HIGHER MONOTONICITY PROPERTIES OF CERTAIN STURM–LIOUVILLE FUNCTIONS

BY

LEE LORCH

University of Alberta, Edmonton, Canada [1]

AND

PETER SZEGO

University of Santa Clara, Santa Clara, Calif., U.S.A. [1]

1. Introduction

An examination of tables [3; 5; 8; 12; 15] of the positive[2] zeros of familiar special functions, such as Bessel functions[3] and various orthogonal polynomials, suggests that sequences of differences constructed from those zeros behave in a regular manner. Indeed, certain heuristically observed regularities are exploited systematically by table-makers as checks on their computations [1; 5, p. 404; 12, esp. pp. liii–liv].[4]

Rigorous study of this useful phenomenon, however, does not appear to have progressed beyond consideration of the second differences of zeros of Sturm–Liouville functions (solutions of the Sturm–Liouville differential equation $y'' + f(x)y = 0$). Here Sturm's comparison theorem [13; 14, pp. 19–21] has been the principal tool.

For instance, denoting by $\{c_{\nu n}\}$, $n = 1, 2, \ldots$, the ascending sequence of positive zeros of an arbitrary Bessel function $C_\nu(x)$ of order ν, Ch. Sturm [13, pp. 173–175] used his comparison theorem to show that the second (forward) differences $\Delta^2 c_{\nu n}$, $n = 1, 2, \ldots$, are all positive if $|\nu| < \frac{1}{2}$ and are all negative if $|\nu| > \frac{1}{2}$. In the same manner, similar results have been established for Hermite, Laguerre and Legendre polynomials and other Sturm–Liouville unctions.

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[2] All quantities discussed throughout this paper are assumed to be real.

[3] By a Bessel function we mean *any* real solution of the Bessel differential equation, not merely J_ν or Y_ν.

[4] The regularities now used to check tables are not the ones discussed in this paper. However, the ones established here can also be used conveniently for this purpose.

Analogous problems can be formulated for the sequence of the areas bounded by the successive arches or waves (having consecutive zeros as end-points) of the graphs of such special functions. The corresponding results, including the information about the second differences of the zeros, were established in a unified and simple manner by E. Makai [11]. His work was subsequently generalized in several directions by I. Bihari [2].

Our main purpose here is to go beyond the second differences and to show that *all* higher differences of certain sequences connected with the zeros and areas of $C_\nu(x)$ have constant sign, alternating from one difference to the next, when $|\nu| > \frac{1}{2}$. Airy functions and certain generalizations of $C_\nu(x)$ are shown to share these properties. The precise formulations are found in §§ 2–4.

In § 5, these results are extended, in part, to the study of the higher differences of sequences whose elements are the (first) differences of the respective zeros of the solutions of *distinct* Sturm–Liouville equations. This generalization is applied in § 6 to obtain information concerning the higher differences of sequences composed of the (first) differences of the respective zeros of arbitrary Bessel functions of different orders.

The general method of proof we employ may perhaps be extensible to yield analogous (although not identical) results for $C_\nu(x)$ with $|\nu| < \frac{1}{2}$ and for other special functions, such as Hermite, Laguerre and Legendre polynomials, the Bateman k-function and, in general, for the confluent hypergeometric function for certain values of the parameter.[1] We list some of our conjectures in § 7.

All the results bearing on Bessel functions which are presented here deal with the behavior of differences of quantities connected with functions of *constant* order. Elsewhere [10] we consider some analogous problems involving instead the differences of zeros of fixed rank, but of functions of varying order.

2. Definitions, notations and general results (one equation)

If $y(x)$ has zeros when x is in an open interval I and is a non-trivial solution of the Sturm–Liouville differential equation

$$y'' + f(x)y = 0, \tag{2.1}$$

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[1] More generally, one may seek a generalization of the Sturm comparison theorem. This theorem shows that the second differences of the sequence of successive zeros of an oscillatory Sturm-Liouville function are all positive if $f'(x) \leq 0$, and are all negative if $f'(x) \geq 0$. Perhaps the signs of the first N differences of these zeros can be inferred from the signs of $f^{(n)}(x)$, $n = 1, \ldots, N$. In particular, it would be interesting to determine if the complete monotonicity of $f'(x)$, $0 < x < \infty$, implies the complete monotonicity of the sequence composed of the differences of successive zeros of an arbitrary solution of $y'' + f(x)y = 0$, $0 < x < \infty$. This is the case for Bessel functions (Theorem 3.1), and for Airy functions (Theorem 4.1).

where $f(x)$ is a given function, we designate *any* increasing sequence of consecutive zeros in I by $x_1, x_2, \ldots$, and define for any fixed $\lambda > -1$

$$M_k = \int_{x_k}^{x_{k+1}} |y(x)|^\lambda dx \quad (k = 1, 2, \ldots). \tag{2.2}$$

When $\lambda = 1$, the M_k are simply the areas, say A_k, of the successive arches of $y(x)$. When $\lambda = 0$, the M_k become the differences, Δx_k, of successive zeros of $y(x)$. Moreover, M_k exists for any $\lambda > -1$, since the zeros of $y(x)$ are all of order 1 (the existence of a multiple zero of a solution of (2.1) would imply that $y(x)$ is identically zero).

Where it is convenient to distinguish any two (non-trivial) solutions of (2.1), linearly independent or not, we denote them by $y(x)$ and $\bar{y}(x)$, with the corresponding zeros and arch-areas, for example, written as $x_k, \bar{x}_k, A_k, \bar{A}_k, k = 1, 2, \ldots$, respectively. When we designate two such solutions by $y_1(x), y_2(x)$ we mean that they are linearly independent.

The symbol $\Delta^n \mu_k$ means, as usual, the n-th (forward) difference of the sequence $\{\mu_k\}$, i.e.,

$$\Delta^0 \mu_k = \mu_k, \quad \Delta \mu_k = \mu_{k+1} - \mu_k, \quad \Delta^n \mu_k = \Delta^{n-1}\mu_{k+1} - \Delta^{n-1}\mu_k \quad (n = 1, 2, \ldots, \quad k = 1, 2, \ldots).$$

For typographical convenience, we frequently use the symbol $D_\xi^n \varphi(\xi)$ to denote the n-th derivative $\varphi^{(n)}(\xi)$.

We prove a general theorem concerning solutions of (2.1) and then specialize in §§ 3–4 to general solutions of the Bessel and Airy differential equations, respectively. Certain preliminary facts are needed in this connection (and again in § 5). Some may be known, but, inasmuch as they occupy a central position in this work, we formulate them as lemmas and supply proofs.

