EMBEDDING THEOREMS FOR LOCAL ANALYTIC GROUPS

BY

S. ŚWIERCZKOWSKI

University of Sussex, Brighton, England

1. Results and fundamental concepts

Results

A Banach space X in which there is defined a continuous Lie multiplication $[x,y]$ will be called a normed Lie algebra. One can assign to every normed Lie algebra X a local group consisting of a sufficiently small neighbourhood of 0 in X in which the multiplication xy is given by the Campbell–Hausdorff–Schur formula

$$xy = x + y + \tfrac{1}{2}[xy] + \tfrac{1}{12}[y[yx]] + \tfrac{1}{12}[x[yx]] + \dots$$

(Birkhoff [3], Cartier [5] and Dynkin [10]). Let us denote this local group by $L(X)$. If X is finite dimensional, then $L(X)$ is of Lie type and therefore it is always locally embeddable in a group (Ado [1], Cartan [4], Pontrjagin [17]). We shall say that a normed Lie algebra X is an E-algebra if $L(X)$ is locally embeddable in a group. Since it has been discovered recently that not all normed Lie algebras are E-algebras (van Est and Korthagen [11]), it is natural to ask which of them are. In this direction we prove

THEOREM 1. *If X is a normed Lie algebra, $Y \subset X$ is a closed ideal and*

a) *the Lie algebra X/Y is abelian,*

b) *Y is an E-algebra,*

then X is an E-algebra.

We shall use this theorem in order to prove that an algebra X which is soluble, or soluble in a generalised sense is always an E-algebra. More precisely, let us say that the normed Lie algebra X is *lower soluble* if there exists an ordinal number α and an ascending sequence

$$\{0\} = X_0 \subset X_1 \subset X_2 \subset \dots X_\beta \subset X_{\beta+1} \subset \dots \subset X_\alpha = X$$

of closed subalgebras of X such that

a) if $\beta \leqslant \alpha$ is not a limit ordinal number, then $X_{\beta-1}$ is an ideal of X_β and the Lie algebra $X_\beta / X_{\beta-1}$ is abelian,

b) if $\beta \leqslant \alpha$ is a limit number, then X_β is the closure of $\bigcup_{\gamma<\beta} X_\gamma$ in X.

If that is so we shall also say that X is lower soluble with sequence $\{X_\beta\}_{\beta \leqslant \alpha}$ and we shall call the smallest ordinal α for which such a sequence exists, the type of X. We shall prove

THEOREM 2. *Every lower soluble normed Lie algebra is an E-algebra.*

One should ask whether there exist lower soluble Lie algebras of arbitrary given type; the answer is positive and it is not hard to construct such algebras modifying Gluškov's construction of lower soluble groups of arbitrary type (cf. [12] where, to begin with one should replace the matrix groups by their Lie algebras).

Added in proof. Using ideas of van Est and Korthagen [11] the author is able to show that the conclusion of Theorem 1 remains valid when the condition that X/Y is abelian is replaced by

a′) *X/Y is of finite dimension.*

Theorem 2 can be generalized correspondingly.

Partial and local groups

If P is a set, $D^{(n)} \subset P^n$ is a subset of the Cartesian product P^n of n copies of P and $f^{(n)}: D^{(n)} \to P$, then $f^{(n)}$ will be called an n-ary partial operation on P. Instead of $\langle x_1, x_2, \ldots, x_n \rangle \in D^{(n)}$ we shall say that $f^{(n)}(x_1, \ldots, x_n)$ exists. A *partial group* is a set P together with a subset $D^{(2)} \subset P \times P$ and a binary partial operation $f^{(2)}: D^{(2)} \to P$ such that if we denote $f^{(2)}(x,y)$ by xy, then

PG.1. If xy and yz exist, then either both $(xy)z$ and $x(yz)$ exist and $x(yz) = (xy)z$ or both $(xy)z$ and $x(yz)$ do not exist.

PG.2. There exists an element e in P such that xe and ex exist for every x in P and $xe = ex = x$.

PG.3. For every x in P there exists a unique x^{-1} in P such that xx^{-1} and $x^{-1}x$ exist and $xx^{-1} = x^{-1}x = e$.

PG.4. If xy exists, then $y^{-1}x^{-1}$ exists and $y^{-1}x^{-1} = (xy)^{-1}$.

The above definition is due to A. I. Malcev [15]. If P, P_1 are partial groups then a mapping $\psi: P \to P_1$ will be called a *homomorphism* if for every x, y in P such that xy exists we have that $\psi(x)\psi(y)$ exists in P_1 and is equal to $\psi(xy)$. A homomorphism ψ will be called an *embedding* if ψ is injective (i.e. if $x \neq y$ implies $\psi(x) \neq \psi(y)$). If P_1 is a group (i.e. if xy

exists for every x, y in P_1) and an embedding $\psi: P \to P_1$ exists, then we shall say that P is embeddable in a group.

Certain n-ary partial operations in a partial group P will be called *words*. These are defined by induction on n as follows

a) There exists exactly one unary partial operation $f^{(1)}$ which is a word, namely the identity operation $f^{(1)}(x) = x$,

b) Assume that $n > 1$ and that for every $k < n$ we have defined what we mean by saying that a k-ary partial operation is a word. Then a partial operation $f^{(n)}$ will be called a word if and only if there exist numbers k, l such that $k + l = n$ and partial operations $f^{(k)}, f^{(l)}$ which are words such that $f^{(n)}(x_1, \ldots, x_n)$ exists if and only if $f^{(k)}(x_1, \ldots, x_k)$, $f^{(l)}(x_{k+1}, \ldots, x_n)$ and $f^{(k)}(x_1, \ldots, x_k) f^{(l)}(x_{k+1}, \ldots, x_n)$ exist and, moreover
$$f^{(n)}(x_1, \ldots, x_n) = f^{(k)}(x_1, \ldots, x_k) f^{(l)}(x_{k+1}, \ldots, x_n).$$

We shall say that P satisfies the general associative law if for every n, for all words $f_1^{(n)}, f_2^{(n)}$ and for every n-tuple $\langle x_1, x_2, \ldots, x_n \rangle \in P^n$ such that both $f_1^{(n)}(x_1, \ldots, x_n)$ and $f_2^{(n)}(x_1, \ldots, x_n)$ exist, we have
$$f_1^{(n)}(x_1, \ldots, x_n) = f_2^{(n)}(x_1, \ldots, x_n).$$

We shall use the following criterion of Malcev [15].

EMBEDDABILITY CRITERION. *A partial group P is embeddable in a group if and only if P satisfies the general associative law.*

By a *local group* we shall mean a set L together with a subset $D^{(2)} \subset L \times L$ and a partial binary operation $f^{(2)}: D^{(2)} \to L$ such that

LG.1. L is a topological Hausdorff space,

LG.2. L is a partial group with respect to $f^{(2)}$,

LG.3. $D^{(2)}$ is an open subset of $L \times L$,

LG.4. The multiplication $f^{(2)}: D^{(2)} \to L$ is continuous,

LG.5. The mapping $x \to x^{-1}$ is continuous.

If L is a local group and $U \subset L$, then we shall say that xy exists in U if xy exists in L and $xy \in U$. If U is open and $U = U^{-1} = \{x^{-1} \mid x \in U\}$ then U together with the partial operation xy is a local group; we shall call U a piece of the local group L. If U is embeddable in a group, we shall say that L is *locally embeddable in a group*.

Analytic mappings and manifolds

In this section we shall define analytic locally Banach manifolds and analytic mappings of one such manifold into another. Similar definitions concerning C_∞-manifolds were given by S. Lang [14]. The analytic mappings of Banach spaces are due to R. S. Martin and A. D. Michal [16].

Analytic mapping. Let X, Y be Banach spaces and let, for every n, $u_n: X^n \to Y$ be a continuous n-linear mapping with norm $\|u_n\|$ (cf. [7], p. 99). For every $\langle x_1, \ldots, x_n\rangle$ in X^n such that all x_i are equal to x, we shall write $u_n x^n$ instead of $u_n(x_1, \ldots, x_n)$.

Let $U \subset X$ be open and let $f: U \to Y$. If $x_0 \in U$, we shall say that f is analytic at x_0, if there exists a sequence $u_1, u_2, u_3, \ldots$ where $u_n: X^n \to Y$ is n-linear and continuous such that for some $\varrho > 0$.

A. 1. $\sum_1^\infty \max_{\|x\| \leqslant \varrho} \|u_n x^n\| < \infty$,

A. 2. $f(x_0 + x) = f(x_0) + \sum_1^\infty u_n x^n$ for every $\|x\| < \varrho$.

We say that $f: U \to Y$ is analytic if f is analytic at every x_0 in U. The series in A.2 will be called the power series of f at x_0, and $\{x \in X \mid \|x\| < \varrho\}$ will be called a ball of analytic convergence of that series.

Analytic manifold. Let X be a Banach space and let M be a set. An analytic X-manifold on M is a set of pairs $\{\langle U_\tau, \phi_\tau\rangle\}_{\tau \in T}$ where T is some index set, such that

M.1. $\bigcup U_\tau = M$,

M.2. $\phi_\tau: U_\tau \to X$ is injective and $\phi_\tau(U_\tau \cap U_\mu)$ is always open in X,

M.3. $\phi_\mu \phi_\tau^{-1}: \phi_\tau(U_\tau \cap U_\mu) \to X$ is always analytic,

and moreover $\{\langle U_\tau, \phi_\tau\rangle\}_{\tau \in T}$ is maximal with respect to these properties (i.e. if

$$\{\langle U_\tau, \phi_\tau\rangle\}_{\tau \in T} \cup \{\langle U, \phi\rangle\}$$

has the properties M.1, M.2 and M.3, then $\langle U, \phi\rangle = \langle U_\nu, \phi_\nu\rangle$ for some $\nu \in T$). We call each pair $\langle U_\tau, \phi_\tau\rangle$ a chart and we define a topology in M by calling a set $U \subset M$ open if and only if $\phi_\tau(U \cap U_\tau)$ is open in X for every τ. When considering only one manifold on M, we shall denote it simply by M.

Let N be another analytic manifold and let $f: N \to M$ be a continuous map such that for any two charts $\langle U_\tau, \phi_\tau\rangle$, $\langle V_\mu, \psi_\mu\rangle$ of M and N respectively, the mapping

$$\phi_\tau f \psi_\mu^{-1}: \psi_\mu(V_\mu \cap f^{-1} U_\tau) \to X$$

is analytic. Then we shall say that $f: N \to M$ is analytic. If an analytic mapping $f: N \to M$ is bijective and the inverse $f^{-1}: M \to N$ is also analytic then we shall call f an analytic homeomorphism. We shall use the following basic facts.

Principle of analytic continuation. If N is a connected analytic manifold, $U \subset N$ is open and $f: N \to M$, $g: N \to M$ are analytic such that $f = g$ on U, then $f = g$ on N.

Composition principle. If Q, M, N are analytic manifolds and the mappings $f: Q \to M$, $g: M \to N$ are analytic, then so is their composite $g \circ f: Q \to N$.

These are easily proved, once they are known for the special case when M, N, Q are Banach spaces. In that latter case they can be shown similarly as for finite dimensional spaces in [8].

