_{\boldsymbol{C}})[\boldsymbol{C}_1\boldsymbol{A}_0] = \mathfrak{J}[\boldsymbol{C}_1] - \eta\mathfrak{F}[\boldsymbol{C}_1] + \dots, \\ &\boldsymbol{A}_n(\eta) = \mathfrak{R}(n+\eta, \mathfrak{T}_{\boldsymbol{C}})\left[\sum_{r=1}^{n} \boldsymbol{C}_r\boldsymbol{A}_{n-r}\right] \quad (n=2, 3, \dots). \end{aligned} \tag{5.44}$$

Thus a solution of (3.18) is

$$\boldsymbol{V}_{(0)}(s) = \lim_{\eta\to 0} \boldsymbol{W}(s, \eta) = \{s\mathfrak{J}[\boldsymbol{C}_1] + s^2\mathfrak{R}(2, \mathfrak{T}_{\boldsymbol{C}})[\boldsymbol{C}_1\mathfrak{J}[\boldsymbol{C}_1]] + \dots\}\, s^{\boldsymbol{C}}, \tag{5.45}$$

and this is not identically zero if $\mathfrak{J}[\boldsymbol{C}_1]\neq\boldsymbol{O}$, and is fundamental if $\mathfrak{J}[\boldsymbol{C}_1]$ is regular. The solution

$$\boldsymbol{V}_{(1)}(s) = \lim_{\eta\to 0} \frac{\partial \boldsymbol{W}(s, \eta)}{\partial\eta} = \{\boldsymbol{E} - s\mathfrak{F}[\boldsymbol{C}_1] + \dots\}\, s^{\boldsymbol{C}} + s\log s\,\{\mathfrak{J}[\boldsymbol{C}_1] + \dots\}\, s^{\boldsymbol{C}}. \tag{5.46}$$

is fundamental.

It is possible to compute $\mathfrak{J}$. Let the residue idempotent of $\boldsymbol{R}(\lambda, \boldsymbol{H})$ at $\lambda=k$ $(k=1, 2, \dots, p)$ be $\boldsymbol{U}_k$. It is not difficult to show that the leading diagonal of $\boldsymbol{U}_k$ has J in the kth position and zeros elsewhere, so that $\boldsymbol{U}_k\neq\boldsymbol{O}$ (thus all poles of $\boldsymbol{R}(\lambda, \boldsymbol{H})$ in fact have order exactly one). Then the elements of $\boldsymbol{U}_k$ can be found by methods akin to those of Lemma 8. The formulae are complicated. The residue idempotent of $\boldsymbol{R}(\lambda, \boldsymbol{C})$ at $\lambda=-kp^{-1}$ is found to be $-pk^{-1}\boldsymbol{U}_k\boldsymbol{C}$. A formula due to Daletsky [1] can be used to show that

$$\mathfrak{J}[\boldsymbol{X}] = -(\boldsymbol{E}+p\boldsymbol{FC})\boldsymbol{X}\boldsymbol{U}_p\boldsymbol{C} \quad (\boldsymbol{X}\in\mathfrak{M}_p).$$

[1] [2]; quoted in [4].

6. Solution of (1.1)

It remains to obtain a solution of (1.1) from a solution of (3.18).

There exist a priori estimates for solutions of (3.18). Suppose that σ (of § 3) is such that, in some open sector Σ with vertex 0, $\boldsymbol{R}(ps, \boldsymbol{A})$ is holomorphic. Write Σ_0 for any closed 'interior' sector. Then

$$M = \sup_{s \in \Sigma_0} \| s \boldsymbol{R}(ps, \boldsymbol{A}) \boldsymbol{B} \| \tag{6.1}$$

exists and is finite. Let s, s_0 lie on the same ray Υ from 0, in Σ_0. Then [1] for any solution $\boldsymbol{V}$ of (3.18) we have

$$\| \boldsymbol{V}(s) \| \leqslant k_1 \| \boldsymbol{V}(s_0) \| \cdot |s|^M \quad \text{for} \quad |s| > |s_0|, \tag{6.2}$$

$$\| \boldsymbol{V}(s) \| \leqslant k_2 \| \boldsymbol{V}(s_0) \| \cdot |s|^{-M} \quad \text{for} \quad |s| < |s_0|, \tag{6.3}$$

where the constants k_1, k_2 depend upon Σ_0.