LEMMA 2.1. *Let $p(x)$ be a positive function such that $p^{(N)}(x)$ exists in an open interval (a,b). Map (a,b) onto an interval of a variable t through the relation $x'(t) = p(x)$. Then, for any $\sigma > 0$ and any $n = 0, 1, \ldots, N$, the derivative $D_t^n\{[x'(t)]^\sigma\}$ is a homogeneous form in $p^{(0)}(x)$, $p^{(1)}(x), \ldots, p^{(n)}(x)$ for $x \in (a,b)$. It has (i) order $(\sigma + n)$, (ii) weight n, (iii) non-negative coefficients and exponents (both dependent only on n and σ), and (iv) integral exponents with the possible exception of the exponent of $p^{(0)}(x)$.*

Proof. The mapping is possible since $p(x) > 0$. The proof proceeds by induction.

When $n = 0$, we have $D_t^n\{[x'(t)]^\sigma\} = [p^{(0)}(x)]^\sigma$; when $n = 1$, we have $D_t^n\{[x'(t)]^\sigma\} = \sigma[p^{(0)}(x)]^\sigma[p^{(1)}(x)]$. In both cases, the assertions of the lemma are fulfilled.

A general term of $D_t^n\{[x'(t)]^\sigma\}$ would appear as

$$S = C[p^{(0)}(x)]^{\alpha_0}[p^{(1)}(x)]^{\alpha_1} \ldots [p^{(n)}(x)]^{\alpha_n},$$

with C, $\alpha_0, ..., \alpha_n$ all non-negative and $\alpha_1, ..., \alpha_n$ all integers. The order would be $\alpha_0 + \alpha_1 + ... + \alpha_n$ and the weight $\alpha_1 + 2\alpha_2 + ... + n\alpha_n$.

The induction is carried through by differentiating S with respect to t. We have

$$\frac{dS}{dt} = p(x)\frac{dS}{dx} = C\alpha_0[p^{(0)}(x)]^{\alpha_0}[p^{(1)}(x)]^{\alpha_1+1}[p^{(2)}(x)]^{\alpha_2} ... [p^{(n)}(x)]^{\alpha_n}$$
$$+ C\alpha_1[p^{(0)}(x)]^{\alpha_0+1}[p^{(1)}(x)]^{\alpha_1-1}[p^{(2)}(x)]^{\alpha_2+1} ... [p^{(n)}(x)]^{\alpha_n}$$
$$+ ...$$
$$+ C\alpha_n[p^{(0)}(x)]^{\alpha_0+1}[p^{(1)}(x)]^{\alpha_1} ... [p^{(n)}(x)]^{\alpha_n-1}[p^{(n+1)}(x)].$$

Thus, $S'(t)$ is a polynomial in $p^{(0)}(x)$, $p^{(1)}(x), ..., p^{(n+1)}(x)$ with non-negative coefficients. None of the exponents that appear become negative; the exponent of $p(x)$ never decreases while the others decrease by steps of one unit, if at all. They may reach zero, but this occurrence would only presage the vanishing of the derivative of the corresponding factor.

The order, ω, and weight, w, of a term in $S'(t)$ other than the first and last, are, respectively,

$$\omega = (\alpha_0 + 1) + \alpha_1 + ... + (\alpha_i - 1) + (\alpha_{i+1} + 1) + ... + \alpha_n = 1 + (\alpha_0 + \alpha_1 + ... + \alpha_n)$$

and

$$w = \alpha_1 + 2\alpha_2 + ... + i(\alpha_i - 1) + (i + 1)(\alpha_{i+1} + 1) + ... + n\alpha_n = 1 + (\alpha_1 + 2\alpha_2 + ... + n\alpha_n).$$

Thus, the order and weight increase by one unit under the process of differentiation of these terms with respect to t. The same is clearly true of the first and last terms as well. Finally, we observe that the coefficients and exponents do not depend on $p(x)$ but only on n and σ.

Thus, $S'(t)$ satisfies the assertion of the lemma and the induction (which can be continued to $n = N$) is complete.

LEMMA 2.2. *Let $p(x)$ satisfy the conditions of Lemma 2.1 and, in addition, the inequalities*

$$(-1)^n p^{(n)}(x) > 0 \quad (n = 0, 1, ..., N; \ x \in (a, b)). \tag{2.3}$$

Let the mapping described in Lemma 2.1 be carried out. Then, for any $\sigma > 0$,

$$(-1)^n D_t^n \{[x'(t)]^\sigma\} > 0 \quad (n = 0, 1, ..., N). \tag{2.4}$$

The lemma remains true if the factor $(-1)^n$ is deleted from both (2.3) and (2.4).

Proof. From Lemma 2.1, we see that a general term of $D_t^n \{[x'(t)]^\sigma\}$ has the form

$$S = C[p^{(0)}(x)]^{\alpha_0} [p^{(1)}(x)]^{\alpha_1} ... [p^{(n)}(x)]^{\alpha_n}.$$

By assumption (2.3) the sign of S is

$$(-1)^{\alpha_1 + 2\alpha_2 + \dots + n\alpha_n}.$$

But the exponent is simply the weight which, according to Lemma 2.1, has the value n. Thus, S has sign $(-1)^n$ and (2.4) is established. The final sentence of the lemma is trivial.

LEMMA 2.3. *Let* y_1, y_2 *be solutions of* (2.1), *normalized so that their Wronskian* $y_1 y_2' - y_1' y_2 = 1$. *Let* $p(x) = y_1^2(x) + y_2^2(x)$ *and define the transformation* $y(x) = [p(x)]^{\frac{1}{2}} u(t)$; $x'(t) = p(x)$. *This transforms* (2.1) *into the differential equation* $u''(t) + u(t) = 0$.

Proof. The function $p(x)$ is an admissible mapping function since it is always positive. The transformation takes (2.1) into $u''(t) + q(t)u(t) = 0$, where

$$q(t) = \tfrac{1}{2} p(x) p''(x) - \tfrac{1}{4}[p'(x)]^2 + [p(x)]^2 f(x) \tag{2.5}$$

so that (with $p(x) = y_1^2(x) + y_2^2(x)$ and $y'' = -f(x)y$),

$$q(t) = p(x) [(y_1')^2 + (y_2')^2 + y_1 y_1'' + y_2 y_2''] - [y_1^2(y_1')^2 + 2y_1 y_1' y_2 y_2' + y_2^2(y_2')^2] + [p(x)]^2 f(x)$$

$$= (y_1^2 + y_2^2)[(y_1')^2 + (y_2')^2] - [y_1^2(y_1')^2 + 2y_1 y_1' y_2 y_2' + y_2^2(y_2')^2] = (y_1 y_2' - y_1' y_2)^2 = 1.$$

The main result of this section can be proved now:

