Bounded holomorphic functions of several variables

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Introduction

A characteristic property of holomorphic functions, in one as well as in several variables, is that they are about as "rigid" as one can demand, without being identically constant. An example of this rigidity is the fact that every holomorphic function element, or germ of a holomorphic function, has associated with it a unique domain, the maximal domain to which the function can be continued. Usually this domain is no longer in euclidean space (a "schlicht" domain), but lies over euclidean space as a many-sheeted Riemann domain.

It is now natural to ask whether, given a domain, there is a holomorphic function for which it is the domain of existence, and for domains in the complex plane this is always the case. This is a consequence of the Weierstrass product theorem (cf. [13] p. 15). In higher dimensions, however, the situation is different, and the domains of existence, usually called domains of holomorphy, form a proper subclass of the class of all domains, which can be characterised in various ways (holomorphic convexity, pseudoconvexity etc.). To obtain a complete theory it is also in this case necessary to consider many-sheeted domains, since it may well happen that the maximal domain to which all functions in a given domain can be continued is no longer "schlicht".

It is possible to go further than this, and ask for quantitative refinements of various kinds, such as: is every domain of holomorphy the domain of existence of a function which satisfies some given growth condition? Certain results in this direction have been obtained (cf. for example [17] and [19]).

An extreme case is to consider the bounded functions, and now a new situation arises. It is for example no longer true that every domain in the complex plane is the domain of definition of a bounded function; one only has to consider the punctured disc and use the Riemann removable singularity theorem. A similar phenomenon occurs in higher dimensions, wich can be seem follows.

Consider a domain of holomorphy and a function which is holomorphic and not identically zero in the domain, but which has zeros there, nevertheless. Then the set of points in the domain where the function is different from zero is also a domain of holomorphy (it can easily be seen to be holomorphically convex). One can now apply the extension of Riemann's theorem (cf. [11] p. 19) to conclude that there is no bounded function for which this new domain of holomorphy is the domain of existence.

The problem now arises to characterise those domains which are the domains of existence of bounded, holomorphic functions, a problem where little progress has been made so far. A major difficulty is that the condition of boundedness is a global condition, so that the usual methods of patching together local elements to get a global function (e.g. the use of cohomology theory for coherent sheaves) do not seem to apply.

The present work has its origin in attempts to obtain the desired characterisation using adaptations of the criteria used in the unbounded case. The complete characterisation has proved elusive, but certain partial results have been obtained. Furthermore, the classes of domains studied here are important in other contexts (such as the Serre problem concerning holomorphic fibre bundles, and the classification of pseudoconvex domains), so that the results relating these classes can be of interest.

Now for a sketch of the contents of the various sections.

Section 1 starts with a short description of the construction of general envelopes of holomorphy. Then follow basic definitions and criteria, and finally examples are given of domains which are domains of existence of a bounded holomorphic function.

Section 2 is concerned with the bounded analogue of holomorphic convexity. It turns out that one gets two different kinds of convexity, and the relation between them is studied, as well as their relation to the domains of existence of bounded holomorphic functions.

Section 3 deals with what can be seen as a generalisation of one of the concepts of bounded convexity, namely completeness with respect to the Carathéodory metric. The relationship between this concept and the previously introduced ones is given.

Section 4, finally, is concerned with a characterisation using a bounded plurisubharmonic function, namely hyperconvexity. It is shown, among other things, that strongly complete domains are hyperconvex, and a certain consequence of this result is stated.

Some words about notation: $\mathcal{O}(D)$ will denote the space of holomorphic functions and $H^\infty(D)$ the space of bounded holomorphic functions on a domain D. By $\|f\|_D$ we denote the supremum of the function f on D.

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1. Domains of bounded holomorphy

As mentioned in the introduction, it is usually necessary to consider domains lying over $\mathbf{C}^n$ and consisting of several sheets when one is looking for the maximal domain to which a holomorphic function (or class of such functions) can be continued. The precise definition is the following. (For fundamental facts on envelopes of holomorphy, see [18].)

Definition 1.1. *Let D be a domain (a non-empty, connected open set) in $\mathbf{C}^n$, let X be a connected Hausdorff space and let $\pi\colon X\to D$ be a local homeomorphism (i.e. every point in X has an open neighbourhood U such that $\pi(U)$ is open in D and π restricted to U is a homeomorphism onto $\pi(U)$). Then (X, π, D) is said to be a Riemann domain over D.*

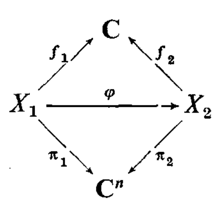
Note that according to the Poincaré—Volterra theorem ([18] p. 25) a Riemann domain has a second countable topology. Furthermore, the map π can be used to endow X with the structure of a complex manifold, and consequently one can talk meaningfully about holomorphic functions on X, and also about holomorphic maps between Riemann domains.

Definition 1.2. *Let $(X_1, \pi_1, \mathbf{C}^n)$ and $(X_2, \pi_2, \mathbf{C}^n)$ be Riemann domains, and let f_1 be a holomorphic function on X_1. A holomorphic function f_2 on X_2 is said to be the analytic continuation of f_1 to X_2 if there is a holomorphic map $\varphi\colon X_1\to X_2$ such that*

i) $\pi_1=\pi_2\circ\varphi$

ii) $f_1=f_2\circ\varphi$.

Graphically i) and ii) can be expressed by saying that the following diagram commutes:



Note that according to the principle of analytic continuation f_2 is uniquely determined by f_1.

One way to obtain the maximal Riemann domain associated with a given holomorphic function is to consider the connected component in the sheaf of germs of holomorphic functions on $\mathbf{C}^n$ which contains a germ of the given function. In this manner one gets the following result (cf. [18] p. 28).

Theorem 1.3. *Given a holomorphic function f_1 on a Riemann domain $(X_1, \pi_1, \mathbf{C}^n)$, there is a Riemann domain $(X_2, \pi_2, \mathbf{C}^n)$ and a continuation f_2 of f_1 to X_2 with the following property.*

For any Riemann domain $(X', \pi', \mathbf{C}^n)$ and continuation f' of f_1 to X' there is a holomorphic map $\varphi\colon X' \to X_2$ such that $\pi_2 \circ \varphi = \pi'$ and $f_2 \circ \varphi = f'$.

The construction given above can be generalised to the case of simultaneous continuation of all holomorphic functions in any given subset $S \subset \mathcal{O}(X)$. One can conceive of the resulting Riemann domain as the intersection of the maximal domains for the individual functions in S. See [20] for an elegant treatment along those lines. The appropriate definition is the following.

