# POLYNOMIALLY AND RATIONALLY CONVEX SETS

BY

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On a Euclidean space of even dimension we can introduce, by a choice of complex-valued coordinate functions, $z_1, \ldots, z_n$, the structure of complex $n$-space, $C^n$. We can then associate with each compact subset, $X$, of our space, its *polynomial convex hull in* $C^n$, denoted hull$(X)$. By definition, hull$(X)$ *is the set of all* $p$ *in* $C^n$ *which satisfy the relation*

$$|f(p)| \leqslant \max_{x \in X} |f(x)|,$$

*for every polynomial*, $f(z_1, \ldots, z_n)$. When $X = \text{hull}(X)$, we say that $X$ *is polynomially convex in* $C^n$.

Our primary object of study here is the polynomial convex hull of $X$. However, we have found it very helpful to consider also, as an intermediary set, $R$-hull$(X)$, *the rational convex hull of* $X$ *in* $C^n$. By definition, $R$-hull$(X)$ *consists of all* $p$ *in* $C^n$ *such that*

$$|g(p)| \leqslant \max_{x \in X} |g(x)|,$$

*for every rational function*, $g$, *which is analytic about* $X$. For our purposes, we often prefer the alternate description of $R$-hull$(X)$, (1.1), as *the set of all* $p$ *in* $C^n$ *for which* $f(p) \in f(X)$, *for every polynomial*, $f$. If $X = R\text{-hull}(X)$, we say that $X$ *is rationally convex in* $C^n$. Notice that

$$X \subset R\text{-hull}(X) \subset \text{hull}(X).$$

These hulls are compact, and both inclusions can be proper.

Our aim is to understand what these hulls look like. In what sense does $X$ "surround" them in $C^n$? Consider first $C^1$, where the complete picture is well known. There, every compact $X$ is rationally convex (obvious), and hull$(X)$ is formed by adjoining to $X$ all the bounded components of its complement (classical, see (1.3)). Thus, in $C^1$, rational convexity

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is no restriction at all and polynomial convexity is purely a topological matter. But, as we shall see, both these facts are very much "one-dimensional accidents". The general situation is extremely complicated and our knowledge of it is only fragmentary. The purpose of this paper is to contribute some new fragments and to replace in its proper setting a beautiful but neglected piece by K. Oka (1.4).

In Chapter 1 we present local descriptions of hull$(X)$ and $R$-hull$(X)$. These may be regarded as generalizations of the setup in $C^1$ in the following way.

*In* $C^1$, *if* $p \in \text{hull}(X)$ *then every curve joining p to the 'point at infinity' crosses* $X$. There is a natural generalization to $C^n$, (see (1.2)). Namely, *in* $C^n$, *if* $p \in \text{hull}(X)$ *then every curve of algebraic hypersurfaces, going through p to the "hyperplane at infinity", crosses* $X$. The local version, (1.4), reads: *In* $C^n$, *if a curve of analytic hypersurfaces, defined locally on some neigborhood of* hull$(X)$, *crosses* hull$(X)$ *and leaves it, then it must also cross* $X$.

This local characterization of hull$(X)$ was demonstrated by K. Oka in 1937 in [21]. Its proof, which we present in Chapter 1, draws on certain fundamental facts in the theory of several complex variables (see (A.3) and (A.8)). As a simple corollary of Oka's theorem we deduce H. Rossi's Local Maximum Modulus Principle, [23], (1.7), which asserts that *if* $S \subset \text{hull}(X)$ *then* $S \subset \text{hull}(\partial S \cup (S \cap X))$, (where $\partial S$ denotes the topological boundary of $S$ in hull$(X)$). Using the proof of Oka's theorem we also obtain, (1.8), a version of Rossi's Local Peak Point Theorem, [23, p. 6].

We then study in a similar way the local structure of $R$-hull$(X)$. We first observe, (1.1′), that *in* $C^n$, *if* $p \in R\text{-hull}(X)$ *then every algebraic hypersurface through p meets* $X$. The expected local version would say that *if an analytic hypersurface, which is defined locally on some neighborhood of* $R$-hull$(X)$, *meets* $R$-hull$(X)$, *then it meets* $X$. However, as example (1.11) shows, this is not always true. What is needed is a topological restriction on $R$-hull$(X)$: namely, that $\check{H}^2(R\text{-hull}(X);Z)=0$ (Čech cohomology with integer coefficients.) In that case, the above local description of $R$-hull$(X)$ does obtain (see (1.9)) and, as a corollary (corresponding to the Local Maximum Modulus Principle), we find that *if* $\check{H}^2(R\text{-hull}(X);Z)=0$ *and* $S \subset R\text{-hull}(X)$ *then* $S \subset R\text{-hull}(\partial S \cup (S \cap X))$, (here $\partial S$ denotes the topological boundary of $S$ in $R$-hull$(X)$).

In Chapter 2 we turn to the problem of showing that, under certain circumstances (closely related to rational convexity), there are topological conditions which insure polynomial convexity. Let us first point out that, for $n > 1$, polynomial convexity is definitely no longer a strictly topological property. In fact, the question of whether a given compact set $X$ is polynomially convex may well depend on the choice of complex coordinate functions used to introduce the structure of $C^n$ on our space. For example, if $C^2$ is given by coordinates $z_1$, $z_2$ then the circle $(e^{i\theta}, e^{-i\theta})$, $0 \leq \theta \leq 2\pi$, is polynomially convex in $C^2$; whereas, if we use instead $z_1$, $\bar{z}_2$ as coordinates, then the polynomial convex hull of that circle is the disk, $|z_1| \leq 1$, $z_1 = \bar{z}_2$. (Similarly, a set can be rationally convex in terms of one choice of coordinates, but not another, (2.10).)

Furthermore, there are compact cells, even arcs, which are not polynomially convex. (Examples were first given by J. Wermer in [33] and [35].) Hence, a purely topological sufficient condition for a set to be polynomially convex in $C^n$ appears unattainable. However, in our work we have found that all the known non-polynomially convex cells are not even rationally convex (see, for example, [31]). Therefore, it is conceivable that for rationally convex subsets of $C^n$ there are topological conditions which imply polynomial convexity. Let us say that $X$ *is simply-coconnected* if every non-vanishing continuous complex-valued function on $X$ has a log. (Note that any contractible $X$ is simply-coconnected.) Then the most general problem we pose is this.

(*) *Is every rationally convex, simply-coconnected subset of $C^n$ necessarily polynomially convex?*

This is apparently a difficult question. An affirmative answer would include the special result that every compact differentiable arc in $C^n$ is polynomially convex. (Wermer [34] has proved this for real analytic arcs and, recently, E. Bishop, on the basis of [9], has extended the proof to all continuously differentiable arcs.) Nevertheless, we have made *some* headway by introducing a certain "generalized argument principle". And, in doing so, we have been able to settle a number of questions which had previously seemed unrelated. As one application of our argument principle we show that *if $X$ is rationally convex and simply-coconnected then there is no one-dimensional analytic variety whose boundary lies entirely in $X$* (see (2.7′)). This suggests the difficulty involved in seeking a counterexample to (*).

What is our argument principle? We say that $X$ *enjoys the generalized argument principle* (*g.a.p.*) provided that every polynomial which has a log on $X$ does not vanish at any point of hull($X$). It is very easily seen, (2.2), that *whenever $X \subset Y$, with $X$ having the g.a.p. and $Y$ simply-coconnected, then* hull$(X) \subset R$-hull$(Y)$. We use this to show

(i) *There are disjoint, contractible, polynomially convex sets, $X$ and $Y$, in $C^3$, such that $f(X)$ intersects $f(Y)$, for every polynomial, $f$.* (2.8)

(ii) *There exist polynomially convex sets which cannot be approximated by analytic polyhedra in such a way that the corresponding Šilov boundaries converge to the Šilov boundary.* (2.16)

(iii) *There is a rational polyhedron whose polynomial convex hull is not even an analytic polyhedron.* (2.17)

For (ii) and (iii) we use an example, (2.15), due to Bishop and Hoffman, of a compact set in $C^2$ which does *not* enjoy the g.a.p. Despite this example, there is a weak form of the generalized argument principle, (2.19), which holds for all compact $X$ in $C^n$. It states that *if a polynomial, $f$, has a log on $X$ and $X$ is open in $f^{-1}(f(X)) \cap \text{hull}(X)$ then $f$ does not vanish at any point of* hull$(X)$. Our proof of this weak g.a.p. uses in an essential way Oka's local characterization of hull$(X)$, (1.4).

The weak g.a.p. has several interesting applications.

(iv) *If $X$ is simply-coconnected and if there is a polynomial, $f$, such that $f(X)$ does not intersect $f(\text{hull}(X) - X)$ then $X$ is polynomially convex.* (2.20)

(v) *Let $S_1, \ldots, S_{n-1}$ be compact subsets of $C^1$, each the boundary of the unbounded component of its complement. Let $f_1, \ldots, f_{n-1}$ be $n-1$ polynomials (of $n$ variables) in general position. If $X$ is any rationally convex, simply-coconnected subset of $C^n$ such that $f_i(X) \subset S_i$ for each $i = 1, \ldots, n-1$, then $X$ is polynomially convex.* (2.22)

In particular, (v) implies

(vi) *If $X$ is contained in a simply-coconnected subset of the $n$-torus, $|z_i| = 1$, $i = 1, \ldots, n$ in $C^n$, then $X$ is polynomially convex.* (2.25)

*Definitions and background theory* from several complex variables and Banach algebras may be found in the appendix which follows Chapter 2. (References to the appendix are denoted, (A, .).)

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## 1. Local descriptions of the hulls

We open this chapter with some elementary global descriptions of $R$-hull$(X)$ and hull$(X)$ which establish our point of view.

(1.1) $R$-hull$(X) = \{p \in C^n : f(p) \in f(X)$, *for all polynomials*, $f\}$.

*Proof.* If $f(p) \notin f(X)$, then $r = (f - f(p))^{-1}$ is a rational function, analytic about $X$, and

$$\infty = r(p) > \max_{x \in X} |r(x)|.$$

Thus, $p \notin R$-hull$(X)$.

Conversely, if $p \notin R\text{-hull}(X)$, there is a rational function, $r$, analytic about $X$, such that

$$1 = r(p) > \max_{x \in X} |r(x)|.$$

Express $r$ as $f/g$ where $f$ and $g$ are relatively prime polynomials. Let $h = f - g$. Then $h(p) = 0 \notin h(X)$. Q.E.D.

In terms of algebraic hypersurfaces, (1.1) says

(1.1′) *A point $p$ in $C^n$ belongs to $R$-hull$(X)$ if and only if every algebraic hypersurface through $p$ meets $X$.*

The corresponding description of hull$(X)$ involves curves of algebraic hypersurfaces.

DEFINITION. Let $F$ be a continuous function on $[s, \infty) \times C^n$ which defines, for each $t \in [s, \infty)$, a non-constant polynomial, $F_t$. Let $H_t$ be the zero-set of $F_t$ in $C^n$. Then, the correspondence, $t \to H_t$, which we denote simply by $(H_t)$, is called *a curve of algebraic hypersurfaces in $C^n$*. If the distance from $H_t$ to the origin tends to infinity as $t \to \infty$, we say that $(H_t)$ *joins the initial hypersurface to the hyperplane at infinity*. We say that $(H_t)$ *passes through the set*, $X$, if some $H_t$ intersects $X$.

(1.2) hull$(X) = \{p \in C^n$: *every curve of algebraic hypersurfaces, which passes through $p$ and joins the initial hypersurface to the hyperplane at infinity, passes through* $X\}$.