THEOREM 2.1. *Suppose that in some closed interval* I^* *there exist two linearly independent solutions* $y_1(x)$, $y_2(x)$ *of* (2.1) *such that*

$$(-1)^n D_x^n \{ [y_1(x)]^2 + [y_2(x)]^2 \} > 0 \quad (n = 0, 1, \dots, N), \tag{2.6}$$

where the N-th derivative exists in the open interval I, *and the lower order derivatives are continuous in its closure* I^*. *Then*[1]

$$(-1)^n \Delta^n M_k > 0 \quad (n = 0, 1, \dots, N; \ k = 1, 2, \dots), \tag{2.7}$$

so that, in particular,

$$(-1)^{n-1} \Delta^n x_k > 0 \quad (n = 1, 2, \dots, N+1; \ k = 1, 2, \dots). \tag{2.8}$$

Moreover, if $x_1 > \bar{x}_1$, *then*

$$(-1)^n \Delta^n (x_k - \bar{x}_k) > 0 \quad (n = 0, 1, \dots, N; \ k = 1, 2, \dots). \tag{2.9}$$

If the factor $(-1)^n$ *is deleted from assumption* (2.6), *then the conclusions* (2.7), (2.8) *and* (2.9) *remain valid provided they are amended by eliminating the factors* $(-1)^n$, $(-1)^{n-1}$ *and* $(-1)^n$, *respectively.*

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[1] The quantities M_k, x_k, $\bar{x}_k$ are defined in the first paragraph of this section, e.g., by (2.2). It should be remembered that they refer to *any* nontrivial solution of (2.1), not merely to $y_1(x)$ and $y_2(x)$.

Proof. Without loss of generality, we normalize the solutions $y_1(x)$, $y_2(x)$ so that their Wronskian is 1. Then we can apply Lemma 2.3 and transform the differential equation (2.1) into the form $u''(t) + u(t) = 0$. Since $p(x) = x'(t) > 0$ in I, there is a one-to-one correspondence between the zeros of $y(x)$ and those of $u(t)$. But $u(t) = A \cos(t-b)$, A, b constants, so that its zeros are equidistant from one another with $\Delta t_k = \pi$, $k = 1, 2, ...$, where t_k is the zero of $u(t)$ corresponding to x_k.

Thus, (2.2) becomes

$$M_k = \int_{t_k}^{t_{k+1}} [x'(t)]^{1+\frac{1}{2}\lambda} |u(t)|^\lambda dt.$$

Recalling that $\Delta t_k = \pi$ $(k = 1, 2, ...)$, and noting that $|u(t+\pi)| = |u(t)|$, we have

$$\Delta M_k = \int_{t_{k+1}}^{t_{k+2}} - \int_{t_k}^{t_{k+1}} = \int_{t_k}^{t_{k+1}} \{[x'(t+\pi)]^{1+\frac{1}{2}\lambda} |u(t+\pi)|^\lambda - [x'(t)]^{1+\frac{1}{2}\lambda} |u(t)|^\lambda\} dt$$

$$= \int_{t_k}^{t_{k+1}} \{\Delta_\pi \{[x'(t)]^{1+\frac{1}{2}\lambda}\}\} |u(t)|^\lambda dt,$$

where $\Delta_\pi \{F(t)\} = F(t+\pi) - F(t)$.

It follows, in the same way, that the higher differences are given by

$$\Delta^n M_k = \int_{t_k}^{t_{k+1}} \{\Delta_\pi^n \{[x'(t)]^{1+\frac{1}{2}\lambda}\}\} |u(t)|^\lambda dt, \tag{2.10}$$

where $\Delta_\pi^n \{F(t)\} = \Delta_\pi \{\Delta_\pi^{n-1} F(t)\}$.

Now, according to a mean-value theorem for higher derivatives and differences [7, p. 73] there exists a θ such that

$$\Delta_\pi^n F(t) = \pi^n F^{(n)}(t + \theta n\pi) \quad (0 < \theta < 1), \tag{2.11}$$

provided $F^{(n)}(t)$ exists in the open interval $(t, t+n\pi)$ and the lower derivatives are continuous in the corresponding closed interval.

Applying the extended mean-value theorem (2.11) to the expression (2.10) for $\Delta^n M_k$, we obtain

$$\Delta^n M_k = \pi^n \int_{t_k}^{t_{k+1}} \{D_t^n \{[x'(t + \theta n\pi)]^{1+\frac{1}{2}\lambda}\}\} |u(t)|^\lambda dt \quad (0 < \theta(t) < 1). \tag{2.12}$$

It should be noted that the argument of x' can fall anywhere in $[t_k, t_k + (n+1)\pi]$. However, this causes no difficulty since the range of t-values occurring in $\Delta^n M_k$ encompasses the same interval.

Lemma 2.2 can now be applied, with $\sigma = 1 + \frac{1}{2}\lambda$, since condition (2.3) of the lemma is fulfilled by virtue of assumption (2.6) in the theorem. Employing the conclusion of the lemma, (2.4), the result (2.7) of the theorem follows from (2.12).

Taking $\lambda = 0$ in (2.7), in turn, yields (2.8), since

$$M_k = \Delta x_k, \Delta^n M_k = \Delta^{n+1} x_k \quad \text{for} \quad \lambda = 0.$$

The proof of (2.9) is slightly different. Let $t_1, t_1 + \pi, t_1 + 2\pi, \ldots$, be the sequence of t-values corresponding to the zeros $x_1, x_2, x_3, \ldots$, and $\bar{t}_1, \bar{t}_1 + \pi, \bar{t}_1 + 2\pi, \ldots$, be the sequence of t-values corresponding to $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots$. Then, with $x_k = x(t_k), \bar{x}_k = x(\bar{t}_k)$,

$$\Delta^n(x_k - \bar{x}_k) = \Delta^n[x(t_k) - x(\bar{t}_k)] = \Delta^n[x(t_k) - x(t_k - \eta)],$$

where $\eta = t_k - \bar{t}_k$ and is independent of k.

Applying the same mean-value theorem (2.11) as before, we have

$$\Delta^n(x_k - \bar{x}_k) = \pi^n[x^{(n)}(t_k + \theta n\pi) - x^{(n)}(t_k + \theta n\pi - \eta)],$$

with $0 < \theta < 1$. This can be rewritten as

$$\Delta^n(x_k - \bar{x}_k) = \pi^n \int_{t_k + \theta n\pi - \eta}^{t_k + \theta n\pi} x^{(n+1)}(t)\, dt.$$

Thus, the sign of $\Delta^n(x_k - \bar{x}_k)$ is determined by the sign of $x^{(n+1)}(t)$. However, Lemma 2.2 (with $\sigma = 1$), applicable in view of hypothesis (2.6), shows this sign to be $(-1)^n$ for $n = 0, 1, \ldots, N$. Hence, the quantity $(-1)^n \Delta^n(x_k - \bar{x}_k)$ is positive or negative according as $\eta > 0$ or $\eta < 0$. But

$$x_k - \bar{x}_k = x(t_k) - x(\bar{t}_k) = \int_{t_k - \eta}^{t_k} x'(t)\, dt,$$

which has the same sign as η since $x'(t) > 0$. Inasmuch as $x_1 > \bar{x}_1$, it is clear that $\eta > 0$, so that (2.9) is established. The final sentence of the theorem follows at once on making the obvious changes in the above proof.