Normed Lie algebra

Let X be a Banach space with norm $\|\cdot\|$ over the field of real numbers. We shall call X a normed Lie algebra if there is a Lie multiplication $[x,y]$ defined in X (cf. Jacobson [12]) such that

$$\|[x,y]\| \leqslant \|x\| \cdot \|y\|$$

holds for every x,y in X.

By saying that Y is a closed subalgebra of X, we shall mean that $Y \subset X$ is a subalgebra in the usual sense and moreover Y is a closed subset of X. If Y is a closed ideal of X, then the coset space $X/Y = \{x+Y \mid x \in X\}$ can be made into a normed Lie algebra by defining the norm and Lie multiplication by

$$\|x+Y\| = \inf\{\|x+y\| \mid y \in Y\}; \quad [x+Y, z+Y] = [x,z]+Y.$$

Notation: Let $m_1, m_2, \ldots, m_k$ and $n_1, n_2, \ldots, n_k$ be two sequences of non-negative integers. Then we shall denote by

$$\langle m_1, n_1, m_2, n_2, \ldots, m_k, n_k; x, y\rangle$$

the sequence $x_1, x_2, \ldots, x_r$ each of whose terms is equal either to x or to y, such that the first m_1 terms are equal to x, the following n_1 terms are equal to y, the following m_2 equal to x, etc. (we then have $r = \sum m_i + n_i$). For this sequence $x_1, \ldots, x_r$ we define

$$[m_1, n_1, m_2, n_2, \ldots, m_k, n_k; x, y] = \frac{1}{r}[x_1[x_2[\ldots[x_{r-1}, x_r]\ldots]]]$$

if $r > 1$ and we put $[m_1, n_1, \ldots, m_k, n_k; x, y] = x_1$ if $r = 1$.

The Campbell–Hausdorff–Schur formula

Let X be a normed Lie algebra and let $B = \{x \in X \mid \exp 2\|x\| < 2\}$. By the SCH-formula (or series) we shall mean the mapping $\langle x,y\rangle \to xy$ of $B \times B$ into X defined by

$$xy = \sum_{k=1}^{\infty} \sum \frac{(-1)^{k+1}}{k}(m_1!\, n_1!\, m_2!\, n_2! \ldots m_k!\, n_k!)^{-1}[m_1, n_1, m_2, n_2, \ldots, m_k, n_k; x, y]$$

$$= x + y + \tfrac{1}{2}[xy] + \tfrac{1}{12}[y[yx]] + \tfrac{1}{12}[x[yx]] + \ldots,$$

where the $\sum$ is over all sequences of k pairs $\langle m_1, n_1\rangle, \ldots, \langle m_k, n_k\rangle$ of non-negative integers satisfying $m_i + n_i \geqslant 1$; $i = 1, \ldots, k$. The above form of the SCH-formula is due to E. B. Dynkin [9].

It is easy to see that the series

$$\sum_{k=1}^{\infty}\sum \max_{\|x\|\leqslant\xi,\,\|y\|\leqslant\eta}\left\|\frac{1}{k}(m_1!\,n_1!\,\dots\,m_k!\,n_k!)^{-1}[m_1,n_1,m_2,n_2,\dots,m_k,n_k;x,y]\right\|$$

is majorised by the expansion of $\sum_{k=1}^{\infty}(\exp\xi\exp\eta-1)^k$ in terms of $\xi^m/m!$ and $\eta^n/n!$. This proves that any ball in $X\times X$ of centre $\langle 0,0\rangle$ whose closure is contained in $B\times B$ is a ball of analytic convergence for the SCH-series. We shall need the following facts.

(i) *The mapping $\langle x,y\rangle\to xy$ of $B\times B$ into X is analytic.*

Indeed, we have above its power series expansion at $\langle 0,0\rangle$. The analyticity of the mapping at any other $\langle x_0,y_0\rangle\in B\times B$ follows from the fact that every such point belongs to a ball of analytic convergence of the SCH-series (cf. [16], Th. II 28, p. 47 and [2]).

(ii) *The multiplication xy defines a local group.*

Let $L(X)$ be the ball $\{x\in X\,|\,\exp 3\|x\|<2\}$. Then, for every $x,y,z\in L(X)$ such that xy and yz are in B we have $(xy)z=x(yz)$. Proofs of this identity can be found in Birkhoff [2] Cartier [5] and Dynkin [9], [10]. Let us say that the product xy exists in $L(X)$ if $xy\in L(X)$. Then $L(X)$, together with the multipliation xy is a local group; the unity e is the 0-vector in X and $x^{-1}=-x$ for every x. We shall denote henceforth the 0 in X by e and we shall write occasionally x^{-1} instead of $-x$. We note that x^2 exists if and only if $2x\in L(X)$ and then $x^2=2x$.

(iii) *If $z\in B$ and $xz=zx$ holds for all sufficiently small x in B, then z commutes with every x in B and $[x,z]=0$ for every x in B.*

To prove this, take an arbitrary x in B and denote by $\mathcal{J}$ the open interval $\{\lambda\,|\,\lambda x\in B\}$ which obviously contains 0 and 1. It is clear that

$$\lambda\to(\lambda x)z \quad\text{and}\quad \lambda\to z(\lambda x)$$

are analytic mappings of $\mathcal{J}$ into X, and as they coincide for small λ, they are identical on $\mathcal{J}$, whence $xz=zx$.

To prove the second part of our assertion we take any sufficiently small x in B so that

$$2(x^{-1}(\tfrac{1}{2}z)x)=(x^{-1}(\tfrac{1}{2}z)x)(x^{-1}(\tfrac{1}{2}z)x)=x^{-1}(\tfrac{1}{2}z)(\tfrac{1}{2}z)x=x^{-1}zx=z,$$

whence $x^{-1}(\frac{1}{2}z)x=\frac{1}{2}z$. This shows that $\frac{1}{2}z$ commutes with all sufficiently small x, and therefore with all x. Iterating this argument, we obtain that $2^{-n}z$ commutes with all x. It is now sufficient to apply the formula (cf. Birkhoff [3], Dynkin [10])

$$[x,z]=\lim_{n\to\infty}2^n\{(2^{-n}x)(2^{-n}z)(2^{-n}x)^{-1}(2^{-n}z)^{-1}\}.$$

Analytic local groups and analytic groups

A local group L with multiplication xy defined on $D^{(2)} \subset L \times L$ will be called analytic if L is an analytic manifold and the mappings $\langle x,y\rangle \to xy$, $x \to x^{-1}$ are analytic ($L \times L$, and hence $D^{(2)}$, has a natural manifold structure). If X is a normed Lie algebra, then the local group $L(X)$ defined in the previous section is an analytic local group. We shall call $L(X)$ the a.l.g. assigned to X.

By an analytic group we shall mean an a.l.g. in which the product of every two elements exists. We shall use the following theorem

Extension of analytic structure. *Let X be a normed Lie algebra, let $L(X)$ be the* a.l.g. *assigned to X and let $Q \subset L(X)$ be an open ball of centre* 0 *which, as an* a.l.g. *is embeddable in a group. Then there exists a simply connected analytic group G and an embedding $\varepsilon : Q \to G$ such that εQ is an open subset of G and the map $\varepsilon : Q \to \varepsilon Q$ is an analytic homeomorphism.*

The proof is the same as for Lie groups (cf. Cohn [7], Theorems 2.6.2, 2.7.1, 7.4.3 and 7.4.5).

If G is an analytic group then by a local analytic endomorphism of G we shall mean an analytic mapping $\psi : V \to G$ where $V \subset G$ is an open neighbourhood of the identity and $\psi(xy) = \psi(x)\psi(y)$ whenever $x, y, xy \in V$. If $V = G$, ψ will be called an analytic endomorphism. We shall use the following theorem (Chevalley [6], p. 49):

Extension of local endomorphism. *If G is a simply connected analytic group, $\psi : V \to G$ is a local analytic endomorphism and V is connected, then ψ can be extended to an analytic endomorphism.*

2. First embedding theorem

In this section we shall prove Theorem 1. We assume throughout that X is a normed Lie algebra and Y is a closed ideal of X such that

a) X/Y is an abelian Lie algebra,
b) Y is an E-algebra.

We shall show that X is an E-algebra.

The SCH-formula defines for all x, y in $B = \{x \in X \mid \exp 2\|x\| < 2\}$ their product xy. Denote by $\bar{x}$ the coset $x + Y \in X/Y$. We shall use only the following consequence of a);

a′) $\overline{xy} = \bar{x} + \bar{y}$ *for every x, y in B.*

To prove a′), note that, as X/Y is abelian, we have $[x, y] \in Y$ for all x, y in X, and as the only term in the SCH-formula which is not a bracket is $x+y$, it follows that xy and $x+y$ are in the same Y-coset.

Let us adopt from now on the convention that all balls are open balls in X with centre $0=e$. If $L(X)$ is the a.l.g. assigned to X then we have from b) that there exists a ball $Q\subset L(X)$ such that $Q \cap Y$ is embeddable in a group. By the principle of extension of analytic structure there exists a simply connected analytic group G and an embedding $\varepsilon: Q \cap Y \to G$ such that $\varepsilon(Q \cap Y)$ is an open subset of G and the map $\varepsilon: Q \cap Y \to \varepsilon(Q \cap Y)$ is an analytic homeomorphism. To simplify the notation, we shall assume that ε is the inclusion map, so that $Q \cap Y \subset G$ is a neighbourhood of e in G.

In the local group Q we have, for every natural n, an open neighbourhood U_n of e such that $f^{(n)}(x_1,\dots,x_n)$ exists, for every word $f^{(n)}$ and every $x_1,\dots,x_n$ in U_n (i.e. if $x_1,\dots,x_n \in U_n$, then $x_1x_2\dots x_n \in Q$ and this product does not depend on the way of placing the brackets). Let V be any ball contained in U_{10}. We shall prove that the local group V can be embedded in a group; this will be done by embedding V in a partial group P (Lemma 4) in which the general associative law is valid. Before doing this, we shall establish some relations between G and V.

The action of V on G

LEMMA 1. *There exists an analytic mapping $\phi: V\times G \to G$ such that*

$$\phi(x,y) = x^{-1}y\,x$$

for every $x\in V$, $y \in V \cap Y$. Moreover, $y \to \phi(x,y)$ is for every x in V an endomorphism of G.