*Proof*. One direction is immediate. If $p \notin \text{hull}(X)$, there is a polynomial, $f$, with $f(p) = 1$ and $|f| < 1$ on $X$. Define $F_t(u) = f(u) - t$ on $[1, \infty) \times C^n$ and let $H_t$ be the zero-set of $F_t$ in $C^n$. Then $(H_t)$ will be the required curve of algebraic hypersurfaces.

On the other hand, suppose $p \in \text{hull}(X)$ and $(H_t)$ is a curve of algebraic hypersurfaces passing through $p$, missing $X$, and joining the initial hypersurface to the hyperplane at infinity. By compactness of hull$(X)$, there is a last point, $r$, in $[s, \infty)$ such that $H_r$ intersects hull$(X)$. Let $T$ be the open interval, $(r, \infty)$, and set

$T_0 = \{t \in T$: $1/F_t$ is a uniform limit of polynomials on hull$(X)\}$. We shall show that $T_0 = T$.

Firstly, $T_0$ is not empty. For, if $t$ is large enough, $H_t$ lies beyond a polycylinder which contains hull$(X)$. Then, $1/F_t$ is analytic on the polycylinder, so it has a power series expansion which converges uniformly on hull$(X)$.

Next, $T_0$ is closed in $T$. For, just observe that as $t_i \to t_0$ in $T_0$, $1/F_{t_i}$ converges uniformly to $1/F_{t_0}$ on hull$(X)$.

Finally, $T_0$ is open in $T$. For, choose $t_0$ in $T_0$. If $t$ is close enough to $t_0$, then $|1 - F_t/F_{t_0}| < 1$ on hull$(X)$. In that case,

$$\frac{1}{F_t} = \frac{1}{F_{t_0}} \left( \sum_{i=0}^{\infty} \left(1 - \frac{F_t}{F_{t_0}}\right)^i \right),$$

with the series converging uniformly on hull($X$). Since $1/F_{t_0}$ is a uniform limit of polynomials on hull($X$), so is $1/F_t$.

Since $T$ is connected, it must be that $T_0 = T$.

If $q \in H_r \cap \text{hull}(X)$, then $|1/F_t(q)| \to \infty$ as $t \to r$. Since no $H_t$ intersects $X$, $1/F$ is bounded on, say, $[r, r+1] \times X$. Hence, there is a $t$ in $T$ for which

$$\left|\frac{1}{F_t}(q)\right| > \max_{x \in X} \left|\frac{1}{F_t}(x)\right|.$$

But $T = T_0$, so there is also a polynomial with this property, contradicting $q \in \text{hull}(X)$. Q.E.D.

As a corollary of (1.2) we obtain the usual description of hull($X$) in $C^1$.

(1.3) *In* $C^1$, $\text{hull}(X) = \{p \in C^1$: *every curve joining* $p$ *to the "point at infinity" crosses* $X\}$. *Consequently, the set* $X$ *is polynomially convex in* $C^1$ *if and only if* $C^1 - X$ *is connected.*

For, in $C^1$, an algebraic hypersurface is a finite set of points. (We remark that our proof of (1.2) is merely an appropriate translation in $C^n$ of the standard proof of (1.3) (which is essentially the proof of Runge's Theorem [26]).)

We are now ready to present Oka's local version of (1.2). This requires both a solution of a Cousin I problem, (A.8), and the Approximation Theorem, (A.4), as a way of passing, in two stages, from local to global information.

DEFINITION. Let $U \subset O$, be open subsets of $C^n$, and let $[r, s]$ be a closed interval on the real line (considered as a subset of $C^1$). Let $F$ be a continuous function on $[r, s] \times U$ which defines, for each $t$ in $[r, s]$, a non-constant analytic function, $F_t$, on $U$. Set $H_t = \{u \in U$: $F_t(u) = 0\}$. If each $H_t$ is closed in $O$, we say that *the correspondence,* $t \to H_t$, *is a curve of analytic hypersurfaces in* $O$. We denote it, simply, by $(H_t)$.

(1.4) OKA'S CHARAKTERIZATION THEOREM. *Let* $O$ *be a neighborhood of* hull($X$) *in* $C^n$. *If* $(H_t)$ *is a curve of analytic hypersurfaces in* $O$ *such that* $H_r$ *intersects* hull($X$), *but* $H_s$ *does not, then some* $H_t$ *must intersect* $X$.

*Proof.* Let $(H_t)$ be defined by the function, $F$, on $[r, s] \times U$. We shall first establish the theorem in the case when $F$ extends to be analytic on $T \times U$, where $T$ is some neighborhood of $[r, s]$ in $C^1$. This was the case considered by Oka in [21] and it suffices for all our applications.

By shrinking, we can arrange that $O$ and $T$ are both open polynomial polyhedra and that $V_F = \{(t, u) \in T \times U : F(u, t) = 0\}$ is closed in $T \times O$. Cover $T \times O$ by the open sets $T \times U$ and $(T \times O) - V_F$, and assign the meromorphic functions, $1/F$ on $T \times U$ and $0$ on $(T \times O) - V_F$.

Since $T \times O$ is again an open polynomial polyhedron, we can apply (A.8) to obtain a meromorphic function, $G$, on all of $T \times O$, such that $G$ is analytic off $V_F$ and, on $T \times U$, $G = 1/F + E$, where $E$ is analytic on $T \times U$.

If no $H_t$ intersects $X$, then $G$ is continuous, hence, bounded, on the compact set $[r, s] \times X$. Let $c$ be the last point in $[r, s]$ such that $H_c \cap \text{hull}(X) \neq \emptyset$, and choose a point, $p$, in this intersection. From the representation of $G$ on $T \times U$, we see that

$$|G(t, p)| \to \infty \quad \text{as} \quad t \downarrow c.$$

Hence, for some $t_1 > c$,

$$|G(t_1, p)| > \max_{x \in X} |G(t_1, x)|.$$

Since $t_1 > c$, $G(t_1, z)$ is analytic about hull$(X)$; so, by (A.4), there must also be a polynomial, $g$, such that

$$|g(p)| > \max_{x \in X} |g(x)|.$$

But this is absurd, because $p \in \text{hull}(X)$. Therefore, some $H_t$ must intersect $X$, and we are done.

A word about the general case. One can apply the Weierstrass Approximation Theorem [17] on $[r, s]$ to obtain a sequence of analytic functions, $F_i$, on $C^1 \times U$, which converges uniformly to $F$ on compact subsets of $[r, s] \times U$, and then deduce the general result from the case already proved. Or, one may appeal to [19, Theorem 1] for a solution of a Cousin I problem on $O$ with data (and solution) depending continuously on the parameter $t \in [r, s]$, and then repeat the above argument.

The following corollary is conceptually clear, but the details are a bit annoying.

(1.5) COROLLARY. *If $S$ is a compact subset of* hull$(X)$, *and if $f$ is analytic on a neighborhood of $S$, then*

$$|f(s)| \leq \max\{|f(y)| : y \in \partial S \cup (S \cap X)\},$$

*for all $s \in S$, (where $\partial = \partial_{\text{hull}(X)}$).* (See (A.5).)

*Proof.* Suppose, on the contrary, that $f$ is analytic about $S$ and there is an $s^*$ in $(\partial S \cup (S \cap X))$, for which

$$1 = f(s^*) = \max_S |f| > \max_{\partial S \cup (S \cap X)} |f|. \tag{**}$$

Choose a compact subset, $\tilde{S}$, of $C^n$, with interior, $\tilde{S}_0$, such that $\tilde{S} \cap \text{hull}(X) = S$, $\tilde{S}_0 \cap \text{hull}(X) = S - \partial S$, and $f$ is analytic on a neighborhood of $\tilde{S}$. Then $f$ is analytic and non-constant on the component of $\tilde{S}_0$ containing $s^*$; so $f(\tilde{S}_0)$ contains a neighborhood of $1 = f(s^*)$ in $C^1$. Hence, for a sufficiently small $\Delta > 0$, $f(\tilde{S}_0)$ contains the interval $[1, 1 + \Delta]$. On $[1, 1 + \Delta] \times \tilde{S}_0$, define

$$F(t, u) = f(u) - i,$$

and set
$$H_t^0 = \{u \in \tilde{S}_0 \colon F(t, u) = 0\}.$$

Express hull$(X)$ as the intersection of a descending chain of relatively compact open sets, $O_i$, such that $O_i \supset \bar{O}_{i+1}$.

We claim that, if $i$ is large enough, then every $H_t^0 \cap O_i$ is closed in $O_i$. Suppose not. Then, for each $i$, there exists $t_i \in [1, 1+\Delta]$ and a point, $p_i$, which belongs to the closure of $H_{t_i}^0 \cap O_i$ in $O_i$, but not to $H_{t_i}^0$. Then $p_i$ belongs to the compact set, $\tilde{S}$, but not to its interior, $\tilde{S}_0$, because $H_{t_i}^0$ is closed in $\tilde{S}_0$. Also, by continuity, $f(p_i) = t_i$. By passing to subsequences, we can arrange that $t_i \to t^* \in [1, 1+\Delta]$ and $p_i \to p^* \in \tilde{S} - \tilde{S}_0$. Then

$$f(p^*) = t^* \geqslant 1.$$

Also,
$$p^* \in \bigcap_i \bar{O}_i = \text{hull}(X).$$

Therefore,
$$p^* \in \text{hull}(X) \cap (\tilde{S} - \tilde{S}_0) = \partial S \text{ and } f(p^*) \geqslant 1,$$

contradicting (**). Hence, for $i$ large enough, every $H_t^0 \cap O_i$ is closed in $O_i$. Choose $i = i_0$ that large, and set $O = O_{i_0}$. Let $U$ be the connected component of $\tilde{S}_0 \cap O$ which contains $s^*$ and set $H_t = H_t^0 \cap U$ for each $t \in [1, 1+\Delta]$. Then $(H_t)$ is a curve of analytic hypersurfaces in $O$ such that

$$s^* \in H_1 \cap \text{hull}(X) \neq \emptyset.$$

Since $|f| \leqslant 1$ on $S$, and every $H_t \cap \text{hull}(X) \subset S$, it follows that

$$H_t \cap \text{hull}(X) = \emptyset \text{ for } t > 1.$$

Therefore, by (1.4), $H_1 \cap X \neq \emptyset$, so there exists $x \in S \cap X$ such that $f(x) = 1$. But this contradicts (**). Q.E.D.

It will be useful to improve (1.5) to

(1.6) COROLLARY. *With $S$ and $f$ as in* (1.5),

$$\partial_{C^1} f(S) \subset f(\partial S \cup (S \cap X)).$$

The proof is the same as the first two sentences of the proof of (A.6) if we replace "$X$" by "$\partial S \cup (S \cap X)$", "hull$(X)$" by "$S$", and then appeal to (1.5).

Corollary (1.5) also yields, for the special case when $f$ is a polynomial, the following theorem of H. Rossi [23].

(1.7) LOCAL MAXIMUM MODULUS PRINCIPLE. *If $S \subset$ hull$(X)$, then $S \subset$ hull $(\partial S \cup (S \cap X))$, (where $\partial = \partial_{\text{hull}(X)}$).*

Next, we shall show how a version of Rossi's Local Peak Point Theorem [23, Theorem 4.1], can easily be obtained using the methods of proof of (1.4) and (1.5) and Bishop's description of peak points, (A.25).

DEFINITION. *A point* $p \in \mathrm{hull}(X)$ *is a local peak point for* $P(X)$, (*the set of all uniform limits of polynomials on* $X$) if there is a neighborhood, $U$, of $p$ in hull$(X)$ and a function $f \in P(X)$ such that $|f(p)| > |f(u)|$ for all $u \in U - \{p\}$.