Remark. It should be kept in mind that the definition of $x_1, x_2, \ldots, \bar{x}_1, \bar{x}_2, \ldots$, provides a considerable amount of flexibility. Neither x_1 nor $\bar{x}_1$ need be the first zero of its respective function, nor is there even any requirement that x_1 should occupy the same relative position among the totality of zeros of $y(x)$ in I as $\bar{x}_1$ does relative to $\bar{y}(x)$. This is useful to know in connection with applications of Theorem 2.1.

3. Application of § 2 to certain Bessel functions

One opportunity to apply Theorem 2.1 to Bessel functions of appropriate order is facilitated by Nicholson's integral [15, p. 444(1)].

Theorem 2.1 applies for all non-negative integers N to the Bessel differential equation in the form

$$y''(x) + \left\{ 1 - \frac{\nu^2 - \frac{1}{4}}{x^2} \right\} y = 0 \quad (x > 0),\tag{3.1}$$

provided $|\nu| > \frac{1}{2}$.

The fundamental solutions y_1, y_2 of (3.1) can be taken to be $(\frac{1}{2}\pi x)^{\frac{1}{2}} J_\nu(x)$, $(\frac{1}{2}\pi x)^{\frac{1}{2}} Y_\nu(x)$ so that

$$p(x) = \tfrac{1}{2}\pi x \{ [J_\nu(x)]^2 + [Y_\nu(x)]^2 \}.$$

For $|\nu| > \frac{1}{2}$, $p(x)$ is analytic, $0 < x < \infty$, and so $p^{(N)}(x)$ exists for all non-negative integers N.

To satisfy the conditions of Theorem 2.1, we need only choose an interval I encompassing all zeros under consideration and with its lower end-point a positive number. This can obviously be done, since each Bessel function has a least positive zero. Condition (2.6) on the sign of $p^{(n)}(x)$ can be verified by using G. N. Watson's manipulation of Nicholson's integral representation for our $p(x)$ [15, p. 446]:

Watson's calculation gives

$$p'(x) = \frac{4}{\pi} \int_0^\infty \{ K_0(2x \sinh T) \} (\tanh T)(\cosh 2\nu T) \cdot [\tanh T - 2\nu \tanh 2\nu T] \, dT,\tag{3.2}$$

where $K_0(\xi)$ is the Bessel function of second kind, imaginary argument and zero order.

As Watson points out, the bracketed factor in the above integrand is negative for $|\nu| > \frac{1}{2}$; everything else is positive. Hence, $p'(x) < 0$. Further differentiation yields

$$p^{(n)}(x) = \frac{4}{\pi} \int_0^\infty \{ K_0^{(n-1)}(2x \sinh T) \} (2 \sinh T)^{n-1} (\tanh T)(\cosh 2\nu T)$$
$$\cdot [\tanh T - 2\nu \tanh 2\nu T] \, dT.\tag{3.3}$$

Now, it is clear from the representation [15, p. 446 footnote]

$$K_0(\xi) = \int_0^\infty e^{-\xi \cosh t} \, dt,$$

that $(-1)^n K_0^{(n)}(\xi) > 0$ for $\xi > 0$, $n = 0, 1, 2, \ldots$, so that $p(x)$ satisfies the hypotheses of Theorem 2.1, and (2.7), (2.8) and (2.9) are therefore valid for Bessel functions of order ν, $|\nu| > \frac{1}{2}$, for all $N = 1, 2, \ldots$.

There is an alternative argument, based on work of P. Hartman [6], available to establish the validity of (2.7), (2.8), (2.9) for these Bessel functions.

Applied directly to equation (3.1), Hartman's Theorem 18.1_n [6, p. 182], with $n = \infty$, asserts that $p(x)$ is completely monotonic in $(0, \infty)$, i.e., it gives (2.6) in the slightly weakened form in which "$>$" is replaced by "$\geqslant$". However, as shown in § 8 below, a function completely monotonic in $(0, \infty)$ must be a constant if any one of its derivatives (including the

zeroth, the function itself) vanishes for any single positive value. Thus, if equality ever prevailed in (2.6), $p(x)$ would have to be a constant. However, this, in turn, is impossible for $|\nu| > \frac{1}{2}$, since, according to a lemma established in § 9 below, this would require the coefficient of y in (3.1) also to be constant, which it is not.

Thus, Hartman's Theorem 18.1_n, with $n = \infty$, implies (2.6) and so also (2.7), (2.8) and (2.9) for Bessel functions of order ν, $|\nu| > \frac{1}{2}$. Therefore, by either argument, we obtain the following results for Bessel functions.

THEOREM 3.1. *Let* $c_{\nu k}$, $\bar{c}_{\nu k}$ *denote, respectively, the k-th positive zeros, arranged in ascending order, of any pair of non-trivial solutions (linearly independent or not) of the Bessel equation* (3.1) *with* $|\nu| > \frac{1}{2}$. *Let* λ *be a constant,* $\lambda > -1$, *and*

$$M_k = \int_{c_{\nu k}}^{c_{\nu,k+1}} x^{\frac{1}{2}\lambda} |C_\nu(x)|^\lambda \, dx. \tag{3.4}$$

Then, for $k = 1, 2. \ldots,$

$$(-1)^n \Delta^n M_k > 0 \quad (n = 0, 1, \ldots), \tag{3.5}$$

$$(-1)^{n-1} \Delta^n c_{\nu k} > 0 \quad (n = 1, 2, \ldots), \tag{3.6}$$

$$(-1)^n \Delta^n (c_{\nu,m+k} - \bar{c}_{\nu k}) > 0 \quad (n = 0, 1, 2, \ldots), \tag{3.7}$$

for any fixed $m = 0, 1, \ldots,$ *provided* $c_{\nu,m+1} > \bar{c}_{\nu 1}$.

In particular (still with $|\nu| > \frac{1}{2}$),

$$(-1)^{n-1} \Delta^n j_{\nu k} > 0, \ (-1)^{n-1} \Delta^n y_{\nu k} > 0 \quad (n = 1, 2, \ldots), \tag{3.8}$$

$$(-1)^n \Delta^n (j_{\nu k} - y_{\nu k}) > 0 \quad (n = 0, 1, 2, \ldots), \tag{3.9}$$

where $j_{\nu k}$, $y_{\nu k}$ *denote the respective k-th positive zeros of* $J_\nu(x)$ *and* $Y_\nu(x)$.

