Asymptotic value sets of bounded holomorphic functions along Green lines

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In this note we shall study asymptotic value sets, along Green lines, of bounded holomorphic functions on hyperbolic Riemann surfaces. As their applications to the case of the open unit disc, we shall obtain some results on radial limits.

Let R be a hyperbolic Riemann surface and let $G(a, b)$ denote the Green function for R with pole at a point $b \in R$. For any number $\alpha > 0$, let $R(\alpha, b)$ denote the region $\{a \in R | G(a, b) > \alpha\}$ and let $B(\alpha, b)$ denote the first Betti number of $R(\alpha, b)$. In this note we assume the following conditions (1) and (2):

(1) The closure of $R(\alpha, b)$ is compact for any $b \in R$ and any $\alpha > 0$.

(2) $\int_0^\infty B(\alpha, b) d\alpha < \infty$ for some $b \in R$.

Let R^* denote the Martin compactification of R and let Δ_1 denote the set of all minimal points in $\Delta = R^* - R$. Let f be a nonconstant bounded holomorphic function on R and let $\hat{f}(q)$ denote the fine limit of f at $q \in \Delta_1$. We consider Green lines issuing from a fixed point chosen arbitrarily in R (see [2], p. 259). The asymptotic value of f along a Green line terminating at a point $q \in \Delta$ is called the radial limit of f at q.

It follows from Theorem 4.2 of [2] that $\hat{f}$ exists almost everywhere on Δ_1, and that f is identical to the solution of the Dirichlet problem for R with the boundary function $\hat{f}$, since R^* is a resolutive compactification. It therefore follows from Theorem 7 of [3] that f has radial limits at almost all points of Δ_1, and that the fine limit and the radial limit at the same point of Δ_1 are identical almost everywhere on Δ_1. Hence we obtain the following extension of classical Fatou's theorem on radial limits:

Lemma 1. *f has radial limits at almost all points in Δ_1.*

By Lemma 1 and Theorem 14.1 of [1] we obtain the following extension of classical Riesz' uniqueness theorem:

Lemma 2. *The set of radial limits of f at all points of a set of positive harmonic measure in Δ_1 is of positive capacity.*

Let p be a point in Δ. The cluster set of f at p is defined as

$$C(f, p) = \bigcap_{r>0} \overline{f(R \cap U(p, r))},$$

where $U(p, r)$ denotes the r-neighborhood of p and where the bar denotes closure. We put $\Delta(p, r) = \Delta \cap U(p, r)$. Let E^* be any set, in Δ, of harmonic measure zero. Let E be the set, lying in Δ and containing E^*, of harmonic measure zero and such that by Lemma 1, f has radial limits at all points of $\Delta_1 - E$. Let $A_E(f, \Delta(p, r))$ denote the set of radial limits of f at all points of $\Delta(p, r) \cap \Delta_1 - E$. We define the set

$$X_E^*(f, p) = \bigcap_{r>0} \overline{A_E(f, \Delta(p, r))}.$$

We call that $w' \in X_{*E}(f, p)$ if for any $U(p, r)$ and for any neighborhood U of $w' \in \{|w| \leqq \infty\}$, $A_E(f, \Delta(p, r)) \cap U$ is of positive capacity. For each $q \in \Delta_1$, let F_q be a filter basis on R with respect to the fine topology. For an open set G, we put

$$\Delta_1(G) = \{q \in \Delta_1 | G \in F_q\}.$$

A region $D \subset R$ is said to be of class SO_{HB} if there is no nonconstant positive bounded harmonic function in D which vanishes continuously at every point on the relative boundary ∂D of D with respect to R.

If there is a function s which satisfies the following conditions (i), (ii) and (iii), then we call s a barrier at p:

(i) s is positive and superharmonic in $R \cap U(p, r)$, where $r > 0$ may be sufficiently small.

(ii) $\lim_{a \to p} s(a) = 0$.

(iii) There is a sequence $r > r_1 > r_2 > \ldots > r_n > \ldots$, $r_n \to 0$ such that $\inf s > 0$ on $\partial U(p, r_n)$ for each n.

Henceforth we assume that for any r, $\Delta(p, r)$ is of positive harmonic measure. Clearly $X_{*E}(f, p) \subset X_E^*(f, p) \subset C(f, p)$. We shall show more detailedly this topological relationship by the Theorem and Corollary 1.

Theorem. *If there is a barrier at p, then*

$$\partial C(f, p) \subset X_{*E}(f, p).$$

Proof. We suppose, on the contrary, that there is a $w_0 \in \partial C(f, p)$ not in $X_{*E}(f, p)$. There are then a $U(p, r_0)$ and a neighborhood U of w_0 such that $A_E(f, \Delta(p, r_0)) \cap U$ is of capacity zero. Let V be an open disc centered at w_0 with radius d and such that $\overline{V} \subset U$. Now $p \in \overline{f^{-1}(V)}$.

We first consider the case that $p \notin \overline{\partial f^{-1}(V)}$. It is possible to find a $U(p, r_1)$, $0<r_1<r_0$, such that $f(R \cap U(p, r_1)) \subset V$. By Lemma 2, $A_E(f, \Delta(p, r_1)) \cap \overline{V}$ is of positive capacity. This is a contradiction.

We next consider the case that $p \in \overline{\partial f^{-1}(V)}$. We take an r_2, $0<r_2<r_0$, and put $G=U(p, r_2) \cap f^{-1}(V)$.

If there is at least one component D of G such that $\Delta_1(D)$ is of positive harmonic measure, then by Lemma 2, $A_E(f, \Delta(p, r_0)) \cap \overline{V}$ is of positive capacity. This is a contradiction.