Proof. For every x in V the mapping $y \to x^{-1}y\,x$ takes $V\cap Y$ into G. Indeed, if $x\in V$ and $y \in V \cap Y$, then

$$\overline{x^{-1}yx} = -\bar{x}+\bar{y}+\bar{x} = \bar{y} = 0$$

by a′). Since $VVV\subset Q$, we conclude that $x^{-1}(V\cap Y)x \subset Q \cap Y \subset G$ for every x in V. From $V\subset U_{10}$ it follows that the product of any six elements of V exists in Q and does not depend on the way of inserting the brackets. Thus it is seen that $y\to x^{-1}yx$ is a local endomorphism of G, defined on $V\cap Y$. This endomorphism is clearly analytic and by one of our remarks above it can be extended to an analytic endomorphism of G. Let us denote the latter by $\phi(x,y)$, i.e. $\phi(x,y)=x^{-1}yx$ for all $\langle x,y\rangle \in V\times(V\cap Y)$ and, for every fixed x in $V, y\to\phi(x,y)$ is an analytic endomorphism of G. It remains to prove that ϕ is analytic on $V\times G$, and for this purpose it is enough to show that the restriction of ϕ to $V\times y_0(V\cap Y)$ is analytic, for every y_0 in G (the product $y_0(V\cap Y)$ is in G). Since G is connected, it is generated by $V\cap Y$. Thus $y_0=y_1y_2\dots y_n$ where $y_i\in V\cap Y$ and hence

$$\phi(x,y)=\phi(x,y_1)\,\phi(x,y_2)\ldots\phi(x,y_n)\,\phi(x,y_0^{-1}y)$$

for every y in G. Using the composition principle for analytic functions, we find that the maps

$$\langle x, y\rangle \to x^{-1}(y_0^{-1}y)\,x=\phi(x,y_0^{-1}y)$$

and
$$\langle x, y\rangle \to x^{-1}y_i\,x=\phi(x, y_i);\quad i=1,\ldots,n$$

are analytic on $V\times y_0(V\cap Y)$ and thus $\phi(x,y)$ is analytic on that manifold, and hence on $V\times G$.

Notation: We shall denote $\phi(x,y)$ by y^x.

A set of representatives of Y-cosets in V

Let V/Y be the image of the ball V under the natural map $X\to X/Y$; the coset $x+Y$ belongs to V/Y if and only if $(x+Y)\cap V\neq\emptyset$.

LEMMA 2. *It is possible to select from every coset α belonging to V/Y a representative $x_\alpha\in\alpha\cap V$ such that if $t\alpha\in V/Y$, then $x_{t\alpha}=tx_\alpha$, for every real t.*

Proof. Let K be the boundary of V so that $V\cup K$ is a closed ball. We call a coset $\beta\in X/Y$ tangent to K if $\beta\cap K\neq\emptyset$ and $\beta\cap V=\emptyset$. We select now from every set $\{\beta,-\beta\}\subset V/Y$ where β and $-\beta$ are tangent to K, one of the two cosets. If $\varepsilon\beta$ is selected ($\varepsilon=1$ or -1), we associate with $\{\beta,-\beta\}$ an arbitrarily chosen element $x_{\varepsilon\beta}\in\varepsilon\beta\cap K$ and we take the elements $tx_{\varepsilon\beta}$; $|t|<1$ as the representatives of the cosets $t\varepsilon\beta$. Since for every coset $\alpha\neq 0$ intersecting V there exists a unique $\tau>1$ such that $\tau\alpha$ is tangent to K (τ is the greatest number such that $\tau\alpha\cap K\neq\emptyset$), we have defined x_α uniquely. Clearly $t\alpha\cap V\neq\emptyset$ implies $x_{t\alpha}=tx_\alpha$.

A formula for the multiplication in V

We shall say that an element x of V is written in normal form if $x=x_\alpha a$, where x_α is one of the representatives defined in the previous section and $a\in G$.

LEMMA 3. *Every x in V has a unique normal form. If $x_\alpha a$, $x_\beta b$ are any two elements of V in normal forms and their product $(x_\alpha a)(x_\beta b)$ also belongs to V, then its normal form is*

$$(x_\alpha a)(x_\beta b)=x_{\alpha+\beta}\,C_{\alpha,\beta}\,a^{x_\beta}\,b,$$

where $C_{\alpha,\beta}=x_{\alpha+\beta}^{-1}x_\alpha x_\beta\in G$.

Proof. If $x\in V$ and α is the Y-coset containing x, then

$$\overline{x_\alpha^{-1}x}=-\overline{x_\alpha}+\bar{x}=-\alpha+\alpha=0$$

by a′), i.e. the element $a = x_\alpha^{-1} x$ is in Y. Clearly a is in $V^2 \cup Y \subset Q \cap Y \subset G$. If $x_\alpha a = x_\gamma c$ are two normal forms of x, then $\overline{x_\alpha} + \bar{a} = \alpha + 0 = \overline{x_\gamma} + \bar{c} = \gamma + 0$ whence $x_\alpha = x_\gamma$ and $a = c$. If $x_\alpha a$, $x_\beta b$ are in normal forms, and $(x_\alpha a)(x_\beta b) \in V$, *then* $\overline{(x_\alpha a)(x_\beta b)} = \alpha + \beta \in V/Y$ and thus $x_{\alpha+\beta}$ exists. Moreover

$$x_\alpha a x_\beta b = (x_{\alpha+\beta} x_{\alpha+\beta}^{-1}) x_\alpha (x_\beta x_\beta^{-1}) a x_\beta b = x_{\alpha+\beta} C_{\alpha,\beta} a^{x\beta} b$$

because the multiplication can be performed in any order, by $V \subset U_{10}$ (note that $a, b \in VV$, thus we have above a product of 10 elements of V). Also

$$\bar{C}_{\alpha,\beta} = \overline{x_{\alpha+\beta}^{-1} x_\alpha x_\beta} = -\bar{x}_{\alpha+\beta} + \bar{x}_\alpha + \bar{x}_\beta = -(\alpha+\beta) + \alpha + \beta = 0,$$

whence $C_{\alpha,\beta} \in Y$. Clearly $C_{\alpha,\beta} \in V^3 \cap Y \subset Q \cap Y \subset G$. This completes the proof.

Remark. If $C_{\alpha,\beta}$ exists, i.e. if $\alpha, \beta, \alpha+\beta \in V/Y$, then the set of all real t for which $C_{t\alpha, t\beta}$ exists is an open interval $\mathfrak{J}_{\alpha,\beta}$ containing 0 *and* 1, *and*

$$t \to C_{t\alpha, t\beta}$$

is an analytic mapping of $\mathfrak{J}_{\alpha,\beta}$ into G.

Indeed, we have $\mathfrak{J}_{\alpha,\beta} = \{t \mid t\alpha, t\beta, t(\alpha+\beta) \in V/Y\}$. By Lemma 2,

$$C_{t\alpha, t\beta} = x_{t(\alpha+\beta)}^{-1} x_{t\alpha} x_{t\beta} = (-tx_{\alpha+\beta})(tx_\alpha)(tx_\beta),$$

which shows that the map $t \to C_{t\alpha, t\beta}$ is the composite of linear maps ($t \to tx_\alpha$, etc.) and of the group multiplication, hence it is analytic.

The partial group P

If $\alpha, \beta, \alpha+\beta \in V/Y$ and $a, b \in G$ are such that $x_\alpha a$, $x_\beta b$ and $(x_\alpha a)(x_\beta b) \in V$, then the normal form of $(x_\alpha a)(x_\beta b)$ is $x_{\alpha+\beta} C_{\alpha,\beta} a^{x\beta} b$. But $C_{\alpha,\beta} a^{x\beta} b$ remains meaningful for arbitrary a, $b \in G$, provided $a^{x\beta}$ is read as $\phi(x_\beta, a)$, as in Lemma 1. This suggests

LEMMA 4. *Let P denote the set $V/Y \times G$ together with the multiplication*

$$\langle \alpha, a \rangle \langle \beta, b \rangle = \langle \alpha+\beta,\ C_{\alpha,\beta} a^{x\beta} b \rangle$$

such that the product $\langle \alpha, a \rangle \langle \beta, b \rangle$ exists if and only if $\alpha+\beta \in V/Y$. Then P is a partial group. The mapping $\mu: V \to P$ given by

$$\mu(x_\alpha a) = \langle \alpha, a \rangle,$$

where $x_\alpha a$ runs over all elements of V in their normal forms, is an embedding of V into P.

Proof. It is easily checked that $\langle 0,e\rangle$ is the identity of P and that the inverse of $\langle\alpha,a\rangle$ is $\langle -\alpha, (a^{-1})^{-x_\alpha}\rangle$. Only the associative law PG.1 is not trivial; we shall prove it now.

It is easily seen that if $\langle\alpha,a\rangle\langle\beta,b\rangle$ and $\langle\beta,b\rangle\langle\gamma,c\rangle$ exist, then

$$(\langle\alpha,a\rangle\langle\beta,b\rangle)\langle\gamma,c\rangle \quad\text{and}\quad \langle\alpha,a\rangle(\langle\beta,b\rangle\langle\gamma,c\rangle)$$

exist or do not exist simultaneously depending whether $\alpha+\beta+\gamma$ is or is not in V/Y. Assume that these products exist. Then it is not hard to see that we have an open interval $\mathcal{J}$ containing 0 and 1 such that

$$(\langle t\alpha,a\rangle\langle t\beta,b\rangle)\langle t\gamma,c\rangle \quad\text{and}\quad \langle t\alpha,a\rangle(\langle t\beta,b\rangle\langle t\gamma,c\rangle)$$

both exist for all $t\in\mathcal{J}$ and are equal to

$$\langle t(\alpha+\beta+\gamma),\, F_1(t,a,b)\cdot c\rangle \quad\text{and}\quad \langle t(\alpha+\beta+\gamma),\, F_2(t,a,b)\cdot c\rangle,$$

where

$$F_1(t,a,b)=C_{t(\alpha+\beta),\,t\gamma}\,(C_{t\alpha,\,t\beta}\,a^{x_{t\beta}}\,b)^{x_{t\gamma}},$$

$$F_2(t,a,b)=C_{t\alpha,\,t(\beta+\gamma)}\,a^{x_{t(\beta+\gamma)}}\,C_{t\beta,\,t\gamma}\,b^{x_{t\gamma}}.$$

The Cartesian product $\mathcal{J}\times G\times G$ is a connected analytic manifold in a natural way. Moreover, the maps $t\to C_{t(\alpha+\beta),\,t\gamma}$, $\langle t,a\rangle\to a^{x_{t\beta}}=\phi(tx_\beta,a)$, etc. are analytic by Lemmas 1, 2 and the Remark following Lemma 3. This implies the analyticity of the maps

$$F_{1,2}\colon \mathcal{J}\times G\times G\to G.$$

But if $\langle t,a,b\rangle\in\mathcal{J}\times G\times G$ is sufficiently near to $\langle 0,e,e\rangle$, then we have, by Lemma 1

$$F_1(t,a,b)=(x_{t(\alpha+\beta+\gamma)}^{-1}\,x_{t(\alpha+\beta)}\,x_{t\gamma})\,(x_{t\gamma}^{-1}\,(x_{t(\alpha+\beta)}^{-1}\,x_{t\alpha}\,x_{t\beta}\,(x_{t\beta}^{-1}\,a x_{t\beta})\,b)\,x_{t\gamma}),$$

$$F_2(t,a,b)=(x_{t(\alpha+\beta+\gamma)}^{-1}\,x_{t\alpha}\,x_{t(\beta+\gamma)})\,(x_{t(\beta+\gamma)}^{-1}\,a x_{t(\beta+\gamma)})\,(x_{t(\beta+\gamma)}^{-1}\,x_{t\beta}\,x_{t\gamma})\,(x_{t\gamma}^{-1}\,b x_{t\gamma}),$$

whence $F_1(t,a,b)=F_2(t,a,b)$. It follows now, by the principle of analytic continuation that $F_1=F_2$ on $\mathcal{J}\times G\times G$. In particular we have $F_1(1,a,b)=F_2(1,a,b)$ which proves that

$$(\langle\alpha,a\rangle\langle\beta,b\rangle)\langle\gamma,c\rangle=\langle\alpha,a\rangle(\langle\beta,b\rangle\langle\gamma,c\rangle).$$

To complete the proof of Lemma 4, we note that μ is evidently an injection. Moreover, the formula $(x_\alpha a)(x_\beta b)=x_{\alpha+\beta}\,C_{\alpha,\beta}\,a^{x_\beta}b$ valid in V (Lemma 3), implies that μ is a homomorphism. Hence μ is an embedding. This completes the proof.

