# ON MAHLER'S CLASSIFICATION OF TRANSCENDENTAL NUMBERS

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## 1. Introduction

According to the well-known classification introduced by Mahler [5] in 1932, the transcendental numbers are divided into three disjoint classes, termed the $S$-numbers, $T$-numbers and $U$-numbers, depending upon which of three possible conditions of approximation the numbers satisfy. A full account of this classification is given in Schneider [11] (Kap. 3) and we refer there for the details. An important feature of the classification is that algebraically dependent numbers belong to the same class. Further subdivisions of the classes have been given, the $S$-numbers having been classified according to "type" (see [11], p. 67), and the $U$-numbers according to their "degree" (see [3]). The existence of $U$-numbers of each degree was proved by LeVeque [3], but it is not known whether there are any $T$-numbers, or even $S$-numbers of type exceeding 1.

It is the main purpose of the present paper to investigate how Mahler's classification for real transcendental numbers is related to the more direct classification in which the numbers are divided into two sets according as the regular continued fraction has bounded or unbounded partial quotients. We show that, in fact, there is little correlation; both sets of real numbers, those with bounded partial quotients and those with unbounded partial quotients, contain $U$-numbers, and also either $T$-numbers or $S$-numbers of arbitrarily high type. It follows, incidentally, that at least one of the two sets, the $T$-numbers or the $S$-numbers of type exceeding 1, is not empty.

In order to obtain the results referred to above we first prove a general theorem concerning the approximation of transcendental numbers by numbers in a fixed algebraic number field. This extends a theorem of LeVeque [4] (Ch. 4) which itself is a generalisation of Roth's Theorem [9]. Let $K$ be an algebraic number field and define the *field height* of any algebraic number $\alpha$ in $K$ as the maximum of the absolute values of the relatively prime integer coefficients in the field equation satisfied by $\alpha$. Then LeVeque's Theorem implies that if $\xi$ is a real or complex number, and $\varkappa > 2$, and $\alpha_1, \alpha_2, \ldots$ are distinct numbers in $K$ with field heights at most $H(\alpha_1)$, $H(\alpha_2), \ldots$ such that, for each $j$,

$$|\xi - \alpha_j| < (H(\alpha_j))^{-\varkappa} \tag{1}$$

then $\xi$ is transcendental. (1) We prove that if we impose the further condition that

$$\limsup_{j \to \infty} \frac{\log H(\alpha_{j+1})}{\log H(\alpha_j)} < \infty \tag{2}$$

then $\xi$ is not a $U$-number. More precisely we prove

THEOREM 1. *Suppose that $\xi$ is a real or complex number and $\varkappa > 2$. Let $\alpha_1$, $\alpha_2, \ldots$ be a sequence of distinct numbers in an algebraic number field $K$ with field heights at most $H(\alpha_1), H(\alpha_2), \ldots$ such that* (1) *and* (2) *hold. Then there is a positive constant $\mu$ such that*

$$|x_0 + x_1 \xi + \ldots + x_n \xi^n| > X^{-\mu_n} \tag{3}$$

*for all positive integers $n$ and all sets of integers $x_0, x_1, \ldots, x_n$, not all zero, where*

$$X = \max(2, |x_0|, |x_1|, \ldots, |x_n|), \tag{4}$$

*and $\mu_n$ is given by*

$$\log \log \mu_n = \mu n^2. \tag{5}$$

The measure of transcendence given by (3) clearly implies that $\xi$ is not a $U$-number, for $\mu_n$ is independent of $X$. We note that without condition (2) $\xi$ could be a $U$-number, the Liouville numbers providing examples with $K$ as the rational field. That conditions (1) and (2), with $K$ as the rational field, imply that $\xi$ is transcendental was proved by Schneider [10] in 1936, before the work of Roth, and later LeVeque [3], using different methods, showed also that $\xi$ could not be a $U$-number of small degree.

The set of real numbers with unbounded partial quotients certainly contains $U$-numbers, for the Liouville numbers are in the set. Theorem 1 shows that the set also contains either $T$-numbers or $S$-numbers of arbitrarily high type, as we prove in the following

(1) To obtain this formulation, note that by a lemma of Siegel [*Math. Zeitschrift*, 10 (1921), p. 176, Hilfssatz III] the absolute height is less than a constant multiple, depending only on $K$, of the field height.

COROLLARY. *Suppose that $N$ is a positive integer and let*

$$\xi = \sum_{n=1}^{\infty} 2^{-(N+2)^n}. \tag{6}$$

*Then $\xi$ has unbounded partial quotients and is either a T-number or an S-number of type at least $N$.*

In order to show that the corresponding results hold for the set of real numbers with bounded partial quotients we use a method due to Maillet, some further developments of which have been given in a recent paper (see [1]). We prove

THEOREM 2. *Consider a quasi-periodic continued fraction*

$$\xi = \left[a_0, a_1, \ldots, a_{n_0-1}, \overleftrightarrow{a_{n_0}, \ldots, a_{n_0+k_0-1}}^{\lambda_0}, \overleftrightarrow{a_{n_1}, \ldots, a_{n_1+k_1-1}}^{\lambda_1}, \ldots\right]$$

*where the notation implies that $n_i = n_{i-1} + \lambda_{i-1} k_{i-1}$, and the $\lambda$'s indicate the number of times a block of partial quotients is repeated.*(1) *Suppose that $a_i \leqslant A$, $k_i \leqslant K$ for all $i$, and let $C$ be given by*

$$\log C = 4 A^K. \tag{7}$$

*Let*

$$L = \limsup_{i\to\infty} \lambda_i/\lambda_{i-1}, \quad l = \liminf_{i\to\infty} \lambda_i/\lambda_{i-1}.$$

*If $L = \infty$ and $l > 1$ then $\xi$ is a U-number of degree 2. If $L < \infty$ and $\phi > 1$ is a constant such that $l > C\phi$ then $\xi$ is either a T-number or an S-number of type at least $\phi$.*

The proof of the first statement is direct and in the proof of the second we use Theorem 1. Theorem 2 may be regarded as a refinement of Theorem 3 of [1], in the sense that it serves to classify certain continued fractions previously known only to be transcendental. One point that emerges is that the quasi-periodic continued fractions considered in Theorem 2 for which $l > C$ cannot include $U$-numbers of degree greater than 2, that is there is a gap in the type of transcendental number given by them.

In the proof of Theorem 1 we use essentially the methods of Roth as generalised by LeVeque. In the usual applications it is assumed that $\xi$ is algebraic and the main object is then to construct a polynomial with a zero of high index at one given point but which is not zero at another suitably chosen point. Here we suppose instead that $\xi$ allows better approximations than is indicated by (3) and this enables

(1) It is understood that two blocks which correspond to consecutive $i$ are not identical.

us to construct two polynomials, one of which has the first of the two usual properties and the sum of which has the second. On using condition (2) we then obtain the required contradiction.

Finally we mention two immediate applications of Theorem 1. In 1937, Mahler [6] proved an interesting theorem to the effect that certain infinite decimals, for example

$$0 \cdot 1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,\ldots,$$

in which the positive integers, represented on the decimal scale, are written as a sequence in the natural order, are transcendental but not Liouville numbers. It is clear from the proof of these results that the hypotheses of Theorem 1 are satisfied, with $K$ as the rational field, for many of the infinite decimals considered by Mahler, including the example mentioned above, and hence these are indeed not $U$-numbers. Secondly, since algebraically dependent numbers belong to the same class, it follows that if $\xi$ satisfies the hypotheses of Theorem 1 and $\eta$ is any $U$-number then all polynomials in $\xi$, $\eta$ with algebraic coefficients, not all zero, are transcendental. Thus we have a method for the construction of transcendental numbers.

I am indebted to Professor Davenport for valuable suggestions in connection with the present work.

## 2. Lemmas

We now give seven lemmas preliminary to the proof of Theorem 1. We use the following notation. If

$$A(x_1, \ldots, x_m) = \sum_{i_1=0}^{r_1} \ldots \sum_{i_m=0}^{r_m} a_{i_1, \ldots, i_m} x_1^{i_1} \ldots x_m^{i_m} \tag{8}$$

is a polynomial in $m$ variables then we denote by $A_{j_1, \ldots, j_m}(x_1, \ldots, x_m)$ the polynomial given by

$$\frac{1}{j_1! \ldots j_m!} \frac{\partial^{j_1+\ldots+j_m}}{\partial x_1^{j_1} \ldots \partial x_m^{j_m}} A(x_1, \ldots, x_m),$$

where $j_1, \ldots, j_m$ are non-negative integers. We note that if $A(x_1, \ldots, x_m)$ has integer coefficients then so also has $A_{j_1, \ldots, j_m}(x_1, \ldots, x_m)$.

