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# On completely invariant Fatou components

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**Abstract.** Completely invariant components of the Fatou sets of meromorphic maps are discussed. Positive answers are given to Baker's and Bergweiler's problems that such components are the only Fatou components for certain classes of meromorphic maps.

## 1. Introduction

Let $f$ be a transcendental meromorphic map defined in the complex plane $\mathbf{C}$. The Fatou set $F(f)$ of $f$ is the largest subset of $\widehat{\mathbf{C}}$ where the iterates $f^n$ of $f$ are well defined and form a normal family. The complement of $F(f)$ is called the Julia set of $f$ and denoted by $J(f)$. It is clear that $F(f)$ is open and completely invariant under $f$, and $J(f)$ is closed and also completely invariant. If $U$ is a component of $F(f)$, then $f^n(U)$ is contained in some component of $F(f)$ which we denote by $U_n$. If $U_n \cap U_m = \emptyset$ for all $n \neq m$, then $U$ is called wandering. Otherwise $U$ is called periodic or preperiodic. In addition, if $f^{-1}(U) \subset U$ and $f(U) \subset U$ for a component $U$ of $F(f)$, then $U$ is called a completely invariant component of $F(f)$. More details of these can be found in [11], [12] and [18].

We define $\mathrm{FV}(f)$ to be the set of Fatou exceptional values of $f$, that is, the points whose inverse orbit

$$O^-(z) = \{w : f^n(w) = z \text{ for some } n \in \mathbf{N}\}$$

is finite. The set $\mathrm{FV}(f)$ contains at most two points. Transcendental meromorphic maps can be divided into the following three classes:

(i) $\mathbf{E} = \{f : f \text{ is entire }\}$;

(ii) $\mathbf{P} = \{f : f \text{ is meromorphic, has exactly one pole, and } \infty \in \mathrm{FV}(f)\}$;

(iii) $\mathbf{M} = \{f : f \text{ is meromorphic, has at least one pole, and } \infty \notin \mathrm{FV}(f)\}$.

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The iteration of maps in $\mathbf{E}$ was studied by Fatou [14], Baker [1], [2], [3], [4], [5], [6], and other authors. If $f$ is a map in $\mathbf{P}$ then we may assume without loss of generality that it has a pole at the point 0, and it then follows that $f$ must be an analytic map of the punctured plane $\mathbf{C}^*=\mathbf{C}\setminus\{0\}$ onto itself. The iteration of such maps was studied first by Rådström [20] and then by others [5], [16] and [17]. In a series of papers [7], [8], [9] and [10], Baker, Kotus and Lü studied the iteration of maps in $\mathbf{M}$.

For a rational function $f$ with degree more than one, it is known that $F(f)$ can have at most two completely invariant components and if $F(f)$ has two such components, then these are the only components of $F(f)$. In [2], Baker showed that if $f\in\mathbf{E}$, then there is at most one completely invariant component of $F(f)$. He also asked whether the existence of a completely invariant component of $F(f)$ precludes the existence of other components or not (see [3]). Eremenko and Lyubich [13, Theorem 6] showed that this is true if $f\in\mathbf{S}\cap\mathbf{E}$, where

$$\mathbf{S}=\{f:f \text{ is meromorphic and has finitely many critical and asymptotic values}\}.$$

Less is known about completely invariant Fatou components of meromorphic maps with at least one pole. Bergweiler [12, Questions 13 and 14] put forward the following questions for meromorphic maps: Let $f$ be a meromorphic map. Can $F(f)$ have more than two completely invariant components? If $F(f)$ has two completely invariant components $U_1$ and $U_2$, does $F(f)$ contain only $U_1$ and $U_2$? Baker, Kotus and Lü [9, Theorem 4.5] showed that if $f\in\mathbf{S}$, then $F(f)$ has at most two completely invariant components.

Our first result shows that the completely invariant components are the only Fatou components for the class $\mathbf{S}$.

**Theorem 1.** *Let $f$ be a meromorphic map in $\mathbf{S}$. If $F(f)$ contains two completely invariant components $V_1$ and $V_2$, then $F(f)=V_1\cup V_2$.*

*Remarks.* 1. We note from [8] that $f(z)=\tan(z)$ $(\in\mathbf{S})$ has exactly two completely invariant domains, the upper and the lower half-plane, separated by $J(f)=\mathbf{R}$.

2. Our proof of the theorem is different from that of Eremenko and Lyubich in [13].

We also consider another class $\mathbf{F}$, where

$$\mathbf{F}=\{f:f(z)=z+r(z)\exp(p(z)), \text{ where } r \text{ is rational and } p \text{ is a polynomial}\}.$$

**Theorem 2.** *Let $f$ be a map in $\mathbf{F}\cap\mathbf{E}$. If $F(f)$ has a completely invariant component $U$, then $F(f)=U$.*

If $f$ is an analytic self-map of $\mathbf{C}^*$, we see from [5] that there are four types of maps $f$:

(a) $f(z)=kz^n$, $k\neq 0$, $n\in\mathbf{Z}$, $n\neq 0,\pm 1$ (we are excluding Möbius transformations);

(b) $f(z)=z^n\exp(g(z))$, $g$ non-constant entire, $n\in\mathbf{N}$;

(c) $f(z)=z^{-n}\exp(g(z))$, $g$ non-constant entire, $n\in\mathbf{N}$ (we note that without loss of generality, $f\in\mathbf{P}$ is just this type);

(d) $f(z)=z^m\exp(g(z)+h(1/z))$, $g$, $h$ non-constant entire maps, $m\in\mathbf{Z}$.