(1.8) LOCAL PEAK POINT THEOREM. *Every local peak point for* $P(X)$ *is actually a peak point for* $P(X)$.

*Proof.* Let $p$ be a local peak point for $P(X)$. Then there is a neighborhood, $U$, of $p$ in hull$(X)$ and an $f \in P(X)$ such that $1 = f(p) > |f(u)|$, for all $u \in U - \{p\}$. Let $F$ map hull$(X)$ into $C^{n+1}$ by

$$F(y_1, \ldots, y_n) = (y_1, \ldots, y_n, f(y_1, \ldots, y_n)).$$

From the fact that hull$(X)$ is the maximal ideal space of $P(X)$, (A.30), it follows easily that

$$F(\mathrm{hull}(X)) = \mathrm{hull}(F(X)).$$

Therefore, it is no loss of generality to assume that $\mathrm{hull}(X) \subset C^{n+1}$ and that $f = z_{n+1}$, the $(n+1)-st$ coordinate function. By (A.25), it suffices to show that for each neighborhood, $N$, of $p$ in hull$(X)$, there is a $g \in P(X)$ such that $|g| \leqslant 1$ on hull$(X)$, $|g| < \frac{1}{4}$ on hull$(X) - N$, and $|g(p)| > \frac{3}{4}$. By (A.4), it suffices to find such a $g$ which is analytic about hull$(X)$.

Proceeding as in the proof of (1.5) we can obtain the following setup. There is an open polynomial polyhedron, $O$, containing hull$(X)$, an open subset, $U_0$, such that $p \in U_0 \cap \mathrm{hull}(X) \subset U$, a number $\Delta > 0$, and a polynomial polyhedral neighborhood, $T$, of $[1, 1+\Delta]$ in $C^1$ such that, for each $t \in T$,

$$H_t = \{u \in U_0 : z_{n+1} - t = 0\}$$

is closed in $O$.

Following the proof of (1.4), we solve a Cousin I problem on $T \times O$ and obtain $G$, meromorphic on $T \times O$, analytic off $T \times U_0$, with a representation,

$$G(t, z) = (z_{n+1} - t)^{-1} + E(t, z) \text{ on } T \times U_0,$$

where $E$ is analytic on $T \times U_0$.

Choose a neighborhood, $N$, of $p$ in hull$(X)$ such that $\bar{N} \subset U_0 \cap \mathrm{hull}(X)$. Since the pole-set of $G$ intersects $[1, 1+\Delta] \times \mathrm{hull}(X)$ only in $\{(1, p)\}$, $|G|$ is bounded on $[1, 1+\Delta] \times (\mathrm{hull}(X) - N)$, and $G(t, z)$ is analytic about hull$(X)$ for each $t > 1$. Also, $\max |G(t, z)|$ on hull$(X)$ tends to infinity as $t \downarrow 1$. Hence, there is a $\delta$, $0 < \delta < \Delta$, such that, for each $t \in (1, 1+\delta]$, $|G(t, z)|$ attains its maximum over hull$(X)$ only on $N$. Also, there is a $K > 0$ such that $|E| < K$ on $[1, \delta] \times \bar{N}$.

Consider the ratio, $$R_t = |G(t,p)|/\max_{\text{hull}(X)}|G(t,z)|,$$

for each $t \in (1, 1+\delta]$. Then $R_t \leqslant 1$ and

$$R_t = \frac{|(1-t)^{-1} + E(t,p)|}{\max_{\bar{N}} |(z_{n+1}-t)^{-1} + E(t,z)|} \geqslant \frac{(t-1)^{-1} - K}{\max_{\bar{N}} |(z_{n+1}-t)^{-1}| + K}.$$

However, on $\bar{N}$, $|z_{n+1}| \leqslant 1$ and $z_{n+1} = 1$ precisely at $p$. Therefore, for $t \in (1, 1+\delta]$, $\max_{\bar{N}} |z_{n+1} - t|^{-1} = (t-1)^{-1}$. (Also, for $t > 1$ close enough to 1, we have $(t-1)^{-1} > K$.) So,

$$1 \geqslant R_t \geqslant \frac{(t-1)^{-1} - K}{(t-1)^{-1} + K} \to 1 \quad \text{as} \quad t \downarrow 1.$$

Hence, for a $t \in (1, 1+\delta]$ sufficiently close to 1, if we define

$$g(z) = G(t,z)/\max_{\text{hull}(X)} |G(t,z)| \quad \text{on} \quad O,$$

then $g$ will be analytic on $O$ and we will have $|g| \leqslant 1$ on hull$(X)$, $|g| < \frac{1}{4}$ on hull$(X) - N$, and $|g(p)| > \frac{3}{4}$. Therefore, (by (A.25) and (A.4)), $p$ is a peak point for $P(X)$. Q.E.D.

*Comment.* Rossi's original proofs of (1.4) and (1.8) involve the solution of a Cousin II problem rather than a Cousin I problem. This is a somewhat more difficult approach, but he obtains a stronger result than (1.8) (see [23, Theorem 4.1]).

For our local description of $R$-hull$(X)$, (1.9), the solution of a Cousin II problem seems essential.

(1.9) THEOREM. *Let $O$ be a neighborhood of* $R$-hull$(X)$ in $C^n$. If $\check{H}^2(R\text{-hull}(X); Z) = 0$ *and $H$ is an analytic subvariety of $O$ which is a hypersurface and which intersects* $R$-hull$(X)$, *then $H$ also intersects $X$.*

*Proof.* By (A.3), we can express $R$-hull$(X)$ as the intersection of a descending chain of rational polyhedra, $O_i$, with $O_i \supset \bar{O}_{i+1}$. By (A.22), for $i$ sufficiently large, there is an $h$ analytic on $O_i$ such that

$$H \cap O_i = \{u \in O_i : h(u) = 0\}.$$

If $H \cap X = \emptyset$, then $\min_{x \in X} |h(x)| > 0$, although $h(p) = 0$ for any $p \in H \cap R\text{-hull}(X)$. By (A.4), there is also a rational function $r$, analytic about $R$-hull$(X)$ such that $0 \leqslant |r(p)| < \min_{x \in X} |r(x)|$. But then $1/r$ is a rational function analytic about $X$ such that $|1/r(p)| > \max_{x \in X} |1/r(x)|$, contradicting $p \in R\text{-hull}(X)$. Q.E.D.

Corresponding to the Local Maximum Modulus Principle, (1.7), we have

(1.10) COROLLARY. *If* $\check{H}^2(R\text{-hull}(X); Z) = 0$ *and* $S \subset R\text{-hull}(X)$, *then* $S \subset R\text{-hull}\,(\partial S \cup (S \cap X))$ (*where* $\partial = \partial_{R\text{-hull}(X)}$).

*Proof.* If $p \in S - R\text{-hull}(\partial S \cup (S \cap X))$, there is a polynomial $f$, such that $0 = f(p) \notin f(\partial S \cup (S \cap X))$, (by (1.1)). Let $H_f = \{u \in C^n : f(u) = 0\}$. Express $R$-hull$(X)$ as the intersection of a descending chain of relatively compact open sets, $O_i$, such that $O_i \supset \bar{O}_{i+1}$, and let $H_i$ be the connected component of $H_f \cap O_i$ through $p$. Then $H_i$ is an analytic hypersurface which is a subvariety of $O_i$. The $H_i$ form a descending chain of analytic varieties whose intersection, $H_\infty$, is the connected component of $H_f \cap \text{hull}(X)$ through $p$. Since $p \in S$ and $H_f \cap \partial S = \emptyset$, $H_\infty \subset S$. Since $H_f \cap (S \cap X) = \emptyset$, we actually have $H_\infty \subset S - (S \cap X)$. By compactness of $X$, there must then be an $i = i_0$ for which $H_{i_0} \cap X = \emptyset$. If we now set $O = O_{i_0}$ and $H = H_{i_0}$, and apply (1.9), we arrive at a contradiction. Q.E.D.

Let us now demonstrate, by an example, that the conclusions of (1.9) and (1.10) do not necessarily hold without the restriction, $\check{H}^2(R\text{-hull}(X); Z) = 0$. We shall use one of Wermer's non-polynomially convex cells and an elementary result from Chapter 2.

(1.11) EXAMPLE. *There exists a compact set, $X$, in $C^3$ and a compact subset, $S$, of* $R$-hull$(X)$ *such that* $R$-hull$(\partial S \cup (S \cap X)) \not\supset S$ (*where* $\partial = \partial_{R\text{-hull}(X)}$).

*Demonstration.* Let $B$ be the bicylinder $|z| \leqslant 1$, $|w| \leqslant 1$ in $C^2$. Map $C^2$ into $C^3$ by

$$\Phi(z, w) = (z, zw, w(zw - 1)),$$

and set $X = \Phi(B)$. Let $V$ be the variety in $C^3$ defined by $z_2 = 1$, $z_3 = 0$ and let $D$ be the disk, $|z_1| \leqslant 1$, on $V$. Then $X \cap V$ is the circle, $|z_1| = 1$ on $V$, so hull$(X \cap V) = D$. Since $\Phi$ is a one–one mapping, $X$ is certainly simply-coconnected. Therefore, by (2.6), $R$-hull$(X) \supset X \cup D$. But Wermer shows in [35] that hull$(X) = X \cup D$. Hence, $R$-hull$(X) = X \cup D$. Let

$$S = \{u \in D : |z_1(u)| \leqslant \tfrac{1}{2}\}.$$

Then $S \cap X = \emptyset$, $\partial S$ is the circle, $|z_1| = \frac{1}{2}$ on $D$, and $R$-hull$(\partial S) = \partial S \not\supset S$.

We conclude this chapter by mentioning some simple applications of our local descriptions, (1.4), 1.9), to hulls which lie on one-dimensional analytic varieties.

(1.12) *Let $V$ be a purely one-dimensional analytic variety in $C^n$ and suppose* hull$(X) \subset V$. *Then*

(i) *$X$ is rationally convex.*

(ii) $\partial_V \text{hull}(X) \subset X$.

(iii) hull$(X) - X$ *is the union of all components of $V - X$ whose closures (in $V$) are compact.*

These facts are generally known. They can be derived, for example, from Behnke's generalization of Runge's Theorem to an open Riemann surface (see [3], [4]). However, they also follow readily from our local descriptions of the hulls. Part (ii) of (1.12) will be used in Chapter 2, so we present a short proof of it. (Note, by the way, that (iii) follows directly from (ii) and the maximum modulus principle on an analytic variety.)

*Proof of (iii).* The singular points of $V$ are isolated and, by (A.7), every isolated point of hull$(X)$ must belong to $X$. Therefore, we need only show that every *regular* point, $p$, in $\partial_V \operatorname{hull}(X)$ is in $X$. For a sufficiently small neighborhood, $U$, of $p$ in $V$ there is a coordinate function, $f$, analytic on a neighborhood of $\bar{U}$, and mapping $\bar{U}$ homeomorphically onto an open set in $C^1$. Since $p \in \partial_V \operatorname{hull}(X)$, $f(p) \in \partial_{C^1} f(\bar{U} \cap \operatorname{hull}(X))$. Therefore, by (1.5), $f(p) \in f(\partial \bar{U})$ or $f(p) \in f(X)$. That means, $p \in \partial \bar{U}$ or $p \in X$. Hence, $p \in X$. Q.E.D.

## 2. The role of the argument principle

Let $X$ be a compact set in $C^n$.