Remark. If $f^{(n)}$ is a word and $\langle\alpha_1,a_1\rangle,\dots,\langle\alpha_n,a_n\rangle\in P$ are such that $f^{(n)}(\langle\alpha,a\rangle,\dots,\langle\alpha_n,a_n\rangle)$ exists, then $\alpha_1+\alpha_2+\dots+\alpha_n\in V/Y$ and $f^{(n)}(\langle\alpha_1,u_1\rangle,\dots,\langle\alpha_n,u_n\rangle)$ exists for all $\langle u_1,\dots,u_n\rangle$ in G^n.

The general associative law in P

The proof of this law is much the same as that of the associative law, but, of course, we do not find the functions F_1, F_2 explicitly. Instead, we use the following

LEMMA 5. *If $\alpha,\beta,\dots,\varkappa$ are arbitrary fixed n elements of V/Y and $f^{(n)}$ is a word such that $f^{(n)}(\langle\alpha,a\rangle, \langle\beta,b\rangle,\dots,\langle\varkappa,h\rangle)$ exists for all $\langle a,b,\dots,h\rangle$ in G^n* (see Remark in previous section) *then there exists an open real interval $\mathcal{J}$ containing* 0 *and* 1 *and an analytic mapping*

$$\langle t,a,b,\dots,h\rangle \to F(t,a,b,\dots,h)$$

of $\mathcal{J}\times G^n$ into G such that $f^{(n)}(\langle t\alpha,a\rangle,\dots,\langle t\varkappa,h\rangle)$ exists for every $\langle t,a,b,\dots,h\rangle$ in $\mathcal{J}\times G^n$ and

$$f^{(n)}(\langle t\alpha,a\rangle, \langle t\beta, b\rangle,\dots,\langle t\varkappa,h\rangle) = \langle t(\alpha+\beta+\dots+\varkappa),\, F(t,a,b,\dots,h)\rangle.$$

Postponing the proof of Lemma 5, we shall deduce now the general associative law in P. Suppose that $f_1^{(n)}$ is another word such that $f_1^{(n)}(\langle\alpha,a\rangle, \langle\beta,b\rangle,\dots,\langle\varkappa,h\rangle)$ is defined for all $\langle a,b,\dots,h\rangle$ in G^n. We have to show that

$$f^{(n)}(\langle\alpha,a\rangle, \langle\beta,b\rangle, \dots ,\langle\varkappa,h\rangle) = f_1^{(n)}(\langle\alpha,a\rangle, \langle\beta,b\rangle, \dots ,\langle\varkappa,h\rangle),$$

which by Lemma 5 is equivalent to

$$F(1,a,b,\dots,c) = F_1(1,a,b,\dots,c),$$

where F_1 is related to $f_1^{(n)}$ in the same way as F to $f^{(n)}$. Let us denote by $\Omega\subset V$ a ball such that if $x_1,\dots,x_{2n}$ are any $2n$ elements of Ω, then the product $x_1x_2\dots x_{2n}$ exists in whichever way we insert brackets and belongs to V. Let $\mathcal{J}_0$ be the real interval such that $tx_\alpha, tx_\beta, \dots, tx_\varkappa \in\Omega$ for all $t\in\mathcal{J}_0$. If $\langle t,a,b,\dots,h\rangle\in\mathcal{J}_0\times\Omega^n$, then $(tx_\alpha)a, (tx_\beta)b,\dots,(tx_\varkappa)h\in\Omega\Omega$ which implies that in the local group V

$$f^{(n)}(x_{t\alpha}a, x_{t\beta}b, \dots, x_{t\varkappa}h) = f_1^{(n)}(x_{t\alpha}a, x_{t\beta}b, \dots, x_{t\varkappa}h).$$

Applying the embedding μ (Lemma 4), we deduce that

$$f^{(n)}(\langle t\alpha,a\rangle, \langle t\beta,b\rangle, \dots, \langle t\varkappa,h\rangle) = f_1^{(n)}(\langle t\alpha,a\rangle, \langle t\beta,b\rangle, \dots, \langle t\gamma,h\rangle)$$

holds for all $\langle t,a,b,\dots,h\rangle$ in $\mathcal{J}_0\times\Omega^n$. It follows now from Lemma 5 that the functions F and F_1 coincide on the open subset $(\mathcal{J}_0\cap\mathcal{J}\cap\mathcal{J}_1)\times\Omega^n$ of the connected manifold $(\mathcal{J}\cap\mathcal{J}_1)\times G^n$. Since they are analytic, they are identical on $(\mathcal{J}\cap\mathcal{J}_1)\times G^n$, in particular they are equal when $t=1$. This proves the general associative law in P.

Proof of Lemma 5. The proof is by induction on n. The Lemma holds trivially for $n=1$. Now let n be any integer and assume that the Lemma holds for every word $f^{(k)}$ with $k<n$.

Let $\alpha,\beta,\ldots,\gamma,\delta,\ldots,\varkappa$ be arbitrary n fixed elements of V/Y and let

$$f^{(n)}(\langle\alpha,a\rangle, \langle\beta,b\rangle, \ldots, \langle\gamma,c\rangle, \langle\delta,d\rangle, \ldots, \langle\varkappa,h\rangle)$$

exist for all $\langle a,b,\ldots,c,d,\ldots,h\rangle$ in G^n. There exist integers $k, l<n$ and words $f_1^{(k)}, f_2^{(l)}$ such that $k+l=n$ and $f^{(n)}(\xi_1,\ldots,\xi_n)$ exists in P if and only if

$$f_1^{(k)}(\xi_1,\ldots,\xi_k),\ f_2^{(l)}(\xi_{k+1},\ldots,\xi_n) \text{ and } f_1^{(k)}(\xi_1,\ldots,\xi_k)\,f_2^{(l)}(\xi_{k+1},\ldots,\xi_n)$$

exist and then

$$f^{(n)}(\xi_1,\ldots,\xi_n)=f_1^{(k)}(\xi_1,\ldots,\xi_k)\,f_2^{(l)}(\xi_{k+1},\ldots,\xi_n). \tag{1}$$

In particular, $f_1^{(k)}(\langle\alpha,a\rangle,\ldots,\langle\gamma,c\rangle)$ and $f_2^{(l)}(\langle\delta,d\rangle,\ldots,\langle\varkappa,h\rangle)$ are defined for all $\langle a,\ldots,c\rangle$ in G^k and all $\langle d,\ldots,h\rangle$ in G^l. Hence, by the inductive assumption, there exist open intervals $\mathcal{J}_1, \mathcal{J}_2$ containing 0 and 1 and analytic mappings $F_1:\mathcal{J}_1\times G^k\to G$, $F_2:\mathcal{J}_2\times G^l\to G$ such that

$$f_1^{(k)}(\langle t\alpha,a\rangle,\ldots,\langle t\gamma,c\rangle)=\langle t(\alpha+\ldots+\gamma),\ F_1(t,a,\ldots,c)\rangle,$$

$$f_2^{(l)}(\langle t\delta,d\rangle,\ldots,\langle t\varkappa,h\rangle)=\langle t(\delta+\ldots+\varkappa),\ F_2(t,d,\ldots,h)\rangle.$$

Let $\mathcal{J}\subset\mathcal{J}_1\cap\mathcal{J}_2$ be an open interval containing 0 and 1 such that

$$t(\alpha+\ldots+\gamma+\delta+\ldots+\varkappa)\in V/Y$$

for every $t\in\mathcal{J}$ (see Remark in previous section). Then for all $\langle t,a,\ldots,c,d,\ldots,h\rangle$ in $\mathcal{J}\times G^n$ the product of the above two words $f_1^{(k)}, f_2^{(l)}$ exists. Using the identity (1) we obtain that $f^{(n)}(\langle t\alpha,a\rangle,\ldots,\langle t\gamma,c\rangle,\langle t\delta,d\rangle,\ldots,\langle t\varkappa,h\rangle)$ exists and from the multiplication formula in Lemma 4 we find that

$$F(t,a,\ldots,c,d,\ldots,h)=C_{t(\alpha+\ldots+\gamma),\,t(\delta+\ldots+\varkappa)}\,(F_1(t,a,\ldots,c))^{x_{t(\delta+\ldots+\varkappa)}}\,F_2(t,d,\ldots,h).$$

By the Remark following Lemma 3, $t\to C_{t(\alpha+\ldots+\gamma),\,t(\delta+\ldots+\varkappa)}$ is an analytic mapping of $\mathcal{J}$ into G. Thus $C_{t(\alpha+\ldots+\gamma),\,t(\delta+\ldots+\varkappa)}$ can be regarded as an analytic function of the variable

$$\langle t,a,\ldots,c,d,\ldots,h\rangle\in\mathcal{J}\times G^n,$$

not depending on $\langle a,\ldots,c,d,\ldots,h\rangle$. Since

$$(F_1(t,a,\ldots,c))^{x_{t(\delta+\ldots+\varkappa)}}=\phi(tx_{\delta+\ldots+\varkappa},\ F_1(t,a,\ldots,c))$$

we conclude from Lemma 1 that this function is analytic on $\mathcal{J}\times G^k$ and hence on $\mathcal{J}\times G^n$ (not depending on $\langle d,\ldots,h\rangle$). Similarly $F_2(t,d,..,h)$ can be regarded as analytic on $\mathcal{J}\times G^n$. It follows now that F is the product of three functions which are analytic on $\mathcal{J}\times G^n$. This completes the proof of Lemma 5 and of Theorem 1.

For the proof of Theorem 2 we shall need the

Remarks: Let us assume that the above embedding $\mu: V \to P$ is an inclusion, so that $V \subset P$. G is generated by $V \cap Y \subset G$, for we have $x^n = nx$ if x, $nx \in Q$, and hence $V \cap Y$ generates $Q \cap Y$ which generates G by assumption. Since we have in P

$$\langle 0, a\rangle \langle 0, b\rangle = \langle 0, ab\rangle, \quad \text{for all } a, b \in G,$$

it follows that $V \cap Y$ generates in P the group $G \subset P$. We have shown above that there exists a group H containing P. It follows that G is the subgroup of H which is generated by $V \cap Y$.

