Function algebras and flows II

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§ 1. When the real line $\mathbf{R}$ acts on a space X there arises a natural notion of analyticity for bounded functions on X. Specifically, we shall say that a bounded function ϕ on X is *analytic* in case the restriction of ϕ to each orbit is a function in $H^\infty(\mathbf{R})$, the space of boundary functions of functions which are bounded and analytic in the upper half plane. Without some global assumptions about the space and the functions, it does not seem possible to say much about the analytic functions. In this paper, which is a sequel to [11], we shall assume that X is a *separable* compact Hausdorff space and that the action of $\mathbf{R}$ on X is continuous. The pair $(X, \mathbf{R})$ will be referred to as a *flow* and for x in X and t in $\mathbf{R}$, the translate of x by t will be denoted by $x+t$. The analytic functions on X considered here are assumed to come from $C(X)$, the space of all continuous complex-valued functions on X, and the algebra which the analytic functions form will be denoted by $\mathfrak{A}$.

Theorem II of [11] asserts that if the flow $(X, \mathbf{R})$ is *strictly ergodic*, meaning that there is a unique probability measure on X which is invariant under the action of $\mathbf{R}$, then $\mathfrak{A}$ is a Dirichlet algebra on X. While the notion of strict ergodicity seems rather special, there is a vague sense in which the strictly ergodic flows are generic among all flows. For example, all minimal almost periodic flows are strictly ergodic; all nil flows are too; and surprisingly it happens that if $\mathbf{R}$ acts measurably on a (standard Borel) measure space Y, if the action preserves a finite measure on Y, and if the action is weakly mixing, then there is a strictly ergodic flow $(X, \mathbf{R})$ which is Borel isomorphic to the action of $\mathbf{R}$ on Y [8]. Our objective in this paper is to identify the maximal ideal space $\mathcal{M}_{\mathfrak{A}}$ of $\mathfrak{A}$ when the flow $(X, \mathbf{R})$ is strictly ergodic. We shall show in Theorem II that if the unique invariant measure is not a point mass then $\mathcal{M}_{\mathfrak{A}}$ is homeomorphic to the quotient space obtained from $X \times [0, 1]$ by identifying the slice $X \times \{0\}$ to a point. This result generalizes the well known theorem of Arens and Singer [1] which describes the maximal ideal spaces for the algebras of analytic almost periodic functions on

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the line, i.e., the algebras of analytic functions associated with minimal almost periodic flows. We shall also handle the case when the unique invariant measure is a point mass, although the description is a bit more complicated, and we shall identify the Gleason parts of $\mathcal{M}_{\mathfrak{A}}$.

While the reasons for assuming that X is separable in this investigation are technical, they appear compelling because of the tools used in our proofs. However, experimental evidence seems to indicate that our results are correct in the non-separable case as well.

In Section 2 we establish some notation, terminology, and elementary facts which will be used throughout, while in Section 3 we develop some properties of "Poisson integrals" of measures. The result of Section 4 (Theorem I) establishes conditions under which the representing measures for certain points in $\mathcal{M}_{\mathfrak{A}}$ are concentrated on orbits. It is there that the assumption that X is separable is really used. The characterization of $\mathcal{M}_{\mathfrak{A}}$ is presented in Section 5, and Section 6 is devoted to concluding remarks.

§ 2. We begin by reminding the reader of our standing assumptions: We shall always assume that X is compact and separable and that the flow $(X, \mathbf{R})$ is strictly ergodic. Although at times one or the other of these assumptions will not be needed, we shall leave it to the reader to decide for himself on questions of generality.

The space of all bounded complex Borel measures on X will be denoted by $M(X)$ and it will often be convenient to denote the integral $\int \phi d\lambda$ of a function ϕ in $C(X)$ with respect to a measure λ in $M(X)$ by $\langle \phi, \lambda \rangle$.

The action of $\mathbf{R}$ on X induces a strongly continuous one-parameter group $\{T_t\}_{t \in \mathbf{R}}$ of automorphisms of $C(X)$ defined by the formula $(T_t\phi)(x) = \phi(x - t)$, $\phi \in C(X)$. The group of adjoints of $\{T_t\}_{t \in \mathbf{R}}$ acting on $M(X)$ will be denoted by $\{T_t^*\}_{t \in \mathbf{R}}$. Observe that for each t in $\mathbf{R}$ and for each λ in $M(X)$, $T_t^*\lambda$ is the measure which assigns to each Borel set E the value $\lambda(E + t)$. Observe also that in general $\{T_t^*\}_{t \in \mathbf{R}}$ is not strongly continuous but is merely weak-$*$ continuous.