We call $f$ a transcendental analytic self-map of $\mathbf{C}^*$ if $f$ has the form (b), (c) or (d). In all cases the set $J(f)$ is closed, non-empty and even perfect in $\mathbf{C}^*$, with the complete invariance property $f(J(f))=f^{-1}(J(f))=J(f)$, thus $f(F(f))=F(f)$. One may ask how about completely invariant domains of maps in $\mathbf{P}$, or more generally, of analytic self-maps of $\mathbf{C}^*$. Considering this problem we have the following results.

**Theorem 3.** *Let $f\in\mathbf{P}$. If $F(f)$ has a completely invariant component, then*

(i) *all components of $F(f)$ are simply connected;*

(ii) *in every other component of $F(f)$, $f$ is either a univalent map or a two-fold map.*

**Theorem 4.** *Let $f$ be a transcendental analytic self-map of $\mathbf{C}^*$. Then $F(f)$ has at most one completely invariant component. In particular, this is the case for $f\in\mathbf{P}$.*

**Corollary 1.** *If $f$ is a transcendental analytic self-map of $\mathbf{C}^*$, then the number of the components of the Fatou set is either $0, 1$ or $\infty$. In particular, this is the case for $f\in\mathbf{P}$.*

In addition, using the same method as in the proof of Theorems 3 and 4, we can obtain a result about Julia sets as Jordan arcs. A Jordan arc $\gamma$ in $\widehat{\mathbf{C}}$ is defined to be the image of the real interval $[0,1]$ under a homeomorphism $\varphi$. If the interval $[0,1]$ is replaced by the unit circle then $\gamma$ is said to be a Jordan curve. Finally, if $f$ is a meromorphic map which is not rational of degree less than two, $\alpha$ is said to be a free Jordan arc in $J(f)$ if there exists a homeomorphism $\psi$ of the open unit disc onto a domain $D$ in $\widehat{\mathbf{C}}$ such that $J(f)\cap D$ is the image of $(-1,1)$ under $\psi$ and $\alpha$ is the image of some real interval $[a,b]$ where $-1<a<b<1$. We are able to prove the following result.

**Theorem 5.** *Let $f$ be a transcendental meromorphic map with at most finitely many poles. If $J(f)$ contains a free Jordan arc, then $J(f)$ must be a Jordan arc passing through $\infty$ and both the endpoints of $J(f)$ are finite.*

## 2. Completely invariant domains for analytic self-maps of $\mathbf{C}^*$

We first prove Theorems 3, 4 and their corollary. We shall need the following lemmas.

**Lemma 2.1.** ([5]) *If $f \in \mathbf{P}$, then $F(f)$ has at most one multiply connected component. Furthermore, if the multiply connected component exists, then it is doubly connected and it separates the pole of $f$ and $\infty$.*

**Lemma 2.2.** *Let $f$ be a transcendental meromorphic function. If $U$ is a completely invariant component of $F(f)$, then*

(i) *$U$ is unbounded;*
(ii) *$\partial U = J(f)$ (we denote the boundary of a domain $D$ by $\partial D$);*
(iii) *$U$ is either simply connected or infinitely connected;*
(iv) *all other components of $F(f)$ are simply connected;*
(v) *$U$ is simply connected if and only if $J(f)$ is connected.*

*Remark.* In this lemma, (i), (ii) can be found in [9], Lemma 4.2 and its proof; (iii) is Lemma 4.1 of [9]; (iv), (v) in Beardon's book [11, pp. 82–83] are shown to be true for the case when $f$ is a rational function, however, the proofs of the rational case apply to the general meromorphic function without further difficulties. For completeness, we give the proofs of (i), (ii), (iv) and (v) here.

*Proofs of Lemma* 2.2(i), (ii), (iv) *and* (v). Since $\infty$ is an essential singularity of $f$, it follows from the big Picard theorem that $f(z)=a$ has infinitely many solutions in any neighborhood of $\infty$ for all $a \in U$ except for at most two points. Since $U$ is completely invariant, all these solutions belong to $U$. Thus $U$ is unbounded and this is (i).

To prove (ii), we need to prove only that $J(f) \subset \partial U$. Let $V$ be a domain in $\mathbf{C}$ such that $V \cap \partial U = \emptyset$. Then either $V \subset U$ or $V \subset \mathbf{C} \setminus U$. In the first case we have $V \subset F(f)$; in the second case, we have $f^m(V) \cap U = \emptyset$ $(m=0,1,...)$. Thus $\{f^m\}_{m=0}^{\infty}$ is normal in $V$, and so, $V \subset F(f)$. Both cases imply $J(f) \subset \partial U$.