Without assuming that all embeddings are inclusions, we can state these remarks as

THEOREM 1'. *Let X, Y satisfy the assumptions of Theorem* 1, *let $Q \subset X$ be a ball and let G be a simply connected analytic group such that there is an embedding $\varepsilon: Q \cap Y \to G$ with the property that $\varepsilon(Q \cap Y)$ is an open subset of G and $\varepsilon: Q \cap Y \to \varepsilon(Q \cap Y)$ is an analytic homeomorphism. Then there exists a ball $V \subset Q$, a group H and an embedding $\eta: V \to H$ such that*

1) *G is the subgroup of H generated by $\eta(V \cap Y)$,*
2) *$\eta = \varepsilon$ on $V \cap Y$.*

3. Second embedding theorem

Let X be a normed Lie algebra which is lower soluble with sequence $\{X_\beta\}_{\beta \leqslant \alpha}$. We shall prove that X is an E-algebra. Our proof is by induction; we show that for every $\beta \leqslant \alpha, X_\beta$ is an E-algebra. This is trivially the case for $\beta = 0$. If β is not a limit number and $X_{\beta-1}$ is an E-algebra, then so is X_β, by Theorem 1. The main difficulty of the proof is in showing that X_β is an E-algebra if β is a limit number and we know that each X_γ with $\gamma < \beta$ is an E-algebra The proof will be prepared in the following three sections and then given in the fourth one.

The universal enveloping group of a partial group

Let P be a partial group and let $\varepsilon: P \to G$ be an embedding in a group G such that the subset $\varepsilon(P) \subset G$ generates G. We shall say that G is a *universal enveloping group* (u.e.g.) for P with embedding $\varepsilon: P \to G$ if every diagram,

$$\begin{tikzcd} & P \arrow[dl, "\varepsilon"'] \arrow[d, "\eta"] \\ G & H \end{tikzcd}$$

where H is a group and $\eta: P \to H$ is a homomorphism can be completed to a commutative diagram,

$$\begin{tikzcd} & P \arrow[dl, "\varepsilon"'] \arrow[d, "\eta"] \\ G \arrow[r, "\nu"'] & H \end{tikzcd}$$

where ν is a homomorphism. An immediate consequence of this definition is the following

LEMMA 1. *If in a commutative diagram*

$$\begin{tikzcd} & P \arrow[dl, "\eta"'] \arrow[d, "\varepsilon"] \\ H \arrow[r, "\mu"'] & G \end{tikzcd}$$

the group G is a u.e.g. *for P with embedding ε, H is a group, $\eta: P \to H$ is a homomorphism, H is generated by $\eta(P)$ and $\mu: H \to G$ is a homomorphism, then μ is an isomorphism and H is a* u.e.g. *for P with embedding η.*

Proof. Combining the above diagram with the preceding one, we obtain

$$\begin{tikzcd} & P \arrow[dl, "\eta"'] \arrow[d, dash, "\varepsilon"] \arrow[dr, "\eta"] & \\ H \arrow[r, "\mu"'] & G \arrow[r, "\nu"'] & H \end{tikzcd}$$

Since $\nu(\mu(\eta(x))) = \eta(x)$ holds for every $x \in P$ and H is generated by $\eta(P)$, $H \xrightarrow[\mu]{} G \xrightarrow[\nu]{} H$ is the identity on H. This, together with the fact that $\mu: H \to G$ is surjective (because $\mu(\eta(P)) = \varepsilon(P)$ generates G) implies that μ is an isomorphism with inverse ν.

LEMMA 2 (*Existence of a* u.e.g.). *Let P be a partial group which is embeddable in a group and let F be the free group with the set of free generators P. Let us call an element u of F an e-element if the following condition holds*

(*) *there exist* $a_1, a_2, \ldots, a_n \in P$, $\omega_1, \omega_2, \ldots, \omega_n \in \{-1, 1\}$ *and a word* $f^{(n)}$ *such that* $a_1^{\omega_1} a_2^{\omega_2} \ldots a_n^{\omega_n} = u$ *holds in* F, $f^{(n)}(a_1^{\omega_1}, \ldots, a_n^{\omega_n})$ *exists in* P *and* $f^{(n)}(a_1^{\omega_1}, \ldots, a_n^{\omega_n}) = e$ *in* P.

Let $N \subset F$ *be the set of all e-elements of* F. *Then* N *is a normal subgroup of* F *and if* $\varepsilon: F \to F/N$ *denotes the natural homomorphism, then the restriction of* ε *to* P *is injective and* F/N *is a* u.e.g. *for* P *with embedding* $\varepsilon: P \to F/N$.

Proof. It is easily seen that N is normal and that $\varepsilon: P \to F/N$ is a homomorphism. Let $\eta: P \to H$ be a homomorphism into a group H. We shall prove that there exists a commutative diagram

$$\begin{tikzcd} P \arrow[d, "\varepsilon"'] \arrow[dr, "\eta"] & \\ \frac{F}{N} \arrow[r, "\nu"'] & H \end{tikzcd}$$

Indeed, since F is freely generated by P, the mapping $\eta: P \to H$ can be extended to a homomorphism $\pi: F \to H$. This gives a commutative diagram

$$\begin{tikzcd} P \arrow[d, "i"'] \arrow[dr, "\eta"] & \\ F \arrow[r, "\pi"'] & H \end{tikzcd}$$

where i is the inclusion map. Since $\eta: P \to H$ is a homomorphism, we have $\pi(u) = e$ for every e-element u in F, thus $\pi(N) = e$. It follows that $F \xrightarrow{\pi} H$ factorises through $F \xrightarrow{\varepsilon} F/N$, so that we have a commutative diagram

$$\begin{tikzcd} F \arrow[r, "\pi"] \arrow[d, "\varepsilon"'] & H \\ \frac{F}{N} \arrow[ur, "\nu"'] & \end{tikzcd}$$

The two diagrams thus obtained imply the required one. Finally to show that $\varepsilon:P\to F/N$ is an embedding it is enough to take any embedding $\eta:P\to H$ and use the commutativity of our diagram.

Partial subgroup. Let P be a partial group and let $P_0\subset P$ be such that

a) if $x,y\in P_0$ and xy exists then $xy\in P_0$,
b) if $x\in P_0$, then $x^{-1}\in P_0$.

Then P_0, together with the multiplication xy will be called a partial subgroup of P (local subgroup, if P is a local group).

LEMMA 3 (*The* u.e.g. *of a dense local subgroup*). *Let L be a local group which is embeddable in a group and let $L_0\subset L$ be a local subgroup such that L_0 is a dense subset of the space L. Let G be a* u.e.g. *for L with embedding $\varepsilon:L\to G$. Then the subgroup $G_0\subset G$ generated by $\varepsilon(L_0)$ is a* u.e.g. *for L_0 with embedding $\varepsilon:L_0\to G_0$ (more precisely $\varepsilon|L_0:L_0\to G_0$).*

Proof. We consider the free groups F and F_0 with sets of free generators L and L_0. Let $N\subset F$ be the normal subgroup consisting of all those elements u which satisfy condition (*) of Lemma 2 where P should be replaced by L. Let $N_0\subset F_0$ be defined similarly (replace P by L_0). Let further

$$\varepsilon:F\to\frac{F}{N}\quad\text{and}\quad\varepsilon_0:F_0\to\frac{F_0}{N_0}$$

be the natural homomomorphisms. Then, by Lemma 2, F/N and F_0/N_0 are u.e.g.'s for L and L_0 with embeddings

$$\varepsilon:L\to\frac{F}{N}\quad\text{and}\quad\varepsilon_0:L_0\to\frac{F_0}{N_0}.$$

We can assume without loss of generality that F/N is the group G mentioned in the Lemma. The subgroup G_0 of $G=F/N$ which is generated by $\varepsilon(L_0)$ is F_0N/N; indeed, L_0 generates the subgroup F_0 in F, thus $\varepsilon(L_0)$ generates the subgroup $\varepsilon(F_0)=F_0N/N$ in $\varepsilon(F)=F/N$. To prove that F_0N/N is a u.e.g. for L_0 with embedding $\varepsilon:L_0\to F_0N/N$ it is enough, by Lemma 1, to find a homomorphism μ: $F_0N/N\to F_0/N_0$ such that the diagram

$$\begin{tikzcd} & L_0 \arrow[dl, "\varepsilon"'] \arrow[d, "\varepsilon_0"] \\ \frac{F_0N}{N} \arrow[r, "\mu"'] & \frac{F_0}{N_0} \end{tikzcd}$$

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commutes. Now suppose that we have proved the equality

$$N_0 = F_0 \cap N.$$

Then $F_0/N_0 = F_0/F_0 \cap N$ and we can take for μ the natural isomorphism μ: $F_0N/N \to F_0/F_0 \cap N$. For every a in L_0 we have $\varepsilon(a) = aN$ and $\mu(aN) = a(F_0 \cap N) = aN_0 = \varepsilon_0(a)$. Hence the diagram commutes.

Proof of $N_0 = F_0 \cap N$. Only the inclusion $N \cap F_0 \subset N_0$ is not evident. Let $u \in N \cap F_0$. Then we have $b_1, b_2, \ldots, b_m$ in L_0 and $\varrho_1, \varrho_2, \ldots, \varrho_m \in \{1, -1\}$ such that $u = b_1^{\varrho_1} b_2^{\varrho_2} \ldots b_m^{\varrho_m}$, and we have also $a_1, a_2, \ldots, a_n \in L$; $\omega_1, \omega_2, \ldots, \omega_n \in \{1, -1\}$ and a word $f^{(n)}$ such that $u = a_1^{\omega_1} a_2^{\omega_2} \ldots a_n^{\omega_n}$ and $f^{(n)}(a_1^{\omega_1}, a_2^{\omega_2}, \ldots, a_n^{\omega_n})$ exists in L and is equal to e. We can assume that in $b_1^{\varrho_1} b_2^{\varrho_2} \ldots b_m^{\varrho_m}$ no cancellations are possible (otherwise we first perform these). Since $u = b_1^{\varrho_1} b_2^{\varrho_2} \ldots b_m^{\varrho_m} = a_1^{\omega_1} a_2^{\omega_2} \ldots a_n^{\omega_n}$ and all the b_i and a_j belong to a set of free generators of F, it follows that after performing all the possible cancellations in $a_1^{\omega_1} \ldots a_n^{\omega_n}$ we shall obtain $b_1^{\varrho_1} \ldots b_m^{\varrho_m}$. This implies that $\{a_1, a_2, \ldots, a_n\}$ can be written as the union of two disjoint sets

$$\{a_1, a_2, \ldots, a_n\} = \{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\} \cup \{a_{j_1}, a_{j_2}, \ldots, a_{j_l}\}$$

where $\{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\} = \{b_1, b_2, \ldots, b_m\}$. It follows further that

1) if $a_i \to a_i'$ is any mapping of the set $\{a_1, \ldots, a_n\}$ into L such that $a_{i_k}' = a_{i_k}$; $k = 1, 2, \ldots, r$, then $(a_1')^{\omega_1} (a_2')^{\omega_2} \ldots (a_n')^{\omega_n} = u$.

We shall use the following consequence of 1):

2) if $a_i \to a_i'$ and $a_i \to a_i''$ are any two mappings of the set $\{a_1, \ldots, a_n\}$ into L such that $a_{i_k}' = a_{i_k}'' = a_{i_k}$ for $k = 1, 2, \ldots, r$, then there exists a word $f^{(2n)}$ such that

$$f^{(2n)}((a_1')^{\omega_1}, (a_2')^{\omega_2}, \ldots, (a_n')^{\omega_n}, (a_n'')^{-\omega_n}, \ldots, (a_1'')^{-\omega_1}) = e$$

in the local group L.