Definition 1.4. *Let $(X_1, \pi_1, \mathbf{C}^n)$ be a Riemann domain and let $S \subset \mathcal{O}(X_1)$. Then the Riemann domain $(X_2, \pi_2, \mathbf{C}^n)$ is an S-envelope of holomorphy of X_1 if the following is true.*

i) *There is a holomorphic map $\varphi\colon X_1 \to X_2$ such that*

a) $\pi_2 \circ \varphi = \pi_1$

b) for every $f_1 \in S$ there is a function $f_2 \in \mathcal{O}(X_2)$ such that $f_2 \circ \varphi = f_1$.

ii) *For every Riemann domain $(X', \pi', \mathbf{C}^n)$ which satisfies* i) *with $\varphi'\colon X_1 \to X'$, there is a holomorphic map $\tau\colon X' \to X_2$ such that*

a) $\pi' = \pi_2 \circ \tau$

b) $\varphi = \tau \circ \varphi'$

c) If f_2 and f' are the continuations to X_2 and X' of f_1, then $f' = f_2 \circ \tau$.

The first part of the definition states that the functions in S can be continued to X_2 in the sense of Definition 1.2, the second part says that if the functions in S can be continued to X', then their continuations can be continued to X_2. If one introduces a preordering on the set of Riemann domains which satisfy i) by defining $X \prec Y$ if the continuations of the functions in S to X can be continued to Y, then ii) asserts that X_2 is a maximal element with respect to this preorder.

The S-envelope is unique up to isomorphism (if it exists).

The following theorem on the existence of S-envelopes was first demonstrated by P. Thullen in 1932. (See [18], p. 91 for a proof.)

Theorem 1.5. *Let $(X, \pi, \mathbf{C}^n)$ be a Riemann domain. For every $S \subset \mathcal{O}(X)$ there is an S-envelope of holomorphy of X.*

In the sequel the case $S = H^\infty(X)$ will be treated. The H^∞-envelope of X will be denoted by $E_\infty(X)$.

Lemma 1.6. *Suppose that all functions in $H^\infty(X)$ can be continued to Y, let $f_1 \in H^\infty(X)$, and let f_2 be the extension of f_1 to Y. Then $\|f_2\|_Y = \|f_1\|_X$.*

Proof. Take $\alpha \in \mathbf{C} \setminus \overline{f_1(X)}$. Then it is clear that $(\alpha - f_1)^{-1} \in H^\infty(X)$ and thus can be continued to Y. By uniqueness of analytic continuation this continuation has to be $(\alpha - f_2)^{-1}$, and since this function is holomorphic in Y it follows that $\alpha \notin f_2(Y)$. Letting α vary, we get $f_2(Y) \subset \overline{f_1(X)}$, and in particular it follows that $\|f_2\|_Y \leqq \|f_1\|_X$.

Remark. If X lies over a bounded domain in $\mathbf{C}^n$, then the lemma shows that the same is true for $E_\infty(X)$, since we can apply it to the coordinate functions on X.

Definition 1.7. *A Riemann domain $(X, \pi, \mathbf{C}^n)$ is a domain of bounded holomorphy (an H^∞-domain for short) if the bounded holomorphic functions separate the points on X and the map of X into its H^∞-envelope is an isomorphism.*

Remark. The statement that the bounded holomorphic functions separate the points on X means that for every pair p, q of points in X there is an $f \in H^\infty(X)$ such that $f(p) \neq f(q)$. This assumption is included in order to avoid certain unpleasant features such as including $\mathbf{C}^n$ among the H^∞-domains. This would be unnatural since the impossibility to continue the bounded holomorphic functions on $\mathbf{C}^n$, i.e. the constants, is a consequence, not of the character of the functions but of the set-theoretic properties of the domain. In the case of bounded domains in $\mathbf{C}^n$ it is of course always true that bounded holomorphic functions separate points.

It is necessary to have some criterion by which one can conclude that a domain is of bounded holomorphy. In order to find such a criterion a description will be given of how to construct a new "larger" Riemann domain to any given one.

The *distance function* on a Riemann domain $(X, \pi, \mathbf{C}^n)$ is defined as follows. If $p \in X$, then $d(p, \complement X)$ is defined as the supremum of the set of real numbers r such that there is a neighbourhood of p which is mapped homeomorphically by π onto a polydisc around $\pi(p)$ with all radii r. One can think of $d(p, \complement X)$ as the distance of p to the "boundary" of X, and since X is connected d is finite unless $X = \mathbf{C}^n$. In case d is finite it is clearly continuous, and in case $X = \mathbf{C}^n$ we already know that X is not an H^∞-domain.

Let $p \in X$, and let D be the polydisc in $\mathbf{C}^n$ with centre at $\pi(p)$ and all radii $2d(p, \complement X)$. One can now construct a new Riemann domain as follows.

Consider first the disjoint union of X and D. Then perform a partial "gluing together" via an equivalence relation in the following way. If $z \in D$ and $q \in X$ we identify them if

a) $\pi(q) = z$,

b) there is a curve in $\pi^{-1}(D)$ joining q to p.

Furthermore every point is identified with itself.

In this way a new Riemann domain is obtained, which will be denoted $X(p)$. Clearly one can think of X as a subdomain of $X(p)$.

The desired criterion can now be stated.

Theorem 1.8. *Let* $(X, \pi, \mathbf{C}^n)$ *be a Riemann domain such that the bounded holomorphic functions separate points on* X, *and such that there is a compact set* K *in* X *so that* $\{p\in X;$ *there is an* $f\in H^\infty(X)$ *which cannot be continued to* $X(p)\}$ *is dense in* $X\setminus K$. *Then* $(X, \pi, \mathbf{C}^n)$ *is an* H^∞*-domain, and there is a function* $f\in H^\infty(X)$ *which cannot be continued beyond* X.

Proof. Assume that X is not an H^∞-domain, and consider the map $\varphi\colon X\to E_\infty(X)$. It is easy to see that since the bounded functions separate points on X, φ is injective. Since it fails to be an isomorphism it also fails to be surjective (as a map between Riemann domains φ is a local homeomorphism), so there is a $q\in E_\infty(X)\setminus\varphi(X)$.

Take $p\in\varphi(X)$ and let γ: $[0, 1]\to E_\infty(X)$ be a continuous curve such that $\gamma(0)=p$ and $\gamma(1)=q$. Let $t_0=\inf\{t; \gamma(t)\notin\varphi(X)\}$.

Since the distance function is continuous and γ is compact, there is a real number $\alpha>0$ such that $d(\gamma(t), \complement E_\infty(X))\geqq\alpha$ for all t, $0\leqq t\leqq 1$. On the other hand, one can choose $t_1<t_0$ so near t_0 that $d(\gamma(t_1), \complement\varphi(X))<\varepsilon$, where ε is a given positive number. Using this it is clear that from the condition in the statement of the theorem one can deduce the existence of a point $r\in\varphi(X)$ such that

$$2d(r, \complement\varphi(x)) < d(r, \complement E_\infty(X)),$$

and such that there is a function $f\in H^\infty(\varphi(X))$ which cannot be continued to $\varphi(X)(r)$. But the inequality above shows that $\phi(X)(r)\subset E_\infty(X)$, and if we identify X with $\varphi(X)$ it is clear that every $f\in H^\infty(\varphi(X))$ can be continued to $E_\infty(X)$, so we have a contradiction.