LEMMA 1. *Suppose that $\xi$ is a real or complex number, $n$ is a positive integer and $u_0, u_1, \ldots, u_n (= u)$ are integers in absolute value at most $X$, where $X$ is an integer. Let*

$$\eta = u_0 + u_1 \xi + \ldots + u_n \xi^n. \tag{9}$$

*Then for each positive integer* $l$ *there are integers* $a_{i,j}^{(l)}$ $(i=0, 1, \ldots, l,\ j=0, 1, \ldots, n-1)$, *in absolute value at most* $(2X)^l$, *such that*

$$(u\,\xi)^l = \sum_{i=0}^{l} \sum_{j=0}^{n-1} a_{i,j}^{(l)}\, \eta^i\, \xi^j. \tag{10}$$

*Proof.* Clearly the lemma is true for $l=1, 2, \ldots, n-1$ with $a_{i,j}^{(l)}=u^l$ if $i=0, j=l$ and 0 otherwise. Let $k$ be an integer $\geqslant n-1$ and assume that the lemma is true for $l=k$. Define $a_{i,j}^{(k)}$ to be the integers given by the lemma for $i=0, 1, \ldots, k$, $j=0, 1, \ldots, n-1$ and to be 0 for all other integral values of $i, j$. From (10), with $l=k$, we obtain

$$(u\,\xi)^{k+1} = \sum_{i=0}^{k} \sum_{j=0}^{n-1} u\, a_{i,j}^{(k)}\, \eta^i\, \xi^{j+1}.$$

Substituting, from (9), for the highest power $\xi^n$ of $\xi$ on the right hand side of this equation it follows that

$$(u\,\xi)^{k+1} = \sum_{i=0}^{k+1} \sum_{j=0}^{n-1} a_{i,j}^{(k+1)}\, \eta^i\, \xi^j,$$

where the $a_{i,j}^{(k+1)}$ $(i=0, 1, \ldots, k+1,\ j=0, 1, \ldots, n-1)$ are integers given by

$$u\, a_{i,j-1}^{(k)} - u_j\, a_{i,n-1}^{(k)} \quad \text{if} \quad j \neq 0 \tag{11}$$

and by

$$a_{i-1,n-1}^{(k)} - u_0\, a_{i,n-1}^{(k)} \quad \text{if} \quad j=0. \tag{12}$$

Since, by hypothesis, the $a_{i,j}^{(k)}$ are in absolute value at most $(2X)^k$ it follows, on estimating (11) and (12), that the $a_{i,j}^{(k+1)}$ are in absolute value at most $(2X)^{k+1}$. Hence, by induction, the lemma is proved.

LEMMA 2. *Let* $m, N, r_1, \ldots, r_m, q_1, \ldots, q_m$ *be positive integers and let* $\delta$ *be a positive number such that*

$$m > (2N+1)^2, \quad \delta < 2^{-2m}, \quad r_m > 10\,\delta^{-1}, \tag{13}$$

$$\log q_1 > 2m(2m+1)\,\delta^{-1} \tag{14}$$

*and, for each* $j=2, 3, \ldots, m$,

$$r_j/r_{j-1} < \delta, \quad r_j \log q_j \geqslant r_1 \log q_1. \tag{15}$$

*Suppose that* $A(x_1, \ldots, x_m)$ *is a polynomial, not identically zero, with integer coefficients in absolute value at most* $q_1^{\delta r_1}$ *and of degree at most* $r_j$ *in* $x_j$. *If* $\zeta_1, \ldots, \zeta_m$ *are m elements in an algebraic number field K of degree N with field heights* $q_1, \ldots, q_m$, *then there are m non-negative integers* $J_1, \ldots, J_m$ *such that*

$$A_{J_1, \ldots, J_m}(\zeta_1, \ldots, \zeta_m) \neq 0 \quad \text{and} \quad \sum_{i=1}^{m} \frac{J_i}{r_i} \leqslant 10^m\, \delta^{(\frac{1}{2})^m}. \tag{16}$$

*Proof.* We use Theorems 4–10, 4–11 and 4–12 on pp. 136–142 of LeVeque [4]. Note that Lemma 2 would follow directly from Theorem 4–12, without the first inequality in (13), if the $q_i$ represented the absolute heights of the $\zeta_i$ and not the field heights.

Since, from (13) and (14),

$$0 < \delta < (m^2 2^m)^{-1} < \{m\, 2^m (2N+1)^2\}^{-1}$$

and

$$\log q_1 > 6N(2N+1)\,\delta^{-1}, \tag{17}$$

it follows that all the hypotheses of Theorem 4–12 are satisfied if we replace the $N$ of the theorem by $2N$, with $N$ given as above, and we reinterpret the $q_i$ of the theorem as the field heights of the $\zeta_i$. We now prove that, with these changes in the hypotheses, the conclusion of the theorem continues to hold for polynomials with rational integer coefficients. This conclusion is equivalent to that of Lemma 2.

First we consider Theorem 4–10. This may be stated as follows. Let $A(x)$ be a polynomial, not identically zero, of degree $r$, with algebraic integer coefficients in an algebraic number field of degree $N_1$, for which all the conjugates are in absolute value at most $B$. Let $\zeta$ be an algebraic number of absolute height $Q$. Suppose that $\theta$ is a non negative number such that $r\theta$ is an integer and $(x-\zeta)^{r\theta}$ divides $A(x)$. Then

$$\theta \leqslant \{3N_1(N_1+1) + N_1 r^{-1} \log B\}\,(\log Q)^{-1}. \tag{18}$$

Suppose that $\zeta$ is contained in an algebraic number field of degree $N$ and that the field height of $\zeta$ is $q$. We prove that if $A(x)$ has rational integer coefficients and

$$\log q > 2N \log (N+1) \tag{19}$$

then (18) holds with $N_1 = 2N$ and $Q = q$.

Let $F$ be the field polynomial of $\zeta$ multiplied by a suitable constant so that it has relatively prime integer coefficients, and $E$ be the defining polynomial of $\zeta$ multiplied by a suitable constant similarly. Then $F$ is some power, at most $N$, of $E$, and the highest power of $\zeta$ in $E$ is at most $N$. Hence the integer coefficients in $F$ are the sum of at most $(N+1)^N$ terms, each a product of at most $N$ coefficients from $E$. It follows that

$$q \leqslant \{(N+1)\,Q\}^N. \tag{20}$$

From (19) and (20) we obtain

$$\log q < 2N \log Q. \tag{21}$$

Since $A(x)$ has rational integer coefficients it follows from (18) with $N_1 = 1$ that

$$\theta \leqslant (6 + r^{-1} \log B)(\log Q)^{-1}.$$

Now using (21) we obtain

$$\theta \leqslant (12N + 2Nr^{-1} \log B)(\log q)^{-1}$$

and hence (18) holds with $N_1 = 2N$, $Q = q$ as required. We note that from (13) and (15)

$$\log q_j \geqslant \log q_1 \quad \text{for } j = 1, 2, \ldots, m$$

so it follows from (17) that (19) holds with $q = q_j$.

Thus the conclusion of Theorem 4–10 continues to hold for polynomials with rational integer coefficients and with $\zeta = \zeta_j$ if we replace $N_1$ by $2N$ and $Q$ by $q_j$. It is then clear from the proofs of Theorems 4–11 and 4–12 that the conclusions of these also continue to hold under similar modifications and hence Lemma 2 is proved.

LEMMA 3. *Suppose $r_1, \ldots, r_m$ are positive integers and $\sigma > 0$. Then the number of sets of integers $j_1, \ldots, j_m$ satisfying*

$$0 \leqslant j_1 \leqslant r_1, \ldots, \quad 0 \leqslant j_m \leqslant r_m, \tag{22}$$

$$\frac{j_1}{r_1} + \ldots + \frac{j_m}{r_m} < \tfrac{1}{2}(m - \sigma) \tag{23}$$

*is at most* $$2m^{\frac{1}{2}}\sigma^{-1}(r_1 + 1) \ldots (r_m + 1).$$

*Proof.* See LeVeque [4], Theorem 4–13, pp. 142–144.