Indeed, we have by 1) that

$$(a_1')^{\omega_1} (a_2')^{\omega_2} \ldots (a_n')^{\omega_n} (a_n'')^{-\omega_n} \ldots (a_1'')^{-\omega_1} = uu^{-1} = e$$

holds in F and since, by axioms PG.2 and PG.3 all products of the form aa^{-1}, $a^{-1}a$, ae, ea

where $a \in L$, exist in L and have the same values as in F, the above product will exist in L after brackets have been suitably inserted and will be equal to e.

To prove now that $u \in N_0$, we shall show that there exists a mapping $a_i \to a_i'$ as in 1) such that $a_1', \ldots, a_n' \in L_0$ and $f^{(n)}((a_1')^{\omega_1}, \ldots, (a_n')^{\omega_n}) = e$ in L_0. From the axioms of a local group it follows easily that there exist neighbourhoods $V_1, V_2, \ldots, V_n$ of $a_1, a_2, \ldots, a_n$ in L such that for every $\langle x_1, \ldots, x_n \rangle \in V_1 \times V_2 \times \ldots \times V_n$, $f^{(n)}(x_1^{\omega_1}, x_2^{\omega_2}, \ldots, x_n^{\omega_n})$ exists. Moreover, by the continuity of multiplication, $f^{(n)}(x_1^{\omega_1}, x_2^{\omega_2}, \ldots, x_n^{\omega_n})$ is near to e if the V_i are chosen small. We can therefore assume that the V_i are such that for every choice of $x_i, y_i \in V_i$ $(i = 1, \ldots, n)$, the product

$$f^{(n)}(x_1^{\omega_1}, \ldots, x_n^{\omega_n})\,(f^{(n)}(y_1^{\omega_1}, \ldots, y_n^{\omega_n}))^{-1}$$

exists in L. Using axiom PG.4 we find that this product is identically equal to a word of the form

$$f_1^{(2n)}(x_1^{\omega_1}, \ldots, x_n^{\omega_n}, y_n^{-\omega_n}, \ldots, y_1^{-\omega_1}).$$

Let us prove that if $a_i \to a_i'$ is a mapping of $\{a_1, \ldots, a_n\}$ into L as in 1) such that $a_i' \in V_i$ holds for $i = 1, \ldots, n$, then $f^{(n)}((a_1')^{\omega_1}, (a_2')^{\omega_2}, \ldots, (a_n')^{\omega_n}) = e$ in L. For assume to the contrary that for some such mapping, $f^{(n)}((a_i')^{\omega_1}, (a_2')^{\omega_2}, \ldots, (a_n')^{\omega_n}) = a \neq e$ and then take a mapping $a_i \to a_i''$ as in 2) such that $a_i'' \in V_i$ and $f^{(n)}((a_1'')^{\omega_1}, \ldots, a_n'')^{\omega_n})$ is near enough to e to ensure $a(f^{(n)}((a_1'')^{\omega_1}, \ldots, (a_n'')^{\omega_n}))^{-1} \neq e$. Then we have

$$f_1^{(2n)}((a_1')^{\omega_1}, \ldots, (a_n')^{\omega_n}, (a_n'')^{-\omega_n}, \ldots, (a_1'')^{-\omega_1}) \neq e$$

in L, contradicting the equality in 2) and the general associative law.

It follows now, in particular, that $f^{(n)}((a_1')^{\omega_1}, \ldots, (a_n')^{\omega_n}) = e$ if $a_i \to a_i'$ is a mapping as in 1) such that $a_i \in V_i \cap L_0$, $i = 1, \ldots, n$. Such a_i exist because L_0 is dense in L. Since then, by 1), $u = (a_1')^{\omega_1} \ldots (a_n')^{\omega_n}$, we see that u satisfies (*) and thus $u \in N_0$.

LEMMA 4 (*The* u.e.g. *of a partial subgroup*). *Let P be a partial group, let $P_0 \subset P$ be a partial subgroup and let $\eta: P \to H$ be an embedding in a group H such that the subgroup $H_0 \subset H$ generated by $\eta(P_0)$ is a* u.e.g. *for P_0 with embedding $\eta: P_0 \to H_0$. Let further G be a* u.e.g. *for P with embedding $\varepsilon: P \to G$ and let $G_0 \subset G$ be the subgroup generated by $\varepsilon(P_0)$. Then G_0 is a* u.e.g. *for P_0 with embedding ε and there exists an isomorphism $v: G_0 \to H_0$ such that*

$$P_0 \xrightarrow{\eta} H_0 = P_0 \xrightarrow{\varepsilon} G_0 \xrightarrow{v} H_0.$$

Proof. Since G is a u.e.g. for P, there exists a homomorphism $\nu: G \to H$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, "\varepsilon"'] \arrow[d, "\eta"] \\ G \arrow[r, "\nu"'] & H \end{tikzcd}$$

commutes. Since $G_0 \subset G$ is the subgroup generated by $\varepsilon(P_0)$ and $H_0 \subset H$ is the subgroup generated by $\eta(P_0)$, it follows that the commutativity of the diagram will be preserved when P, G and H are replaced by P_0, G_0 and H_0. But from the commutativity of

$$\begin{tikzcd} & P_0 \arrow[dl, "\varepsilon"'] \arrow[d, "\eta"] \\ G_0 \arrow[r, "\nu"'] & H_0 \end{tikzcd}$$

where H_0 is a u.e.g. for P_0 with embedding η, it follows by Lemma 1 that G_0 is a u.e.g. for P_0 with embedding ε, and that $\nu: G_0 \to H_0$ is an isomorphism.

LEMMA 5 (*The* u.e.g. *of a union*). *Let P be a partial group and let $\{P_\beta\}_{\beta<\alpha}$ be an ascending sequence of partial subgroups such that $P = \bigcup_{\beta<\alpha} P_\beta$. Let G be a group and let $\{G_\beta\}_{\beta<\alpha}$ be an ascending sequence of subgroups such that $G = \bigcup_{\beta<\alpha} G_\beta$. Let further $\varepsilon: P \to G$ be an embedding such that $\varepsilon(P_\beta)$ generates the subgroup $G_\beta \subset G$ for every β, and G_β is a* u.e.g. *for P_β with embedding ε. Then G is a* u.e.g. *for P with embedding ε.*

Proof. Let $\eta: P \to H$ be an arbitrary homomorphism of P in a group H. We have to find a homomorphism $\nu: G \to H$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, "\varepsilon"'] \arrow[d, "\eta"] \\ G \arrow[r, "\nu"'] & H \end{tikzcd}$$

commutes. Let $H_\beta \subset H$ denote the subgroup generated by $\eta(P_\beta)$. Then $\eta: P_\beta \to H_\beta$ is an embedding and hence there is a homomorphism $\nu_\beta: G_\beta \to H_\beta$ which makes

$$\begin{tikzcd} & P_\beta \arrow[dl, "\varepsilon"'] \arrow[d, "\eta"] \\ G_\beta \arrow[r, "\nu_\beta"'] & H_\beta \end{tikzcd}$$

commute. In this diagram the homomorphism $\nu_\beta: G_\beta \to H_\beta$ is unique, for we must have $\nu_\beta(\varepsilon(x)) = \eta(x)$ for every $\varepsilon(x) \in \varepsilon(P_\beta)$ and $\varepsilon(P_\beta)$ generates G_β. But if $\gamma < \beta$, then $G_\gamma \subset G_\beta$ is the subgroup generated by $\varepsilon(P_\gamma)$ and $H_\gamma \subset H_\beta$ is the subgroup generated by $\eta(P_\gamma)$ whence it follows that the commutativity of the above diagram will be preserved if we replace P_β, G_β and H_β by P_γ, G_γ and H_γ. Hence, by the uniqueness of $\nu_\gamma: G_\gamma \to H_\gamma$, it follows that $\nu_\gamma = \nu_\beta$ on G_γ. Consequently there exists a mapping $\nu: G \to H$ such that $\nu = \nu_\beta$ for each β. It is clear that ν is the required homomorphism.

Extensions of embeddings

In this section we shall prove three lemmas which will allow us to deduce that a local group L is embeddable in a group if we know that a dense local subgroup of L is embeddable (Lemma 6) or that L is analytic and a certain piece of L is embeddable (Lemmas 7 and 8).

LEMMA 6. *Let L be a local group and let $L_0 \subset L$ be a local subgroup which is dense in L and embeddable in a group. Then L is also embeddable in a group.*

Proof. By assumption, the general associative law is valid in L_0; we assert that it is also valid in L. For suppose to the contrary that we have elements $a_1, a_2, \ldots, a_n$ in L and two words $f_1^{(n)}$, $f_2^{(n)}$ such that $f_1^{(n)}(a_1, \ldots, a_n)$ and $f_2^{(n)}(a_1, \ldots, a_n)$ exist and

$$f_1^{(n)}(a_1, \ldots, a_n) \neq f_2^{(n)}(a_1, \ldots, a_n).$$

Then it follows from the axioms of a local group that there exist neighbourhoods $V_1, \ldots, V_n$ of $a_1, \ldots, a_n$ respectively such that $f_i^{(n)}(x_1, \ldots, x_n)$ exist if $x_j \in V_j$; $j = 1, \ldots, n$. Moreover, since $f_i^{(n)}$ are continuous on $V_1 \times V_2 \times \ldots \times V_n$, the V_j can be chosen sufficiently small to ensure

$$f_1^{(n)}(x_1, \ldots, x_n) \neq f_2^{(n)}(x_1, \ldots, x_n)$$

for all $\langle x_1, \ldots, x_n \rangle$ in $V_1 \times V_2 \times \ldots \times V_n$. But since L_0 is dense in L, all these x_i can be chosen from L_0, and we have a contradiction, by the general associative law in L_0. Thus L is embeddable in a group.

LEMMA 7. *Let X be a normed Lie algebra, let $L(X)$ be the* a.l.g. *assigned to X and let U, $U_0 \subset L(X)$ be balls such that*

$$UU \subset U_0 \quad and \quad U_0U_0U_0U_0 \subset L(X).$$

Then, if $V \subset U$ is any ball and $\eta: V \to H$ is a homomorphism into a group, η can be extended to a homomorphism $\tilde{\eta}: U \to H$.

Proof. We shall show first that there exists a mapping $\tilde{\eta}: U_0 \to H$ such that $\tilde{\eta} = \eta$ on V and

$$\tilde{\eta}(x)\,\tilde{\eta}(v) = \tilde{\eta}(xv) \tag{1}$$

provided $v \in V$ and $x(tv) \in U_0$ for all $0 \leqslant t \leqslant 1$. To obtain $\tilde{\eta}$, we consider the diagonal

$$D = \{\langle x, x\rangle \mid x \in U_0\} \subset U_0 \times U_0$$

and the neighbourhood Ω of D in $U_0 \times U_0$ consisting of all pairs $\langle x, xv\rangle$ such that $v \in V$ and $x(tv) \in U_0$ for all $0 \leqslant t \leqslant 1$. It is easy to see that Ω is open and connected. To every $\langle x, y\rangle$ in Ω we assign the permutation $\tau_{x,y}: H \to H$ which takes an arbitrary z in H into $z(\eta(x^{-1}y))$. We have $\tau_{y,z} \circ \tau_{x,y} = \tau_{x,z}$ provided $\langle x,y\rangle$, $\langle y,z\rangle$, $\langle x,z\rangle \in \Omega$. Indeed, $(x^{-1}y)\,(y^{-1}z) = x^{-1}z$ since the product of four elements of U_0 does not depend on the way the brackets are inserted. But as $x^{-1}y, y^{-1}z, x^{-1}z \in V$, applying η we get $\eta(x^{-1}y)\eta(y^{-1}z) = \eta(x^{-1}z)$ which we had to show.