If, for every component D' of G which is not relatively compact on R, $\Delta_1(D')$ is of harmonic measure zero, then by Lemma 4 of [4], D' is of class SO_{HB}. Further every component of G which is relatively compact on R is also of class SO_{HB}. It will be next shown that we have a contradiction.

There is a $w^* \notin C(f, p)$ with $|w_0 - w^*| < d/2$. It is possible to take an r_3, $0<r_3<r_0$, such that $w^* \notin \overline{f(G')}$, where $G'=U(p, r_3) \cap f^{-1}(V)$. The function $v=1/|f_{G'}-w^*|$, where $f_{G'}$ denotes the restriction of f to G', is bounded above and subharmonic in G'. We put $m=\overline{\lim}_{\partial G' \ni a \to p} v(a)$. For any positive $\varepsilon<1$, we choose an r^*, $0<r^*<r_3$, such that $v \leqq m+\varepsilon$ on $U(p, r^*) \cap \partial G'$. There is a positive superharmonic function s in $R \cap U(p, r^*)$ with the properties that $\lim_{a \to p} s(a)=0$, and that $c=\inf s>0$ on $\partial U(p, r_N)$ for some r_N, $0<r_N<r^*$. We choose an $M>m+1$ with $v \leqq M$ in G'. Then $v^*=v-(M-(m+\varepsilon))s/c$ is subharmonic in $G_N=G' \cap U(p, r_N)$. It is seen that $v^* \leqq m+\varepsilon$ on ∂G_N, and that $v^* \leqq M$ in G_N. Since each component of G_N is of class SO_{HB}, it holds that

$$v^* - m - \varepsilon \leqq 0 \quad \text{in } G_N.$$

Since $\lim_{a \to p} s(a)=0$, $a \in G_N$, it holds that

$$\overline{\lim_{G' \ni a \to p}}\, v(a) \leqq \overline{\lim_{\partial G' \ni a \to p}}\, v(a).$$

Thus

$$\begin{aligned} |w_0 - w^*| &\geqq \underline{\lim}_{a \to p} |f_{G'}(a) - w^*| \\ &\geqq \underline{\lim}_{\partial G' \ni a \to p} |f(a) - w^*| \\ &\geqq d/2. \end{aligned}$$

This is a contradiction.

The assertion of the Theorem is proved.

The following Corollary 1 is immediately obtained:

Corollary 1. *If there is a barrier at p, then*

$$\partial C(f, p) \subset X_E^*(f, p).$$

We call p a singular point if $C(f, p)$ contains at least one nondegenerate continuum lying in $f(R)$. From Corollary 1, the following Corollary 2 is obtained:

Corollary 2. *Let p be a singular point of f and let there be a barrier at p and an $r>0$ such that*

$$A_E(f, \Delta(p, r)) \subset \partial f(R).$$

Then

$$C(f, p) = \overline{f(R)}.$$

The following result of W. Seidel is obtained from Corollaries 1 and 2 (see [5], p. 211): Let $w=h(z)$ be a holomorphic function in $\{|z|<1\}$ with $|h(z)|<1$. Let $0\leqq c_1<\Theta<c_2<2\pi$, $r=1$, be an arc of $\{|z|=1\}$ such that $\lim_{r\to 1}|h(re^{i\Theta})|=1$ for almost all values of Θ in $\{c_1<\Theta<c_2\}$. If $e^{i\Theta^*}$ is a singular point of $h(z)$ lying in $\{c_1<\Theta<c_2\}$, then it holds that $C(h, e^{i\Theta^*})=\{|w|\leqq 1\}$. Indeed, since $X_E^*(h, e^{i\Theta^*})\subset \{|w|=1\}$, it follows from Corollary 1 that $\overline{f(\{|z|<1\})}=\{|w|\leqq 1\}$. From Corollary 2, we immediately obtain the result of W. Seidel.

To illustrate the Theorem and Corollary 1 in the case that $R=\{|z|<1\}$, we consider the Blaschke product

$$B(z) = \Pi(|a_n|/a_n)(a_n-z)/(1-\bar{a}_n z)$$

with $a_n=1-1/n^2$. $B(z)$ has a radial limit of modulus 1 everywhere on $\{|z|=1\}$ except at $z=1$, and has the radial limit zero at $z=1$. Now $z=1$ is a singular point of $B(z)$. We choose $\{z=1\}$ as E^*. As stated above, it holds that $C(B, 1)=\{|w|\leqq 1\}$. By the Theorem and Corollary 1, it is seen that $X_{*E}(B, 1)=X_E^*(B, 1)=\{|w|=1\}$. If we choose an empty set as E^*, then it holds that $X_{*E}(B, 1)=\{|w|=1\}$ and $X_E^*(B, 1)=\{|w|=1\}\cup\{w=0\}$.

References

1. Constantinescu, C. und Cornea, A., *Ideale Ränder Riemannscher Flächen,* Springer-Verlag, Berlin—Göttingen—Heidelberg (1963).
2. Hasumi, M., Invariant subspaces on open Riemann surfaces, *Ann. Inst. Fourier, Grenoble,* **24**, 4 (1974), 241—286.
3. Hasumi, M., Invariant subspaces on open Riemann surfaces II, *Ann. Inst. Fourier, Grenoble,* **26**, 2 (1976), 273—299.
4. Nagasaka, Y., Cluster sets on Riemann surfaces, *Hokkaido Math. J.,* **7** (1978), 169—177.
5. Seidel, W., On the distribution of values of bounded analytic functions, *Trans. Amer. Math. Soc.,* **36** (1934), 201—226.

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