In order to prove the existence of a function which cannot be continued beyond X, we consider the restriction maps

$$\tau_i\colon\ H^\infty(X(p_i)) \to H^\infty(X),$$

where $\{p_i\}$ is chosen as a denumerable dense subset of the dense subset of $X\setminus K$ mentioned in the statement of the theorem. These maps are linear continuous maps between Banach spaces ($H^\infty(X)$ with the supremum norm is a Banach space). The condition in the theorem implies that none of the maps is surjective. It is then a consequence of the open mapping theorem that all images are of the first category in $H^\infty(X)$ and the same is true of their union. The Baire category theorem finally gives the existence of a function

$$f\in H^\infty(X)\setminus \bigcup_i \tau_i H(X(p_i)).$$

The same kind of reasoning as in the first part of the proof shows that if we could extend f to a Riemann domain which contains X, then it must be possible to extend f as a (possibly unbounded) holomorphic function on some $X(p_i)$. But the denseness of the p_i shows that there must be a p_j so that f is bounded on $X(p_j)$, which is impossible. The theorem is proved.

In [21] a definition of "schlicht" H^∞-domains was given, which was a straightforward adaption of the definition of domains of holomorphy (cf. [13] p. 36) in $\mathbf{C}^n$. It is important to see how this definition is connected with the present one. First we recall the alternative definition.

Definition 1.9. *A domain Ω in $\mathbf{C}^n$ is an H_s^∞-domain (s for schlicht) if there are no open sets Ω_1 and Ω_2 in $\mathbf{C}^n$ such that*

i) $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$
ii) *Ω_2 is connected and not contained in Ω*
iii) *For every $f \in H^\infty(\Omega)$ there is an $f_2 \in H^\infty(\Omega_2)$ such that $f = f_2$ on Ω_1.*

Note that the above definition does not require the domain to have the property that bounded holomorphic functions separate points on it. It is clear that for example $\mathbf{C}^n$ is an H_s^∞-domain.

The relation between the two definitions is described in the following theorem.

Theorem 1.10. *Let Ω be a domain in $\mathbf{C}^n$. Then Ω is an H^∞-domain if and only if Ω is an H_s^∞-domain and bounded holomorphic functions separate points on Ω.*

Proof. Suppose first that Ω is not an H^∞-domain, but that bounded functions separate points. Theorem 1.8 shows the existence of a point $z \in \Omega$ such that all functions in $H^\infty(\Omega)$ can be continued to $\Omega(z)$. If we take Ω_1 to be the polydisc around z with radius $d(z, \complement \Omega)$ and Ω_2 the polydisct with radius $d(z, \complement \Omega)$, we have sets satisfying i)—iii) in Definition 1.9 and consequently Ω is not an H_s^∞-domain.

Conversely, assume that Ω is not an H_s^∞-domain. Then there are Ω_1 and Ω_2 with the properties i)—iii). If a new Riemann domain X is constructed by considering the disjoint union of Ω and Ω_2 and identifying the points in them which correspond to the points in the connected component (or components) of $\Omega \cap \Omega_2$ which contain Ω_1, then the functions in $H^\infty(\Omega)$ can be extended to X and it is now easy to see that Ω cannot be an H^∞-domain.

Corollary 1.11. *An H^∞-domain in $\mathbf{C}^n$ is a domain of holomorphy.*

This is of course true also for the Riemann domains which satisfy the conditions in Theorem 1.8.

In the examples in the introduction it was shown that certain domains of holomorphy in $\mathbf{C}^n$ failed to be H^∞-domains, by continuation of the bounded functions to the interior of the closure of the original domain. One may now ask whether a domain which is equal to the interior of its closure is in fact an H^∞-domain; we will call such a domain a *full* domain.

For domains in the complex plane the answer is that a bounded full domain is an H^∞-domain, and this is a consequence of the following theorem.

Theorem 1.12. *Let Ω be relatively compact in the Riemann domain $(X, \pi, \mathbf{C})$. If Ω is full then Ω is an H^∞-domain.*

Proof. To begin with we note that a Riemann domain over $\mathbf{C}$ is a Stein manifold (cf. [8] p. 182). Hence holomorphic functions separate points on X and since Ω is relatively compact in X, bounded holomorphic functions separate points on Ω.

Since Ω is relatively compact in X there is a real number $\alpha>0$ such that $d(p, \complement X)>\alpha$ for all p in Ω. It is now possible to find a compact set K in Ω such that for every point q in $\Omega\setminus K$ it is the case that $d(q, \complement \Omega)<\alpha/2$. For every q in $\Omega\setminus K$ one can therefore identify $\Omega(q)$ with a subdomain of X, containing points in $X\setminus\Omega$. Since Ω is full, every $\Omega(q)$ must contain a neighbourhood of an exterior point s to Ω. The Mittag—Leffler theorem (cf. [8] p. 181) can now be used to prove the existence of a meromorphic function on X with the only pole at s. This function is bounded on Ω and cannot be continued over $\Omega(q)$, so Theorem 1.8 shows that Ω is an H^∞-domain.

The situation is different in higher dimensions. This is a consequence of a remarkable example of Sibony [21]. This example is very useful to disprove the analogues for bounded holomorphic functions of several well known characterisations of domains of holomorphy and will occur frequently in the sequel, so we will give a sketch of the construction.

One starts by constructing a subharmonic function V defined in the unit disc and continuous there, satisfying $0\leqq V(z)<1$. The function V takes the value zero on a discrete infinite sequence in the unit disc with the property that every boundary point of the disc is a non-tangential limit of a subsequence.

One then defines the domain

$$M(V) = \{(z, w);\ |z| < 1,\ |w|e^{V(z)} < 1\}.$$

Since $|w|e^{V(z)}$ is plurisubharmonic in the unit bi-disc it follows that $M(V)$ is pseudoconvex and hence a domain of holomorphy. Furthermore, the continuity of V implies that $M(V)$ is a full domain.

If $g\in H^\infty(M(V))$ (one may assume that $|g|\leqq 1$ on $M(V)$), it is possible to develop g in a power series in w with coefficients $h_\nu(z)$ which are bounded and holomorphic in the unit disc. Using Cauchy's inequality one deduces that $|h_\nu|\leqq 1$ on the above mentioned discrete sequence. Making use of Fatou's theorem on the non-tangential limit of bounded holomorphic functions in the unit disc, one finds that $|h_\nu|\leqq 1$ in the disc and consequently the power series for g converges for $|w|<1$

and we get an extension of g to the whole of the bi-disc. Lemma 1.6 shows that the extended functions are still bounded on the bi-disc. It is easy, using power series expansions as mentioned and approximating the coefficients by polynomials, to show that $M(V)$ is a Runge domain, i.e. polynomials approximate the holomorphic functions in $M(V)$, uniformly on compact subsets.

In order to illustrate the usefulness of the criteria for H^∞-domains previously obtained, some examples will now be given (Cf. also [21] p. 210.) Let us first make the following definitions.