LEMMA 4. *Suppose the hypotheses of Lemma 1 hold. Let $m, r_1, \ldots, r_m$ be positive integers such that $r_{j-1} > r_j$ for $j = 2, 3, \ldots, m$, and let $\sigma = 6nm^{\frac{1}{2}}$. Then there is a polynomial $W(x_1, \ldots, x_m)$, not identically zero, the sum of two polynomials $U(x_1, \ldots, x_m)$ and $V(x_1, \ldots, x_m)$, all of degree at most $r_j$ in $x_j$ for $j = 1, 2, \ldots, m$, with the following properties.*

(i) *$W(x_1, \ldots, x_m)$ has integer coefficients in absolute value at most $(8X)^{mr_1}$.*

(ii) *For each set of non-negative integers $j_1, \ldots, j_m$,*

$$|U_{j_1, \ldots, j_m}(\xi, \ldots, \xi)| < \{32X(1 + |\xi|)\}^{mr_1}, \tag{24}$$

*and* $$U_{j_1, \ldots, j_m}(\xi, \ldots, \xi) = 0 \quad \textit{if} \quad \sum_{i=1}^{m} \frac{j_i}{r_i} < \tfrac{1}{2}(m - \sigma). \tag{25}$$

(iii) *Each* $u^{mr_1} V_{j_1, \ldots, j_m}(\xi, \ldots, \xi)$ *has the form*

$$\sum_{i=1}^{mr_1} \sum_{j=0}^{n-1} f_{i,j} \eta^i \xi^j, \tag{26}$$

*where the* $f_{i,j}$ *are integers in absolute value at most* $(8X)^{3mr_1}$.

*Proof*. We consider all polynomials $A(x_1, \ldots, x_m)$ of the form (8) with integer coefficients satisfying

$$0 \leqslant a_{i_1, \ldots, i_m} \leqslant B,$$

where $B$ is an integer $>1$. The number of such polynomials is $(B+1)^r$ where

$$r = (r_1 + 1) \ldots (r_m + 1). \tag{27}$$

For any such polynomial $A(x_1, \ldots, x_m)$, each $A_{j_1, \ldots, j_m}(\xi, \ldots, \xi)$ has the form

$$\sum_{l=0}^{mr_1} b_l(j_1, \ldots, j_m) \xi^l,$$

where the $b_l(j_1, \ldots, j_m)$ are integers. Since the coefficients in each derived polynomial $A_{j_1, \ldots, j_m}(x_1, \ldots, x_m)$ have absolute value at most

$$\binom{r_1}{j_1} \ldots \binom{r_m}{j_m} B \leqslant 2^{r_1 + \cdots + r_m} B \leqslant 2^{mr_1} B,$$

and the total number of terms is at most $r$, it follows that the $b_l(j_1, \ldots, j_m)$ have absolute value at most

$$r 2^{mr_1} B = (r_1 + 1) \ldots (r_m + 1) 2^{mr_1} B \leqslant 4^{mr_1} B.$$

We now use Lemma 1. From (10) we obtain

$$\begin{aligned} u^{mr_1} A_{j_1, \ldots, j_m}(\xi, \ldots, \xi) &= \sum_{l=0}^{mr_1} b_l(j_1, \ldots, j_m) u^{mr_1 - l} (u\xi)^l \\ &= \sum_{l=0}^{mr_1} \sum_{i=0}^{l} \sum_{j=0}^{n-1} b_l(j_1, \ldots, j_m) u^{mr_1 - l} a_{i,j}^{(l)} \eta^i \xi^j \\ &= \sum_{i=0}^{mr_1} \sum_{j=0}^{n-1} c_{i,j}(j_1, \ldots, j_m) \eta^i \xi^j, \end{aligned} \tag{28}$$

where
$$c_{i,j}(j_1, \ldots, j_m) = \sum_{l=i}^{mr_1} u^{mr_1 - l} a_{i,j}^{(l)} b_l(j_1, \ldots, j_m).$$

Since, from Lemma 1, the $a_{i,j}^{(l)}$ are integers in absolute value at most $(2X)^l$ and $|u| \leqslant X$, it follows that the $c_{i,j}(j_1, \ldots, j_m)$ are integers in absolute value at most

$$(mr_1+1)\,X^{mr_1}\,(2\,X)^{mr_1}\,4^{mr_1}\,B \leqslant (4\,X)^{2mr_1}\,B.$$

Hence, from Lemma 3, the total number of possible sets of integers $c_{o,j}(j_1, \ldots, j_m)$, where $j=0, 1, \ldots, n-1$ and $j_1, \ldots, j_m$ are integers for which (22) and (23) hold, is at most

$$S=\{2\,(4\,X)^{2mr_1}\,B+1\}^{2nm^{\frac{1}{2}}\sigma^{-1}r},$$

where $r$ is given by (27).

Now let $B=(8\,X)^{mr_1}$. Then, using $2\,nm^{\frac{1}{2}}\,\sigma^{-1}=\frac{1}{3}$, we obtain

$$(B+1)^r>(8\,X)^{\frac{2}{3}mr_1 r}\,B^{\frac{1}{3}r}\geqslant\{4\,(4\,X)^{2mr_1}\,B\}^{\frac{1}{3}r}>S.$$

Thus there are more polynomials $A(x_1, \ldots, x_m)$ than there are possible sets of integers $c_{o,j}(j_1, \ldots, j_m)$, where $j=0, 1, \ldots, n-1$ and $j_1, \ldots, j_m$ are integers for which (22) and (23) hold, and hence there are two different polynomials $A^{(1)}(x_1, \ldots, x_m)$ and $A^{(2)}(x_1, \ldots, x_m)$ with the same set of values. Let

$$W(x_1, \ldots, x_m)=A^{(1)}(x_1, \ldots, x_m)-A^{(2)}(x_1, \ldots, x_m).$$

Then $W(x_1, \ldots, x_m)$ is not identically zero, it has integer coefficients in absolute value at most $B$, and, from (28), for each set of integers $j_1, \ldots, j_m$ for which (22) and (23) hold, $u^{mr_1}\,W_{j_1, \ldots, j_m}(\xi, \ldots, \xi)$ has the form (26), where the $f_{i,j}$ are integers in absolute value at most

$$2\,(4\,X)^{2mr_1}\,B\leqslant(8\,X)^{3mr_1}.$$

We define $w_{j_1, \ldots, j_m}$ to be 0 if $j_1, \ldots, j_m$ are integers such that (22) and (23) hold and 1 otherwise. Put

$$U(x_1, \ldots, x_m)=\sum_{i_1=0}^{r_1}\cdots\sum_{i_m=0}^{r_m} w_{i_1, \ldots, i_m}\,W_{i_1, \ldots, i_m}\,(x_1-\xi)^{i_1}\ldots(x_m-\xi)^{i_m},$$

where, for brevity, we write $W_{i_1, \ldots, i_m}$ for $W_{i_1, \ldots, i_m}(\xi, \ldots, \xi)$. Then clearly

$$U_{j_1, \ldots, j_m}(\xi, \ldots, \xi)=w_{j_1, \ldots, j_m}\,W_{j_1, \ldots, j_m} \tag{29}$$

and this is 0 if $j_1, \ldots, j_m$ satisfy (23), that is (25) holds. Since the coefficients in each derived polynomial $W_{j_1, \ldots, j_m}(x_1, \ldots, x_m)$ are in absolute value at most $2^{mr_1}\,(8\,X)^{mr_1}$ and the total number of terms is at most $r$, it follows that

$$|\,W_{j_1, \ldots, j_m}|<4^{mr_1}\,(8\,X)^{mr_1}\,(1+|\,\xi\,|)^{mr_1}$$

for all sets of non negative integer $j_1, \ldots, j_m$. Hence (24) follows from (29).

Finally we define

$$V(x_1, \ldots, x_m) = W(x_1, \ldots, x_m) - U(x_1, \ldots, x_m).$$

Then clearly
$$V_{j_1, \ldots, j_m}(\xi, \ldots, \xi) = W_{j_1, \ldots, j_m}$$

if $j_1, \ldots, j_m$ are integers such that (22) and (23) hold, and 0 otherwise. It follows, in both cases, that $u^{mr_1} V_{j_1, \ldots, j_m}(\xi, \ldots, \xi)$ has the form (26) and hence Lemma 4 is proved.