Remark. (3.7) implies (3.9), since $j_{\nu 1} > y_{\nu 1}$ [15, p. 487 (10)].

4. Application of § 2 to Airy functions

The Airy functions satisfy the differential equation [14, p. 18]

$$y''(x) + \tfrac{1}{3} x y = 0. \tag{4.1}$$

A broader class of functions, including the Airy functions, satisfy the differential equation [15, p. 97 (9)]

$$y''(x) + \beta^2 \gamma^2 x^{2\beta-2} y = 0. \tag{4.2}$$

These functions are closely related to Bessel functions. Indeed, $y = x^{\frac{1}{2}} C_{1/(2\beta)} (\gamma x^\beta)$ satisfies (4.2).

When $1 < \beta \leqslant \frac{3}{2}$, equation (4.2) meets Hartman's conditions [6, p. 183 (Theorem $20.1_n)$] with $n = \infty$, so that the special case (4.1), where $\beta = \frac{3}{2}$, is covered as well. Accordingly, we have an analogue of Theorem 3.1 for Airy functions and their zeros. In particular, the following result holds:

THEOREM 4.1. *If* $1 < \beta \leqslant \frac{3}{2}$, *then*

$$(-1)^{n-1}\Delta^n\{[c_{1/(2\beta),k}]^{1/\beta}\} > 0 \quad (n, k = 1, 2, \dots).$$

If $\{a_k\}$, $\{\bar{a}_k\}$ *are the sequences of positive zeros, arranged in ascending order, of any pair of Airy functions (solutions of* (4.1)), *and* $a_1 > \bar{a}_1$, *then*

$$(-1)^{n-1}\Delta^n a_k > 0, \quad (-1)^n\Delta^n(a_k - \bar{a}_k) > 0 \quad (n, k = 1, 2, \dots). \tag{4.3}$$

This theorem follows from Hartman's Theorem 20.1_n, with $n = \infty$, in precisely the same way as Theorem 3.1 was shown to follow from his Theorem 18.1_n, with $n = \infty$, and the subordinate results in §§ 8–9 below.

For this theorem, unlike Theorem 3.1, we have no alternative proof to offer.

5. Higher monotonicity of sequences arising from zeros of two Sturm–Liouville functions

Previous sections discussed higher monotonicity of various sequences connected with, *inter alia*, the areas under successive arches of the graphs of certain Sturm–Liouville functions and, more particularly, with their zeros. Here we supply a partial generalization (restricted to zeros) of some of the previous results by relating the zeros of an arbitrary solution of one Sturm–Liouville equation to those of an arbitrary solution of a *different* Sturm–Liouville equation, under certain circumstances. This will imply, for example, the complete monotonicity of certain sequences, such as $\{j_{\mu, r+k} - y_{\nu, m+k}\}_{k=1}^{\infty}$, for appropriate μ, ν, r and m (cf. Theorem 6.1 for details).

To this end, we need the following extension of our first two lemmas:

LEMMA 5.1. *Let an open interval* I_t *of the variable t be mapped onto corresponding open intervals* I_x *and* I_X *of the variables x and X by the mappings* $x'(t) = p(x)$. $X'(t) = P(X)$, *respectively, where* $p(x)$, $P(X)$ *are prescribed positive functions such that* $p^{(N)}(x)$, $P^{(N)}(X)$ *exist in the open interval* $I_x \cap I_X$ *(say* I_{xX}). *Suppose, furthermore, that* $p(x)$, $P(X)$ *satisfy the inequalities*

$$(-1)^n D_X^n\{P(X)\} \geqslant (-1)^n D_x^n\{p(x)\} > 0 \quad (n = 0, 1, \dots, N), \tag{5.1}$$

for $x, X \in I_{xX}$, *where x and X in* (5.1) *correspond to the same t.*

Then, for any $\sigma > 0$,

$$(-1)^n D_t^n \{[X'(t)]^\sigma\} \geqslant (-1)^n D_t^n \{[x'(t)]^\sigma\} > 0, \tag{5.2}$$

$n = 0, 1, ..., N,$ *for* $t \in I_t$.

Again, if both factors $(-1)^n$ *are eliminated from assumption* (5.1), *then conclusion* (5.2) *remains valid once the same deletions are made.*

Proof. The second inequality of (5.2) is identical with the conclusion of Lemma 2.2. The first inequality follows from a term by term comparison of the homogeneous forms for $D_t^n \{[X'(t)]^\sigma\}$ and $D_t^n \{[x'(t)]^\sigma\}$: According to Lemma 2.1, a typical term of $(-1)^n D_t^n \{[x'(t)]^\sigma\}$ is

$$S_x = (-1)^n C [p(x)]^{\alpha_0} [p'(x)]^{\alpha_1} ... [p^{(n)}(x)]^{\alpha_n} = C [p(x)]^{\alpha_0} [(-1) p'(x)]^{\alpha_1} ... [(-1)^n p^{(n)}(x)]^{\alpha_n},$$

since $\alpha_1 + 2\alpha_2 + ... + n\alpha_n = n$ (this being the weight), with C positive, and with each remaining factor positive by (5.1).

The corresponding typical term of $(-1)^n D_t^n \{[X'(t)]^\sigma\}$ is

$$S_X = C [P(X)]^{\alpha_0} [(-1) P'(X)]^{\alpha_1} ... [(-1)^n P^{(n)}(X)]^{\alpha_n},$$

with the same values for $C, \alpha_0, ..., \alpha_n$ as for $(-1)^n D_t^n \{[x'(t)]^\sigma\}$, by Lemma 2.1.

Thus, the proof of (5.2) will be complete once it is established that $S_X \geqslant S_x$. This follows from (5.1), since each factor in brackets in S_x is positive and not more than the corresponding factor of S_X, while $C > 0$. The proof of the final assertion follows on making the obvious changes in the foregoing.

Remark. If the symbol " $\geqslant$ " in (5.1) is replaced by " $>$ ", then (5.2) can be strengthened correspondingly.

From this lemma we obtain the principal result of the section.