Definition 1.13. *Let* $(X, \pi, \mathbf{C}^n)$ *be a Riemann domain. A subdomain* Y *of* X *is said to have a Stein basis if the closure of* Y *is the intersection of a family of subdomains of* X, *which are domains of holomorphy, such that* Y *is relatively compact in each member of the family.*

Definition 1.14. *If* $(X, \pi, \mathbf{C}^n)$ *is a Riemann domain, then a subdomain* Y *of* X *is an analytic polyhedron in* X *if* Y *is a connected component of a set*

$$\{p; p\in X, \ |f_j(p)| < 1, \ j = 1, \ldots, m, \ f_j\in\mathcal{O}(X)\}$$

which is supposed to be relatively compact in X.

Theorem 1.15. *We have that*

i) *A Riemann domain which is full and has a Stein basis is an* H^∞*-domain.*

ii) *An analytic polyhedron in a domain of holomorphy is an* H^∞*-domain.*

iii) *If* $\{\Omega_\alpha\}_{\alpha\in A}$ *is a set of* H^∞*-domains in* $\mathbf{C}^n$, *then the interior of their intersection is an* H^∞*-domain.*

iv) *The Cartesian product of a finite number of* H^∞*-domains in* $\mathbf{C}^n$ *is an* H^∞*-domain.*

Proof. i) Let Y be a full Riemann domain with a Stein basis, contained in $(X, \pi, \mathbf{C}^n)$. By definition Y is relatively compact in a domain of holomorphy, so bounded functions separate points on Y. Since Y is relatively compact in X, there is a positive real number α such that $d(p, \complement X)\geqq\alpha$ for $p\in Y$. This means that there is a compact subset K of Y such that for $p\in Y\setminus K$, it is the case that $2d(p, \complement Y)<\alpha$, which implies that one can think of the domains $Y(p)$ as subdomains of X. It is clear that these $Y(p)$ contains points in $X\setminus Y$, and since Y is full they contain exterior points to Y. Since each domain in the Stein basis is the domain of existence of a holomorphic function (cf. [18] p. 114), which is bounded in Y since Y is relatively compact in the domains, it follows that for every $Y(p)$ with $p\in Y\setminus K$ there is a function in $H^\infty(Y)$ which cannot be continued to $Y(p)$. Theorem 1.8 now gives the result.

ii) Let Y be a component of $\{p\in X; \ |f_j(p)|<1, j=1, \ldots, m, f_j\in\mathcal{O}(X)\}$ where X is a domain of holomorphy. As in i) bounded functions separate points in Y. Also as in i) one can find K such that for $p\in Y\setminus K$, $Y(p)\subset X$. Every $Y(p)$ must contain a point q such that one of the f_j satisfies $|f_j(q)|=1$ there. Now a branch of the function $(1-f_j)^{1/2}$ can be defined in Y, and it is clearly bounded and impossible to continue over $Y(p)$. Theorem 1.8 again gives the result.

iii) Since bounded functions separate points on Ω_α for all $\alpha \in A$ they do so on their intersection, Ω. Assume now that there are Ω_1 and Ω_2 as in Definition 1.9, and choose Ω_α such that Ω_2 is not contained in Ω_α. It is then clear that Ω_1 and Ω_2, having the assumed properties, contradicts the fact that Ω_α is an H^∞-domain.

iv) It is sufficient to consider the case of two domains, D_1 and D_2. To begin with, it is clear that if we have two different points in $D_1 \times D_2$, their projection in one of the components must be different and one can then use a bounded function in that component to separate them.

If we have Ω_1 and Ω_2 as in Definition 1.9, then Ω_2 must intersect the boundary of $D_1 \times D_2$, and hence it must intersect either the boundary of D_1 or that of D_2. In either case there is a function in $H^\infty(D_1 \times D_2)$ which one cannot continue to Ω_2.

Further examples of H^∞-domains will be obtained in Sections 2 and 3, as byproducts of the results there.

Remark. It is known that an arbitrary domain of holomorphy can be exhausted by an increasing sequence of analytic polyhedra (cf. [4] p. 25), and so by an increasing sequence of H^∞-domains. Since, as we have seen above, not every domain of holomorphy is an H^∞-domain, we can draw the conclusion that not every union of an increasing sequence of H^∞-domains is an H^∞-domain. This is contrary to the case of domains of holomorphy, since (at least in $\mathbf{C}^n$) every increasing sequence of domains of holomorphy is again such a domain ("the theorem of Behnke—Stein", cf. [4] p. 130).

In [6] it is shown that a bounded domain of holomorphy in $\mathbf{C}^n$ with smooth real-analytic boundary has a Stein basis. In view of Theorem 1.15 and the fact that a domain with the described properties is necessarily full, it is an H^∞-domain.

2. Bounded convexity

Let $(X, \pi, \mathbf{C}^n)$ be a Riemann domain, and let $F \subset \mathcal{O}(X)$ be a family of holomorphic functions on X. Then X is said to be *F-convex* if, for every compact set K in X, the set

$$\hat{K}_F = \{p \in X;\ |f(p)| \leqq \|f\|_K \text{ for all } f \in F\}$$

is compact in X.

A classical theorem, due to H. Cartan and P. Thullen (cf. for example [18] p. 110), states that X is a domain of holomorphy if and only if it is $\mathcal{O}(X)$-convex (holomorphically convex).

It seems natural to ask whether this result is true with domain of holomorphy replaced by H^∞-domain and H^∞-convexity instead of holomorphic convexity, and this is indeed the case when X is a domain in the complex plane (cf. [1]). In higher dimensions, however, the result is not true in either direction.

To begin with, the set $M(V)$ in Sibony's construction is as mentioned a Runge domain. Hence, according to a well known theorem (cf. [13] p. 53) it is convex with respect to the polynomials, and so, being bounded, it is a fortiori H^∞-convex. As we saw it is not, however, an H^∞-domain.

Conversely, the "Hartogs' triangle"

$$\Delta = \{(z, w);\ |z| < |w| < 1\} \subset \mathbf{C}^2,$$

is easily seen, using Definition 1.9 and Theorem 1.10, to be an H^∞-domain. On the other hand it is not H^∞-convex, since if we consider the set

$$\Gamma = \{z = 0,\ |w| = 1/2\} \subset \Delta,$$

then every function in $H^\infty(\Delta)$, restricted to $\Delta \cap \{z=0\}$, can be continued over the origin, and hence they are all holomorphic in $\{|z|=0,\ |w|<1\}$. The maximum principle then shows that $|f(0,0)| \leqq \|f\|_\Gamma$ and this implies that $\hat{\Gamma}_{H^\infty}$ is not compact in Δ.

There is, however, another way to consider convexity with respect to the bounded holomorphic functions.

In the case of "ordinary" holomorphic convexity the following theorem is valid.