LEMMA 5. *Suppose the hypotheses of Lemma 1 hold. Let $m, N, r_1, \ldots, r_m, q_1, \ldots, q_m$ be positive integers and $\delta$ be a positive number such that* (13), (14) *and* (15) *hold. Let $\zeta_1, \ldots, \zeta_m$ be $m$ elements in an algebraic number field $K$ of degree $N$ with field heights $q_1, \ldots, q_m$. Suppose that*

$$\log q_1 > m\delta^{-1} \log (8X) \tag{30}$$

*and let*

$$\sigma = 6nm^{\frac{1}{2}}, \quad \varrho = 10^m \delta^{(\frac{1}{2})^m}. \tag{31}$$

*Then there are two polynomials $P(x_1, \ldots, x_m)$ and $Q(x_1, \ldots, x_m)$, with sum $R(x_1, \ldots, x_m)$, all of degree at most $r_j$ in $x_j$ for $j = 1, 2, \ldots, m$, having the following properties.*

(i) *$R(x_1, \ldots, x_m)$ has integer coefficients in absolute value at most $(16X)^{mr_1}$.*

(ii) *$R(\zeta_1, \ldots, \zeta_m)$ is not zero.*

(iii) *For each set of non negative integers $j_1, \ldots, j_m$*

$$|P_{j_1, \ldots, j_m}(\xi, \ldots, \xi)| < \{64X(1 + |\xi|)\}^{mr_1}, \tag{32}$$

*and*

$$P_{j_1, \ldots, j_m}(\xi_1, \ldots, \xi) = 0 \ \text{if} \ \sum_{i=1}^{m} \frac{j_i}{r_i} < \tfrac{1}{2}(m - \sigma) - \varrho. \tag{33}$$

(iv) *Each $u^{mr_1} Q_{j_1, \ldots, j_m}(\xi, \ldots, \xi)$ has the form* (26) *where the $f_{i,j}$ are integers in absolute value at most $(16X)^{3mr_1}$.*

*Proof.* The hypotheses of Lemma 4 hold and we suppose that $U(x_1, \ldots, x_m)$, $V(x_1, \ldots, x_m)$ and $W(x_1, \ldots, x_m)$ are the polynomials given by the lemma. From (30) we obtain

$$q_1^{\delta r_1} > (8X)^{mr_1}$$

so that the polynomial $W(x_1, \ldots, x_m)$ satisfies the hypotheses of Lemma 2 in place of $A(x_1, \ldots, x_m)$. It follows that there are $m$ non-negative integers $J_1, \ldots, J_m$ such that

$$W_{J_1, \ldots, J_m}(\zeta_1, \ldots, \zeta_m) \neq 0 \quad \text{and} \quad \sum_{i=1}^{m} \frac{J_i}{r_i} \leqslant \varrho. \tag{34}$$

Let
$$P(x_1, \ldots, x_m), \ Q(x_1, \ldots, x_m), \ R(x_1, \ldots, x_m)$$

be taken as
$$U_{J_1, \ldots, J_m}(x_1, \ldots, x_m), \ V_{J_1, \ldots, J_m}(x_1, \ldots, x_m), \ W_{J_1, \ldots, J_m}(x_1, \ldots, x_m)$$

respectively. Then the coefficients in $R(x_1, \dots, x_m)$ are in absolute value at most

$$\binom{r_1}{J_1} \cdots \binom{r_m}{J_m} (8X)^{mr_1} \leqslant (16X)^{mr_1}$$

so that (i) holds. Clearly (ii) is equivalent to the first part of (34), and (33) of (iii) follows from (25) and the second part of (34). Since for each set of non-negative integers $j_1, \dots, j_m$

$$P_{j_1, \dots, j_m}(x_1, \dots, x_m) = \binom{J_1 + j_1}{j_1} \cdots \binom{J_m + j_m}{j_m} U_{J_1+j_1, \dots, J_m+j_m}(x_1, \dots, x_m)$$

and this is identically zero unless $J_i + j_i \leqslant r_i$ for all $i$, it follows from (24) that (32) of (iii) holds. Similarly from (iii) of Lemma 4, each $u^{mr_1} Q_{j_1, \dots, j_m}(\xi, \dots, \xi)$ has the form (26) where the $f_{i,j}$ are integers in absolute value at most $2^{mr_1}(8X)^{3mr_1}$ and this proves Lemma 5.

LEMMA 6. *Suppose that $K$ is an algebraic number field of degree $N$ and that $\zeta$ is an algebraic number in $K$ with field height $H(\zeta)$. Let the field conjugates of $\zeta$ be $\zeta^{(1)} = \zeta, \zeta^{(2)}, \dots, \zeta^{(N)}$ and let the coefficient of $x^N$ in the field equation of $\zeta$, with relatively prime integer coefficients, be $h$. Then*

$$h \prod_{i=1}^{N} (1 + |\zeta^{(i)}|) < 6^N H(\zeta). \tag{35}$$

*Further, if $j_1, \dots, j_s$ are $s$ distinct integers between 1 and $N$ inclusive then*

$$h\, \zeta^{(j_1)} \cdots \zeta^{(j_s)}$$

*is an algebraic integer.*

*Proof.* See LeVeque [4], Theorem 4–2, pp. 124–125 and Theorem 2–21, pp. 63–65.

LEMMA 7. *Suppose that the hypotheses of Theorem 1 hold. For each positive integer $j$, let $F(\alpha_j)$ be the exact field height of $\alpha_j$. Then there is an increasing sequence of positive integers $n_1, n_2, \dots$ such that*

$$F(\alpha_{n_i}) < F(\alpha_{n_{i+1}}), \tag{36}$$

$$|\xi - \alpha_{n_i}| < (F(\alpha_{n_i}))^{-\varkappa}, \tag{37}$$

*for all $i$, and*

$$\limsup_{i \to \infty} \frac{\log F(\alpha_{n_{i+1}})}{\log F(\alpha_{n_i})} < \infty. \tag{38}$$

*Proof.* We first prove that

$$\min\{\log H(\alpha_j),\ \log H(\alpha_{j+1})\} < \max\{\log F(\alpha_j),\ \log F(\alpha_{j+1})\} \tag{39}$$

for all sufficiently large $j$. Let $N$ be the degree of the algebraic number field $K$ and, for each $j$, let $\alpha_j^{(1)} = \alpha_j, \alpha_j^{(2)}, \ldots, \alpha_j^{(N)}$ be the field conjugates of $\alpha_j$. Let $h_j$ be the coefficient of $x^N$ in the field equation of $\alpha_j$ with relatively prime integer coefficients. We put

$$\Xi = h_j\, h_{j+1} \operatorname{Norm}(\alpha_j - \alpha_{j+1}), \tag{40}$$

where

$$\operatorname{Norm}(\alpha_j - \alpha_{j+1}) = \prod_{i=1}^{N} (\alpha_j^{(i)} - \alpha_{j+1}^{(i)}). \tag{41}$$

Since the $\alpha_j$ are distinct, it follows from (40) that $\Xi$ is not zero. From (41), $\Xi$ is the sum of products of conjugates of $\alpha_j$ and $\alpha_{j+1}$, all multiplied by $h_j\, h_{j+1}$. It follows from Lemma 6 that $\Xi$ is a rational integer and hence we obtain

$$|\Xi| \geqslant 1. \tag{42}$$

We now calculate an upper bound for $|\Xi|$. Since

$$|\alpha_j^{(i)} - \alpha_{j+1}^{(i)}| < |\alpha_j^{(i)}| + |\alpha_{j+1}^{(i)}| < (1 + |\alpha_j^{(i)}|)\,(1 + |\alpha_{j+1}^{(i)}|)$$

and, from (35) of Lemma 6,

$$h_j \prod_{i=1}^{N} (1 + |\alpha_j^{(i)}|) \leqslant 6^N F(\alpha_j),$$

it follows from (40) and (41) that

$$\begin{aligned} |\Xi| &= |\alpha_j - \alpha_{j+1}| \left| h_j\, h_{j+1} \prod_{i=2}^{N} (\alpha_j^{(i)} - \alpha_{j+1}^{(i)}) \right| \\ &< |\alpha_j - \alpha_{j+1}|\, 6^{2N} F(\alpha_j)\, F(\alpha_{j+1}) \\ &\leqslant |\alpha_j - \alpha_{j+1}|\, 6^{2N} (\max\{F(\alpha_j),\ F(\alpha_{j+1})\})^2. \end{aligned} \tag{43}$$