THEOREM 5.1. *Suppose that in some closed interval* I^* *there exist two pairs of linearly independent solutions* $\{y_1(x), y_2(x)\}$, $\{Y_1(x), Y_2(x)\}$ *(each pair normalized so that the Wronskian is* 1*) of the respective differential equations*

$$y''(x) + f(x) y = 0; \quad Y''(x) + F(x) Y = 0, \tag{5.3}$$

(where f, F *are given functions) such that*

$$(-1)^n P^{(n)}(x) \geqslant (-1)^n p^{(n)}(x) > 0 \quad (n = 0, 1, ..., N), \tag{5.4}$$

where the N-*th derivatives exist in the open interval* I, *and the lower derivatives are continuous in its closure* I^*, *and where*

$$p(x) = [y_1(x)]^2 + [y_2(x)]^2; \quad P(x) = [Y_1(x)]^2 + [Y_2(x)]^2.$$

Then, if $x_1 > X_1$, we have

$$(-1)^n \Delta^n (x_k - X_k) > 0, \quad (n = 0, 1, ..., N; \ k = 1, 2, ...), \tag{5.5}$$

for the zeros x_k, X_k of y and Y, respectively, in the intersection $I^* \cap J$, where J is an interval defined to be either (X_1, ξ_0) or (X_1, ∞) according as there does or does not exist a solution ξ for the equation

$$\int_{x_1}^{\xi} \frac{ds}{p(s)} = \int_{X_1}^{\xi} \frac{ds}{P(s)}, \tag{5.6}$$

ξ_0 *being the least such solution.*

If the symbol "$\geqslant$" in (5.4) is replaced by "$>$" for $n = 0$, then the solution of (5.6), if any, is unique.

Proof. As in § 2, Lemma 2.3, the transformations

$$
\begin{aligned}
y(x) &= [x'(t)]^{\frac{1}{2}} u(t); \quad x'(t) = [y_1(x)]^2 + [y_2(x)]^2 \\
Y(X) &= [X'(t)]^{\frac{1}{2}} U(t); \quad X'(t) = [Y_1(X)]^2 + [Y_2(X)]^2
\end{aligned}
\tag{5.7}
$$

take equations (5.3) into, respectively,

$$u''(t) + u(t) = 0; \quad U''(t) + U(t) = 0. \tag{5.8}$$

The mapping functions (5.7) are not unique; an integration constant is left unspecified. In order to particularize (5.7) completely, we take

$$t = \int_{x_1}^{x} \frac{ds}{p(s)} = \int_{X_1}^{X} \frac{ds}{P(s)}. \tag{5.9}$$

The transformations (5.7) now establish a one-to-one correspondence between the zeros of $y(x)$ and $u(t)$ and between those of $Y(X)$ and $U(t)$, respectively, $x'(t)$ and $X'(t)$ being positive.

Let us designate by $t_1, t_2, ...,$ the zeros of $u(t)$ corresponding, respectively, to $x_1, x_2, ...$ and by $T_1, T_2, ...,$ the zeros of $U(t)$ corresponding to $X_1, X_2, ...$. Since all solutions of (5.8) are of the form $A \cos(t - b)$, it follows that $\Delta t_k = \Delta T_k = \pi$ for all k.

Now, $t_1 = T_1 = 0$, from (5.9), so that $t_k = T_k = (k-1)\pi$, $k = 1, 2, ...$. Hence

$$(-1)^n \Delta^n (x_k - X_k) = (-1)^n \Delta_\pi^n [x(t_k) - X(t_k)], \tag{5.10}$$

where the subscript π on the right-hand difference operator signifies that these differences are to be taken with increment π.

According to a mean-value theorem for higher differences [7, p. 73], (cf. (2.11)) there exists θ such that

$$\Delta_\pi^n F(t) = \pi^n F^{(n)}(t + n\pi\theta) \quad (0 < \theta < 1), \tag{5.11}$$

under conditions on $F(t)$ which are satisfied here by $x(t) - X(t)$. Applying this to (5.10), we have, for such a θ,

$$(-1)^n \Delta^n(x_k - X_k) = (-\pi)^n[x^{(n)}(t_k + n\pi\theta) - X^{(n)}(t_k + n\pi\theta)] \quad (0 < \theta < 1; \ n = 0, 1, ..., N; \ k = 1, 2, ...). \tag{5.12}$$

The argument in (5.12) may fall anywhere in the interval $[t_k, t_k + n\pi]$; this is precisely the range implied in $\Delta_\pi^n F(t_k)$. Now, (5.5) will follow from (5.12) once it is established that

$$(-1)^n x^{(n)}(t) > (-1)^n X^{(n)}(t) \quad (n = 0, 1, ..., N), \tag{5.13}$$

for t in the appropriate interval. The case $n = 0$ requires that we show

$$x(t) > X(t), \tag{5.14}$$

while the other cases require

$$(-1)^n D_t^n \{X'(t)\} > (-1)^n D_t^n \{x'(t)\} \quad (n = 0, 1, 2, ..., N - 1). \tag{5.15}$$

In view of assumption (5.4), it is clear that Lemma 5.1 is applicable, so that (5.15) would follow from

$$(-1)^n P^{(n)}(X) > (-1)^n p^{(n)}(x) \quad (n = 0, 1, ..., N - 1). \tag{5.16}$$

But this is equivalent to

$$(-1)^n[P^{(n)}(X) - P^{(n)}(x)] > (-1)^n[p^{(n)}(x) - P^{(n)}(x)] \quad (n = 0, 1, ..., N - 1). \tag{5.17}$$

The right side of (5.17) is non-positive (from (5.4)). Showing that the left side is positive will prove the inequality (5.17). That is, it will suffice to show that

$$(-1)^n[P^{(n)}(X) - P^{(n)}(x)] = (-1)^n \int_x^X P^{(n+1)}(s)\,ds > 0 \quad (n = 0, 1, ..., N - 1).$$

However, hypothesis (5.4) states that $(-1)^n P^{(n+1)}(s) < 0$, so that to prove (5.17), it is sufficient to establish that $x(t) > X(t)$, i.e., inequality (5.14). In other words, the validity of (5.5) is coextensive with that of (5.14).

Now, we have assumed that

$$x(0) - X(0) = x_1 - X_1 > 0.$$

Hence, if (5.14) is ever violated, then there must exist at least one value of t, say τ, for which $x(\tau) = X(\tau) \equiv \xi > X_1$.

If no such ξ exists, then the proof is complete, and the interval J of the theorem is simply (X_1, ∞).

Otherwise, employing the restriction on t, (5.9), the condition on ξ becomes (5.6). Accordingly, we define the differentiable function $g(v)$ by

$$g(v) = \int_{X_1}^{v} \frac{ds}{P(s)} - \int_{x_1}^{v} \frac{ds}{p(s)}, \quad v \geqslant X_1. \tag{5.18}$$

Obviously, $g(X_1) > 0$, since $x_1 > X_1$, and

$$g'(v) = \frac{1}{P(v)} - \frac{1}{p(v)} \leqslant 0,$$

from (5.4).