To prove (iv), observe that from (ii), $J(f) \cup U$ is the closure of $U$ and so is connected ([11, Proposition 5.1.1]). By [11, Proposition 5.1.5], the components of its complement are simply connected and as these components are just the components of $F(f)$ other than $U$, (iv) follows. Finally, (v) is a direct consequence of Lemma 2.2(ii) and [11, Proposition 5.1.4]. □

By Lemma 2.2(iv), we can immediately obtain the following result.

**Corollary 2.** *Let $f$ be a meromorphic function. If $F(f)$ has two or more completely invariant components, then each component of $F(f)$ is simply connected.*

*Proof of Theorem* 3. (i) The result follows immediately from Lemma 2.1 and Lemma 2.2(iii) and (iv).

(ii) Let $U$ be a completely invariant component. Then by Lemma 2.2(i) and Theorem 3(i), $U$ is unbounded and all components are simply connected. Suppose that there is a component $V \neq U$ of $F(f)$ in which $f$ is neither a univalent map nor a two-fold map. Let $K$ be a component of $F(f)$ such that $f(V) \subset K$. Then $K \neq U$.

Take a value $a$ in $K$ such that $f(z)=a$ has infinitely many simple roots ($f'(z)=0$ at only countably many $z$ so we have to avoid only countably many choices of $a$), and take three distinct points $p, q, r \in V$ with $f'(p) \neq 0$, $f'(q) \neq 0$ and $f'(r) \neq 0$ such that $f(p)=f(q)=f(r)=a$. Thus there are three different branches $z=P(w)$, $z=Q(w)$ and $z=R(w)$ of the inverse $f^{-1}$ of $w=f(z)$, which are regular at $w=a \in K$ and satisfy $p=P(a)$, $q=Q(a)$ and $r=R(a)$.

By Gross' star theorem (see e.g. [19]), we may continue $P(w)$, $Q(w)$ and $R(w)$ analytically to $\infty$ along almost any ray starting at $a$, in particular along some ray $L$ which meets $U$. Denote by $\gamma$ the segment of $L$ from $a$ to a certain point $b \in U$. Then as $w$ moves along $\gamma$ the functions $P(w)$, $Q(w)$ and $R(w)$ trace out curves $P(\gamma)$, $Q(\gamma)$ and $R(\gamma)$, which are disjoint and join $p \in V$ to $p'=P(b) \in U$, $q \in V$ to $q'=Q(b) \in U$ and $r \in V$ to $r'=R(b) \in U$, respectively.

Join $p$ to $q$ by a simple arc $\alpha \subset V$, $q$ to $r$ by $\beta \subset V$ and $r$ to $p$ by $\delta \subset V$. Also join $p'$ to $q'$ by a simple arc $\alpha' \subset U$, $q'$ to $r'$ by $\beta' \subset U$ and $r'$ to $p'$ by $\delta' \subset U$. Let $\bar{p}$ be the last intersection of $\alpha$ with $P(\gamma)$ and $\bar{q}$ be the first intersection with $Q(\gamma)$. Let $\bar{\alpha}$ be the subarc of $\alpha$ which joins $\bar{p}$ to $\bar{q}$. Similarly define $\bar{p}'$ as the last intersection of $\alpha'$ with $P(\gamma)$, $\bar{q}'$ as the first intersection with $Q(\gamma)$ and $\bar{\alpha}'$ as the subarc $\bar{p}'\bar{q}'$ of $\alpha'$. Denote by $\pi_1$ the subarc $\bar{p}\bar{p}'$ of $P(\gamma)$, by $\varkappa_1$ the subarc $\bar{q}\bar{q}'$ of $Q(\gamma)$. Then $\pi_1\bar{\alpha}'\varkappa_1^{-1}\bar{\alpha}^{-1}$ is a Jordan curve $C_1$. In the same way we can obtain Jordan arcs $C_2 = \pi_2\bar{\beta}'\varkappa_2^{-1}\bar{\beta}^{-1} \subset Q(\gamma) \cup \beta' \cup R(\gamma) \cup \beta$ and $C_3 = \pi_3\bar{\delta}'\varkappa_3^{-1}\bar{\delta}^{-1} \subset R(\gamma) \cup \delta' \cup P(\gamma) \cup \delta$, where $\pi_2, \bar{\beta}', \varkappa_2, \bar{\beta}, \pi_3, \bar{\delta}', \varkappa_3$ and $\bar{\delta}$ are subarcs of $Q(\gamma), \beta', R(\gamma), \beta, R(\gamma), \delta', P(\gamma)$ and $\delta$, respectively, as in the construction of $C_1$. Denote by $D_i$ the interior of $C_i$ ($i=1,2,3$). Since none of $P(\gamma)$, $Q(\gamma)$ and $R(\gamma)$ contains a pole of $f$ and $f$ has only one pole, we can see that there exists at least one number $j \in \{1,2,3\}$ such that $D_j$ contains no pole of $f$. Without loss of generality we assume that $D_1$ contains no pole of $f$. Then $D_1$ is mapped by $f$ into a bounded region $f(D_1)$ whose boundary is contained in $f(C_1) \subset f(\alpha) \cup f(\alpha') \cup \gamma$.