We take the contour c in (2.4) to be a loop coming from infinity along Υ, encircling the origin once in a counterclockwise sense, and returning to infinity along Υ. If z is such that

$$\operatorname{re}(sz^p) < 0 \quad \text{for } s \text{ on } \Upsilon, \tag{6.4}$$

the estimate (6.2) can be used to show that this choice of c fulfils the requirements of § 2. Then any non-zero solution $\boldsymbol{V}(s)$ of (3.18) determines by (2.4) a solution $\boldsymbol{Y}(z)$ of (2.2), and in turn a solution $\boldsymbol{W}(z)$ of (1.1); $\boldsymbol{W}(z)$ is valid for z lying in some sector determined by the requirement that a ray Υ exists in Σ for which (6.4) holds. It is sufficient for our purposes if the solution $\boldsymbol{V}$ is not identically zero: it need not be fundamental. Let $\boldsymbol{V}$ be such a solution, with non-zero jth column

$$\boldsymbol{v}_j = \boldsymbol{V}\boldsymbol{e}_j \quad (\boldsymbol{e}_j = j\text{th column of } \boldsymbol{E}),$$

and write $\zeta^T = (z^{p-1}, z^{p-2}, \ldots, z, 1)$. The corresponding solution of (1.1) is

$$W_{(j)}(z) = z^{\pi_p + \theta_p} \exp\left(\frac{\varkappa}{p} z^p + \frac{\pi_1 + \theta_1}{p-1} z^{p-1} + \ldots + (\pi_{p-1} + \theta_{p-1}) z\right) Y_{(j)}(z), \tag{6.5}$$

with
$$Y_{(j)}(z) = \zeta^T \int_c e^{sz^p} \boldsymbol{V}(s)\, ds\, \boldsymbol{e}_j.$$

[1] Cf. [4], § 4.

7. Asymptotic expansion for the solution

We obtain an asymptotic formula for $Y(z)$ when $|z|$ is large, for the case where the solution $V(s)$ contains no logarithmic terms. The method comes from Horn [6].

Take
$$V(s)=\sum_{n=0}^{\infty} A_n s^{C+nE}=S(s)s^{C}, \quad S(s)=\sum_{n=0}^{\infty} A_n s^n, \tag{7.1}$$

the A's being supposed determined, and the series converging absolutely for $0<|s|\leqslant\varrho$, and write

$$\begin{aligned}\int_c e^{sz^p} V(s)\,ds &= \int_c e^{sz^p}\left\{V(s)-\sum_{n=0}^{N} A_n s^{C+nE}\right\}ds+\sum_{n=0}^{N} A_n \int_c e^{sz^p} s^{C+nE}\,ds \\ &= T_n + T_n', \text{ say.}\end{aligned} \tag{7.2}$$

To simplify the discussion, we suppose Υ chosen so that sz^p is real and negative for s on Υ. Thus if Σ_0 is the sector $\alpha\leqslant\theta\leqslant\beta$, the discussion applies to points z in the sector

$$\Psi\colon \frac{\pi-\beta}{p}\leqslant \arg z \leqslant \frac{\pi-\alpha}{p}; \tag{7.3}$$

also
$$\int_c e^{sz^p} s^{C+nE}\,ds = z^{-p(C+(n+1)E)}\int_{c'} e^{w} w^{C+nE}\,dw,$$

where c' is a loop contour from infinity along the negative real axis.

We are thus led to consider gamma functions of elements of $\mathfrak{M}_p$. Definitions for these are obtainable as follows from the operational calculus for a general Banach algebra with identity, described in [5], Chapter V. Let $X\in\mathfrak{M}_p$, and suppose that none of $0, -1, -2, \dots$ belong to $\mathrm{Sp}\,(X)$; then

$$\Gamma(X)=\frac{1}{2\pi i}\int_\gamma \Gamma(\xi)R(\xi, X)\,d\xi,$$

γ being an oriented envelope of $\mathrm{Sp}\,(X)$, defines $\Gamma(X)$ as a locally analytic function, and $\Gamma(\alpha E)=\Gamma(\alpha)E$ for scalar α. If $1\notin \mathrm{Sp}\,(X)$, then also $\Gamma(X)=(X-E)\Gamma(X-E)$. Again,