Thus, $g(v)$ is a continuous, non-increasing function which is positive at X_1, so that the set of its zeros, i.e., of the values of ξ violating (5.14), would, if non-empty, be a closed interval. If there be such ξ, then there is a least such, say ξ_0, and (5.14) is satisfied throughout the open interval $J \equiv (X_1, \xi_0)$.

This completes the proof of the theorem, except for its final sentence. This, in turn, follows immediately on observing that assumption (5.4) becomes, in this case, the inequality $P(x) > p(x)$, so that the function $g(v)$ is now strictly decreasing and hence cannot vanish more than once, if at all.

Remark 1. If the functions $p(s)$, $P(s)$ are analytic (or even merely members of the same quasi-analytic class), then also the solution of (5.6), if any, is unique. For, otherwise, there would exist an interval of solutions ξ, so that, differentiating both sides of (5.6) with respect to ξ, we would have $p(\xi) = P(\xi)$ for all ξ in the solution interval. Then, from (quasi) analyticity, it follows that $p(s) = P(s)$ everywhere. Since $x_1 > X_1$, (5.6) clearly could not be satisfied, a contradiction.

Remark 2. The inequality (2.8) and its consequences, but not the whole of Theorem 2.1, follow from Theorem 5.1, on taking the two differential equations (5.3) to be identical, so that (5.4) is satisfied with "$\geqslant$" reducing to "$=$".

Remark 3. As in § 2, if assumption (5.4) is altered by deleting both factors $(-1)^n$, then conclusion (5.5) will remain valid provided the factor $(-1)^n$ is deleted.

Finally, we note the additional corollary below which, in essence, assures us that Theorem 5.1 has non-vacuous content in certain specific circumstances.

COROLLARY 5.1. *Let all the conditions of Theorem 5.1 be satisfied and suppose, further, that* (i) $I^* = [X_1, \infty]$, (ii) *there are infinitely many* x_k, X_k *in* I^*, *and* (iii)

$$\int_{X_1}^{\infty} \left\{ \frac{1}{p(s)} - \frac{1}{P(s)} \right\} ds < \infty. \tag{5.19}$$

Then, for any fixed choice of X_1, there always exists an x_1 sufficiently large so that (5.5) holds in (X_1, ∞).

Proof. It suffices to show that there exists no ξ satisfying (5.6), and for this we need establish only that $g(v) > 0$ for all v where the function $g(v)$ is defined by (5.18). This function can be put in the form

$$g(v) = \int_{X_1}^{x_1} \frac{ds}{P(s)} - \int_{x_1}^{v} \left\{ \frac{1}{p(s)} - \frac{1}{P(s)} \right\} ds \quad (v \geqslant X_1). \tag{5.20}$$

Now, $g'(v) \leqslant 0$, so that the positivity of $g(v)$, $v \geqslant X_1$, can be shown by proving that $g(\infty) > 0$, i.e., by demonstrating that

$$\int_{X_1}^{x_1} \frac{ds}{P(s)} > \int_{x_1}^{\infty} \left\{ \frac{1}{p(s)} - \frac{1}{P(s)} \right\} ds, \tag{5.21}$$

for all sufficiently large x_1. Both sides of (5.21) are non-negative, since $P(s) \geqslant p(s) > 0$ and $x_1 > X_1$. However, the left side is an increasing function of x_1, while the right side, which is non-increasing, approaches zero as $x_1 \to \infty$. This proves the corollary.

6. An application of § 5 to Bessel functions

The conditions of Corollary 5.1 will be verified for certain Bessel functions of orders μ, ν, respectively.

THEOREM 6.1. *Given three numbers μ, ν, m, with $\frac{1}{2} \leqslant \mu < \nu$, m a positive integer, let $c_{\mu m}$, $\gamma_{\nu m}$ denote the m-th positive zero of any solutions of the Bessel equations of order μ and ν, respectively. Then there exists a positive integer r such that $c_{\mu r} > \gamma_{\nu m}$ and*

$$(-1)^n \Delta^n \{ c_{\mu, r+k} - \gamma_{\nu, m+k} \} > 0 \quad (n, k = 0, 1, \ldots). \tag{6.1}$$

In particular, c and γ can be chosen to be j and y in either order.

Proof. With $P(x) = \frac{1}{2} \pi x [J_\nu^2(x) + Y_\nu^2(x)]$ and $p(x) = \frac{1}{2} \pi x [J_\mu^2(x) + Y_\mu^2(x)]$, we have to show that (5.4) and (5.19) are satisfied.

To verify (5.4) we obtain from [16, p. 446]

$$\frac{\partial (-1)^n p^{(n)}(x)}{\partial \nu} = \frac{4}{\pi} \int_0^\infty \left\{ (-1)^n [K_0^{(n-1)}(2x \sinh t)] (2 \sinh t)^{n-1} \tanh t \right\}$$

$$\times \left\{ (\cosh 2\nu t) (-2 \tanh 2\nu t - 4\nu t \operatorname{sech}^2 2\nu t) + (2t \sinh 2\nu t) (\tanh t - 2\nu \tanh 2\nu t) \right\} dt.$$

Regarding this integral, we note that the expression in the first set of braces is negative, since $(-1)^n K_0^{(n)}(T) > 0$ for all positive T, and that the expression in the second set of braces is also negative, since $v \geqslant \frac{1}{2}$ and $\tanh T \uparrow 1$ as $T \to \infty$. Hence

$$\frac{\partial (-1)^n p^{(n)}(x)}{\partial v} > 0 \quad \text{for} \quad v \geqslant \tfrac{1}{2}. \tag{6.2}$$

and so, since $\frac{1}{2} \leqslant \mu < v$, we see that $(-1)^n P^{(n)}(x) > (-1)^n p^{(n)}(x) > 0$, which is (5.4).

To establish (5.19), we note that both $p(x)$ and $P(x)$ equal $1 + O(x^{-2})$ as $x \to \infty$ [15, p. 449 (1)], so that

$$\frac{1}{p(s)} - \frac{1}{P(s)} = O(s^{-2}) \quad \text{as} \quad s \to \infty.$$

Thus, (5.19) is satisfied, since

$$\int_v^\infty \left\{ \frac{1}{p(s)} - \frac{1}{P(s)} \right\} ds = O(v^{-1}) \quad \text{as} \quad v \to \infty.$$

This being the case, Corollary 5.1 applies and the theorem is proved.

7. Remarks. Open Problems

We comment here on some variants of our results and call attention to some further problems.