From (1) we obtain

$$\begin{aligned} |\alpha_j - \alpha_{j+1}| &\leqslant |\xi - \alpha_j| + |\xi - \alpha_{j+1}| \\ &< (H(\alpha_j))^{-\varkappa} + (H(\alpha_{j+1}))^{-\varkappa} \\ &\leqslant 2\,(\min\{H(\alpha_j),\ H(\alpha_{j+1})\})^{-\varkappa}, \end{aligned}$$

so that from (42) and (43) it follows that

$$(\min\{H(\alpha_j),\ H(\alpha_{j+1})\})^{\varkappa} < 2.6^{2N} (\max\{F(\alpha_j),\ F(\alpha_{j+1})\})^2. \tag{44}$$

There are only a finite number of elements of $K$ with bounded field height and hence

$$\max \{F(\alpha_j), F(\alpha_{j+1})\} \to \infty \text{ as } j \to \infty.$$

Thus from (44), on taking logarithms and noting that $\varkappa > 2$, it follows that (39) holds for all sufficiently large $j$.

We may suppose that
$$H(\alpha_j) < H(\alpha_{j+1}) \tag{45}$$
for all $j$, for otherwise we have only to replace the sequence $\alpha_j$ $(j = 1, 2, \ldots)$ by a subsequence $\alpha_{j_i}$ $(i = 1, 2, \ldots)$, where $j_1 = 1$ and, for each integer $i \geqslant 1$, $j_{i+1}$ is defined inductively as the least integer $> j_i$ for which

$$H(\alpha_{j_i}) < H(\alpha_{j_{i+1}}).$$

Then
$$\log H(\alpha_{j_{i+1}}) / \log H(\alpha_{j_i}) \leqslant \log H(\alpha_{j_{i+1}}) / \log H(\alpha_{j_{i+1}-1})$$
so that (2) holds for the subsequence $\alpha_{j_i}$ $(i = 1, 2, \ldots)$ and clearly (1) also holds for this subsequence.

Next we define inductively a sequence $\Lambda$ of positive integers $k_1, k_2, \ldots$ such that

$$|\xi - \alpha_{k_i}| < (F(\alpha_{k_i}))^{-\varkappa} \tag{46}$$

for all $i$, and
$$\limsup_{i \to \infty} \frac{\log F(\alpha_{k_{i+1}})}{\log F(\alpha_{k_i})} < \infty. \tag{47}$$

Let $k_1 = 1$ and let $i$ be a positive integer. We supppose that $k_i$ has been defined and we take $k_{i+1}$ as $k_i + 1$ or $k_i + 2$ according as $F(\alpha_{k_i+1})$ is or is not greater than $F(\alpha_{k_i+2})$. Then by definition,

$$\max \{\log F(\alpha_{k_{i-1}+1}),\ \log F(\alpha_{k_{i-1}+2})\} = \log F(\alpha_{k_i}). \tag{48}$$

From (2) there is a constant $c > 1$ such that

$$\log H(\alpha_{j+1}) < c \log H(\alpha_j) \tag{49}$$

for all $j$. Hence, from (45) and the definition of $k_i$,

$$\min \{\log H(\alpha_{k_{i-1}+1}), \log H(\alpha_{k_{i-1}+2})\} = \log H(\alpha_{k_{i-1}+1}) > c^{-1} \log H(\alpha_{k_i}). \tag{50}$$

From (39), (48) and (50) we obtain

$$\log F(\alpha_{k_i}) > c^{-1} \log H(\alpha_{k_i}) \tag{51}$$

for all sufficiently large $i$. Since $F(\alpha_j) \leqslant H(\alpha_j)$ for all $j$, it follows from (45), (49) and (51) that

$$\frac{\log F(\alpha_{k_{i+1}})}{\log F(\alpha_{k_i})} < c\,\frac{\log H(\alpha_{k_i+1})}{\log H(\alpha_{k_i})} < c^2\,\frac{\log H(\alpha_{k_i+2})}{\log H(\alpha_{k_i+1})}.$$

Hence (47) follows from (2) and clearly (46) is a direct deduction from (1).

Finally we define a subsequence $n_1, n_2, \ldots$ of $\Lambda$ in a similar manner to that in which we defined the sequence $j_1, j_2, \ldots$ above, that is we take $n_1 = 1$ and, for each integer $i \geqslant 1$, we take $n_{i+1}$ as the least integer in $\Lambda$ greater than $n_i$ for which $F(\alpha_{n_i})$ is less than $F(\alpha_{n_{i+1}})$. Then (36), (37) and (38) hold, and Lemma 7 is proved.

## 3. Proof of Theorem 1

We begin by defining explicitly a positive constant $\mu$ and proceed to prove that it has the property stated in the theorem. We suppose that $H(\alpha_j)$ is the exact field height of $\alpha_j$, for each positive integer $j$, and that $H(\alpha_1), H(\alpha_2), \ldots$ is an increasing sequence. In virtue of Lemma 7 we may make this supposition without loss of generality. From (2), there is a constant $c > 3$ such that

$$H(\alpha_{j+1}) < \{H(\alpha_j)\}^c \tag{52}$$

for all $j$. Let $N$ be the degree of the algebraic number field $K$ and let

$$\lambda = \min(1, \varkappa - 2). \tag{53}$$

We put

$$\nu = (20\,\lambda^{-1})^2 + (2N+1)^2 \log\{3(1+|\xi|)\} + \log H(\alpha_1) + \log\log c \tag{54}$$

and then define $\mu = 2\nu$.

Suppose that $\mu$ does not have the property stated in Theorem 1. Then there is a positive integer $n$, and there are integers $v_0, v_1, \ldots, v_n$, $v_n \neq 0$, such that

$$|v_0 + v_1\xi + \ldots + v_n\xi^n| \leqslant Y^{-\mu_n}, \tag{55}$$

where

$$Y = \max(2, |v_0|, |v_1|, \ldots, |v_n|)$$

and $\mu_n$ is given by (5). Let $\nu_n$ be given by

$$\log\log \nu_n = \nu n^2. \tag{56}$$

We show that there are integers $u_0, u_1, \ldots, u_n$, $u_n \neq 0$, such that

$$|u_0 + u_1\xi + \ldots + u_n\xi^n| < X^{-\nu_n}, \tag{57}$$

where
$$X = \max(|u_0|, |u_1|, \ldots, |u_n|) > e^{5\nu n^2}. \tag{58}$$

If $Y > e^{5\nu n^2}$ then, since $\nu_n < \mu_n$, we have only to take $u_i = v_i$ for $i = 0, 1, \ldots, n$. Suppose therefore that $Y \leqslant e^{5\nu n^2}$. Let
$$w = [e^{5\nu n^2}] + 1$$
and take $u_i = w v_i$ for $i = 0, 1, \ldots, n$. Then clearly (58) holds and from (55) we obtain
$$|u_0 + u_1 \xi + \ldots + u_n \xi^n| \leqslant w\, Y^{-\mu_n} < e^{5\nu n^2}\, 2^{-\mu_n + 1}. \tag{59}$$

However,
$$X \leqslant w\, Y < 2 e^{10\nu n^2} < e^{10\nu n^2 + 1}$$
and hence, from (5) and (56), it follows that
$$2^{-\mu_n} = 2^{-\nu_n^{\log \nu_n}} < e^{-\frac{1}{2}\nu_n^{\log \nu_n}} < e^{-\nu_n(2\nu_n+1)-\nu_n-1} < \tfrac{1}{2} e^{-5\nu n^2(2\nu_n+1)-\nu_n} < \tfrac{1}{2} e^{-5\nu n^2} X^{-\nu_n}.$$

Thus (57) follows from (59). With the integers $u_0, u_1, \ldots, u_n\,(=u)$ defined as above, let $\eta$ be given by (9) of Lemma 1. Then from (57)
$$|\eta| < X^{-\nu_n}. \tag{60}$$