DEFINITION. If $\Phi$: $X \to C^1 - \{0\}$ is of the form $\Phi = \exp(\Psi)$ for some map $\Psi$: $X \to C^1$ we say that $\log(\Phi)$ *is defined and that* $\Psi$ *is a branch of* $\log(\Phi)$.
One basic property of "log" is the following.

(2.1) *Let $X$ be a compact subset of an analytic variety and let $f$ be analytic on a neighborhood of $X$. If* $\log(f|_X)$ *is defined, then there exists a neighborhood, $U$, of $X$ such that* $\log(f|_U)$ *is defined and analytic.*

The proof is elementary, using the fact that if $f(x) \neq 0$ then, for some neighborhood $U_x$ of $x$, $\log(f|_{U_x})$ is defined.

DEFINITION. $X$ *is simply-coconnected* if, for every map $\Phi$: $X \to C^1 - \{0\}$, $\log(\Phi)$ is defined.

*Remark.* $X$ is simply-coconnected if and only if $\check{H}^1(X; Z) = 0$. Note that $X$ can be simply-connected ($\pi_1(X) = 0$) without being simply-coconnected (and conversely). However, if $X$ is contractible, it is certainly simply-coconnected.

DEFINITION. $X$ *enjoys the generalized argument principle* (*g.a.p.*) provided that, if $f$ is any polynomial such that $\log(f|_X)$ is defined, then $0 \notin f(\operatorname{hull}(X))$.

The relevance of the generalized argument principle lies in the following elementary fact.

(2.2) *If $X \subset Y$ with $X$ enjoying the g.a.p. and $Y$ simply-coconnected, then* hull$(X) \subset$ $R$-hull$(Y)$.

*Proof.* If $p \notin R$-hull$(Y)$, then by (1.1) there is a polynomial $f$ such that $0 = f(p) \notin f(Y)$. Since $Y$ is simply-coconnected, $\log(f|_Y)$ is defined. Since $X \subset Y$, $\log(f|_X)$ is also defined. But $X$ enjoys the g.a.p., so $0 \notin f(\operatorname{hull}(X))$. Hence, $p \notin \operatorname{hull}(X)$. Q.E.D.

(2.3) COROLLARY. *If $X$ is simply-coconnected and enjoys the g.a.p. then* $R$-hull$(X)=$ hull$(X)$.

Now we bring in the classical argument principle.

(2.4) *Let $X$ be a compact subset of a purely one-dimensional analytic variety $V$ in $C^n$. If $h$ is analytic on a neighborhood of $X$ and* $\log(h|_{\partial X})$ *is defined* $(\partial=\partial_V)$, *then* $0\notin h(X)$.

*Proof.* Consider first the case where $V=G$, a Riemann surface. By (2.1) there is a neighborhood of $\partial X$ on which $\log(h)$ is defined. Therefore, for any $p\notin X-\partial X$, we can find a compact set $Y$ such that $p\in Y-\partial Y$, $\log(h|_{\partial Y})$ is defined, and $\partial Y$ is a finite system of piecewise smooth Jordan curves. Then (if we suitably orient the components of $\partial Y$), the classical argument principle [29, p. 176], implies for the number of zeros of $h$ in $Y$:

$$\#\text{ (zeros of } h \text{ in } Y)=\frac{1}{2\pi i}\int_{\partial Y} h'/h=\frac{1}{2\pi i}\int_{\partial Y} d(\log(h))=0.$$

Therefore, $h(p)\neq 0$ for all $p\in X-\partial X$. And since $\log(h|_{\partial X})$ is defined, also $0\notin h(X)$.

Now, if $V$ is any purely one-dimensional analytic variety there is a Riemann surface $G$ and a proper analytic map $\sigma$ from $G$ onto $V$. (For example, let $G$ be the normalization of $V$. See [24, p. 443].) Then $\sigma^{-1}(X)$ is compact and

$$\partial_G\sigma^{-1}(X)\subset\sigma^{-1}(\partial X).$$

If $\log(h|_{\partial X})$ is defined then

$$\log(h\circ\sigma|_{\sigma^{-1}(\partial X)})=\log(h|_{\partial X})\circ\sigma$$

is also defined. Therefore, $\log(h\circ\sigma|_{\partial_G\sigma^{-1}(X)})$

is defined, so by the first part of our proof, $0\notin h\circ\sigma(\sigma^{-1}(X))=h(X)$. Q.E.D.

DEFINITION. *An analytic variety $V$ in $C^n$ is a Runge variety* if hull$(X)\subset V$ whenever $X\subset V$.

(2.5) THEOREM. *If $V$ is a purely one-dimensional Runge variety in $C^n$ then every compact $X\subset V$ satisfies the g.a.p.*

*Proof.* Since $V$ is Runge, hull$(X)\subset V$. Since $V$ is one-dimensional, (1.12) (ii) applies, so $\partial_V$hull$(X)\subset X$. The theorem now follows immediately from (2.4).

*Comment.* By using a theorem of Wermer [34, Theorem 1.3], (generalized by Bishop in [9]) one can show, actually,

*If $X$ is a compact subset of any one-dimensional analytic variety in $C^n$ then $X$ satisfies the g.a.p.* (†)

However, the proof is quite difficult, involving the construction of a one-dimensional analytic variety containing hull($X$). We shall not use (†) in this paper.

A direct consequence of (2.2) and (2.5) is

(2.6) THEOREM. *If $X$ is simply-coconnected in $C^n$ and $V$ is a one-dimensional Runge variety with $X \cap V$ compact then* hull$(X \cap V) \subset R$-hull$(X)$.

There are a number of interesting elementary applications of (2.6).

DEFINITION. *$X$ is polynomially convex in dimension one* if, for every one-dimensional (Runge) variety $V$ such that $X \cap V$ is compact, $X \cap V$ is polynomially convex.

(2.7) COROLLARY. *Every simply-coconnected rationally convex set is polynomially convex in dimension one.*

Actually, by combining (2.2) and (2.4) directly, we can make a stronger assertion. Namely,

(2.7′) *If $X$ is rationally convex and simply-coconnected then there is no one-dimensional analytic variety $V$ whose "boundary", $\overline{V} - V$, lies entirely in $X$.*

From this it follows directly that Wermer's non-polynomially convex cells [33], [35], are not even rationally convex.

We next apply (2.6) to answer a question of E. Bishop. Can two disjoint polynomially convex sets always be separated by the modulus of a polynomial? The answer is "no". In fact

(2.8) *There exist $S$ and $T$, disjoint polynomially convex subsets of $C^3$ such that $f(S)$ intersects $f(T)$ for every polynomial, $f$.*

*Remark.* To demonstrate (2.8) it suffices to verify

(2.9) *There exist $S$ and $T$ disjoint connected polynomially convex subsets of $C^3$, such that $S \cup T$ is not rationally convex.*

For, if $S \cup T$ is not rationally convex, then, by (A.7) (iv), $R$-hull$(S \cup T)$ is connected. Therefore, for any polynomial, $f$, $f(R\text{-hull}(S \cup T))$ is connected. But, by (1.1),

$$f(R\text{-hull}(S \cup T)) = f(S \cup T) = f(S) \cup f(T).$$

Hence,
$$f(S) \cap f(T) \neq \emptyset.$$

We shall now settle (2.9). (Note, however, that in the example we use, it is apparent, without appeal to (A.7) (iv), that some component of $R$-hull$(S \cup T)$ intersects both $S$ and $T$; so that $S$ and $T$ cannot be separated by any polynomial, $f$.)

*Proof of (2.9).* With coordinates $z$, $w$, $t$ on $C^3$, set

$S=\{(z, w, t): |z| \leqslant 1, |w| \leqslant 1, t=0\}$ and

$T=\{(z, w, t): (zw-1)(t-1)=0, |z| \leqslant 2, |w| \leqslant \frac{1}{2}, |t| \leqslant 1\}$.

Then $S \cap T = \emptyset$; for $t=0$ and $|z| \leqslant 1$ on $S$, while on $T \cap \{t=0\}$, $|z|=2$. Also $S$ and $T$ are certainly polynomially convex. In fact, each is a polynomial polyhedron. Furthermore, $S$ and $T$ are simply-coconnected (in fact, contractible). For $S$ is just a bicylinder (product of disks); and $T=T_1 \cup T_2$, where

$$T_1=\{(z, w, t): zw=1, |z|=2, |t| \leqslant 1\}$$
$$=\{(2e^{i\theta}, \tfrac{1}{2}e^{-i\theta}, t): 0 \leqslant \theta \leqslant 2\pi, |t| \leqslant 1\},$$

and
$$T_2=\{(z, w, t): |z| \leqslant 2, |w| \leqslant \tfrac{1}{2}, t=1\}.$$

Then we can contract $T_1$ onto

$$T_1 \cap T_2=\{(2e^{i\theta}, \tfrac{1}{2}e^{-i\theta}, 1): 0 \leqslant \theta \leqslant 2\pi\} \subset T_2,$$

and then contract the bicylinder, $T_2$, to a point.

Since $S$ and $T$ are disjoint, compact, simply-coconnected sets, $S \cup T$ is also simply-coconnected. Let

$$V=\{(z, w, t): zw=1, t=0\}.$$

Then
$$(S \cup T) \cap V=\{v \in V: |z(v)|=1 \text{ or } |z(v)|=2\}.$$

So
$$\operatorname{hull}((S \cup T) \cap V)=\{v \in V: 1 \leqslant |z(v)| \leqslant 2\}.$$

Therefore, by (2.6),

$$R\text{-hull}(S \cup T) \supset S \cup T \cup \operatorname{hull}((S \cup T) \cap V) \supsetneqq S \cup T. \quad \text{Q.E.D.}$$

In the same line, we have

(2.10) *There is a compact set, $X$, in $C^2$ which is polynomially convex with respect to the coordinate system $z_1$, $\bar{z}_2$; but which is not even rationally convex with respect to $z_1$, $z_2$.*

Namely, set
$$X=X_1 \cup X_2,$$

where
$$X_1=\{(z_1, z_2): |z_1| \leqslant 2, z_1-4\bar{z}_2=0\}$$

and
$$X_2=\{(z_1, z_2): |z_1| \leqslant 1, z_1-\bar{z}_2=0\}.$$

We leave it to the reader to apply (2.6) to verify our assertion.

Now we are going to use (2.5) to show that certain nice $X$ do satisfy the generalized argument principle. Firstly, we have

(2.11) THEOREM. *If* hull$(X)$ *is an analytic polyhedron in* $C^n$, *then* $X$ *satisfies the generalized argument principle.*

*Proof.* Express the analytic polyhedron, hull$(X)$, as

$$\{u \in U: |f_i(u)| \leqslant k_i,\ i=1,\dots,t\},$$

where $U$ is an open polynomial polyhedron, $f_1,\dots,f_t$ are analytic on $U$, and the $k_i$ are non-negative constants. Let $h$ be a polynomial and let

$$H=\{u \in U: h(u)=0\}.$$

LEMMA. *If* $H \cap X = \emptyset$, *but* $H \cap \text{hull}(X) \neq \emptyset$, *there exists* $V_1$, *a purely one-dimensional analytic subvariety of* $U$, *such that* $H \cap \text{hull}(X \cap V_1) \neq \emptyset$.

*Proof.* For each $y \in H \cap \text{hull}(X)$, define $\#(y)=$ the number of $i \in \{1,\dots,t\}$ for which $|f_i(y)|=k_i$. Choose $p \in H \cap \text{hull}(X)$ such that $\#(p)$ is maximal. By renumbering, we can arrange that $|f_i(p)|=k_i$ if and only if $i \leqslant \#(p)$. If $\#(p)>0$, define

$$V=\{u \in U: f_i(u)-f_i(p)=0,\ i=1,\dots,\#(p)\}.$$

If $\#(p)=0$, define $V=U$.