Theorem 2.1. *A Riemann domain $(X, \pi, \mathbf{C}^n)$ is holomorphically convex if and only if, for every infinite, discrete sequence in X there is a holomorphic function in X which is unbounded on the sequence.*

This result is true even for (not necessarily reduced) complex spaces, for a proof see [10] p. 118.

An analogous concept for the bounded case can now be introduced.

Definition 2.2. *A Riemann domain $(X, \pi, \mathbf{C}^n)$ is sequentially H^∞-convex if for every discrete sequence $(p_k)_{k\in\mathbf{N}}$ in X there is a function $f \in H^\infty(X)$, not identically constant, such that* $\sup_k |f(p_k)| = \|f\|_X$.

Remark. Making use of a conformal map of a disc onto itself, one realises that it is no restriction to assume that all functions f in the definition have a common zero in X.

It is now necessary to make clear the relation between the two types of H^∞-convexity. To begin with we have the following result.

Theorem 2.3. *A sequentially H^∞-convex Riemann domain $(X, \pi, \mathbf{C}^n)$, on which bounded holomorphic functions separate points, is an H^∞-domain.*

Proof. Assume that X is not an H^∞-domain. Theorem 1.8 shows that there must exist a point $p \in X$ such that all $f \in H^\infty(X)$ can be extended to $X(p)$, and from Lemma 1.6 we have that the extension F of f satisfies $\|F\|_{X(p)} = \|f\|_X$.

Take a sequence $(p_k)_{k\in\mathbf{N}}$, with p_k in the inverse image around p of a disc with centre $\pi(p)$ and radius $d(p, \complement X)$, without accumulation point in X, such that $p_k \to p_\infty \in X(p)$.

If X is sequentially H^∞-convex there is a non-constant $f \in H^\infty(X)$ such that $\sup_k |f(p_k)| = \|f\|_X = \|F\|_{X(p)}$, where F is the extension of f to $X(p)$. It follows that $|F(p_\infty)| = \|F\|_{X(p)}$, and hence F is identically constant, a contradiction.

The two concepts of bounded convexity can now be seen to be related in the following way.

Theorem 2.4. *A sequentially H^∞-convex Riemann domain is H^∞-convex. The converse implication does not hold.*

Proof. Let K be compact in the Riemann domain $(X, \pi, \mathbf{C}^n)$. If $\hat{K}_{H^\infty}$ is not compact in X there will be a discrete sequence $(p_k)_{k\in\mathbf{N}}$ in $\hat{K}_{H^\infty}$, and on this sequence every non-constant function in $H^\infty(X)$ will satisfy $|f(p_k)| \leqq \|f\|_K < \|f\|_X$. Hence X is obviously not sequentially H^∞-convex.

On the other hand the domain $M(V)$ of Sibony is H^∞-convex. Since it is not an H^∞-domain and the bounded holomorphic functions separate points on it, Theorem 2.3 shows that it cannot be sequentially H^∞-convex.

Remark. Failure to distinguish between the two kinds of bounded convexity has led to erroneous statements, cf. for example [7] p. 130 and 131 where it is stated that bounded Runge domains (for example $M(V)$) are sequentially H^∞-convex.

Some examples of classes of sequentially H^∞-convex domains will now be given, they are then also H^∞-convex, and to the extent that bounded holomorphic functions separate points on them they are also H^∞-domains.

Theorem 2.5. *The following classes of domains are sequentially H^∞-convex:*

i) *Non-compact domains on the Riemann sphere without discrete boundary components.*

ii) *Simply connected Riemann surfaces on which there exists a non-constant holomorphic function.*

iii) *Analytic polyhedra in complex manifolds.*

iv) *Bounded strictly pseudoconvex domains with C^2 boundary.*

v) *Pseudoconvex domains in $\mathbf{C}^2$ with smooth real analytic boundary.*

vi) *Convex domains.*

Proof. i) A discrete sequence contains a subsequence which tends to a component of the boundary. The complement of this component can then be mapped conformally onto the interior of the unit disc, and the mapping function is then bounded and tends to its supremum along the subsequence.

ii) A simply connected Riemann surface with a non-constant holomorphic function is conformally equivalent to the interior of the unit disc (cf. [8] p. 187). Then reason as in i).

iii) One only has to observe that a discrete sequence has an accumulation point on the boundary of the polyhedron, and at least one of the function used to define it takes the absolute value one there.

iv) A theorem due to Rossi (cf. [11] p. 275) says that for a bounded strictly pseudoconvex domain Ω with C^2 boundary there exists for every $z_0 \in \partial\Omega$ a function f holomorphic on a neighbourhood of $\bar{\Omega}$ (hence bounded on Ω) such that $f(z_0)=1$ and $|f(z)|<1$ whenever $z\bar{\Omega}\in\setminus\{z_0\}$. This implies sequential H^∞-convexity.

v) This follows as in iv) from a result of Bedford and Fornaess [2].

vi) Let Ω be convex in $\mathbf{C}^n$ with $z_0\in\partial\Omega$. The convexity of Ω shows that there is a function

$$L(z) = a_1 z_1 + \ldots + a_n z_n + a_{n+1},$$

with $a_1, \ldots, a_{n+1} \in \mathbf{C}$, such that $L(z_0)=0$ and $\operatorname{Re} L(z)<0$ for $z\in\Omega$. Putting $g(z)=\exp(L(z))$ one realises that Ω is sequentially convex.

Remark. In the proof of part iv) and v) we used that boundary points z_0 were *peak points* for the algebra $A(\Omega)$ of functions holomorphic in the domain and continuous in its closure, i.e. there is an $f\in A(\Omega)$ such that $f(z_0)=1$ and $|f|<1$ on $\bar{\Omega}\setminus\{z_0\}$. Clearly the fact that every point on the boundary of a certain domain, bounded in $\mathbf{C}^n$, is a peak point shows that the domain is sequentially H^∞-convex.

3. The Carathéodory metric

In this section a notion is going to be discussed, which can be considered as a generalisation of sequential H^∞-convexity (cf. Theorem 3.3).

To begin with, recall that the *Poincaré metric* in the unit disc is defined by

$$ds^2 = (1-|z|^2)^{-2} dz d\bar{z}.$$

With this metric the unit disc becomes a Riemannian space of curvature -4. The most important property of the Poincaré metric in this context is that it is decreased under holomorphic maps from the disc to itself, a statement which is equivalent to the Schwarz lemma (cf. [14] for more details).

Definition 3.1. *Let M be a connected complex manifold, and let ϱ denote the Poincaré distance in the unit disc U. The Carathéodory pseudometric is defined by* $C_M(p, q)=\sup_F \varrho(f(p), f(q))$ for p, q in M, *where F is the family of holomorphic maps from M into U.*

Remark. It is immediate from this definition that C_M is a *metric*, i.e. $C_M(p, q)=0$ if and only if $p=q$, exactly when the functions in $H^\infty(M)$ separate points in M. For domains in the complex plane this is equivalent to the existence of a non-constant, bounded holomorphic function on the domain. (If a and b are different points in the domain and $f(a)=f(b)$, then the function $g(z)=(f(z)-f(a))/(z-a)^m$, where m is the order of the zero of $f(z)-f(a)$ at $z=a$, satisfies $g(b)=0$, $g(a)\neq 0$.) If one is interested in properties of domains which are related to the behaviour of the bounded holomorphic functions, then it is natural to assume that C_M is a metric.