Now $f(\alpha)$ ($\subset K$) and $f(\alpha')$ ($\subset U$) are closed, bounded and disjoint curves passing through $a$ and $b$, respectively. Denote by $M$ the unbounded component of their complement. Since $U$ and $K$ are simply connected, $M$ contains $J(f)$. Thus $M$ meets $\gamma$, since $J(f)$ does. Now $f(\pi_1)$ is a segment of $\gamma$ which joins $f(\alpha)$ to $f(\alpha')$. If $t$ is the last point of intersection of $\gamma$ with $f(\alpha)$ and $t'$ the first intersection with $f(\alpha')$, then the segment $tt'$ of $\gamma$ is a cross-cut of $M$ whose ends belong to different components of the boundary of $M$. Thus $tt'$ does not disconnect $M$. Since $tt'$ belongs to $f(\pi_1)$ every point of $tt'$ is a boundary value of $f(D_1)$. Thus $f(D_1)$ must contain the whole of $M\backslash tt'$, i.e. an unbounded set. This contradicts the boundedness of $D_1$ and the result is proved. $\square$

The following result is a generalization of Gross' star theorem and can be found in Stallard [22, Lemma 2.11].

**Lemma 2.3.** *If $R$ is a branch, analytic at $z_0$, of the inverse of a function $g$ that is meromorphic in $\mathbf{C}$ or in $\mathbf{C}\backslash\{0\}$ then $R$ can be continued analytically along almost every ray from $z_0$ to $\infty$.*

**Lemma 2.4.** ([11, p. 108, Proposition 4.6]) *Let $f$ be a continuous map of a topological space $X$ onto itself, and suppose that $X$ has only a finite number of components. Then for some integer $m$, each component is completely invariant under $f^m$.*

**Lemma 2.5.** *Let $f$ be a transcendental analytic self-map of $\mathbf{C}^*$ not in class* (b). *If $U$ is a completely invariant component of $F(f)$, then*

(i) *$U$ is unbounded*;

(ii) *for any neighborhood $D$ of zero, $D\cap U\neq\emptyset$, hence $0\in\partial U$*;

(iii) *$\partial U=J(f)$ in $\widehat{\mathbf{C}}$*;

(iv) *all other components of $F(f)$ are simply connected.*

*Remark.* In Lemma 2.2 we have shown that (i), (iii) and (iv) in Lemma 2.5 are true when $f$ is a meromorphic map, however, since 0 and $\infty$ are essential singularities of $f^2$ for a map $f$ of the form (c) or (d), the proofs of the meromorphic case apply to the transcendental analytic self-map of $\mathbf{C}^*$ in the classes (c) and (d) without further difficulties. We omit the proof.

By Lemma 2.5(iv), we can immediately obtain the following corollary.

**Corollary 3.** *Let $f$ be a transcendental analytic self-map of $\mathbf{C}^*$ not in class* (b). *If $F(f)$ has two or more completely invariant components, then all components of $F(f)$ are simply connected.*

**Lemma 2.6.** ([2]) *If $f$ is a transcendental entire map, then $F(f)$ has at most one completely invariant component.*

*Proof of Theorem* 4. We distinguish between two cases.

(I). Suppose that $f$ has the form (b).

In this case 0 is a removable singularity for $f$. Let $f(0)=0$. Then $f$ is extended to the complex plane as a transcendental entire map, denoted by $f_1$. It follows from Lemma 2.6 that the Fatou set of $f_1$ has at most one completely invariant component. Since the normality is a local property, $F(f)$ and $F(f_1)$ are the same except possibly at 0, and $J(f)$ and $J(f_1)$ are also the same except possibly at 0. Since $f(z)=f_1(z)$ for all $z\neq 0$, we see that $F(f)$ also has at most one completely invariant component.

(II). Let $f$ be a map in the class (c) or (d). Suppose on the contrary that $F(f)$ has two mutually disjoint completely invariant components $U$ and $V$. Then by Lemma 2.5 and Corollary 3, $U$ and $V$ are simply connected and unbounded.

Take a value $a$ in $V$ such that $f(z)=a$ has infinitely many simple roots ($f'(z)=0$ at only countably many $z$ so we have to avoid only countably many choices of $a$), and take three distinct points $p,q,r\in V$ with $f'(p)\neq 0$, $f'(q)\neq 0$ and $f'(r)\neq 0$ such that $f(p)=f(q)=f(r)=a$. Thus there are three different branches $z=P(w)$, $z=Q(w)$ and $z=R(w)$ of the inverse $f^{-1}$ of $w=f(z)$, which are regular at $w=a\in V$ and satisfy $p=P(a)$, $q=Q(a)$ and $r=R(a)$.