(i) The results stated in § 2 and § 5 assert strict positivity of certain quantities. It may be useful for other applications to record parallel results in which, instead, non-negativity is assumed and inferred. This involves replacing " > " by " ⩾ " in the lemmas and theorems of these sections. It is essential to observe, however, that for the case $n = 0$ of (2.3) and (5.4), i.e., for the mapping function $p(x)$ itself, we must retain strict positivity, so that the mapping exists, even though for $n = 1, 2, ..., N$, only non-negativity is assumed. (Suppose $p(x_0) = 0$. Then $y_1(x_0) = y_2(x_0) = 0$ and the Wronskian of y_1, y_2 would vanish for $x = x_0$, contrary to the assumption that y_1, y_2 are linearly independent.)

(ii) We have dealt explicitly only with those zeros of solutions of (2.1) and (5.3) which lie inside the interval of definition. This was primarily for ease of statement.

Excluding end-point zeros, if any, assured us that the correspondence between the zeros of $y(x)$ and $u(t)$ (as given by Lemma 2.3) is one-to-one. A variety of other restrictions would do the same. For instance, if $x'(t)$ is bounded away from 0 as t approaches an end-point, then the requisite one-to-one correspondence would persist even if we include possible end-point zeros. This particular condition is satisfied for $J_v(x)$, $v > \frac{1}{2}$, so that our results

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for this function can be extended to cover its non-negative zeros rather than, as in Theorem 3.1, only its positive zeros.

(iii) We have been unable to establish an analogue for the case $|\nu| < \frac{1}{2}$ of Theorem 3.1. The Sturm comparison theorem shows that the theorem must be modified for this instance and numerical calculations suggest the following (which is known for $n=2$ [11; 13; 14]):

CONJECTURE. *If* $|\nu| < \frac{1}{2}$, *then* $(-1)^n \Delta^n c_{\nu k} > 0$, $k = 1, 2, \ldots$, $n = 2, 3, \ldots$.

(iv) Many similar conjectures can be advanced; an examination of various tables of zeros will produce them. Appropriate analogues or extensions of Nicholson's integral or Hartman's theorems, used in conjunction with an analogue or extension of Theorem 2.1 or Theorem 5.1 may settle some of them.

In some instances, such as the θ-zeros of the Legendre polynomials $P_n(\cos\theta)$, and the Hermite and Laguerre polynomials, the positive zeros appear to form absolutely (rather than completely) monotonic sequences; that is, we conjecture for these cases that *all* differences of the zeros are non-negative.

In this connection, we are happy to thank a number of persons who have aided us by making numerical checks of these conjectures (and of our theorems while they were still conjectures): (1) Messrs. Samuel C. Arthur, Lovell Moore and Robert Robinson for hand calculations involving the differences of zeros of Bessel and Airy functions, done in 1957 while they were students at Philander Smith College; (2) Professor D. H. Lehmer for recomputing certain zeros of Bessel functions which turned out to have been recorded erroneously in [15] (Corrections noted also in [5, vol. 2, p. 925]); (3) Drs. P. J. Davis and P. Rabinowitz [4(a), p. 435; (b), p. 619] for verifying, at our request, that all the differences of the θ-zeros of the Legendre polynomials $P_{63}(\cos\theta)$ and $P_{64}(\cos\theta)$ are non-negative, and for noting that the higher differences of the zeros of $P_{63}(x)$ do not follow this pattern; (4) the University of Alberta Computing Centre for verifying that all the differences of the positive zeros of the Laguerre polynomials of degrees 6 through 15, inclusive, and of the Hermite polynomials of degrees 6 through 20, inclusive, are non-negative.

The problem of generalizing the Sturm comparison theorem has been mentioned in footnote (^{1}), p. 56.

8. Appendix I: A remark on completely monotonic functions

A function $p(x)$ is said to be *completely monotonic* in (a, b) [16, p. 145] if

$$(-1)^n p^{(n)}(x) \geqslant 0, \quad a < x < b \quad (n = 0, 1, 2, \ldots). \tag{8.1}$$

In connection with the application of Hartman's work to the proofs of Theorems 3.1 and 4.1, we used the following result, which we prove now:

If $p(x)$ is completely monotonic in $(0,\infty)$, then $(-1)^n p^{(n)}(x) > 0$, $0 < x < \infty$, $n = 0, 1, 2, ...$, unless $p(x)$ is a constant.

Proof. Suppose there exists a non-negative integer N and a positive number ξ such that $p^{(N)}(\xi) = 0$. Then $p^{(N)}(x) = 0$ for all $x \geq \xi$, since the function $(-1)^N p^{(N)}(x)$ is non-negative and non-increasing.

Now, $p(x)$ and, with it, $p^{(N)}(x)$ are analytic in $(0, \infty)$ [16, p. 146], so that $p^{(N)}(x) = 0$ for $0 < x < \infty$. Hence, $p(x)$ is a polynomial for $0 < x < \infty$. But $p(x) \geq 0$, so that the highest degree term in this polynomial has a positive coefficient (unless $p(x)$ is identically zero, in which case the theorem is obvious).

If $p(x)$ be not constant, then the highest degree term in $p'(x)$ also would have a positive coefficient, and $p'(x)$ would assume positive values for all large x, contradicting the definition (8.1) when $n = 1$. This proves the result.

Remark. The above result can be shown to be equivalent to the following: If $\{\mu_k\}_0^\infty$ is a completely monotonic sequence (i.e., if $(-1)^n \Delta^n \mu_k \geq 0$, $n, k = 0, 1, 2, ...$), then $(-1)^n \Delta^n \mu_k > 0$ for $n, k = 0, 1, 2, ...$, unless $\mu_1 = \mu_2 = ... = \mu_n = ...$.

A direct, elementary proof of this latter statement is provided in [9].

9. Appendix II: A remark on Sturm–Liouville equations

In applying Hartman's work to the proofs of Theorems 3.1 and 4.1, we made use also of the following lemma whose proof we supply now:

Let $y_1(x)$, $y_2(x)$ be two linearly independent solutions of the Sturm–Liouville differential equation $y'' + f(x)y = 0$. If $p(x) \equiv [y_1(x)]^2 + [y_2(x)]^2$ is identically a constant, say χ, then $f(x)$ is identically a constant also, namely χ^{-2}.

Proof. Clearly $\chi > 0$. Without loss of generality, we suppose the solutions y_1, y_2 such that their Wronskian is 1. Then, from Lemma 2.3, we transform the given differential equation into the differential equation $u''(t) + u(t) = 0$, with $x'(t) = p(x) = \chi$, by writing $y(x) = [x'(t)]^{\frac{1}{2}} u(t)$. The solution of this latter equation is $u(t) = A \cos(t - b)$, A, b constants.

Now, $x(t) = \chi t + \delta$, and so

$$y(x) = \chi^{\frac{1}{2}} u(t) = \chi^{\frac{1}{2}} A \cos(\chi^{-1} x - b - \delta \chi^{-1}).$$

Substituting this in the given differential equation then yields the desired result.

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