We now define the numbers $m, \delta, \sigma, \varrho, r_1, \ldots, r_m, q_1, \ldots, q_m, \zeta_1, \ldots, \zeta_m$ with the object of applying Lemma 5. First let $m$ be the integer given by
$$m = [(20 n \lambda^{-1})^2 + (2N + 1)^2], \tag{61}$$
and define $\delta$ by
$$\log \delta^{-1} = m\, 2^m \log 10. \tag{62}$$

Then clearly $m, \delta$ satisfy the first and second inequalities in (13). Let $\sigma, \varrho$ be given by (31) of Lemma 5. From (62) it follows that $\varrho = 1$. We now prove, as in [2], that
$$\frac{2m(1 + 4\delta)}{m - \sigma - 2} < 2 + \lambda. \tag{63}$$

The fraction on the left-hand side increases when $m$ decreases and thus it suffices to prove (63) when $m$ is replaced by $(20 n \lambda^{-1})^2$. The inequality is then equivalent to
$$\frac{3200\,\delta + 240\,\lambda + 4\,\lambda^2 n^{-2}}{400 - 120\,\lambda - 2\,\lambda^2 n^{-2}} < \lambda.$$

Since $\lambda \leqslant 1$ and $n \geqslant 1$, the denominator is at least $400 - 120 - 2 = 278$ and the numerator is at most $3200\,\delta + 240\,\lambda + 4\,\lambda < 3200\,(2^{-400\lambda^{-1}}) + 244\,\lambda < 245\,\lambda$. Hence (63) holds as required.

Now define $$r_1 = [10(2c)^{m-1}\delta^{-m}] + 1 \tag{64}$$

and let $$\theta = 4cm^2\delta^{-2}r_1. \tag{65}$$

We prove that $$\theta < \nu_n. \tag{66}$$

From (64) and (65) $$\theta < 2^{m+5}c^m m^2 \delta^{-(m+2)}$$

and, since $m > 400$, it follows that

$$\log\theta < m\log c + (m+5) + 2\log m + (m+2)\log\delta^{-1} < m\log c + 2m(1+\log\delta^{-1}). \tag{67}$$

From (62) we obtain $$\log\delta^{-1} < \tfrac{1}{4}m^{-1}e^{m-1}$$

and hence, from (67),

$$\log\theta < m\log c + e^{m-1} < (1+\log c)\,e^{m-1} < e^m \log c.$$

Thus, from (54) and (61), it follows that

$$\log\log\theta < m + \log\log c < \nu n^2 = \log\log\nu_n$$

and hence (66) is satisfied.

Next we select a subsequence $\alpha_{j_1}, \alpha_{j_2}, \ldots$ of the $\alpha_j$, where $1 = j_1 < j_2 < \ldots$, such that

$$\log H(\alpha_{j_i}) > 2\delta^{-1}\log H(\alpha_{j_{i-1}}) \geqslant \log H(\alpha_{j_i - 1}) \tag{68}$$

for $i = 2, 3, \ldots$. Then, from (68) and (52),

$$\{H(\alpha_{j_{i-1}})\}^{2c\delta^{-1}} \geqslant \{H(\alpha_{j_i-1})\}^c > H(\alpha_{j_i}), \tag{69}$$

and $$H(\alpha_{j_{i-1}}) < \{H(\alpha_{j_i})\}^{\frac{1}{2}\delta}. \tag{70}$$

We choose $k$ such that $$H(\alpha_{j_{k-1}}) \leqslant (8X)^{m\delta^{-1}} < H(\alpha_{j_k}). \tag{71}$$

This is possible since $H(\alpha_1), H(\alpha_2), \ldots$ is an increasing sequence and, from (58) and (54),

$$(8X)^{m\delta^{-1}} > X > e^{\nu} > e^{\log H(\alpha_1)} = H(\alpha_1).$$

For $i = 1, 2, \ldots, m$, take $$\zeta_i = \alpha_{j_{k+i-1}} \tag{72}$$

and put $$q_i = H(\zeta_i). \tag{73}$$

Then, from (71), $$\log q_1 = \log H(\alpha_{j_k}) > m\delta^{-1}\log(8X)$$

so that (30) of Lemma 5 holds. Further, from (58), (54) and (61) it follows that

$$\log q_1 > m\delta^{-1} \log X > 5\nu n^2 m\delta^{-1} > 2m(2m+1)\delta^{-1},$$

and hence (14) of Lemma 2 holds. From (69), (72) and (73) we obtain, for each $j = 2, 3, \ldots, m$

$$\log q_j/\log q_{j-1} < 2c\delta^{-1}, \tag{74}$$

and, from (70),

$$\log q_j/\log q_{j-1} > 2\delta^{-1}. \tag{75}$$

Finally we define integers $r_2, \ldots, r_m$ such that

$$r_1 \log q_1/\log q_j \leqslant r_j < 1 + r_1 \log q_1/\log q_j, \tag{76}$$

for $j = 2, 3, \ldots, m$. Then clearly the second part of (15) holds. From (76), (74) and (64) we obtain for each $j = 2, 3, \ldots, m$

$$r_j \geqslant r_1 \log q_1/\log q_j > r_1 (2c\delta^{-1})^{-(j-1)} > 10(2c)^{m-j}\delta^{-m+j-1}. \tag{77}$$

For $j = m$ this gives the third inequality in (13). For each $j = 2, 3, \ldots, m$, (77) gives

$$r_1 \log q_1/\log q_j > 10\delta^{-1} > 1 \tag{78}$$

and hence, from (76), it follows that

$$r_j \log q_j/(r_{j-1} \log q_{j-1}) < 1 + \log q_j/(r_1 \log q_1) < 2.$$

Then, using (75), we obtain

$$2\delta^{-1} r_j/r_{j-1} < 2$$

so that the first inequality in (15) is satisfied.

Hence we have verified all the hypotheses of Lemma 5. Let $P(x_1, \ldots, x_m)$, $Q(x_1, \ldots, x_m)$ and $R(x_1, \ldots, x_m)$ be the polynomials given by the lemma. For each $i = 1, 2, \ldots, m$, let $\zeta_i^{(1)} = \zeta_i, \zeta_i^{(2)}, \ldots, \zeta_i^{(N)}$ be the field conjugates of $\zeta_i$ and $h_i$ be the coefficient of $x^N$ in the field equation of $\zeta_i$ with relatively prime integer coefficients. Then

$$\Psi = h_1^{r_1} \ldots h_m^{r_m} \operatorname{Norm} R(\zeta_1, \ldots, \zeta_m)$$

is the sum of products of powers of the $\zeta_i^{(j)}$ with integer coefficients, and in each such product a factor $\zeta_i^{(j)}$ occurs to the power at most $r_i$. Hence, from Lemma 6, $\Psi$ is a rational integer and, from (ii) of Lemma 5, it is non zero. It follows that

$$|\Psi| \geqslant 1. \tag{79}$$

We now calculate an upper bound for $|\Psi|$. First we consider

$$\Phi = h_1^{r_1} \dots h_m^{r_m} \prod_{j=2}^{N} R(\zeta_1^{(j)}, \dots, \zeta_m^{(j)}).$$

From (i) of Lemma 5, the coefficients in $R(x_1, \dots, x_m)$ are in absolute value at most $(16X)^{mr_1}$ and hence, using (35) of Lemma 6, we obtain

$$|\Phi| \leq |h_1^{r_1} \dots h_m^{r_m}| (16X)^{mr_1(N-1)} \prod_{i=1}^{m} \prod_{j=2}^{N} (1+|\zeta_i^{(j)}|)^{r_i}$$
$$< (16X)^{mr_1 N} \prod_{i=1}^{m} \prod_{j=1}^{N} \{|h_i| (1+|\zeta_i^{(j)}|)\}^{r_i}$$
$$< (16X)^{mr_1 N} 6^{mr_1 N} q_1^{r_1} \dots q_m^{r_m}.$$

From (76) and the first inequality in (78)

$$q_j^{r_j} < q_1^{r_1(1+\delta/10)}.$$

Hence, from (14) and (30), noting that $m > 4N$, it follows that

$$|\Phi| < (96X)^{mr_1 N} q_1^{mr_1(1+\delta/10)} < 12^{mr_1 N} q_1^{mr_1(1+\delta)} < q_1^{mr_1(1+2\delta)}. \tag{80}$$