By Lemma 2.3, we may continue $P(w)$, $Q(w)$ and $R(w)$ analytically to $\infty$ along almost any ray starting at $a$, in particular along some ray $L$ which meets $U$. Denote by $\gamma$ the segment of $L$ from $a$ to a certain point $b\in U$. Then as $w$ moves along $\gamma$ the functions $P(w)$, $Q(w)$ and $R(w)$ trace out curves $P(\gamma)$, $Q(\gamma)$ and $R(\gamma)$, which are disjoint and join $p\in V$ to $p'=P(b)\in U$, $q\in V$ to $q'=Q(b)\in U$ and $r\in V$ to $r'=R(b)\in U$, respectively. Following the same deduction as in the proof of Theorem 3(ii) we can obtain a contradiction. Thus $f$ has at most one completely invariant component and Theorem 4 is proved. □

*Proof of Corollary* 1. Suppose that $F(f)$ has only finitely many components $U_1, \dots, U_k$. For the transcendental analytic map $f$ of $\mathbf{C}^*$ to itself, $f(F(f))=F(f)$, then by Lemma 2.4, each $U_j$ is completely invariant under some iterate $f^d$. But $f^d$ is a transcendental analytic map of $\mathbf{C}^*$ to itself, and so it follows from Theorem 4 that $f^d$ has at most one completely invariant component. So we deduce that $k=1$ and the proof is complete. □

## 3. Completely invariant domains for $f\in S$

Next we prove Theorem 1.

**Lemma 3.1.** ([9]) *Suppose that $f$ is a transcendental meromorphic map, $f\in S$ and that $F(f)$ has a simply connected completely invariant component $U_0$. Then $\infty$ is an accessible point of $\partial U_0$.*

*Proof of Theorem* 1. Since by Lemma 2.6 and Theorem 4, $F(f)$ has at most one completely invariant component when $f$ is transcendental entire or $f\in\mathbf{P}$, and the result is known for rational functions (see, for example, [11, Theorem 9.4.3]), we only need to consider the case $f\in\mathbf{M}$. It follows from Lemma 2.2 and Corollary 2 that $\partial V_1=\partial V_2=J(f)$ and all components of $F(f)$ are simply connected.

Suppose that $F(f)$ has another component $U$, $U\neq V_1$, $U\neq V_2$. Then $U$ is simply connected and $\partial U\subset J(f)=O^-(\infty)'$. Let $z_1,z_2\in\partial U$, $z_1\neq z_2$. Then $z_1,z_2\in J(f)=\partial V_1=\partial V_2$ and we can choose two neighborhoods $D_1$ and $D_2$, $z_1\in D_1$, $z_2\in D_2$, such that $D_1\cap D_2=\emptyset$. Then there are four points $a_1,a_2\in D_1$ and $b_1,b_2\in D_2$ such that $a_1,b_1\in V_1$ and $a_2,b_2\in V_2$. We join $a_1$ to $b_1$ in $V_1$ by a Jordan arc $\delta_1$, and $a_2$ to $b_2$ in $V_2$ by a Jordan arc $\delta_2$. We also join $a_1$ to $z_1$, $a_2$ to $z_1$ in $D_1$, $b_1$ to $z_2$, $b_2$ to $z_2$ in $D_2$ by Jordan arcs $\sigma_1$, $\theta_1$, $\sigma_2$ and $\theta_2$, respectively, such that $\Lambda=\sigma_1\cup\delta_1\cup\sigma_2\cup\theta_2\cup\delta_2\cup\theta_1$ forms a Jordan curve in $\mathbf{C}$. The curve $\Lambda$ separates $\widehat{\mathbf{C}}\backslash\Lambda$ into two components $N_1$ and $N_2$. Let $N_1$ be the bounded component in $\mathbf{C}$. Take any points $q$ on $\delta_1$, $q\neq a_1$, $q\neq b_1$, and $r$ on $\delta_2$, $r\neq a_2$, $r\neq b_2$. Join $q$ and $r$ in $N_1$ by a cross-cut $\eta$. Then $\eta$ goes from $V_1$ to $V_2$, and hence must meet $J(f)$. Let $z_0\in\eta\cap J(f)$. Then $N_1$ is a neighborhood of $z_0$ and contains a point $p\in O^-(\infty)$. Thus $p$ is a pole of $f^k$ for some positive integer $k$. By Lemma 3.1 there is a curve $\gamma$ in $V_1$ such that $\gamma\to\infty$. Thus there is an image $\gamma'=f^{-k}(\gamma)$ which tends to $p$ and lies in $V_1$, i.e. $p$ is accessible in $V_1$ along $\gamma'$. We can therefore find a cross-cut $\Gamma_1$ of $V_1$ which has two ends at $p$ and $\infty$, and meets $\Lambda$ only at $q$, for $\delta_1$ is in the domain $V_1$ and $q\in\delta_1$.