Then, by (A.27) and (A.4), $V \cap \text{hull}(X)$ is a maximum set for $(P(X)$. Therefore, by (A.31),

$$V \cap \text{hull}(X)=\text{hull}(X \cap V).$$

Let $V_1$ be the union of all irreducible branches of $V$ which contain $p$. We shall show that $\dim(V_1)=1$.

Firstly, if $\dim(V_1)=0$ then $p$ is isolated in $V$ and, therefore, in hull$(X \cap V)$. By (A.7), $p$ must be in $X \cap V$. But $p \in H$ and $H \cap X=\emptyset$. Contradiction.

Next, suppose some irreducible branch, $V_1^0$, of $V_1$, has dimension $\geqslant 2$. Then, by (A.17), every irreducible branch of $V_1^0 \cap H$ has dimension $\geqslant 1$. Let $W$ be the union of all irreducible branches of $V_1^0 \cap H$ which contain $p$. Then $W$ is closed in $V$, it is connected, and, by (A.18), it is not compact. Therefore, $W$ must intersect $\partial_V \text{hull}(X \cap V)$ at some point $q$ (which is not isolated in $V$). Since

$$\text{hull}(X \cap V)=V \cap \text{hull}(X)=\{v \in V: |f_i(v)| \leqslant k_i,\ i=\#(p)+1,\dots,t\},$$

there must be some $i_0 \in \{\#(p)+1,\dots,t\}$ for which $|f_{i_0}(q)|=k_{i_0}$. In that case, $\#(q) \geqslant \#(p)+1$, contradicting our choice of $p$.

Hence, $V_1$ is purely one-dimensional. To complete the proof of the lemma we shall show that $p \in H \cap \text{hull}(X \cap V_1)$. If $V=V_1$ we are done (because $p \in H \cap \text{hull}(X \cap V)$). If $V \neq V_1$ let $V_2$ be the union of all irreducible branches of $V$ which are not contained in $V_1$.

Then $V = V_1 \cup V_2$ and $p \in V_1 - V_2$. Hence, by (A.19), there exists $f$ analytic on $U$ such that $f(p) = 1$ and $f|_{V_2} = 0$.

If $p \notin \mathrm{hull}(X \cap V_1)$, there is a polynomial, $g$, such that $g(p) = 1$ and $|g| < 1$ on $X \cap V_1$. Since

$$X \cap V = (X \cap V_1) \cup (X \cap V_2),$$

if we choose a large enough positive integer, $r$, we will have $(g^r f)(p) = 1$ while $|g^r f| < 1$ on $X \cap V$. But this contradicts the fact that $p \in \mathrm{hull}(X \cap V)$, (because $g^r f \in P(X \cap V)$, by (A.4)). Therefore, $p \in \mathrm{hull}(X \cap V_1)$. Since $p \in H$, the lemma is established.

To prove the theorem, (2.11), observe that if $\log(h|_X)$ is defined, then $H \cap X = \emptyset$. We have to show that, also, $H \cap \mathrm{hull}(X) = \emptyset$. (That is, $0 \notin h(\mathrm{hull}(X))$.) But if $H \cap \mathrm{hull}(X) \neq \emptyset$, then there is a $V_1$ as in the lemma. However, $\log(h|_{X \cap V_1})$ is also defined; so, by (2.5) and (A.21), $0 \notin h(\mathrm{hull}(X \cap V_1))$. That is, $H \cap \mathrm{hull}(X \cap V_1) = \emptyset$; contradicting the lemma. Therefore, $H \cap \mathrm{hull}(X) = \emptyset$, and we are done. Q.E.D.

We want to show that (2.11) persists in the limit. This is made precise in (2.13) below.

The compact subsets of $C^n$ form a metric space, $\mathcal{K}$, by defining, for $X$ and $Y$, compact subsets of $C^n$,

$$\text{distance } (X, Y) = \max_{x \in X} \min_{y \in Y} |x - y| + \max_{y \in Y} \min_{x \in X} |y - x|.$$

Moreover, for any $Z \in \mathcal{K}$, $\{X \in \mathcal{K}: X \subset Z\}$ is compact in the metric topology (see [15]). If a sequence, $X_i$, of compact subsets of $C^n$ converges to a compact set, $X$, in this topology, we write $X_i \to X$.

The following result is elementary.

(2.12) *Let $X_i$ be a sequence of compact subsets of $C^n$, each of which satisfies the g.a.p. If $X_i \to X$ and* $\mathrm{hull}(X_i) \to Y$ *then, for every polynomial, $h$, such that* $\log(h|_X)$ *is defined,* $0 \notin h(Y)$. *In particular, if* $\mathrm{hull}(X_i) \to \mathrm{hull}(X)$, *then $X$ satisfies the g.a.p.*

*Proof.* Choose $y \in Y$ and a sequence of points $y_i \in \mathrm{hull}(X_i)$, such that $y_i \to y$. For each $i$, let $T_i$ be the motion of $C^n$ defined by $T_i(p) = p + y - y_i$ (coordinatewise addition). Observe that, for each $i$, $y \in T_i(\mathrm{hull}(X_i))$ and that

$$T_i(\mathrm{hull}(X_i)) = \mathrm{hull}(T_i(X_i)).$$

Moreover, we still have $T_i(X_i) \to X$ and each $T_i(X_i)$ satisfies the g.a.p. (This is very easy to see.) By (2.1), if $h$ is a polynomial such that $\log(h|_X)$ is defined, then, for $i$ large enough, $\log(h|_{T_i(X_i)})$ is defined. But we have seen that $y \in \mathrm{hull}(T_i(X_i))$ and each $T_i(X_i)$ satisfies the g.a.p. Therefore, $h(y) \neq 0$; for all $y \in Y$. Q.E.D.

(2.13) COROLLARY. *Let* $X_i \to X$ *and* $\operatorname{hull}(X_i) \to \operatorname{hull}(X)$, *with each* $\operatorname{hull}(X_i)$ *an analytic polyhedron. Then* $X$ *satisfies the g.a.p. Consequently, if* $X$ *is also simply-coconnected, then* $\operatorname{hull}(X) = R\text{-}\operatorname{hull}(X)$.

*Proof.* The first assertion follows directly from (2.11) and (2.12); the second, from these and (2.3). Q.E.D.

Of course, by (A.2), for any compact, $X$, there are sequences $\operatorname{hull}(X_i) \to \operatorname{hull}(X)$, where each $\operatorname{hull}(X_i)$ is an analytic (in fact, polynomial) polyhedron. Also, it is easily seen that, under these circumstances, the limit set of the $X_i$ must contain $X$. The problem is then the following.

(2.14) QUESTION. *If* $X$ *is a compact subset of* $C^n$ *must there exist a sequence of analytic polyhedra of the form* $\operatorname{hull}(X_i)$ *with* $\operatorname{hull}(X_i) \to \operatorname{hull}(X)$ *and* $X_i \to X$?

If the answer were "yes" then every $X$ would satisfy the g.a.p. (by (2.13)). However,

(2.15) (*Bishop-Hoffman*). *There exists a compact set,* $E$, *in* $C^2$ *which does not satisfy the g.a.p. In fact,* $\log(z_1|_E)$ *is defined, although* $z_1$ *vanishes at a point of* $\operatorname{hull}(E)$.

*Demonstration.* (Such an example, in some $C^n$, was first constructed by K. Hoffman using [27]. Later, E. Bishop found the following very simple example in $C^2$.)

Let
$$E = E_1 \cup E_2,$$
where
$$E_1 = \{(e^{i\theta}, z_2)\colon 0 \leqslant \theta \leqslant \pi,\ |z_2| = 1\}$$
and
$$E_2 = \{(e^{i\theta}, 0)\colon \pi \leqslant \theta \leqslant 2\pi\}.$$

Notice that $\log(z_1|_{E_i})$ is defined for $i = 1, 2$ (because $z_1(E_i)$ is an arc in $C^1 - \{0\}$). Since $E_1$ and $E_2$ are compact and disjoint, also $\log(z_1|_E)$ is defined. However,

$$\operatorname{hull}(E) = \operatorname{hull}(E_1 \cup E_2) = \operatorname{hull}(\{(e^{i\theta}, z_2)\colon 0 \leqslant \theta \leqslant \pi,\ |z_2| \leqslant 1\} \cup \{(e^{i\theta}, 0)\colon \pi \leqslant \theta \leqslant 2\pi)\})$$
$$\supset \operatorname{hull}(\{(e^{i\theta}, 0)\colon 0 \leqslant \theta \leqslant 2\pi\}) = \{(z_1, 0)\colon |z_1| \leqslant 1\}.$$

Therefore, $(0, 0) \in \operatorname{hull}(E)$, at which point $z_1$ certainly vanishes. Q.E.D.

(2.16) COROLLARY. *The answer to Question* (2.14) *is "no".*

*Proof.* By (2.13) and (2.15).

Bishop's example, together with (2.11), also yields

(2.17) EXAMPLE. *There exists a rational polyhedron,* $R$, *in* $C^2$ *such that* $\operatorname{hull}(R)$ *is not an analytic polyhedron.*

*Demonstration.* Consider $E=E_1 \cup E_2$ as in (2.15). Evidently $E_1$ and $E_2$ are both connected and rationally convex. However,

$$z_2(R\text{-hull}(E_1 \cup E_2)) = z_2(E_1) \cup z_2(E_2)$$

is not connected, because $|z_2|=1$ on $E_1$ while $z_2|_{E_2}=0$. Hence, $R$-hull$(E_1 \cup E_2)$ is not connected; so, by (A.7) (iv), $E=E_1 \cup E_2$ is rationally convex. By (A.2), for any neighborhood, $U$, of $E$ in $C^2$ there is a rational polyhedron, $R_U$, with $E \subset R_U \subset U$. Hence, by (2.1), there is a rational polyhedron, $R$, such that $E \subset R$ and $\log(z_1|_R)$ is defined. Then,

$$(0, 0) \in \text{hull}(E) \subset \text{hull}(R).$$

Therefore, also $R$ does not satisfy the g.a.p.; so, by (2.11), hull$(R)$ is not an analytic polyhedron. Q.E.D.

*Comments.* 1. The example of (2.15) should be compared with the assertion of (2.23).

2. A trivial modification of Bishop's example shows also

(2.18) *There is a simply-coconnected (in fact, contractible) compact set, $F$, in $C^2$ such that* $R$-hull$(F) \nsubseteq$ hull$(F)$.

Namely, let $$E = E_1 \cup E_2$$
be as in (2.15) and let $$D = \{(1, z_2)\colon |z_2| \leqslant 1\}.$$
If we set $$F = E \cup D,$$
then $F$ is easily seen to be contractible. But

$$(0, 0) \in \text{hull}(F) - R\text{-hull}(F).$$

Note, however, that $F$ is not rationally convex. (Theorem (2.6) shows that $R$-hull$(F) \supset \{(e^{i\theta}, z_2)\colon 0 \leqslant \theta \leqslant \pi,\ |z_2| \leqslant 1\}$.)

Although (2.13) does imply that certain simply-coconnected rationally convex sets are necessarily polynomially convex, the condition given therein seems difficult to verify in any particular case. However, there *is* a weak version of the generalized argument principle, (2.19) below, valid for all $X$, which can be applied effectively in some interesting situations to show that various simply-coconnected sets must be polynomially convex. To establish (2.19) we shall use the local descriptions of hull$(X)$.