We shall apply now the principle of monodromy (cf. Chevalley [6], p. 46) to the simply connected space U_0, the connected neighbourhood Ω of the diagonal $D \subset U_0 \times U_0$ and the mappings $\tau_{x,y}$. The principle implies tht there exists a mapping $\tilde{\eta}: U_0 \to H$ such that $\tilde{\eta}(e) = e$ and $\tau_{x,y}(\tilde{\eta}(x)) = \tilde{\eta}(y)$ for every $\langle x, y\rangle$ in Ω. In other words,

$$\tilde{\eta}(x)\,\eta(x^{-1}y) = \tilde{\eta}(y) \text{ provided } \langle x, y\rangle \in \Omega.$$

Putting $x = e$, we deduce that $\eta = \tilde{\eta}$ on V. Thus (1) is proved.

We shall now show that the restriction of $\tilde{\eta}$ to U is a homomorphism. Let $y \in U$. Then, for every natural n

$$\left(\frac{y}{n}\right)^k \left(t\frac{y}{n}\right) \in U, \quad \text{for } k = 0, 1, \ldots, n-1 \text{ and } 0 \leqslant t \leqslant 1 \tag{2}$$

since $(y/n)^k\,(t\,y/n) = ((k+t)/n)\,y$. Thus, by (1),

$$\tilde{\eta}\left(\left(\frac{y}{n}\right)^k\right)\tilde{\eta}\left(\frac{y}{n}\right) = \tilde{\eta}\left(\left(\frac{y}{n}\right)^{k+1}\right) \quad \text{for } k = 0, 1, \ldots, n-1, \tag{3}$$

provided $y/n \in V$, which is so when n is large enough. Combining these equalities, we arrive at $\tilde{\eta}(y) = (\tilde{\eta}(y/n))^n$, for sufficiently large n.

Now let $x, y \in U$. Then, for every n, by (2),

$$x\left(\frac{y}{n}\right)^k\left(t\frac{y}{n}\right) \in U_0 \quad \text{for } k=0,1,\ldots,n-1 \text{ and } 0 \leqslant t \leqslant 1.$$

Thus, by (1),
$$\tilde{\eta}\left(x\left(\frac{y}{n}\right)^k\right)\eta\left(\frac{y}{n}\right)=\tilde{\eta}\left(x\left(\frac{y}{n}\right)^{k+1}\right) \quad \text{for } k=0,1,\ldots,n-1,$$

provided n is large enough to ensure $y/n \in V$. Combining these n equalities we obtain $\tilde{\eta}(xy)=\tilde{\eta}(x)\,(\tilde{\eta}(y/n))^n$. Since $(\tilde{\eta}(y/n))^n=\tilde{\eta}(y)$, if n is large enough, this gives us finally

$$\tilde{\eta}(xy)=\tilde{\eta}(x)\,\tilde{\eta}(y) \quad \text{for every } x, y \text{ in } U.$$

Remark. If $\eta: U \to H$ is a homomorphism, then $\eta(y)=(\eta(y/n))^n$ holds for every y in U and $n=0,1,2,\ldots$.

Indeed, (2) above, taken for $t=1$, implies (3) for η and every n. Combining the equalities (3) we obtain $\eta(y)=(\eta(y/n))^n$.

LEMMA 8. *Let X and $L(X)$ be as Lemma 7, let*

$$C=\{c \in L(X)\,|\, xc=cx \text{ for all } x \in L(X) \text{ such that } xc \text{ and } cx \text{ exist}\}$$

be the centre of the local group $L(X)$ and let V, Q, $U \subset L(X)$ be open balls such that

$$V \subset Q \quad \text{and} \quad QQ \subset U.$$

Let further $\eta: U \to H$ be a homomorphism into a group H such that

a) *$\eta: V \to H$ is an embedding,*
b) *if $c \in V \cap C$ and $c \neq e$, then $\eta(c)$ is of infinite order in H.*

Then $\eta: Q \to H$ is an embedding.

Proof. Suppose that under the above assumptions we have $x \neq y$ and $\eta(x)=\eta(y)$ for some x,y in Q. Then, as $\eta(x^{-1}y)=\eta(x^{-1})\eta(y)=(\eta(x))^{-1}\eta(y)$ holds in U, we get $\eta(c)=e$ where $c=x^{-1}y \neq e$ belongs to U. Let us show that $c \in C$. We note that, by a), the kernel $K=\{x \in U\,|\,\eta(x)=e\}$ is discrete in U. But if $c \in K$, then for all z sufficiently near to e,

$$\eta(z^{-1}cz)=\eta(z^{-1})\eta(c)\,\eta(z)=(\eta(z))^{-1}\eta(z)=e.$$

Thus $z^{-1}cz \in K$, and since K is discrete, we must have $z^{-1}cz=c$ for all z sufficiently near to e.

Hence $c \in C$ (see Remark (iii) about the SCH-series). Taking n sufficiently large, we shall have $e \neq c/n \in V \cap C$, thus $(\eta(c/n))^n \neq e$, by b). But as $\eta(c) = (\eta(c/n))^n$ by the remark after Lemma 7, we obtain now $\eta(c) \neq e$, a contradiction. Thus η is injective on Q.

Lower soluble Lie algebras

Let X be a normed Lie algebra which is lower soluble with sequence $\{X_\beta\}_{\beta \leqslant \alpha}$. Let Z_β denote the centre of X_β and let $Z = \bigcup_{\beta \leqslant \alpha} Z_\beta$. Every two elements of Z commute, for if $z_1 \in Z_\beta$, $z_2 \in Z_\gamma$ and $\beta \leqslant \gamma$, then $z_1 \in X_\gamma$, $z_2 \in Z_\gamma$ and $[z_1, z_2] = 0$. Clearly $X_1 = Z_1 \subset Z$. We shall call the sequence $\{X_\beta\}_{\beta \leqslant \alpha}$ reduced if $X_1 = Z$.

LEMMA 9. *Every lower soluble normed Lie algebra X is lower soluble with a reduced sequence $\{X_\beta\}_{\beta \leqslant \alpha}$.*

Proof. We shall define, for every ordinal number δ a sequence $\{X_\beta^\delta\}_{\beta \leqslant \alpha}$ of closed subalgebras of X so that

1) $X_0^\delta = \{0\}$, $X_\alpha^\delta = X$,
2) if β is not a limit number, then $X_{\beta-1}^\delta$ is an ideal in X_β^δ and $X_\beta^\delta / X_{\beta-1}^\delta$ is abelian,
3) if β is a limit number, then $\bigcup_{\gamma < \beta} X_\gamma^\delta$ is dense in X_β^δ,
4) $X_\beta^\gamma \subset X_\beta^\delta$ if $\gamma < \delta$,
5) $Z^{\delta-1} \subset X_1^\delta$ if δ is not a limit number,

where $Z^\delta = \bigcup_{\beta \leqslant \alpha} Z_\beta^\delta$ and Z_β^δ is the centre of X_β^δ. This will suffice to prove our Lemma, for from 4) we have that $X_1^{\delta-1} = X_1^\delta$ for some sufficiently large δ whence by 5) it follows that $Z^{\delta-1} \subset X_1^{\delta-1}$. But as $X_1^{\delta-1} \subset Z_1^{\delta-1}$, we have $X_1^{\delta-1} \subset Z^{\delta-1}$ and thus $Z^{\delta-1} = X_1^{\delta-1}$.

The definition of the sequence $\{X_\beta^\delta\}_{\beta \leqslant \alpha}$ is by induction on δ. We put $\{X_\beta^0\}_{\beta \leqslant \alpha} = \{X_\beta\}_{\beta \leqslant \alpha}$. If δ is not a limit number, and we have already defined $\{X_\beta^\mu\}_{\beta \leqslant \alpha}$ for every $\mu < \delta$ so that 1)–5) are satisfied (with δ replaced by μ), we define

$$X_0^\delta = \{0\} \quad \text{and} \quad X_\beta^\delta = \overline{Z^{\delta-1} + X_\beta^{\delta-1}} \text{ for } \beta \geqslant 1,$$

i.e. X_β^δ is the closure of the subspace of X spanned by the subspaces $Z^{\delta-1}$ and $X_\beta^{\delta-1}$. If $z \in Z^{\delta-1}$, then $[z, X_\beta^{\delta-1}] \subset X_\beta^{\delta-1}$, for we have that either z belongs to a subalgebra of $X_\beta^{\delta-1}$ or else it commutes with $X_\beta^{\delta-1}$. This shows that X_β^δ is a subalgebra. It is easy to check that conditions 1), 3), 4) and 5) are satisfied, provided they are true if δ is replaced by $\delta-1$. To check 2) assume that $\beta \geqslant 1$ is not a limit number. We have to prove that

a) $Z^{\delta-1} + X_{\beta-1}^{\delta-1}$ is an ideal in $Z^{\delta-1} + X_\beta^{\delta-1}$,
b) $(Z^{\delta-1} + X_\beta^{\delta-1}) / (Z^{\delta-1} + X_{\beta-1}^{\delta-1})$ is abelian,

for then analogous statements will be valid for the closures of these algebras. Let

$$Z_{+\beta}^{\delta-1}=\bigcup_{\gamma\geqslant\beta} Z_\gamma^{\delta-1} \quad\text{and}\quad Z_{-\beta}^{\delta-1}=\bigcup_{\gamma<\beta} Z_\gamma^{\delta-1}.$$

Then clearly $[Z_{+\beta}^{\delta-1}, X_\beta^{\delta-1}]=0$ and since $X_\beta^{\delta-1}/X_{\beta-1}^{\delta-1}$ is abelian,

$$[Z_{-\beta}^{\delta-1}, X_\beta^{\delta-1}]\subset[X_\beta^{\delta-1}, X_\beta^{\delta-1}]\subset X_{\beta-1}^{\delta-1}.$$

Thus, $$[Z^{\delta-1}+X_\beta^{\delta-1}, Z^{\delta-1}+X_\beta^{\delta-1}]=[Z_{-\beta}^{\delta-1}, X_\beta^{\delta-1}]+[X_\beta^{\delta-1}, X_\beta^{\delta-1}]\subset X_{\beta-1}^{\delta-1}.$$

This proves a) and b).

It remains now to define X_β^δ under the assumption that δ is a limit ordinal and all X_β^μ with $\mu<\delta$ are already defined and satisfy 1)–5). We define

$$X_\beta^\delta=\overline{\bigcup_{\mu<\delta} X_\beta^\mu}.$$

Then it is easy to check that 1), 3), 4) and 5) are satisfied. To prove 2) note that from the inductive assumption

a′) $\bigcup_{\mu<\delta} X_{\beta-1}^\mu$ is an ideal in $\bigcup_{\mu<\delta} X_\beta^\mu$,

b′) $\bigcup_{\mu<\delta} X_\beta^\mu/\bigcup_{\mu<\delta} X_{\beta-1}^\mu$ is abelian,

hence analogous statements are valid for the closures of these spaces. This completes the proof of Lemma 9.