$$\psi(X)=\frac{1}{2\pi i}\int_\gamma [\Gamma(\xi)]^{-1}R(\xi, X)\,d\xi \tag{7.4}$$

defines a locally analytic function for all $X\in\mathfrak{M}_p$; and $0, -1, -2, \dots \notin \mathrm{Sp}\,(X)$, we have $\psi(X)=[\Gamma(X)]^{-1}$. The integral

$$\omega(X)=\frac{1}{2\pi i}\int_{c'} e^{w} w^{X}\,dw \tag{7.5}$$

is also defined for all $X \in \mathfrak{M}_p$; it can be verified (by expressing w^X as a series and using the properties of the operational calculus) that $\omega(X) = \psi(-X)$.

Since $\mathrm{Sp}\,(-C - nE) = \{-n,\ -(n-p^{-1}), \ldots, -(n-1)\}$, none of $\Gamma(-C-nE)$ $(n = 0, 1, 2, \ldots)$ is defined. However, we do have, for all $X \in \mathfrak{M}_p$,

$$\omega(X) = -X\omega(X-E) = -\omega(X-E)\,X, \tag{7.6}$$

and $\omega(C-E) = [\Gamma(E-C)]^{-1}$. Then (7.4) gives

$$\omega(C-E) = \frac{1}{2\pi i}\int_\gamma [\Gamma(1-\eta)]^{-1} R(\eta, C)\, d\eta = -E + J - \sum_{k=1}^{p} \frac{U_k C}{\dfrac{k}{p}\Gamma\left(1+\dfrac{k}{p}\right)},$$

where U_k is the residue idempotent of $R(\lambda, H)$ at $\lambda = k$; and using (7.6) we obtain, after some calculation,

$$\omega(C) = -\sum_{k=1}^{p} \frac{U_k C}{\dfrac{k}{p}\Gamma\left(\dfrac{k}{p}\right)}, \quad \omega(C+nE) = -\sum_{k=1}^{p-1} \frac{U_k C}{\dfrac{k}{p}\Gamma\left(\dfrac{k}{p}-n\right)} \quad (n = 1, 2, 3, \ldots). \tag{7.7}$$

The asymptotic expansion to be derived is: *For fixed* N $(> \|C\| - 2)$,

$$|z|^{p(N-\|C\|)+1} \left\| Y_{(j)}(z) - \zeta^T \sum_{n=0}^{N} A_n z^{-p(C+(n+1)E)}\, \omega(C+nE)\, e_j \right\| \to 0 \tag{7.8}$$

as z *tends to infinity along a ray in the sector* Ψ.

Let the contour c constitute the union of the several portions

c_1: those parts of the two arms along Υ for which $|s| > \frac{1}{2}\varrho$,
c_2: the parts of the arms along Υ for which $\delta < |s| < \frac{1}{2}\varrho$,
c_3: a counterclockwise circuit of the origin along $|s| = \delta$.

We may assume without loss of generality that $\frac{1}{2}\varrho \leqslant 1$. On c_2 and c_3 we can write

$$S(s) - \sum_{n=0}^{N} A_n s^n = \frac{s^{N+1}}{2\pi i}\int_{c_0} \frac{S(\zeta)\, d\zeta}{\zeta^{N+1}(\zeta - s)}, \quad c_0 = \{\zeta : |\zeta| = \tfrac{3}{4}\varrho\},$$

$\|S(s)\| \leqslant M_1 < \infty$, for some constant M_1, and deduce that

$$\left\| \int_{c_2 \cup c_3} e^{sz^p} \left(S(s) - \sum_{n=0}^{N} A_n s^n\right) s^C\, ds \right\| \leqslant \frac{3M_1}{(\frac{3}{4}\varrho)^{N+1}} \int_{c_2 \cup c_3} |e^{sz^p}|\, |s|^{N+1} \|s^C\|\, |ds|. \tag{7.9}$$