(2.19) THEOREM. *Let $X$ be a compact subset of $C^n$ and let $f$ be analytic on a neighborhood of* hull$(X)$. *If* $\log(f|_X)$ *is defined and $X$ is open in* $f^{-1}(f(X)) \cap$ hull$(X)$, *then*

$$0 \notin f(\text{hull}(X)).$$

*In particular,*

*if* $f(X) \cap f(\text{hull}(X) - X) = \emptyset$ *and* $\log(f|_X)$ *is defined, then* $0 \notin f(\text{hull}(X))$.

*Proof*. There is a neighborhood, $U$, of $X$ in $C^n$ such that $\log(f|_U)$ is defined and analytic, (2.1), and

$$\bar{U} \cap f^{-1}(f(X)) \cap \text{hull}(X) = X.$$

Let $f_U = f|_U$. Since $f_U$ is analytic, $f_U(U)$ contains a neighborhood of $f(X)$ in $C^1$. We assert that there is a neighborhood, $N$, of $f(X)$ in $C^1$, with $\bar{N} \subset f_U(U)$ and $f_U^{-1}(\bar{N}) \cap \text{hull}(X)$ compact. For, if not, there is a sequence, $N_i$, of neighborhoods of $f(X)$, with $f_U(U) \supset N_i \supset \bar{N}_{i+1}$, $\bigcap_i \bar{N}_i = f(X)$, and with points, $p_i$, in $f^{-1}(\bar{N}_i) \cap \text{hull}(X) \cap \partial_{C^n} U$. Passing to a subsequence, the $p_i$ converge to some $p \in \bigcap_i f^{-1}(\bar{N}_i) \cap \text{hull}(X) \cap \partial_{C^n} U$. But

$$\bigcap_i f^{-1}(\bar{N}_i) = f^{-1}(\bigcap_i \bar{N}_i) = f^{-1}(f(X)),$$

so
$$p \in f^{-1}(f(X)) \cap \text{hull}(X) \cap \bar{U} = X.$$

But also, $p \in \partial_{C^n} U$, and $X \cap \partial_{C^n} U = \emptyset$. Contradiction. Thus the assertion is true.

Set
$$S = f_U^{-1}(\bar{N}) \cap \text{hull}(X).$$

Then
$$f_U(S) = \bar{N} \cap f(\text{hull}(X)).$$

Furthermore, $S$ is a compact subset of the open set, $U$, so $\partial S$ (with $\partial = \partial_{\text{hull}(X)}$) is contained in $\partial_U f_U^{-1}(\bar{N})$. But

$$\partial_U f_U^{-1}(\bar{N}) \subset f_U^{-1}(\partial_{C^1} \bar{N}),$$

so
$$\partial S \subset f_U^{-1}(\partial_{C^1} \bar{N}).$$

Also, $S \cap X = X$ because $X \subset f_U^{-1}(\bar{N})$. By (1.6),

$$\partial_{C^1} \log(f_U)(S) \subset \log(f_U)(\partial S \cup (S \cap X)).$$

Therefore,
$$\partial_{C^1} \log(f_U)(S) \subset \log(f_U)(f_U^{-1}(\partial_{C^1} \bar{N})) \cup \log(f_U)(X).$$

Let Log be the (one-many) correspondence from $C^1 - \{O\}$ to $C^1$ which associates with each $z_0 \in C^1 - \{O\}$ all the values of $\log(z_0)$. Thus,

$$\text{Log}(z_0) = \{w \in C^1 \colon e^w = z_0\}.$$

Evidently, $\log(f_U)(f_U^{-1}(\partial_{C^1} \bar{N})) \subset \text{Log}(\partial_{C^1} \bar{N})$ and $\log(f_U)(X) \subset \text{Log}(f_U(X))$. Therefore, combining the above remarks,

(i)
$$\partial_{C^1} \log(f_U)(S) \subset \text{Log}(\partial_{C^1} \bar{N} \cup f_U(X)).$$

To complete the proof of the theorem we suppose, on the contrary, that $0 \in f(\text{hull}(X))$. Let $N_0$ be another neighborhood of $f(X)$ in $C^1$ with $\bar{N}_0 \subset N$. By (A.6),

$$\partial_{C_1} f(\text{hull}(X)) \subset f(X).$$

Therefore, $$G = f(\mathrm{hull}(X)) - (N_0 \cap f(\mathrm{hull}(X)))$$

is an open subset of $C^1$, with

$$\partial_{C^1} G \subset (\partial_{C^1} \bar{N}_0) \cap f(\mathrm{hull}(X)).$$

Hence, $\partial_{C^1} G$ does not intersect $\partial_{C^1} \bar{N} \cup f_U(X)$. Also, $\bar{N}_0 \subset f_U(U)$, and $\log(f_U)$ is defined, so $0 \notin \bar{N}_0$. Therefore, $0 \in G$.

We now appeal to the classical argument principle [1, p. 123], to conclude that, for some component, $\Delta$, of $\partial_{C^1} G$, $\log(z|_\Delta)$ is *not* defined. By the nature of the correspondence' Log, this implies that $\mathrm{Log}(\Delta)$ is a connected unbounded subset of $C^1$. Also,

(ii) $$\mathrm{Log}(\Delta) \cap \mathrm{Log}(\partial_{C^1} \bar{N} \cup f_U(X)) = \emptyset,$$

because $\Delta \cap (\partial_{C^1} \bar{N} \cup f_U(x)) = \emptyset$).

However, $$\Delta \subset \bar{N} \cap f(\mathrm{hull}(X)) = f_U(S),$$

so $\mathrm{Log}(\Delta)$ does intersect $\log(f_U)(S)$. But $\mathrm{Log}(\Delta)$ is connected and unbounded, while $\log(f_U)(S)$ is compact. Hence, $\mathrm{Log}(\Delta)$ intersects $\partial_{C^1} \log(f_U)(S)$. Therefore, by (i), $\mathrm{Log}(\Delta)$ also intersects $\mathrm{Log}(\partial_{C^1} \bar{N} \cup f_U(X))$. But this contradicts (ii). Thus, $0 \notin f(\mathrm{hull}(X))$. Q.E.D.

(2.20) COROLLARY. *Let $X$ be a compact subset of $C^n$.*

*If there is a simply-coconnected set, $Y$, containing $X$ and a function, $g$, continuous on $Y \cup \mathrm{hull}(X)$ and analytic about $\mathrm{hull}(X)$, for which $X$ is open in $g^{-1}(g(Y)) \cap \mathrm{hull}(X)$, then $X$ is polynomially convex.*

*In particular, if $X$ is simply-coconnected and there is a function, $g$, analytic about $\mathrm{hull}(X)$ (for instance, a polynomial) such that $g(X) \cap g(\mathrm{hull}(X) - X) = \emptyset$, then $X$ is polynomially convex.*

*Proof.* (The argument is similar to that of (2.2).) First we prove that $g(\mathrm{hull}(X)) \subset g(Y)$. If not, there is $p \in \mathrm{hull}(X)$ with $g(p) \notin g(Y)$. Setting $f = g - g(p)$, we have $0 = f(p) \notin f(Y)$. Then $\log(f|_X)$ is defined (because $X \subset Y$ and $Y$ is simply-coconnected.) Since $0 = f(p) \in f(\mathrm{hull}(X))$, (2.19) implies that $X$ cannot be open in $f^{-1}(f(X)) \cap \mathrm{hull}(X)$. Therefore, $X$ cannot be open in $g^{-1}(g(X)) \cap \mathrm{hull}(X)$ (because $f$ and $g$ differ by a constant). Hence, since $X \subset Y$, $X$ cannot be open in $g^{-1}(g(Y)) \cap \mathrm{hull}(X)$, contradicting our hypothesis.

Therefore, $$g(\mathrm{hull}(X)) \subset g(Y);$$

so $$g^{-1}(g(Y)) \cap \mathrm{hull}(X) = \mathrm{hull}(X).$$

Thus $X$ is open in $\mathrm{hull}(X)$, which implies, by (A.7) that $X = \mathrm{hull}(X)$. Q.E.D.

Now we are going to apply (2.20).

DEFINITION. Let $O$ be an open subset of $C^n$ and let $f_1, \dots, f_k$ be analytic functions on $O$. For each $p \in O$ we define *the level set of* $\{f_1, \dots, f_k\}$ *through* $p$ to be $\{u \in O\colon f_i(u) = f_i(p), i = 1, \dots, k\}$. If each level set of $f_1, \dots, f_k$ has dimension, at most, $\max\{0, n-k\}$, we say that $\{f_1, \dots, f_k\}$ *is in general position*. If $k \leq n$, and $\{f_1, \dots, f_k\}$ is in general position then, by (A.17), every subset of $\{f_1, \dots, f_k\}$ is also in general position. Evidently, for one analytic function, $f$, on $O$, $\{f\}$ is in general position if and only if it is not constant on any component of $O$.

(2.21) THEOREM. *Let $X$ be a compact subset of $C^n$ and let $O$ be a neighborhood of* hull$(X)$. *Suppose there are $n-1$ analytic functions, $f_1, \dots, f_{n-1}$ on $O$, such that*

(i) *$\{f_1, \dots, f_{n-1}\}$ is in general position, and*
(ii) *for each $j = 1, \dots, n-1$, $f_j(X) \subset M_{f_j(\mathrm{hull}(X))}$, (the minimal boundary for the uniform limits of functions analytic about $f_j$*(hull$(X)$)*, see* (A.25)).

*Under these circumstances, if $X$ is simply-coconnected and is polynomially convex in dimension one, then $X$ is polynomially convex.*

*Proof.* By (A.2), we can take $O$ to be an open polynomial polyhedron, so that, by (A.21), every analytic subvariety of $O$ is a Runge variety.

For each $j = 0, \dots, n-1$, define $L_j$ to be the set of all level sets, $V$, of all subsets, $\{f_1, \dots, f_{i_j}\}$ (with $j$ distinct elements) such that $V \cap \mathrm{hull}(X) = \mathrm{hull}(X \cap V)$. Note that $L_0 = \{O\}$, and that, by (i), $f_1, \dots, f_{n-1}$ must be distinct. Every $V \in L_{n-1}$ is a level set of $\{f_1, \dots, f_{n-1}\}$ and, therefore, is purely one-dimensional. Since $X$ is polynomially convex in dimension one we have, for $V \in L_{n-1}$, $\mathrm{hull}(X \cap V) = X \cap V$.

Suppose now that for all $V \in L_j$ we have $\mathrm{hull}(X \cap V) = X \cap V$, for some $j \in \{1, \dots, n-1\}$. We shall show that this implies that $\mathrm{hull}(X \cap V) = X \cap V$ for all $V \in L_{j-1}$.

Therefore, we choose $V \in L_{j-1}$. Say $V$ is a level set of $\{f_{i_1}, \dots, f_{i_{j-1}}\}$. Choose some *other* $f = f_{i_j} \in \{f_1, \dots, f_{n-1}\}$. By (2.20), to prove $\mathrm{hull}(X \cap V) = X \cap V$, it suffices to show that $f(X) \cap f(\mathrm{hull}(X \cap V) - (X \cap V)) = \emptyset$.