Proof of Theorem 2

We assume that the normed Lie algebra X is lower soluble with a reduced (cf. Lemma 9) sequence $\{X_\beta\}_{\beta\leqslant\alpha}$. We consider the a.l.g. $L(X)$ assigned to X and we wish to prove that a piece of $L(X)$ is embeddable in a group.

Let Q, U, U_0 be balls satisfying

$$QQ\subset U\subset UU\subset U_0\subset U_0U_0U_0U_0\subset L(X);$$

we shall show that Q is embeddable in a group. We observe first that for any subalgebra X_β we have

$$\begin{aligned}(Q\cap X_\beta)(Q\cap X_\beta)&\subset U\cap X_\beta\subset(U\cap X_\beta)(U\cap X_\beta)\\ &\subset(U_0\cap X_\beta)(U_0\cap X_\beta)(U_0\cap X_\beta)(U_0\cap X_\beta)\subset L(X_\beta).\end{aligned}$$

Let us prove that for every $\beta\leqslant\alpha$

a) *the local group* $Q\cap X_\beta$ *is embeddable in a group,* and it is possible to assign to every $Q \cap X_\beta$ a u.e.g. G_β with embedding $\varepsilon_\beta : Q \cap X_\beta \to G_\beta$ so that

b) *if $\gamma < \beta$, then G_γ is the subgroup of G_β generated by $\varepsilon_\beta(Q \cap X_\gamma)$ and $\varepsilon_\gamma = \varepsilon_\beta$ on $Q \cap X_\gamma$,*

c) *if $x \in Q \cap X_1$ and $x \neq e$, then $\varepsilon_1(x)$ is of infinite order in G_1.*

1. We use induction on β. Since $X_0 = \{e\}$, we can take $G_0 = \{e\}$, and then $\varepsilon_0(e) = e$. Since X_1 is abelian, $xy = x + y$ holds for any x, y in $Q \cap X_1$, whence we have for the group X_1 (with respect to vector addition) an embedding $\eta : Q \cap X_1 \to X_1$ such that $\eta(x) = x$. Let G_1 be a u.e.g. for $Q \cap X_1$ with embedding $\varepsilon_1 : Q \cap X_1 \to G_1$. Then there is a homomorphism $\nu : G_1 \to X_1$ such that the diagram

$$\begin{tikzcd} & Q \cap X_1 \arrow[dl, "\varepsilon_1"'] \arrow[d, "\eta = i"] \\ G_1 \arrow[r, "\nu"'] & X_1 \end{tikzcd}$$

commutes. Hence, if $x \in Q \cap X_1$ and $x \neq e$, then $\nu(\varepsilon_1(x)) = x$ and $\nu((\varepsilon_1(x))^n) = nx \neq e$ for $n = 1, 2, 3, \ldots$. Consequently $(\varepsilon_1(x))^n \neq e$ in G_1 which proves c).

2. Suppose now that we have a non-limit ordinal δ such that every $\beta < \delta$ satisfies a) and b). In particular, we have the embedding $\varepsilon_{\delta-1} : Q \cap X_{\delta-1} \to G_{\delta-1}$. Let us introduce in $G_{\delta-1}$ the structure of an $X_{\delta-1}$-manifold such that $\varepsilon_{\delta-1}(Q \cap X_{\delta-1})$ is open in $G_{\delta-1}$ and $\varepsilon_{\delta-1} : Q \cap X_{\delta-1} \to \varepsilon_{\delta-1}(Q \cap X_{\delta-1})$ is an analytic homeomorphism (cf. Th. 2.6.2 and Th. 2.7.1 in Cohn [7]). We assert that then $G_{\delta-1}$ is simply connected. To show this, we apply the principle of extension of analytic structure, by which there exists a simply connected analytic group $G'_{\delta-1}$ and an embedding $\varepsilon'_{\delta-1} : Q \cap X_{\delta-1} \to G'_{\delta-1}$ such that $\varepsilon'_{\delta-1}(Q \cap X_{\delta-1})$ is an open subset of $G'_{\delta-1}$ and the map $\varepsilon'_{\delta-1} : Q \cap X_{\delta-1} \to \varepsilon'_{\delta-1}(Q \cap X_{\delta-1})$ is an analytic homeomorphism. But $G_{\delta-1}$ is a u.e.g. for $Q \cap X_{\delta-1}$ whence there is a homomorphism $\nu : G_{\delta-1} \to G'_{\delta-1}$ such that the diagram

$$\begin{tikzcd} & Q \cap X_{\delta-1} \arrow[dl, "\varepsilon_{\delta-1}"'] \arrow[d, "\varepsilon'_{\delta-1}"] \\ G_{\delta-1} \arrow[r, "\nu"'] & G'_{\delta-1} \end{tikzcd}$$

commutes. Therefore $\nu = \varepsilon'_{\delta-1}\varepsilon_{\delta-1}^{-1}$ holds on $\varepsilon_{\delta-1}(Q \cap X_{\delta-1})$ which shows that ν is a local topological homeomorphism between $G_{\delta-1}$ and $G'_{\delta-1}$. Thus $\nu : G_{\delta-1} \to G'_{\delta-1}$ is continuous, and since it is surjective, as $\varepsilon'_{\delta-1}(Q\cap X_{\delta-1})$ generates $G'_{\delta-1}$, we obtain that $G_{\delta-1}$ is simply connected. Applying Theorem 1′, we find that there exists a ball $V\subset Q$, a group H and an embedding $\eta: V\cap X_\delta \to H$ such that

1) $G_{\delta-1}$ is the subgroup of H generated by $\eta(V\cap X_{\delta-1})$,

2) $\eta=\varepsilon_{\delta-1}$ on $V\cap X_{\delta-1}$.

3. Let us show first that the embedding $\eta: V\cap X_\delta\to H$ can be extended to an embedding $Q\cap X_\delta\to H$. Let C_δ be the centre of $L(X_\delta)$. We show first that if $c\in V\cap C_\delta$ and $c\neq e$ then $\eta(c)$ is of infinite order in H. Let Z_δ be the centre of the Lie algebra X_δ. Applying our Remark (iii) about the SCH-formula (Chapter 1), we find that $C_\delta\subset Z_\delta$, and since $Z_\delta\subset X_1$ by assumption (as $\{X_\beta\}_{\beta\leqslant\alpha}$ is reduced), we obtain

$$V\cap C_\delta\subset V\cap X_1\subset V\cap X_{\delta-1}.$$

Hence, if $c\in V\cap C_\delta$ then by 2) and by the inductive hypothesis b) with $\beta=\delta-1, \gamma=1$

$$\eta(c)=\varepsilon_{\delta-1}(c)=\varepsilon_1(c)\in G_1\subset G_{\delta-1}\subset H.$$

Hence, if $c\neq e$ then by c), $\eta(c)$ is of infinite order in H. Applying now Lemmas 7 and 8 we obtain that the embedding $\eta: V\cap X_\delta\to H$ can be extended to an embedding $\eta: Q\cap X_\delta\to H$. Part a) of our inductive assumption is now proved for $\beta=\delta$.

4. It is clear that the local group $Q\cap X_{\delta-1}$ is generated by its piece $V\cap X_{\delta-1}$ (we have $x^n=nx$ if x, $nx\in Q$; n integral). Therefore the subgroup of H generated by $\eta(Q\cap X_{\delta-1})$ is the same as the subgroup generated by $\eta(V\cap X_{\delta-1})$, i.e. it is $G_{\delta-1}$. Moreover, since η and $\varepsilon_{\delta-1}$ coincide on $V\cap X_{\delta-1}$, they must coincide on $Q\cap X_{\delta-1}$. It follows that the subgroup of H generated by $\eta(Q\cap X_{\delta-1})$ is a u.e.g. for $Q\cap X_{\delta-1}$ with embedding η. Let G_δ be a u.e.g. for $Q\cap X_\delta$ with embedding ε_δ. By the above and by Lemma 4 we obtain that the subgroup $G^*_{\delta-1}$ of G_δ which is generated by $\varepsilon_\delta(Q\cap X_{\delta-1})$ is a u.e.g. for $Q\cap X_{\delta-1}$ with embedding ε_δ, moreover, there exists an isomorphism $\nu: G^*_{\delta-1}\to G_{\delta-1}$ such that the diagram

$$\begin{tikzcd}
 & Q\cap X_{\delta-1} \arrow[dl, "\varepsilon_\delta"'] \arrow[d, "\eta=\varepsilon_{\delta-1}"] \\
G^*_{\delta-1} \arrow[r, "\nu"'] & G_{\delta-1}
\end{tikzcd}$$

commutes.

If we now identify $G^*_{\delta-1}$ with $G_{\delta-1}$ taking ν to be the identity map, we obtain that $\varepsilon_\delta = \varepsilon_{\delta-1}$ on $Q \cap X_{\delta-1}$, and that $G_{\delta-1}$ is the subgroup of G_δ generated by $\varepsilon_\delta(Q \cap X_{\delta-1})$. Thus b) is shown for $\beta = \delta$.

5. Now suppose that δ is a limit number such that a) and b) hold for all $\beta < \delta$. Let

$$P = Q \cap X_\delta, \quad P_0 = \bigcup_{\beta<\delta} (Q \cap X_\beta), \quad G = \bigcup_{\beta<\delta} G_\beta.$$

It is clear that there exists an embedding $\varepsilon: P_0 \to G$ such that $\varepsilon(Q \cap X_\beta)$ generates the subgroup G_β of G and $\varepsilon = \varepsilon_\beta$ on $Q \cap X_\beta$, for all $\beta < \delta$. It follows from Lemma 5 that G is a u.e.g. for P_0 with embedding ε. Since P_0 is dense in P, Lemma 6 implies that P is embeddable in a group. Thus part a) of the inductive hypothesis is proved for $\beta = \delta$.

Let G_δ be a u.e.g. for P with embedding ε_δ. Let H be the subgroup of G_δ generated by $\varepsilon_\delta(P_0)$. The map $\varepsilon_\delta: P_0 \to H$ is an embedding, hence there exists a homomorphism $\nu: G \to H$ such that the diagram

$$\begin{tikzcd} & P_0 \arrow[dl, "\varepsilon"'] \arrow[d, "\varepsilon_\delta"] \\ G \arrow[r, "\nu"'] & H \end{tikzcd}$$

commutes. But by Lemma 3, H is a u.e.g. for P_0 with embedding ε_δ, whence by Lemma 1, $\nu: G \to H$ is an isomorphism. Identifying H and G via ν we obtain that $G \subset G_\delta$ and $\varepsilon_\delta = \varepsilon$ on P_0. Thus $\varepsilon_\delta(Q \cap X_\beta)$ generates the subgroup $G_\beta \subset G \subset G_\delta$ and $\varepsilon_\delta = \varepsilon_\beta$ on $Q \cap X_\beta$, for all $\beta < \delta$. Hence part b) of the inductive hypothesis is shown for $\beta = \delta$, and the proof is now complete.

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