Similarly we can find a cross-cut $\Gamma_2$ of $V_2$ which has two ends at $p$ and $\infty$, and meets $\Lambda$ only at $r$. Then $\Gamma=\Gamma_1\cup\Gamma_2$ forms a Jordan curve in $\widehat{\mathbf{C}}$, $\Gamma$ separates $\widehat{\mathbf{C}}\backslash\Gamma$ into two components $E_1$ and $E_2$, and $\Lambda\backslash\Gamma$ separates into two Jordan arcs $\mu_1$ and $\mu_2$ which both have deleted ends $q$ and $r$. Suppose $z_1\in\mu_1$. If $z_2\in\mu_1$, then as $\mu_1=(\delta_1\cap\mu_1)\cup\sigma_1\cup\theta_1\cup(\delta_2\cap\mu_1)$ or $\mu_1=(\delta_1\cap\mu_1)\cup\sigma_2\cup\theta_2\cup(\delta_2\cap\mu_1)$ and $z_1,z_2\in J(f)$, we have $z_1,z_2\in\sigma_1\cup\theta_1\subset D_1$ or $z_1,z_2\in\sigma_2\cup\theta_2\subset D_2$, i.e. $D_1\cap D_2\neq\emptyset$, which is a contradiction. Therefore, if $z_1\in\mu_1$, then $z_2\in\mu_2$. Since $\Gamma$ meets $\Lambda$ only at $q$ and $r$, we have $\mu_i\cap\Gamma=\emptyset$ $(i=1,2)$. It follows from the connectivity of $\mu_1$ and $\mu_2$ that $\mu_i\subset E_1$ or $\mu_i\subset E_2$ $(i=1,2)$. If $\mu_1$ and $\mu_2$ are in the same component, say $E_1$, then $N_1\subset E_1$ as $N_1$ is bounded in $\mathbf{C}$ and $E_1$ and $E_2$ are both unbounded in $\mathbf{C}$. Thus $p\in E_1$, a contradiction. Hence $\mu_1$ and $\mu_2$ are in the different components $E_1$ and $E_2$. Since $z_1\in\mu_1$ and $z_2\in\mu_2$, the points $z_1$ and $z_2$ are in the different components $E_1$ and $E_2$. Therefore by $z_i\in\partial U$ $(i=1,2)$, $U$ contains both points of $E_1$ and $E_2$, which contradicts the connectivity of $U$ since $\Gamma\cap U=\emptyset$. $\square$

## 4. Completely invariant domains of $f\in\mathbf{F}\cap\mathbf{E}$

In this section, we will prove Theorem 2. To this end, we need the following lemmas.

**Lemma 4.1.** ([21]) *If $f\in\mathbf{F}$, then $f$ does not have wandering domains.*

**Lemma 4.2.** ([21, Lemma 4.3]) *For a specified $K>1$, and a function $f$ in the class* $\mathbf{F}$, *let*

$$G_K(f)=\{f_\phi=\phi f\phi^{-1}:\phi \text{ is } K\text{-quasiconformal fixing } 0,\ 1,\ \infty,\ f_\phi \text{ is meromorphic}\}.$$

*Then the family $G_K(f)$ can be expressed uniquely in terms of a finite set of complex parameters $X_1, \dots, X_{n(K,f)}$.*

**Lemma 4.3.** *If $f\in\mathbf{F}$, then every periodic cycle of simply connected Baker domains of $f$ contains a singularity of $f^{-1}$.*

*Proof.* Let $U$ be a simply connected Baker domain of $f$ with period $p$ and suppose that $U, U_1, U_2, \dots, U_{p-1}$ do not contain singularities of $f^{-1}$, where $U_n$ ($n\in\mathbf{N}$) is a component of $F(f)$ containing $f^n(U)$. It follows that $U_n$ is simply connected and that $f|_{U_n}$ is univalent for all $n$. As observed by Herman [15, p. 609], this implies that the space of quasiconformal deformations of $f$ is infinite dimensional. But by Lemma 4.2, $G_K(f)$, the quasiconformal deformation family of $f$, depends only on finitely many parameters, which is a contradiction. □

The following result is due to Eremenko and Lyubich [13, Lemma 11].

**Lemma 4.4.** *Let $f$ be a transcendental entire function. If $F(f)$ has a completely invariant component $U$, then all the critical values and logarithmic singularities of $f^{-1}$ are contained in $U$.*

We denote the set of all singularities of $f^{-1}$ by $\operatorname{sing} f^{-1}$ and define

$$P(f)=\overline{\bigcup_{n=0}^{\infty} f^n(\operatorname{sing} f^{-1})}.$$

**Lemma 4.5.** ([12, Theorem 7]) *Let $f$ be a meromorphic map, and let $G=\{U_0, U_1, \dots, U_{p-1}\}$ be a periodic cycle of components of $F(f)$.*

(i) *If $G$ is a cycle of immediate attractive basins or Leau domains, then we have $U_j\cap\operatorname{sing} f^{-1}\neq\emptyset$ for some $j\in\{0,1,\dots,p-1\}$.*

(ii) *If $G$ is a cycle of Siegel discs or Herman rings, then $\partial U_j\subset P(f)$ for all $j\in\{0,1,\dots,p-1\}$.*