If this is not so, there is a $p \in \mathrm{hull}(X \cap V) - (X \cap V)$ with $f(p) \in f(X)$. By assumption (ii),

$$f(p) \in M_{f(\mathrm{hull}(X))}.$$

Hence, by (A.25), there is an $h \in A(f(\mathrm{hull}(X)))$ such that $1 = h(f(p)) > |h(\lambda)|$ for all $\lambda \in f(\mathrm{hull}(X)) - \{f(p)\}$. Then

$$g = h \circ f \in A(\mathrm{hull}(X)) = P(X)$$

and $$\{u \in \mathrm{hull}(X)\colon g(u) = \max_{x \in X} |g(x)|\} = \{u \in \mathrm{hull}(X)\colon f(u) = f(p)\}.$$

Thus, if we set $$H = \{u \in O\colon f(u) = f(p)\},$$

then $H \cap \operatorname{hull}(X)$ is a maximum set for $P(X)$. In particular, $H \cap \operatorname{hull}(X \cap V)$ is a maximum set for $P(X \cap V)$. Thus, by (A.31),

$$H \cap \operatorname{hull}(X \cap V) = \operatorname{hull}(X \cap V \cap H).$$

But $V \in L_{j-1}$, so
$$\operatorname{hull}(X \cap V) = V \cap \operatorname{hull}(X).$$

Therefore, if we let $V_1 = V \cap H$, then we have

$$V_1 \cap \operatorname{hull}(X) = \operatorname{hull}(X \cap V_1).$$

Also, $V_1$ is a level set of $\{f_{i_1}, \ldots, f_{i_j}\}$. Therefore, $V_1 \in L_j$. Thus, by our assumption on $L_j$, we have

$$X \cap V_1 = \operatorname{hull}(X \cap V_1) = V_1 \cap \operatorname{hull}(X).$$

However,
$$p \in V_1 \cap \operatorname{hull}(X) - (V_1 \cap X),$$

so we have arrived at a contradiction. Hence,

$$f(X) \cap f(\operatorname{hull}(X \cap V) - (X \cap V)) = \emptyset$$

and, by (2.20),
$$\operatorname{hull}(X \cap V) = X \cap V, \text{ for all } V \in L_{j-1}.$$

By induction, since we have already remarked that $\operatorname{hull}(X \cap V) = X \cap V$ for all $V \in L_{n-1}$, we obtain, for $L_0 = \{O\}$, $\operatorname{hull}(X) = X$. Q.E.D.

One considerable defect in the hypothesis of (2.21) is that, for (ii), we are required to know something about $\operatorname{hull}(X)$ to begin with. However, there are interesting situations where this information is built-in. Namely, we have

(2.22) THEOREM. *Let $S_1, \ldots, S_{n-1}$ be compact subsets of $C^1$, each with connected complement. Let $f_1, \ldots, f_{n-1}$ be $n-1$ polynomials on $C^n$ such that $\{f_1, \ldots, f_{n-1}\}$ is in general position (on $C^n$). Let $X$ be any compact subset of*

$$f_1^{-1}(\partial S_1) \cap \ldots \cap f_{n-1}^{-1}(\partial S_{n-1}) \ (\textit{where } \partial = \partial_{C^1}).$$

*Under these conditions, if $X$ is simply-coconnected and is polynomially convex in dimension one, then $X$ is polynomially convex. In particular, if $X$ is simply-coconnected and rationally convex (or, if $X$ is an arc), then $X$ is polynomially convex.*

*Proof.* The last statement follows from the preceding one by (2.7) and (1.12). The proof of the main part is based on the following lemma. This proof of the lemma is due to E. Bishop.

LEMMA. *If $S$ is a compact subset of $C^1$ such that $C^1 - S$ is connected, then $M_S = \partial S$.*

*Proof.* (Following Bishop.) Since $\partial S$ is a boundary for $A(S)$ (see (A.29)) and $M_S$ is the minimal boundary for $A(S)$ we have $M_S \subset \partial S$.

On the other hand, suppose $p \in \partial S - M_S$. By (A.25), there is a positive Borel measure, $\lambda_p$, vanishing off $M_S$, with $h(p) = \int h d\lambda_p$, for all $h \in A(S)$. Let $m_p$ be the unit point mass at $p$ and set

$$\mu_p = m_p - \lambda_p.$$

Then $\mu_p$ is a finite Borel measure on $\partial S$ and, since $p \notin M_S$, we have

$$\mu_p(\{p\}) = m_p(\{p\}) - \lambda_p(\{p\}) = 1 - 0 = 1.$$

Also, $\int h d\mu_p = h(p) - h(p) = 0$ for all $h \in A(S)$. Therefore, by Lemmas 9 and 10 of [5] we can decompose $\mu_p$ into $\Sigma_i \mu_i$ where each $\mu_i$ vanishes on points; so $\mu_p(\{p\}) = 0 \neq 1$. Contradiction. Hence, $p \in M_S$ for all $p \in \partial S$. Q.E.D. Lemma.

To complete the proof of (2.22) we first observe that

$$f_j(\mathrm{hull}(X)) \subset \mathrm{hull}(f_j(X)) \subset \mathrm{hull}(\partial S_j) = S_j, \quad j = 1, \ldots, n-1.$$

Therefore, $$M_{S_j} \cap f_j(\mathrm{hull}(X)) \subset M_{f_j(\mathrm{hull}(X))}.$$

By the lemma $$M_{S_j} = \partial S_j,$$

and, by assumption, $$f_j(X) \subset \partial S_j.$$

Hence, $$f_j(X) \subset M_{S_j} \cap f_j(\mathrm{hull}(X)) \subset M_{f_j(\mathrm{hull}(X))}, \quad j = 1, \ldots, n-1.$$

The theorem now follows directly from (2.21). Q.E.D.

We shall conclude by stating explicitly some special cases of (2.22).

(2.23) COROLLARY. *Let $X$ be a compact subset of $C^2$. If $X$ is simply-coconnected and rationally convex (or, if $X$ is an arc) and if there is a non-constant polynomial, $f$, such that $|f| = 1$ on $X$, then $X$ is polynomially convex.*

DEFINITION. $T(n) = \{p \in C^n: |z_i(p)| = 1, i = 1, \ldots, n\}$.

(2.24) COROLLARY. *Let $X$ be a compact subset of $T(n-1) \times C^1$. If $X$ is simply-coconnected and rationally convex, (or, if $X$ is an arc), then $X$ is polynomially convex.*

(2.25) *Let $X$ be a compact subset of a simply-coconnected subset of $T(n)$. Then* (i) *$X$ is polynomial convex and* (ii) *every complex-valued continuous function on $X$ is a uniform limit of polynomials in $z_1, \ldots, z_n$.*

*Proof.* Clearly, every compact subset, $X$, of $T(n)$ is rationally convex, and $\mathrm{hull}(X) - X$ does not intersect $T(n)$. By applying (2.24) we find that, for $X \subset Y \subset T(n)$ with $Y$ simply-coconnected, we have $X = \mathrm{hull}(X)$. This proves (i).

To obtain (ii) from (i) first observe that, on $T(n)$, $\bar{z}_i = 1/z_i$, $i = 1, \dots, n$. Thus, for $X$, a polynomially convex subset of $T(n)$, $\bar{z}_i = 1/z_i \in P(X)$ (by (A.4)). Therefore, $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n \in P(X)$. Hence, by the Stone-Weierstrass Theorem [17], $P(X)$ contains every complex-valued continuous function on $X$. Q.E.D.

## Appendix. Definitions and background theory

Many important properties of hulls are derived by approximating the hulls by analytic polyhedra.

(A.1) DEFINITION. *An analytic polyhedron in $C^n$ is a compact set, $K$, of the form*

$$K = \{u \in U: |f_j(u)| \leq k_j,\ j = 1, \dots, r\},$$

*where $U$ is an open subset of $C^n$, the $f_j$ are analytic on $U$, and the $k_j$ are non-negative constants.*

If the $f_j$ are polynomials [rational functions] we say that $K$ *is a polynomial* [*rational*] *polyhedron.*

By *an open analytic polyhedron in $C^n$* we mean the interior (in $C^n$) of an analytic polyhedron. (Note that the $U$ in the definition of an analytic polyhedron can be taken to be an open analytic polyhedron.)

(A.2) *Let $X$ be a compact subset of $C^n$. Then* hull$(X)$ [$R$-hull$(X)$] *can be represented as the intersection of a descending chain of open polynomial* [*rational*] *polyhedra, $O_i$, with $O_i \supset \bar{O}_{i+1}$.*

This follows directly from the definitions of hull$(X)$ and $R$-hull$(X)$.

We must draw upon the theory of analytic functions of several variables for certain deep theorems concerning analytic polyhedra. Firstly, there is the Oka-Weil Approximation Theorem [20].

(A.3) *If $O$ is an open polynomial* [*rational*] *polyhedron, then each function which is analytic on $O$ is a uniform limit of polynomials* [*rational functions*] *on every compact subset of $O$.*

By (A.2) this implies

(A.4) *If $X$ is polynomially* [*rationally*] *convex in $C^n$, then every function analytic about $X$ is a uniform limit, on $X$, of polynomials* [*rational functions which are analytic about $X$*].

(A.5) NOTATION. *If $A$ is a topological space and $B \subset A$, then $\partial_A B$ denotes the topological boundary of $B$ in $A$. When there is only one $A$ under consideration we write simply $\partial B$ for $\partial_A B$.*

One corollary of (A.4) is

(A.6) *If $f$ is analytic about* hull($X$), *then* $f(X) \supset \partial f(\text{hull}(X))$, (*where* $\partial = \partial_{C^1}$).

*Proof.* If $p \in \partial f(\text{hull}(X)) - f(X)$, there is a point, $q$, in the complement of $f(\text{hull}(X))$ so close to $p$ that

$$|(p-q)^{-1}| > \max\{|(z-q)^{-1}| : z \in f(X)\}.$$

Then $(f-q)^{-1}$ is analytic about hull($X$), but $|(f-q)^{-1}|$ does not attain its maximum over hull($X$) on $X$. By (A.4), there must also be a polynomial, $g$, such that $|g|$ does not attain its maximum over hull($X$) on $X$; which is absurd. Therefore, $\partial f(\text{hull}(X) \subset f(X)$. Q.E.D.

Some other useful consequences of (A.4) are collected in

(A.7) (i) *A polynomial [rational] polyhedron is polynomially [rationally] convex.*

(ii) *An open and closed subset of a polynomially [rationally] convex set is polynomially [rationally] convex.*

(iii) *If $X$ is connected, then $R$-*hull($X$) *and* hull($X$) *are also connected.*

(iv) *Let $X$ and $Y$ be disjoint, connected, polynomially [rationally] convex subsets of $C^n$. If $X \cup Y$ is not polynomially [rationally] convex, then* hull($X \cup Y$) [$R$-hull($X \cup Y$)] *is connected.*

These results are well-known. They all follow very easily from the fact that if $S$ is an open and closed subset of hull($X$) [$R$-hull($X$)], then by (A.4), the function which is 1 on $S$ and 0 on *hull*$(X)-S$ [$R$-hull$(X)-S$] is a uniform limit on hull($X$) [$R$-hull($X$)] of polynomials [rational functions analytic about $R$-hull($X$)].

Next, there is the solution of the Cousin I problem [11], [20].

(A.8) *Let $O$ be an open analytic polyhedron. Then every Cousin 1 problem on $O$ admits a solution. That is, if $O_i$ is an open cover of $O$ and we assign meromorphic functions, $G_i$, on $O_i$, such that $G_i - G_j$ is analytic on $O_i \cap O_j$ (for all $i$, $j$), then there exists $G$, meromorphic on $O$, such that, for each $i$, $G - G_i$ is analytic on $O_i$.*

Theorems (A.3) and (A.8) are the two tools used by Oka in [21] to obtain his local description of hull($X$), (1.4).

We also need the concept of an analytic variety in $C^n$.