Definition 3.2. *A connected complex manifold is strongly pseudocomplete with respect to the Carathéodory pseudometric (strongly pseudocomplete for short) if every subset of the manifold which is bounded in the Carathéodory pseudometric is relatively compact in the manifold. In the case when bounded holomorphic functions separate points we talk about strong completeness in the corresponding situation.*

Lemma 3.3. *Let M be a connected complex manifold, let $p\in M$ be fixed, and put*

$$F_p = \{f\in H^\infty(M);\ f(p) = 0,\ |f| \leqq 1 \text{ on } M\}.$$

Then M is strongly (pseudo-)complete if and only if every subset of M which is bounded by a constant less than one in the (pseudo-)metric

$$C_0(p, q) = \sup_{F_p} |f(q)|,$$

is relatively compact in M.

Proof. It is known that

$$C_M(p, q) = \sup_{F_p} \left\{1/2 \log \frac{1+|f(q)|}{1-|f(q)|}\right\} \quad \text{(cf. [12] p. 209)},$$

so that a subset of M is bounded in C_M if and only if it is bounded by a constant less than one in C_0.

Carathéodory-completeness can now be shown to generalise sequential H^∞-convexity.

Theorem 3.4. *A connected complex manifold M is strongly pseudocomplete if and only if for every discrete sequence $(p_k)_{k\in\mathbf{N}}$ in M there is a sequence of functions $(f_k)_{k\in\mathbf{N}}$ with $f_k\in F_p$, such that* $\sup_k |f_k(p_k)|=1$.

Proof. Assume to begin with that M is strongly pseudocomplete. If $(p_k)_{k\in\mathbf{N}}$ is a discrete sequence in M then it follows that $C_M(p, p_k)$ must be unbounded, and by Lemma 3.3, $C_0(p, p_k)$ tends to one (maybe after taking a subsequence). This clearly implies that there are $f_k \in F_p$ such that $\sup_k |f_k(p_k)| = 1$.

Conversely, if there is a subset in M which is bounded but *not* relatively compact in M, then one can find a discrete sequence $(p_k)_{k \in \mathbf{N}}$ in this subset which is bounded by a constant less than one in C_0, so for every $f \in F_p$ we have that $|f(p_k)| \leqq \alpha < 1$ for some constant α. No sequence $(f_k)_{k \in \mathbf{N}}$ with the desired properties can then exist.

Corollary 3.5. *A sequentially H^∞-convex manifold is strongly pseudocomplete.*

The following two theorems show that strongly complete manifolds share certain properties with sequentially H^∞-convex ones. The results are not new (cf. [14] p. 55 and [21] p. 223 resp.) but the given proofs are more in the spirit of the present work.

Theorem 3.6. *A strongly pseudocomplete manifold is H^∞-convex.*

Proof. Let M be strongly pseudocomplete with K compact in M, and assume that $\hat{K}_{H^\infty}$ fails to be compact in M. Then there is a discrete sequence $(p_k)_{k \in \mathbf{N}} \subset \hat{K}_{H^\infty}$, and the strong pseudocompleteness guarantees the existence of a corresponding sequence $(f_k)_{k \in \mathbf{N}} \subset F_p$, where p is fixed in M, such that $\sup_k |f_k(p_k)| = 1$.

Now F_p is a normal family, and hence there is a subsequence of $(f_p)_{p \in \mathbf{N}}$ which converges uniformly on compact subsets of M. The limit function f will clearly satisfy $|f| \leqq 1$ on M, $f(p) = 0$. Because of the maximum principle one has

$$(*) \qquad \|f\|_K = \alpha < 1.$$

But $\hat{K}_{H^\infty} \supset (p_k)_{k \in \mathbf{N}}$ shows that

$(**)$ $\|f_k\| \geqq |f_k(p_k)|$, which tends to 1 as $k \to \infty$ (possibly after an earlier choice of subsequence).

It is clearly impossible to reconcile $(*)$ and $(**)$ with the fact that a subsequence of $(f_k)_{k \in \mathbf{N}}$ tends uniformly to f on compact subsets of M.

Theorem 3.7. *A strongly complete Riemann domain is an H^∞-domain.*

Proof. Assume that the Riemann domain $(X, \pi, \mathbf{C}^n)$ is not an H^∞-domain, and take $p \in X$ such that the functions in $H^\infty(X)$ can be extended to $H^\infty(X(p))$. Now take a sequence $(p_k)_{k \in \mathbf{N}}$ just as in Theorem 2.3, and let $(f_k)_{k \in \mathbf{N}}$ be the corresponding sequence of functions in F_p which exists since X is strongly complete.

Lemma 1.6 shows that $(f_k)_{k \in \mathbf{N}}$ is a normal family in $X(p)$ and so there is a subsequence converging on compact subsets of $X(p)$ to f, which satisfies $f(p) = 0$, $|f| < 1$ in $X(p)$.

By taking a compact set in $X(p)$ containing p_∞ and using $|f(p_\infty)|=\alpha<1$, $|f_n(p_n)|\to 1$ as $n\to\infty$, one gets, as in the previous proof, a contradiction to the uniform convergence.

Remark. As for the reverse of the implications in Theorems 3.6 and 3.7, the domain $M(V)$ of Sibony and the Hartogs' triangle show that in general they do not hold. In the case of domains which are bounded in the complex plane, the two theorems actually coincide, and it is shown in [1] that the reverse is not true in general here either.

Sibony has given an example of a bounded domain in $\mathbf{C}^n$, which is H^∞-convex as well as an H^∞-domain, but which is not strongly complete (cf. [21] p. 219).

4. Hyperconvexity

Progress in complex analysis during the last thirty years has largely been due to the use of plurisubharmonic functions, and in this context the most important problem has been to show that domains of holomorphy can be characterised using such functions, the so-called Levi problem. A positive answer has been obtained in a large number of cases. See [22] for a recent review.

A Riemann domain $(X, \pi, \mathbf{C}^n)$ is said to be *pseudoconvex* if there is a plurisubharmonic function p on X such that the set $\{z\in X;\ p(z)<\alpha\}$ is relatively compact in X for all real α. One way to express the solution is then the following (cf. [11] p. 283).

Theorem 4.1. *A Riemann domain is a domain of holomorphy if and only if it is pseudoconvex.*

The concept of hyperconvexity, introduced by Stehlé in [23] can be considered as a bounded version of pseudoconvexity.