Secondly we deduce an upper bound for $|Q(\zeta_1, \dots, \zeta_m)|$. From (1),

$$|\xi - \zeta_i| \leq 1 \quad \text{for} \quad i = 1, 2, \dots, m. \tag{81}$$

Now using (iv) of Lemma 5 and (60), we obtain for each set of non-negative integers $j_1, \dots, j_m$

$$|u^{mr_1} Q_{j_1, \dots, j_m}(\xi, \dots, \xi)| < (16X)^{3mr_1} \sum_{i=1}^{mr_1} \sum_{j=0}^{n-1} |\eta|^i |\xi|^j < (16X)^{3mr_1} X^{-\nu_n} mr_1 \sum_{j=0}^{n-1} |\xi|^j$$
$$< X^{-\nu_n} \{2(16X)^3 (1+|\xi|)\}^{mr_1}.$$

Hence, from (81), on expanding $Q(x_1, \dots, x_m)$ about the point $(\xi, \dots, \xi)$ by Taylor's Theorem, we obtain

$$|Q(\zeta_1, \dots, \zeta_m)| \leq \sum_{i_1=0}^{r_1} \dots \sum_{i_m=0}^{r_m} |Q_{i_1, \dots, i_m}(\xi, \dots, \xi)(\zeta_1 - \xi)^{i_1} \dots (\zeta_m - \xi)^{i_m}|$$
$$\leq \sum_{i_1=0}^{r_1} \dots \sum_{i_m=0}^{r_m} |Q_{i_1, \dots, i_m}(\xi, \dots, \xi)|$$
$$< X^{-\nu_n} \{2(16X)^3 (1+|\xi|)\}^{mr_1} |u|^{-mr_1} 2^{mr_1}$$
$$< X^{-\nu_n} \{(32X)^3 (1+|\xi|)\}^{mr_1}.$$

From (58) and (54), $\qquad \log X > 5\nu > 45 \log \{3(1+|\xi|)\}$

so that
$$X > 3^{45}(1+|\xi|). \tag{82}$$

Hence from (65) and (66) we obtain

$$|Q(\zeta_1, \ldots, \zeta_m)| < X^{-\nu_n} X^{4mr_1} < X^{-\theta+4mr_1} < X^{-\frac{3}{4}\theta}. \tag{83}$$

Thirdly we find an upper bound for $|P(\zeta_1, \ldots, \zeta_m)|$. From (33) of Lemma 5, $P(\zeta_1, \ldots, \zeta_m)$ is the sum of at most $2^{mr_1}$ terms, each of the form

$$P_{j_1, \ldots, j_m}(\xi, \ldots, \xi)(\zeta_1 - \xi)^{j_1} \ldots (\zeta_m - \xi)^{j_m},$$

where $j_1, \ldots, j_m$ are integers such that

$$0 \leqslant j_1 \leqslant r_1, \ldots, \; 0 \leqslant j_m \leqslant r_m, \tag{84}$$

and
$$\sum_{i=1}^{m} \frac{j_i}{r_i} \geqslant \tfrac{1}{2}(m-\sigma) - \varrho. \tag{85}$$

From (1),
$$|\xi - \zeta_i| < q_i^{-\varkappa} \quad \text{for} \quad i = 1, 2, \ldots, m$$

and hence, using (32) of Lemma 5, we obtain

$$|P(\zeta_1, \ldots, \zeta_m)| < 2^{mr_1}\{64\,X(1+|\xi|)\}^{mr_1} q_1^{-\varkappa J_1} \ldots q_m^{-\varkappa J_m},$$

where $J_1, \ldots, J_m$ are integers which satisfy (84) and (85) above in place of $j_1, \ldots, j_m$. From (76),
$$q_j^{r_j} \geqslant q_1^{r_1} \quad \text{for} \quad j = 1, 2, \ldots, m$$
and thus it follows that

$$|P(\zeta_1, \ldots, \zeta_m)| < \{128\,X(1+|\xi|)\}^{mr_1} q_1^{-\varkappa r_1\{\frac{1}{2}(m-\sigma)-\varrho\}}.$$

Now $\varrho = 1$ and hence from (53) and (63) we obtain

$$q_1^{-\varkappa r_1\{\frac{1}{2}(m-\sigma)-\varrho\}} < q_1^{-mr_1(1+4\delta)}.$$

From (82),
$$\{128\,X(1+|\xi|)\}^{mr_1} < X^{2mr_1}$$

and hence, from (30), it follows that

$$|P(\zeta_1, \ldots, \zeta_m)| < X^{2mr_1} q_1^{-mr_1(1+4\delta)} < \tfrac{1}{2} q_1^{-mr_1(1+2\delta)}. \tag{86}$$

We now combine the estimates (80), (83) and (86). From Lemma 5,

$$R(\zeta_1, \ldots, \zeta_m) = P(\zeta_1, \ldots, \zeta_m) + Q(\zeta_1, \ldots, \zeta_m)$$

and hence, from (83) and (86),

$$|R(\zeta_1, \dots, \zeta_m)| < X^{-\frac{3}{4}\theta} + \tfrac{1}{2} q_1^{-mr_1(1+2\delta)}.$$

Then, using (80), it follows that

$$|\Psi| = |\Phi|\, |R(\zeta_1, \dots, \zeta_m)| < q_1^{mr_1(1+2\delta)} X^{-\frac{3}{4}\theta} + \tfrac{1}{2}.$$

From (69) and (71),

$$q_1 = H(\alpha_{j_k}) < \{H(\alpha_{j_{k-1}})\}^{2c\delta^{-1}} \leqslant (8X)^{2cm\delta^{-2}},$$

so that, from (65),

$$|\Psi| < (8X)^{\frac{1}{2}\theta(1+2\delta)} X^{-\frac{3}{4}\theta} + \tfrac{1}{2}.$$

Since $\delta < 2^{-2m} < \frac{1}{8}$, and, from (82), $X > 2.8^5$, we obtain finally

$$|\Psi| < (8X)^{\frac{5}{8}\theta} X^{-\frac{3}{4}\theta} + \tfrac{1}{2} = (8^5 X^{-1})^{\frac{1}{8}\theta} + \tfrac{1}{2} < (\tfrac{1}{2})^{\frac{1}{8}\theta} + \tfrac{1}{2} < 1.$$

However, this contradicts (79), and the contradiction proves the theorem.

## 4. Proof of Corollary

The result follows by an immediate application of Theorem 1 with $K$ as the rational field. We define integers

$$p_j = 2^{(N+2)^j} \sum_{n=1}^{j} 2^{-(N+2)^n}, \quad q_j = 2^{(N+2)^j}$$

for $j = 1, 2, \dots,$ and put $\alpha_j = p_j/q_j$. Then the field height of $\alpha_j$, which, in this case, is the same as the absolute height, is given by

$$H(\alpha_j) = \max\,(p_j, q_j) = q_j = 2^{(N+2)^j}.$$

Clearly (2) holds and since, for all $j$,

$$|\xi - p_j/q_j| = \sum_{n=j+1}^{\infty} 2^{-(N+2)^n} < 2^{-(N+2)^{j+1}+1} < q_j^{-(N+\frac{3}{2})} \tag{87}$$

it follows that (1) is satisfied with $\varkappa = N + \frac{3}{2} > 2$. Hence, from Theorem 1, $\xi$ is neither algebraic nor a $U$-number.

As is well known, (87) implies that the partial quotients in the regular continued fraction of $\xi$ are unbounded. Further, since (87) can be written in the form

$$|q_j \xi - p_j| < H^{-(N+\frac{1}{2})},$$

where $H = \max\,(p_j, q_j,)$, it follows, by definition, that $\xi$ cannot be an $S$-number of type $\leqslant N$. This proves the corollary.