*Proof of Theorem* 2. At first, since $f\in\mathbf{E}$ and by Lemma 2.2(i), $U$ is unbounded. We see from [4, Theorem 3.1] that all components of $F(f)$ are simply connected. By Lemma 4.1, $f$ has no wandering domains. Thus every component of $F(f)$ is (pre)periodic. Now suppose that $D$ is a periodic component of $F(f)$ with $D\neq U$. Since $f\in\mathbf{F}$, it follows from Stallard [21, pp. 218–219] that there are no transcendental singularities of $f^{-1}$. Thus by Lemma 4.4, all singularities of $f^{-1}$ are contained in $U$. It follows from Lemmas 4.3 and 4.5 that $D$ can only be a Siegel disc. On the other hand, since $f$ is transcendental entire and $f^{-1}(U)\subset U$, we see that $f|_U$ cannot be a univalent map, then $U$ is neither a Siegel disc nor a Herman ring. Thus the set $\bigcup_{n\geq 0} f^n(\operatorname{sing} f^{-1})$ has only one limit point (possibly $\infty$). Consequently $D$ cannot be a Siegel disc in view of Lemma 4.5(ii). Thus $U$ is the only periodic component of $F(f)$.

If $F(f)$ has a preperiodic component $V$, then there exists a positive integer $n$ such that $f^n(V)$ is periodic. Thus $f^n(V)\subset U$. However, $U$ is completely invariant, hence $V=U$.

We have proved that $F(f)$ has only one component $U$ so that $F(f)=U$. □

## 5. Julia sets as Jordan arcs

Finally, we prove Theorem 5. We begin with some lemmas.

**Lemma 5.1.** ([22, Theorem A]) *Let $f$ be a meromorphic map which is not rational of degree less than two. If $J(f)$ contains a free Jordan arc, then $J(f)$ is a Jordan arc or a Jordan curve.*

**Lemma 5.2.** ([22, Lemma 3.1]) *If $f$ is a map in class* **E** *or* **P** *then $J(f)$ cannot contain a free Jordan arc.*

**Lemma 5.3.** ([22, Lemma 4.1]) *Suppose that $f$ is a map in class* **M** *and that $J(f)$ is a Jordan arc with precisely one finite endpoint $a$. Put $P(z)=z^2+a$. For some $z_1$ such that $fP(z_1)=\alpha\neq a,\infty$, take a fixed branch of $P^{-1}(w)=(w-a)^{1/2}$ at $w=\alpha$. Then $F=P^{-1}fP$ continues analytically to a function in class* **M** *and $J(F)$ is a Jordan curve.*

*Proof of Theorem* 5. Since $f$ is transcendental meromorphic, $J(f)$ must be unbounded. It follows from Lemma 5.1 that $J(f)$ must be one of the following cases:

(I) a Jordan curve containing $\infty$;

(II) a Jordan arc with precisely one finite endpoint $a$;

(III) a Jordan arc passing through $\infty$ and with both endpoints finite;

Thus we need only prove that Cases I and II are impossible.

In Case I, since $J(f)$ must pass through $\infty$, $F(f)$ has precisely two components, $U_1$ and $U_2$, both of which are simply connected. We have either

(IA) $f(U_1)\subset U_1$ and $f(U_2)\subset U_2$, or

(IB) $f(U_1)\subset U_2$ and $f(U_2)\subset U_1$.

In Case IA, we also have $f^{-1}(U_1)\subset U_1$ and $f^{-1}(U_2)\subset U_2$, that is, $U_1$ and $U_2$ are completely invariant components of $F(f)$. Suppose $f(z)$ has $n$ poles (when $f(z)$