Definition 4.2. *A Riemann domain $(X, \pi, \mathbf{C}^n)$ is hyperconvex if there is a plurisubharmonic function p on X such that p is negative on X and the set $\{z\in X;\ p(z)<\alpha\}$ is relatively compact in X for all negative real α.*

Remark. A hyperconvex Riemann domain is necessarily pseudoconvex since if we put $V(z)=-p(z)^{-1}$, where p satisfies the conditions of Definition 4.2, it follows that V is plurisubharmonic in X and $\{z\in X;\ V(z)<\alpha\}$ is relatively compact in X for all real α. It follows from this and Theorem 4.1 that a hyperconvex domain is a domain of holomorphy. It will be shown later that the Hartog's triangle is not hyperconvex, so the hyperconvex domains form a proper subclass of the domains of holomorphy.

When Stehlé in [23] defined hyperconvexity in the more general context of complex analytic spaces, he assumed, in addition to the existence of a plurisubharmonic function with the properties in Definition 4.2, that the space should be a *Stein space*. This concept is usually considered to be the proper generalisation of domains of holomorphy, and a Stein space is characterised by the fact that it is holomorphically convex and holomorphic functions separate points. A Riemann domain is a domain of holomorphy if and only if it is Stein. (Cf. [10] and [11] for more information on Stein spaces.)

An example will now be given which shows that the assumption of Stein-ness is not superfluous, i.e. that the conclusion in the above remark does not extend to the general case. (The manifold below was constructed by H. Grauert and used for other purposes, see Narasimhan [16].)

Theorem 4.3. *There exists a non-compact complex manifold M on which there is a plurisubharmonic function p such that $p<0$ on M and $\{z\in M;\ p(z)<\alpha\}$ is relatively compact in M for all negative α, and such that all holomorphic functions on M are identically constant.*

Proof. Let Γ be the lattice in $\mathbf{C}^2$ generated by $w_1=(1,0)$, $w_i=(z_{i1}, z_{i2})$, $i=2,3,4$, where $\{w_i;\ i=1,\dots,4\}$ are linearly independent over $\mathbf{R}$. It is also assumed that $\operatorname{Re} z_{i1}=0$ for $i\geqq 2$, and that $\{\operatorname{Im} z_{i1};\ i\geqq 2\}$ are linearly independent over the integers.

Now consider the torus $T=\mathbf{C}^2/\Gamma$ with the projection $\pi\colon \mathbf{C}^2\to\mathbf{C}^2/\Gamma$, and let M be the open submanifold of T defined as $\pi(D)$, where

$$D=\{z\in\mathbf{C}^2;\ 0<\operatorname{Re} z_1<1/2\}.$$

Put

$$p(z)=(1-\operatorname{Re} z_1)^{-1}+(1/2+\operatorname{Re} z_1)^{-1}-3.$$

This defines p as a function on M since it is clearly invariant under those transformations in Γ which identify points in D. Furthermore, p is a convex function of the harmonic function $\operatorname{Re} z_1$ and so plurisubharmonic in D. But π is locally biholomorphic and consequently p can be considered as a plurisubharmonic function in M. It is easy to see that $p(z)<0$ in M and also that $p(z)\to 0$ as $z\to\partial M$.

The fact that the only holomorphic functions on M are the constants can be seen as follows. The set $K=\pi(\operatorname{Re} z_1=1/4)$ is compact in M, so that if $f\in\mathcal{O}(M)$, then f assumes its supremum on K at a point p_0. Now take $z_0=(z_1,z_2)$ in D such that $\pi(z_0)=p_0$. The image under π of those points in D whose first coordinate is z_1 is a connected proper analytic subset of M which contains p_0. Because of the assumption on the imaginary parts of the coordinates of the points in the lattice it follows that this analytic subset is dense in K. This implies that, since the maximum principle is valid on analytic sets (cf. [11] p. 272), the function f is constant on K. But K has real dimension 3, so f is constant on M.

Hyperconvex domains have certain features in common with domains of bounded holomorphy. To begin with one realises that the whole of $\mathbf{C}^n$ cannot be hyperconvex since a bounded plurisubharmonic function on $\mathbf{C}^n$ is necessarily a constant. Furthermore, a domain obtained by removing analytic subsets from a given domain can not be a domain of bounded holomorphy, nor can it be hyperconvex. This follows from the fact that one can extend bounded plurisubharmonic functions over analytic subsets.

An example of a domain of bounded holomorphy which is *not* hyperconvex is the Hartogs' triangle. The proof is practically identical to the one given at the beginning of Section 2 where it was shown that this domain is not H^∞-convex, since the maximum principle and the possibility to extend bounded functions over isolated singularities is valid for subharmonic functions too.

On the other hand, if one assumes that the boundary of a bounded domain of holomorphy is twice continuously differentiable, then a theorem of Diederich and Fornaess [5] says that it is in fact hyperconvex.

The Hartogs' triangle shows that an H^∞-domain does not have to be hyperconvex. Using Sibony's domain $M(V)$, described in Section 1, one can see that a hyperconvex domain does not have to be an H^∞-domain either, and so any hope of getting a bounded version of Theorem 4.1 by replacing domain of holomorphy by H^∞-domain and pseudoconvexity by hyperconvexity must fail.

The hyperconvexity of $M(V)$ is seen as follows. From the definition in Section 1 one has

$$M(V) = \{(z, w);\ |z| < 1,\ |w|e^{V(z)} < 1\}.$$

It is easy to see that

$$\partial M(V) = \{(z, w);\ |z| = 1,\ |w| < 1\} \cup \{(z, w);\ |z| < 1,\ |w| = e^{-V(z)}\}.$$

Now put $p(z, w) = \sup\{|z|-1, \log|w| + V(z)\}$. As the supremum of two plurisubharmonic functions p is plurisubharmonic in $M(V)$ and the set $\{(z, w);\ p(z, w) < \alpha\}$ is relatively compact in $M(V)$ for all negative α in view of the continuity of p.

Note that this example also shows that it is possible for a hyperconvex domain not to have a Stein basis (cf. Theorem 1.15). Stehlé has conjectured (see [23] p. 167) that a full Stein domain, relatively compact in $\mathbf{C}^n$ with a Stein basis, is hyperconvex.

In order to relate hyperconvexity to some of the other notions introduced above it is now going to be shown that a strongly complete domain is hyperconvex. This is going to be done in a more general context than the one used up till now and is a generalisation of a result in [3].

The definition of Carathéodory metric and strong completeness can be used without change for complex analytic spaces. A complex analytic space (cf. [11])

is locally isomorphic to an analytic set in some $\mathbf{C}^n$, and the holomorphic functions on the complex space correspond via the isomorphism to equivalence classes of holomorphic functions on the analytic set and such a function is by definition holomorphic on some neighbourhood in $\mathbf{C}^n$ of each point in the set; two functions are equivalent if their difference is zero on the set (we are only considering so called "reduced" complex spaces).

Plurisubharmonic functions on a complex space can be defined in the same manner, so they can locally be considered as plurisubharmonic functions on some neighbourhood in $\mathbf{C}^n$ of a point in the analytic set, "pulled back" via the above mentioned isomorphism. (For more information on this matter, see [15].)