Using $\{T_t\}_{t \in \mathbf{R}}$ it is possible to convert $C(X)$ and $M(X)$ into $L^1(\mathbf{R})$ modules as follows: If ϕ is a function in $C(X)$ and if f is in $L^1(\mathbf{R})$, then $\phi * f$ is defined to be the Bochner integral $\int_{-\infty}^{\infty} (T_t\phi) f(t) dt$. On the other hand, if λ is in $M(X)$ and if f is in $L^1(\mathbf{R})$, then $\lambda * f$ is defined to be the measure such that

$$\langle \phi, \lambda * f \rangle = \langle \phi * \tilde{f}, \lambda \rangle$$

for all ϕ in $C(X)$ where $\tilde{f}$ is the function whose value at t is $f(-t)$. Equivalently, $\lambda * f$ may be expressed as the weak-$*$ convergent integral $\int_{-\infty}^{\infty} (T_{-t}^*\lambda) f(t) dt$. Observe that with respect to these operations of convolution $C(X)$ and $M(X)$ are indeed converted into $L^1(\mathbf{R})$ modules and, moreover, the following inequalities hold for all ϕ in $C(X)$, λ in $M(X)$, and f in $L^1(\mathbf{R})$:

$$\|\phi * f\| \leq \|\phi\| \, \|f\| \qquad \text{and} \qquad \|\lambda * f\| \leq \|\lambda\| \, \|f\|.$$

Because of these inequalities the annihilator of a function in $C(X)$ or of a measure in $M(X)$ is a closed ideal in $L^1(\mathbf{R})$. The *spectrum* of a function ϕ in $C(X)$ or of a measure λ in $M(X)$, in the sense of spectral synthesis, is then defined to be the hull of its annihilator and will be denoted by $\operatorname{sp}(\phi)$ or $\operatorname{sp}(\lambda)$. Equivalently, $\operatorname{sp}(\phi)$ (resp., $\operatorname{sp}(\lambda)$) may be regarded as the closed support of the distributional Fourier transform of the $C(X)$-valued function $T_t\phi$ (resp., the $M(X)$-valued function $T_t^*\lambda$). We note that a function ϕ in $C(X)$ is analytic if and only if $\operatorname{sp}(\phi)$ is nonnegative (see [11, Proposition 2.1]). Our reference for the basic facts about spectra is [5].

Recall that a measure is said to be *quasi-invariant* in case every translate of each null set is also a null set. We note that if μ is a positive quasi-invariant measure on X, then by Proposition 1 of [6] $\{T_t\}_{t\in\mathbf{R}}$ can be extended uniquely in the obvious fashion to be a weak-$*$ continuous one parameter group of automorphisms of $L^\infty(\mu)$. Recall also that a quasi-invariant measure is said to be *ergodic* in case each invariant measurable subset of X is either negligible or has negligible complement.

If m is a representing measure for a point in $\mathfrak{M}_{\mathfrak{A}}$ then $L^p(m)$ and $H^p(m)$ will denote the Lebesgue and Hardy spaces associated with m. The usual Lebesgue and Hardy spaces on the unit circle $\mathbf{T}$ will be denoted by $L^p(\mathbf{T})$ and $H^p(\mathbf{T})$.

§ 3. For the remainder of this paper P_z will denote the Poisson kernel for evaluation at z in the upper half plane; that is, $P_z(t) = y/(\pi(y^2 + (x-t)^2)$ where $z = x + iy$ with $y > 0$. Our objective in this section is to establish certain facts about Poisson integrals of measures on X which will be used later.

PROPOSITION 3.1. (i) *Let $\{m_n\}_{n=1}^\infty$ be a sequence of measures in $M(X)$ which converges to a measure m in the weak-$*$ topology on $M(X)$ and let $\{y_n\}_{n=1}^\infty$ be a convergent sequence of positive real numbers with finite limit y. Then in the weak-$*$ topology on $M(X)$, $\lim_{n\to\infty} m_n * P_{iy_n} = m * P_{iy}$ if $y > 0$ and $\lim_{n\to\infty} m_n * P_{iy_n} = m$ if $y = 0$.*