## 5. Proof of Theorem 2

We assume that $0<\xi<1$ as we may without loss of generality. Let $p_n/q_n$ be the $n$th convergent to $\xi$. For each $i=1, 2, \dots$ we define

$$\eta_i = \Big[a_0, a_1, \dots, a_{n_0-1}, \overset{\longleftarrow\ \lambda_0\ \longrightarrow}{a_{n_0}, \dots, a_{n_0+k_0-1}}, \dots, \overline{a_{n_i}, \dots, a_{n_i+k_i-1}}\Big]$$

where the block of partial quotients indicated by the bar is repeated infinitely many times. Then (see [1], Lemmas 1 and 2) $\eta_i$ is a quadratic irrational of absolute height less than $2\,q^2_{n_i+k_i-1}$ and, since the first $n_{i+1}$ partial quotients of $\eta_i$ are the same as those of $\xi$,

$$|\xi-\eta_i|<q^{-2}_{n_{i+1}-1}. \tag{88}$$

We note that all the $\eta_i$ are distinct. Next, let

$$U=\tfrac{1}{2}(1+5^{\frac{1}{2}}), \quad V=\tfrac{1}{2}(A+(A^2+4)^{\frac{1}{2}}). \tag{89}$$

Then (see [1], Lemma 3)

$$U^{n-1}\leqslant q_n \leqslant V^n \quad \text{for all } n. \tag{90}$$

Suppose that the defining equation of $\eta_i$ with relatively prime integer coefficients is

$$P_i\,x^2+Q_i\,x+R_i=0. \tag{91}$$

From (90) and (91), the absolute height of $\eta_i$ is given by

$$\max\,(|P_i|, |Q_i|, |R_i|)\leqslant X_i, \tag{92}$$

where

$$X_i=2\,V^{2(n_i+k_i-1)}. \tag{93}$$

Let the root of (91) conjugate to $\eta_i$ be $\eta_i^{(1)}$. Either $|\eta_i^{(1)}|\leqslant 1$ or, from (91) and (92),

$$|P_i\,(\eta_i^{(1)})^2| = |Q_i\,\eta_i^{(1)}+R_i| < 2\,X_i\,|\eta_i^{(1)}|,$$

so that, in both cases,

$$|\eta_i^{(1)}|<2\,X_i. \tag{94}$$

We now deduce an upper bound for $|P_i\,\xi^2+Q_i\,\xi+R_i|$. From (88) and (90),

$$|\xi-\eta_i|<U^{-2(n_{i+1}-2)}. \tag{95}$$

From (94) and our assumption that $0<\xi<1$,

$$|\xi-\eta_i^{(1)}|<1+2\,X_i<4\,X_i. \tag{96}$$

Hence from (92), (93), (95) and (96) it follows that

$$|P_i \xi^2 + Q_i \xi + R_i| = |P_i|\,|\xi - \eta_i|\,|\xi - \eta_i^{(1)}| < 4\, X_i^2\, U^{-2(n_{i+1}-2)} = X_i^{-\psi_i}, \tag{97}$$

where $\psi_i = \sigma_i / \varrho_i$ and $\sigma_i, \varrho_i$ are given by

$$\sigma_i = 2\, n_{i+1} \log U - 4\,(n_i + k_i) \log V + 4 \log (V/2\,U), \tag{98}$$

$$\varrho_i = 2\,(n_i + k_i - 1) \log V + \log 2. \tag{99}$$

Similarly we obtain from (95)

$$|\xi - \eta_i| < X_i^{-\psi_i}. \tag{100}$$

We now distinguish two cases as in the statement of the theorem.

(i) First we suppose that $L = \infty$ and $l > 1$. Then there is a positive integer $j$ and a positive number $\zeta$ such that

$$\lambda_{i+1}/\lambda_i > 1 + \zeta \text{ for all } i \geqslant j.$$

It follows that, if $i \geqslant j$,

$$n_{i+1} = \lambda_i k_i + \lambda_{i-1} k_{i-1} + \dots + \lambda_j k_j + n_j \leqslant K(\lambda_i + \lambda_{i-1} + \dots + \lambda_j) + n_j$$
$$< K \lambda_i \{1 + (1+\zeta)^{-1} + \dots + (1+\zeta)^{j-i}\} + n_j < K(1+\zeta^{-1})\, \lambda_i + n_j < c_1 \lambda_i,$$

where $c_1$ is a positive constant independent of $i$. Clearly $n_{i+1} > \lambda_i$ for all $i > 1$, and hence, since $L = \infty$, it follows that

$$\limsup_{i \to \infty} n_{i+1}/n_i = \infty. \tag{101}$$

From (98) and (99) we see that $\varrho_i / n_i$ $(i = 1, 2, \dots)$ is bounded and that there is a positive constant $c_2$ such that

$$\sigma_i / n_i > c_2\, n_{i+1}/n_i$$

for all sufficiently large $i$. Hence, using (101), we obtain

$$\limsup_{i \to \infty} \psi_i = \infty. \tag{102}$$

Then, by definition, (97) and (102) imply that $\xi$ is a $U$-number of degree 2 and the first part of Theorem 2 is proved.

(ii) Secondly we suppose that $L < \infty$ and $\phi > 1$ is a constant such that $l > C\phi$, where $C$ is given by (7). Since $l > 2$, there is an integer $j$ such that

$$\lambda_{i+1}/\lambda_i > 2 \quad \text{for all } i \geqslant j.$$

We suppose that $i$ is sufficiently large. Then, as in (i), it follows that

$$\lambda_i < n_{i+1} < 2\,K\,\lambda_i + n_j < 3\,K\,\lambda_i.$$

Hence, using $l > C\,\phi$, we obtain

$$n_{i+1}/n_i > \lambda_i/(3\,K\,\lambda_{i-1}) > \tfrac{1}{3}\,C\,\phi/K,$$

and, since $L < \infty$, there is a positive constant $c_3$ such that

$$n_{i+1}/n_i < 3\,K\,\lambda_i/\lambda_{i-1} < c_3. \tag{103}$$

It follows from (98) and (99) that

$$\sigma_i/n_i > (n_{i+1}/n_i)\log U - 4\log V > (\tfrac{1}{3}\,C\,\phi\log U - 4\,K\log V)/K,$$

$$\varrho_i/n_i < 3\log V$$

and hence
$$\psi_i > (C\,\phi\log U)/(9\,K\log V) - \tfrac{4}{3}.$$

Noting that $\log U > 9/20$ and $\log V < A$, we obtain

$$\psi_i > C\,\phi/(20\,A\,K) - \tfrac{4}{3} \tag{104}$$

for all sufficiently large $i$.

For each $i$ there are at most $A$, $K$ different possible values for $a_i$, $k_i$. Hence there are at most $A^K$ different sets of integers

$$a_{n_i},\ a_{n_i+1},\ \ldots.\ a_{n_i+k_i-1}.$$

Let $F$ be the algebraic number field generated by all the quadratic irrationals

$$[\overline{a_{n_i},\ a_{n_i+1},\ \ldots,\ a_{n_i+k_i-1}}].$$

Then $F$ has degree at most

$$N = 2\,A^K,$$

and all the $\eta_i$ are elements of $F$. As in the proof of Lemma 2, the field height of $\eta_i$, with respect to $F$, is at most

$$H(\eta_i) = \{(N+1)\,X_i\}^N. \tag{105}$$

From (100) and (105) we obtain

$$|\,\xi - \eta_i\,| < (H(\eta_i))^{-\psi_i/(N+1)} \tag{106}$$

for all sufficiently large $i$. Since $A \geqslant 2$, $K \geqslant 1$, we deduce from (7) and (104) that

$$\psi_i + \tfrac{4}{3} > e^4 A^K/(20\,AK) > N\,.\,2^4 A^K/(20\,AK) > 5\,N,$$

and hence, noting that $N \geqslant 4$,

$$\psi_i/(N+1) > (5N - \tfrac{4}{3})/(N+1) > 3$$

for all sufficiently large $i$. It follows from (106) that (1) is satisfied for all sufficiently large $j$, with $\alpha_j = \eta_j$ and $\varkappa = 3$. From (93) and (105) we obtain

$$\frac{\log H(\eta_{i+1})}{\log H(\eta_i)} = \frac{c_4 + 2(n_{i+1} + k_{i+1}) \log V}{c_4 + 2(n_i + k_i) \log V},$$

where $c_4$ is a constant independent of $i$. From (103) it follows that (2) holds with $\alpha_j = \eta_j$. Thus, for all sufficiently large $j$, the hypotheses of Theorem 1 are satisfied and hence $\xi$ is neither algebraic nor a $U$-number.

Finally, from (7) and (104) we obtain

$$\psi_i > \phi\, e^4 A^K/(20\,AK) - \tfrac{4}{3} > 4\,\phi - \tfrac{4}{3} > 2\,\phi$$

for all sufficiently large $i$, and, in virtue of (97), $\xi$ cannot be an $S$-number of type $\leqslant \phi$. This completes the proof of Theorem 2.

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