As mentioned above, a complex space X is said to be hyperconvex if it is a Stein space, and if there is a negative plurisubharmonic function p on X such that $\{x\in X;\ p(x)<\alpha\}$ is relatively compact in X for all $\alpha<0$. The existence of such a function is going to be expressed by saying that X has a *bounded plurisubharmonic exhaustion function.*

Theorem 4.4. *A strongly pseudocomplete complex space has a bounded plurisubharmonic exhaustion function.*

Proof. Let X be strongly pseudocomplete, and let p be a point in X. As above we have that

$$F_p = \{f\in\mathcal{O}(X);\ f(p) = 0,\ |f| \leqq 1\},$$

and we put $\varphi(x)=\sup_{F_p} |f(x)|$.

Let x_0 be a point in X, let π be an isomorphism of a neighbourhood of x_0 onto an analytic set V in some open set in $\mathbf{C}^n$, and let $\pi(x_0)=z_0$. In order to prove that φ is plurisubharmonic on X, it is sufficient to show that there is a neighbourhood of z_0 in $\mathbf{C}^n$, and a plurisubharmonic function ψ in this neighbourhood, such that $\varphi=\psi\circ\pi$ on the corresponding neighbourhood of x_0.

Let U_0 be a Stein neighbourhood of z_0. This implies that all holomorphic functions on $V\cap U_0$ can be extended to U_0 (cf. [11] p. 245). If Y is a complex space, then $\mathcal{O}(Y)$ is a Fréchet space with seminorms taken as the supremum on compact subsets, so the extension property can be expressed by saying that there is a surjective map

$$\tau\colon\ \mathcal{O}(U_0) \to \mathcal{O}(U_0\cap V),$$

obtained by restricting the holomorphic functions on U_0 to $U_0\cap V$. As a surjective map between Fréchet spaces τ is open.

This means that if $(K_i)_{i\in\mathbf{N}}$ is an increasing sequence of compact subsets of U_0 such that $U_0=\bigcup_i K_i$, then for every K_i there is a K_j and a constant c_i such that for every $f\in\mathcal{O}(V\cap U_0)$ there is an extension $F\in\mathcal{O}(U_0)$ of f with the property

$$(*)\qquad \|F\|_{K_i} \leqq c_i\|f\|_{K_j\cap V}.$$

In particular this is true for the functions in $\pi^*(F_p)$, i.e. the functions on V which via the isomorphism correspond to the functions in F_p.

Let now W be relatively compact in U_0, and choose K_i such that $W \subset\subset K_i$. If G_p denotes those functions in $\mathcal{O}(U_0)$ which have been obtained by extending the functions in $\pi^*(F_p)$ under the condition $(*)$, and if one puts

$$\psi(z) = \sup_{G_p} |f(z)|,$$

then clearly $\varphi = \psi \circ \pi$, and $\psi \leqq c_i$ on W. In order to prove that ψ is plurisubharmonic on W, it is sufficient to show that it is continuous, since the continuous supremum of a uniformly bounded family of plurisubharmonic functions is itself plurisubharmonic (cf. [13] p. 16).

Fix the point $w_0 \in W$ and let W' and P be neighbourhoods of w_0 contained in W with $W' \subset\subset P$ and P a polydisc. It is now possible to use Cauchy's integral formula in the polydisc to obtain estimates for the derivatives in W' of the functions in $\mathcal{O}(U_0)$, and then, using the mean value theorem, one can deduce the existence of a constant K such that for all $w \in W'$ and all $f \in G_p$ one has

$$(**) \qquad |f(w) - f(w_0)| \leqq K\|w - w_0\|,$$

where $\| \ \|$ is the euclidean distance in $\mathbf{C}^n$.

Let now v be a point in W'. Using Montel's theorem, which applies since G_p is a normal family, one obtains that

$$\psi(v) = |g(v)|$$

for a certain function $g \in G_p$. Using $(**)$ this leads to

$$\psi(v) = |g(v)| \leqq |g(v) - g(w_0)| + |g(w_0)| \leqq K\|v - w_0\| + \psi(w_0).$$

In exactly the same way one gets

$$\psi(w_0) \leqq K\|v - w_0\| + \psi(v),$$

and combining these formulae one has

$$|\psi(v) - \psi(w_0)| \leqq K\|v - w_0\|.$$

So ψ is continuous, and it is proved that φ is plurisubharmonic on X.

Since the maximum principle for plurisubharmonic functions applies to complex spaces (cf. [15]) it is obviously the case that $\varphi < 1$ on X.

The fact that X is strongly pseudocomplete means that for every discrete sequence $(x_k)_{k \in \mathbf{N}}$ there is a sequence $(f_k)_{k \in \mathbf{N}}$ of functions in F_p such that $\sup |f_k(x_k)| = 1$, and hence $\varphi(x_k) \to 1$.

If we put $\Psi(x) = \varphi(x) - 1$ then it is clear that Ψ is a bounded plurisubharmonic exhaustion function on X.

Corollary 4.5. *A strongly complete complex space is hyperconvex.*

Proof. It only remains to be shown that the space is a Stein space. But by the definition of strong completeness holomorphic functions separate points on the space, and Theorem 3.6 can be used in this more general context and shows that the space is holomorphically convex.

The concept of hyperconvexity is interesting partly because of its connection with holomorphic fibre bundles and the "problem of Serre".

Let X and B be complex spaces, and π a holomorphic map from X to B. Then X is the total space of a *locally trivial holomorphic fibre bundle with fibre Y and base B* if there is an open cover $\{U_\alpha\}_{\alpha\in A}$ of B such that $\pi^{-1}(U_\alpha)$ is isomorphic to $U_\alpha\times Y$. The problem which Serre posed in 1953 is to decide whether the fact that Y and B are Stein spaces implies that X is a Stein space. This is not true in general (see [22] for a review and the relevant references).

Using hyperconvexity Stehlé in [23] was able to give some positive results using Narasimhan's solution of the Levi problem for complex spaces [15]. If X is the total space and Y the fibre of a locally trivial holomorphic fibre bundle, he proved among other things the following.

i) If there is a strictly plurisubharmonic, bounded function on Y, then there is a strictly plurisubharmonic, positive function on X.

ii) If Y is hyperconvex then there is a plurisubharmonic function φ on X such that $\{x\in X;\ \varphi(x)<\alpha\}\subset\subset X$ for all real α.

If one adds the two functions obtained in this way, one has a strictly plurisubharmonic function on X with the exhaustion property of φ in ii), and then Narasimhan's result shows that X is a Stein space.

A natural manner in which one can guarantee that there exists a strictly plurisubharmonic bounded function on the fibre is to assume that the fibre is relatively, compact in a Stein space. Using Corollary 4.5 we then have the following result proved earlier by Hirschowitz and Sibony (see references in [22]) using the concept of Banach—Stein space.

Theorem 4.6. *A locally trivial holomorphic fibre bundle with Stein base and with a fibre which is relatively compact in a Stein space and strongly complete is Stein.*

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