(ii) *Let $\{m_n\}_{n=1}^\infty$ be a sequence in $M(X)$ and let $\{y_n\}_{n=1}^\infty$ be a sequence of positive numbers such that $\lim_{n\to\infty} y_n = \infty$. If $\lim_{n\to\infty} m_n * P_{iy_n}$ exists in the weak-$*$ topology on $M(X)$, then the limit is invariant.*

(iii) *A measure m in $M(X)$ is invariant if and only if $m * P_{iy} = m$ for some $y > 0$.*

Proof. (i) Since $\lim_{n\to\infty} m_n$ exists in the weak-$*$ topology on $M(X)$, the principle of uniform boundedness implies that there is a K such that $\|m_n\| \leq K$ for all n. Suppose y is positive and let ϕ be in $C(X)$. Then since $\lim_{n\to\infty} \|P_{iy_n} - P_{iy}\| = 0$ in $L^1(\mathbf{R})$ and since $\phi * \tilde{P}_{iy}$ is also in $C(X)$, the conclusion follows from the inequality

$$|\langle \phi, m_n * P_{iy_n}\rangle - \langle \phi, m * P_{iy}\rangle|$$
$$\leq |\langle \phi, m_n * P_{iy_n}\rangle - \langle \phi, m_n * P_{iy}\rangle| + |\langle \phi, m_n * P_{iy}\rangle - \langle \phi, m * P_{iy}\rangle|$$
$$\leq K\|\phi\| \, \|P_{iy_n} - P_{iy}\| + |\langle \phi * \tilde{P}_{iy}, m_n - m\rangle|.$$

To handle the case when $y = 0$, let ϕ be in $C(X)$ as before and observe that since $\{P_{iy}\}_{y>0}$ is an approximate identity for $L^1(\mathbf{R})$, $\lim_{n\to\infty} \|\phi * \tilde{P}_{iy_n} - \phi\| = 0$ by Lemma 1 of [5]. Consequently the following inequality yields the result.

$$|\langle \phi, m_n * P_{iy_n}\rangle - \langle \phi, m\rangle| \leq |\langle \phi * \tilde{P}_{iy_n} - \phi, m_n\rangle| + |\langle \phi, m_n\rangle - \langle \phi, m\rangle|$$
$$\leq K\|\phi * \tilde{P}_{iy_n} - \phi\| + |\langle \phi, m_n\rangle - \langle \phi, m\rangle|.$$

(ii) Let m be the limit in question. Then for each t in $\mathbf{R}$,

$$T_t^* m = \lim_{n\to\infty} T_t^*(m_n * P_{iy_n}) = \lim_{n\to\infty} m_n * P_{-t+iy_n}.$$

But an easy calculation reveals that for t fixed $\lim_{n\to\infty} \|P_{-t+iy_n} - P_{iy_n}\| = 0$ in $L^1(\mathbf{R})$. Consequently, $\lim_{n\to\infty} m_n * P_{-t+iy_n} = m$ also, and this shows that m is invariant.

(iii) If m is invariant, then a calculation shows that

$$m * P_{iy} = m \int_{-\infty}^{\infty} P_{iy}(t)dt = m$$

for all $y > 0$. The converse follows from (ii) and the observation that if $y > 0$, then $(m * P_{iy}) * P_{iy} = m * (P_{iy} * P_{iy}) = m * P_{i2y}$.

PROPOSITION 3.2. *If m is a representing measure for a point in $\mathcal{M}_{\mathfrak{A}}$ and if z is a point in the upper half plane, then $m * P_z$ is also a representing measure for a point in $\mathcal{M}_{\mathfrak{A}}$.*

Proof. First note that if ϕ is in $\mathfrak{A}$ and if $F(t) = \langle T_{-t}\phi, m\rangle$, then by the analysis presented on page 50 of [5], the spectrum of F as a bounded continuous function on $\mathbf{R}$ is contained in sp $(\phi) \cap$ sp (m) which in turn is contained in $[0, \infty)$. That is, F lies in $H^\infty(\mathbf{R})$. If ϕ and ψ lie in $\mathfrak{A}$, then the following equation yields the result.

$$\langle \phi\psi, m * P_z\rangle = \int_{-\infty}^{\infty} \langle T_{-t}(\phi\psi), m\rangle P_z(t)dt \tag{3.1}$$

$$= \int_{-\infty}^{\infty} \langle T_{-t}\phi, m\rangle\langle T_{-t}\psi, m\rangle P_z(t)dt \tag{3.2}$$

$$= \left[\int_{-\infty}^{\infty} \langle T_{-t}\phi, m\rangle P_z(t)dt\right]\left[\int_{-\infty}^{\infty} \langle T_{-t}\psi, m\rangle P_z(t)dt\right]$$
$$= \langle \phi, m * P_z\rangle \langle \psi, m * P_z\rangle. \tag{3.3}$$