is entire, we let $n=0$). Since $f$ is transcendental meromorphic, we can take a point $a\in U_1$ which is neither a Picard exceptional value nor a critical value of $f(z)$. Since $U_1$ is completely invariant and $f^{-1}(a)$ is an infinite set, $f^{-1}(a)\subset U_1$, and we can take $n+2$ branches $g_k(z)$ $(k=1,\dots,n+2)$ of the inverse function of $f(z)$ which are regular at $a$ and satisfy $g_i(a)\neq g_j(a)$, $f'(g_i(a))\neq 0$ $(i,j=1,\dots,n+2, i\neq j)$. By Gross' star theorem one can continue $g_k(z)$ $(k=1,\dots,n+2)$ analytically to infinity along almost all rays emanating from $a$. We can therefore pick such a ray $L$ which meets $U_2$. Denote by $\gamma$ the segment of $L$ joining $a$ to a certain point $b$ in $U_2$ and directed from $a$ to $b$. Then as $z$ moves along $\gamma$ the functions $g_k(z)$ $(k=1,\dots,n+2)$ trace out curves $g_k(\gamma)$ $(k=1,\dots,n+2)$, which are disjoint, for none of $g_k(z)$ $(k=1,\dots,n+2)$ has a singularity on $\gamma$. Thus all $g_k(\gamma)$ $(k=1,\dots,n+2)$ intersect the boundaries of $U_1$ and $U_2$. If $g_k(\gamma)$ is oriented from $g_k(a)$ to $g_k(b)$, let $t_k$ denote its first intersection with $\partial U_1=J(f)$ $(k=1,2,\dots,n+2)$. Then there exist at least $n+1$ mutually disjoint open subarcs of $J(f)$, each of which has one deleted endpoint at $t_i$ and the other at $t_j$ $(i\neq j)$, since $J(f)$ is a Jordan curve. Now that $f(z)$ has $n$ poles and they are all on $J(f)$, we can see that among these arcs, there is an arc that contains no poles of $f$. We denote it by $\eta$ and its deleted endpoints by $t$ and $t'$. Without loss of generality we can suppose that $t\in g_1(\gamma)$ and $t'\in g_2(\gamma)$. Thus $g_1(\gamma)$ joins $u_1=g_1(a)$ in $U_1$ to $u_2=g_1(b)$ in $U_2$ and similarly $g_2(\gamma)$ joins $v_1=g_2(a)$ in $U_1$ to $v_2=g_2(b)$ in $U_2$. Now we join $u_1$ to $v_1$ by a simple arc $\beta_1\subset U_1$ and join $u_2$ to $v_2$ by a simple arc $\beta_2\subset U_2$. For $i=1,2$, if $\beta_i$ is oriented from $u_i$ to $v_i$, let $u_i'$ denote its last intersection with $g_1(\gamma)$ and $v_i'$ its first intersection with $g_2(\gamma)$. Let $\beta_i'$ denote the subarc of $\beta_i$, whose endpoints are $u_i'$ and $v_i'$, oriented from $u_i'$ to $v_i'$ and let $\pi$ and $\varkappa$ denote the arcs $u_1'u_2'$ and $v_1'v_2'$ of $g_1(\gamma)$ and $g_2(\gamma)$, respectively, oriented from $u_1'$ to $u_2'$ and from $v_1'$ to $v_2'$. Then $\pi\beta_2'\varkappa^{-1}(\beta_1')^{-1}$ is a simple closed curve. Denote this curve by $\Gamma$, and the interior of $\Gamma$ by $D$. Now that $\overline{D}$ contains no poles of $f$ according to our choices of $g_1(z)$ and $g_2(z)$. Hence $f(z)$ is analytic in $\overline{D}$ and hence $f(D)$ is a bounded region. Moreover the boundary of $f(D)$ is contained in $f(\Gamma)$ and hence in $\gamma\cup f(\beta_1)\cup f(\beta_2)$.

For $i=1,2$, the curve $f(\beta_i)$ is closed, bounded and lies in $U_i$. Since $U_1$ and $U_2$ are unbounded and simply connected, it follows that $f(\beta_1)$ and $f(\beta_2)$ are mutually disjoint and exterior to one another. Consider the unbounded component $H$ of the complement of $f(\beta_1)\cup f(\beta_2)$. The component $H$ meets $\gamma$ and in fact if $r$ is the last point of intersection of $\gamma$ with $f(\beta_1)$ and $s$ the first point of intersection of $\gamma$ with $f(\beta_2)$, then the segment $rs$ of $\gamma$ is a cross-cut of $H$ whose endpoints belong to different components of the boundary of $H$. It follows that $rs$ does not disconnect $H$. Now in fact a point $w$ of $rs$ $(\neq r,s)$ is the image $f(z)$ of an interior point $z$ in the arc $\pi$ of $\Gamma$. In the neighborhood of $z$ and inside $\Gamma$ the function $f(z)$ take an open set of values near $w$, some of which lie off $\gamma$ and in $H\backslash rs$. Then since the boundary of $f(D)$ is contained in $\gamma\cup f(\beta_1)\cup f(\beta_2)$, we see that $f(D)$ must contain the whole of $H\backslash rs$. But this contradicts the boundedness of $f(D)$.

In Case IB, we have $f^{-1}(U_1)\subset U_2$ and $f^{-1}(U_2)\subset U_1$. As in Case IA, we can take a point $a\in U_1$ which is neither a Picard exceptional value nor a critical value of $f(z)$. Since $f^{-1}(U_1)\subset U_2$, we have $f^{-1}(a)\subset U_2$. Following the same deduction as in Case IA, just substituting $U_1$ by $U_2$, and $U_2$ by $U_1$, we also obtain a contradiction. Hence $J(f)$ cannot be a Jordan curve as described in Case I.

In Case II, $J(f)$ is a Jordan arc with one end at $\infty$ and one finite endpoint $a$. Let $P(z)=z^2+a$. For some $z_0$ such that $fP(z_0)=\alpha\neq a,\infty$, take a fixed branch of $P^{-1}(w)=(w-a)^{1/2}$ at $w=\alpha$. We consider the function $h=P^{-1}fP$. Since by Lemma 5.2, $f\in\mathbf{M}$, it follows from Lemma 5.3 that $h$ continues analytically to a function in class $\mathbf{M}$ and $J(h)$ is a Jordan curve. We also see that $h$ has only finitely many poles. Thus $J(h)$ is a curve as described in Case I, which is impossible.

Therefore $J(f)$ must be in Case III, i.e. $J(f)$ is a Jordan arc passing through $\infty$ and both endpoints of $J(f)$ are finite. $\square$

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