DEFINITIONS. (A.9) *A subset, $V$, of $C^n$ is an analytic variety if, for each $v \in V$, there is a neighborhood, $U_v$, of $v$ in $C^n$ and a collection of analytic functions on $U_v$ whose set of common zeros is $V \cap U_v$.* (In particular, an open subset of $C^n$ is an analytic variety.)

(A.10) *An analytic variety, $V$, in $C^n$ is an analytic hypersurface if, for each $v \in V$, there is a neighborhood, $U_v$, of $v$ in $C^n$ and one function, $f_v$, analytic on $U_v$ such that*

$$V \cap U_v = \{u \in U_v : f_v(u) = 0\}.$$

(A.11) *If $V_1$ and $V_2$ are analytic varieties and $V_2$ is a closed subset of $V_1$, then we say that $V_2$ is an analytic subvariety of $V_1$.*

(A.12) *An analytic variety is irreducible if it is not the union of any two proper analytic subvarieties.*

Here are some facts from the local theory of analytic varieties which we need. (They can be extracted from [2] or [25].)

(A.13) *An irreducible analytic variety, $V$, contains an open dense connected subset which is a complex manifold.*

(A.14) DEFINITION. $\dim(V)$ *= the (complex) dimension of this manifold.*

(A.15) *An analytic variety, $V$, can be expressed, in a unique way, as a denumerable union of irreducible analytic subvarieties, $V_i$, such that $V_i \not\subset \bigcup_{j \neq i} V_j$.*

(A.16) DEFINITION. *We call each $V_i$ an irreducible branch of $V$, and set*

$$\dim(V) = \max_i \dim(V_i).$$

*If every irreducible branch of $V$ has dimension, $d$, we say that $V$ is purely $d$-dimensional.*

(A.17) *If an irreducible analytic variety, $V$, intersects an analytic hypersurface, $H$, then every irreducible branch of $V \cap H$ has dimension, at least,* $\dim(V) - 1$.

(A.18) *Every compact analytic variety in $C^n$ consists of a finite set of points.*

In Chapter 2, we use the following "global" result.

(A.19) *Let $O$ be an open analytic polyhedron in $C^n$ and let $V$ be an analytic subvariety of $O$. If $p \in O - V$, there exists a function, $f$, analytic on $O$, such that $f|_V = 0$ but $f(p) \neq 0$.*

This is proved in [11] and [18].

(A.20) DEFINITION. *Let $V$ be an analytic variety in $C^n$. We say that $V$ is a Runge variety in $C^n$ provided that* $\mathrm{hull}(X) \subset V$ *for every $X \subset V$.*

(A.21) *If $V$ is an analytic subvariety of an open polynomial polyhedron in $C^n$, then $V$ is a Runge variety.*

*Proof.* This follows easily from (A.7) (i) and (A.19).

For our local description of $R$-hull$(X)$, (1.9), we require, in place of (A.8), a solution of a Cousin II problem [11, Chapter XX], but only in the following weak form.

(A.22) *Let $Y$ be a compact subset of $C^n$ which is the intersection of a descending chain of open analytic polyhedra, $O_i$, with $O_i \supset \bar{O}_{i+1}$. Let $H$ be an analytic hypersurface which is a subvariety of some neighborhood of $Y$. If $\check{H}^2(Y; Z)=0$, then, for $i$ large enough, there exists an analytic function, $h$, on $O_i$, such that*

$$H \cap O_i = \{u \in O_i\colon h(u)=0\}.$$

See [22] and [28] for a general discussion of the Cousin II problem.

*Comment.* An open analytic polyhedron is certainly a Stein manifold (defined in [12], [8]). The theorems concerning analytic polyhedra mentioned above can be formulated and are valid on any Stein manifold (as is shown in [11] and [8]). Moreover, if $X$ is a compact subset of a Stein manifold, $M$, we can form its associated holomorphic convex hull,

$$\mathrm{hull}_M(X) = \{m \in M : |h(m)| \leqslant \max_{x \in X} |h(x)|, \quad \text{for all } h \text{ analytic on } M\},$$

and study hulls in this more general setting. However, by Remmert's Imbedding Theorem (proved in [7] and [18]) every Stein manifold, M, can be realized as a (closed) analytic submanifold of some $C^n$. Then, (see [11] or [8]), the analytic functions on $M$ will just be the restrictions to $M$ of the analytic functions on $C^n$ and, for any compact $X \subset M$, we will have $\mathrm{hull}_M(X) = \mathrm{hull}(X)$. For this reason, we have restricted our attention to polynomial convexity.

Finally, from the theory of Banach algebras, (see [36] and [17]), we have the following setup.

Let $A$ be an algebra of continuous complex-valued functions on a compact Hausdorff space, $X$. We assume that $A$ contains the constants, separates the points of $X$, and is complete in the norm, $\max_X|\ldots|$. The maximal ideal space of $A$, denoted $\mathfrak{M}(A)$, can be described as the largest space containing $X$ to which $A$ extends as a Banach algebra of functions with the norm, $\max_X|\ldots|$. (See [36] or [17] for a precise definition.) Every algebra homomorphism of $A$ into $C^1$ is of norm one and corresponds to evaluation at a point of $\mathfrak{M}(A)$.

(A.23) Definition. *A point $p \in \mathfrak{M}(A)$ is a peak point for $A$ provided there exists an $f \in A$ such that $|f(p)| > |f(m)|$, for all $m \in \mathfrak{M}(A) - \{p\}$.*

(A.24) Definition. *A boundary for $A$ is a subset, $b$, of $\mathfrak{M}(A)$ such that*

$$|f(m)| \leqslant \max_b |f|,$$

*for all $m \in \mathfrak{M}(A)$, all $f \in A$.*

Clearly, $X$ is a boundary for $A$. Šilov showed that there exists a unique smallest closed boundary for $A$, [17]. We denote it by $S_A$ and refer to it as *the Šilov boundary for $A$*.

The following remarkable theorem concerning minimal boundaries is due to Bishop [6] and Bishop-de Leeuw [10].

(A.25) *Let $X$ be metrizable. Then*

(i) *The set of peak points for $A$ is a boundary for $A$. Evidently, this is the unique minimal (not necessarily closed) boundary for $A$. We denote it by $M_A$.*

(ii) *For each $m \in \mathfrak{M}(A)$, there is a positive Borel measure, $\lambda_m$, of mass 1, on $X$, which vanishes on $X - M_A$, such that*

$$f(m) = \int_X f d\lambda_m \text{ for all } f \in A.$$

(iii) *Let $m \in \mathfrak{M}(A)$. If, for each neighborhood, $U_m$, of $m$, there is an $f \in A$ such that $|f| \leq 1$ on $\mathfrak{M}(A)$, $|f| < \frac{1}{4}$ on $\mathfrak{M}(A) - U_m$, and $|f(m)| > \frac{3}{4}$, then $m \in M_A$.*

Note that $S_A$ is obviously the closure of $M_A$.

(A.26) DEFINITION. *A subset, $T$, of $\mathfrak{M}(A)$ is a maximum set for $A$ if there is an $f \in A$ such that*

$$T = \{m \in \mathfrak{M}(A): f(m) = \max_X |f|\}.$$

Obviously,

(A.27) *If $f \in A$ and we choose a $p$ for which $|f(p)| = \max_X |f|$, then $T = \{m \in \mathfrak{M}(A): f(m) = f(p)\}$ is a maximum set for $A$.*

The following useful fact is well known (and easy). (See [16, p. 227].)

(A.28) *If $T$ is a maximum set for $A$, then $T$ is the maximal ideal space of $A_T$ (the completion of $A|_T$ in the norm $\max_T |\ldots|$) and $T \cap X$ is a boundary for $A_T$.*

*Proof.* $T = \{m \in \mathfrak{M}(A): f(m) = \max_X |f|\}$, for some $f \in A$. We can assume $f \neq 0$ and $\max_X |f| = 1$. Let $g = \frac{1}{2}(1 + f)$. Then

$$T = \{m \in \mathfrak{M}(A): g(m) = 1 = \max_X |g|\} = \{m \in \mathfrak{M}(A): |g(m)| = 1\}.$$

The first part of (A.28) follows readily from the fact that $g - 1$ vanishes precisely on $T$.

To see that $T \cap X$ is a boundary for $A_T$, assume the contrary. Then there must be an $h \in A$ and a $p \in T - (T \cap X)$ such that $h(p) = 1$ and $|h| < 1$ on $T \cap X$. Let

$$H = \{x \in X: |h(x)| \geq 1\}.$$

Then $H$ is a compact set disjoint from $T$. Thus, $|g| < 1$ on $H$, so for a large enough positive integer, $N$, we will have $|h \cdot g^N| < 1$ on $H$. But also $|h \cdot g^N| < 1$ on $X - H$. Hence, $h \cdot g^N \in A$ and we have

$$(h \cdot g^N)(p) = 1 > \max_X |h \cdot g^N|$$

which contradicts $p \in \mathfrak{M}(A)$. Therefore, $T \cap X$ is a boundary for $A_T$. Q.E.D.

We apply these results in the special case when $X$ is a compact subset of $C^n$ and $A$ is one of the following:

(A.29) $P(X)$ = *the uniform limits on* $X$ *of the polynomials in* $z_1, \ldots, z_n$.
$R(X)$ = *the uniform limits on* $X$ *of the rational functions which are analytic about* $X$.
$A(X)$ = *the uniform limits on* $X$ *of all functions which are analytic about* $X$.

NOTATION. We shall write $M_X$ for $M_{A(X)}$.

By (A.4), if $X$ is rationally convex, then $A(X) = R(X)$; and if $X$ is polynomially convex, then $A(X) = P(X)$. In general, $P(X) \subset R(X) \subset A(X)$.

From the definitions of the hulls it is easy to see that we can identify $P(X)$ with $P(\text{hull}(X))$ and $R(X)$ with $R(R\text{-hull}(X))$. Moreover, it is also easy to verify

(A.30) (i) $\text{hull}((X) = \mathfrak{M}(P(X))$ *and*

(ii) $R\text{-hull}(X) = \mathfrak{M}(R(X))$,

(*via the correspondence*, $m \to$ "*evaluation at* $m$").

*Proof.* (i) is proved in [36]. To deduce (ii) from (i), choose $m \in \mathfrak{M}(R(X))$. By (i), there is a $p \in \text{hull}(X)$ such that $f(m) = f(p)$ for all $f \in P(X) \subset R(X)$. If $0 \notin f(X)$, then $1/f \in R(X)$ and $1 = (f \cdot 1/f)(m) = f(p) \cdot 1/f(m)$. This shows, firstly, that if $0 \notin f(X)$ then $0 \neq f(p)$. Hence, by (1.1), $p \in R\text{-hull}(X)$. Secondly, it shows that $1/f(m) = 1/f(p)$, for all $f \in P(X)$ such that $0 \notin f(X)$ But every rational function which is analytic on a neighborhood of $X$ can be expressed in the form $g/f$ where $g$ and $f$ are polynomials and $0 \notin f(X)$. Then $(g/f)(m) = g(p) \cdot 1/f(p) = (g/f)(p)$. It follows that $m$ corresponds to evaluation at $p$ for $R(X)$. Therefore, $\mathfrak{M}(R(X)) \subset R\text{-hull}(X)$ and, hence, $\mathfrak{M}(R(X)) = R\text{-hull}(X)$. Q.E.D.

Notice that by (A.28) and (A.30) (i) we have

(A.31) *If* $T \subset \text{hull}(X)$ *is a maximum set for* $P(X)$, *then*

$$T = \text{hull}(T \cap X).$$

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