Write $|s| = \sigma$. Then on $c_2 \cup c_3$

$$\|s^{C}\| \leqslant e^{\|C\|(|\log \sigma|+2\pi)} = e^{2\pi\|C\|}\sigma^{-\|C\|}.$$

On c_2, $e^{sz^p} = e^{-\sigma|z|^p}$, while on c_3, $|e^{sz^p}| \leqslant e^{\delta|z|^p}$. Thus the contributions of the integral on the right-hand side of (7.9) are

$$\left.\begin{aligned} \int_{c_2} &\leqslant 2e^{2\pi\|C\|}|z|^{-p(N+2-\|C\|)}\Gamma(N+2-\|C\|), \\ \int_{c_3} &\leqslant 2\pi e^{\delta|z|^p+2\pi\|C\|}\delta^{N+2-\|C\|}. \end{aligned}\right\} \tag{7.10}$$

The first bound is independent of δ, so we may let $\delta \to 0$ in the second when N is large.

The contribution of c_1 to T_N is dominated by

$$\int_{c_1} |e^{sz^p}|\,\|V(s)\|\,|ds| + \sum_{n=0}^{N} \|A_n\| \int_{c_1} |e^{sz^p}|\,\|s^{C}\|\,|s|^n\,|ds| = Q_1 + Q_2, \quad \text{say.}$$

Let s_0 be the intersection of Υ with $|s| = \frac{1}{2}\varrho$. By (6.2),

$$Q_1 \leqslant 2k_1 \|V(s_0)\| \int_{\frac{1}{2}\varrho}^{\infty} e^{-\sigma|z|^p}\sigma^M\,d\sigma,$$

$$Q_2 \leqslant 2e^{2\pi\|C\|}\sum_{n=0}^{N}\|A_n\|\int_{\frac{1}{2}\varrho}^{\infty} e^{-\sigma|z|^p}\max(\sigma^{n-\|C\|},\ \sigma^{n+\|C\|})\,d\sigma.$$

Now for positive ξ and τ, and real α,

$$\int_{\tau}^{\infty} e^{-\xi\sigma}\sigma^{\alpha}\,d\sigma < 2\tau^{\alpha}\xi^{-1}e^{-\tau\xi} \text{ if } \xi > 2\alpha\max(1, \tau^{-1}).$$

Therefore

$$Q_1 \leqslant 4k_1\|V(s_0)\|(\tfrac{1}{2}\varrho)^M|z|^{-p}e^{-\frac{1}{2}\varrho|z|^p} \tag{7.11}$$

and

$$Q_2 \leqslant 4e^{2\pi\|C\|}|z|^{-p}\sum_{n=0}^{N}\|A_n\|\{(\tfrac{1}{2}\varrho)^{n-\|C\|}e^{-\frac{1}{2}\varrho|z|^p} + e^{-|z|^p}\}, \tag{7.12}$$

if $|z|$ is sufficiently large.

From (7.10), (7.11) and (7.12) it follows that, given any $\varepsilon > 0$ and any fixed $N(>\|C\|-2)$, we can find a K_N such that

$$|z|^{p(N+1-\|C\|)}\|T_N\| < \varepsilon \quad \text{for} \quad |z| > K_N. \tag{7.13}$$

Now when the norms are defined by (2.13) and (2.15), we have

$$\|f^T X g\| \leqslant p |f| \cdot |X| \cdot |g|$$

for any $f, g \in \mathfrak{V}_p$, $X \in \mathfrak{M}_p$. Therefore finally

$$\| Y_{(j)}(z) - \zeta^T \sum_{n=0}^{N} A_n z^{-p(C+(n+1)E)} \omega (C + nE) e_j \| \leqslant \|\zeta^T T_N e_j\| \leqslant p |z|^{p-1} |T_N|.$$

The formula (7.8) follows from this and (7.13).

Added in proof. It should be remarked that Birkhoff was in error in believing (1.1) to be a canonical form: see Gantmacher, F. R., *Theory of Matrices*, Vol. II, p. 147. I am grateful to Mr W. A. Coppel for drawing this to my attention.

Results related to the reduction in § 2 and to Lemma 1 are announced in Turrittin, H. L., Reducing the rank of ordinary differential equations, *Duke Math. J.*, 30 (1963), 271–274.

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