The passage from (3.1) to (3.2) is justified by the fact that m is multiplicative on $\mathfrak{A}$ and that $\{T_t\}_{t\in\mathbf{R}}$ leaves $\mathfrak{A}$ invariant (see Lemma 3 of [5]). The passage from (3.2) to (3.3) is justified by the fact just mentioned that the expressions inside the angular brackets are functions in $H^\infty(\mathbf{R})$ and by the fact that the measure $P_z dt$ is multiplicative on $H^\infty(\mathbf{R})$.

As a consequence of Propositions 3.1 and 3.2, we obtain

COROLLARY 3.3. *If m represents a point in $\mathcal{M}_{\mathfrak{A}}$ and if m is neither a point mass nor the unique invariant measure, then the Gleason part containing the point represented by m is nontrivial.*

Proof. It suffices to produce a representing measure for another point in $\mathcal{M}_{\mathfrak{A}}$ which is absolutely continuous with respect to m (see [7, p. 144]). Choose $y > 0$ and consider $m * P_{iy}$. By Proposition 3.2, $m * P_{iy}$ is a representing measure and it is distinct from m by the hypothesis that m is not invariant and Proposition 3.1 (iii). On the other hand, since m is quasi-invariant by Theorem III of [11], it follows that $m * P_{iy}$ is absolutely continuous with respect to m. With this the proof is complete.

§ 4. Let x be a point in X which is not fixed by the action of $\mathbf{R}$ and let y be positive. Then by Proposition 3.2, $\delta_x * P_{iy}$ is a representing measure for a point in $\mathcal{M}_{\mathfrak{A}}$ and, by Proposition 3.1 and the hypothesis on x, $\delta_x * P_{iy}$ is not a point mass. Our objective in this section is to show that the representing measure for almost every point in $\mathcal{M}_{\mathfrak{A}}$ can be written in this form.

THEOREM I. *Let m represent a point in $\mathcal{M}_{\mathfrak{A}}$ and assume that m is neither the unique invariant probability measure on X nor a point mass on X. Then there is a unique x in X and a unique positive y such that $m = \delta_x * P_{iy}$.*

Proof. The proof is divided into two steps. First we show that each representing measure m satisfying the hypotheses of the theorem is concentrated on an orbit. Once this is done, then we show that m has the indicated representation.

Step I. By Corollary 3.3 the Gleason part containing the point represented by m is nontrivial. Also, by Theorem VI and Corollary 3.1 of [11], $H^\infty(m)$ is a maximal weak-* closed subalgebra of $L^\infty(m)$. Therefore, by a theorem of Merrill [10], which is a sharpening of the Wermer imbedding theorem, we may find a Hilbert space isomorphism W from $L^2(m)$ *onto* $L^2(\mathbf{T})$ such that $WH^2(m) = H^2(\mathbf{T})$, $WL^\infty(m)W^{-1} = L^\infty(\mathbf{T})$ and such that $WH^\infty(m)W^{-1} = H^\infty(\mathbf{T})$. As was noted in Section 2, the fact that m is quasi-invariant allows us to extend $\{T_t\}_{t\in\mathbf{R}}$ to a weak-* continuous one-parameter group of automorphisms of $L^\infty(m)$. This we shall do, but we shall keep the same notation for the extended group. Since $\{T_t\}_{t\in\mathbf{R}}$ leaves $\mathfrak{A}$ invariant by Lemma 3 of [5], $\{T_t\}_{t\in\mathbf{R}}$ leaves $H^\infty(m)$ invariant also. For each t in $\mathbf{R}$ and each ϕ in $L^\infty(\mathbf{T})$, define $\tilde{T}_t\phi$ to be $T_t(W^{-1}\phi W)$. Then $\{\tilde{T}_t\}_{t\in\mathbf{R}}$ is a weak-$*$ continuous one-parameter group of automorphisms of $L^\infty(\mathbf{T})$ which leaves $H^\infty(\mathbf{T})$ invariant. Appealing to a theorem of de Leeuw, Rudin, and Wermer [2], we find that there is a continuous one-parameter group $\{\alpha_t\}_{t\in\mathbf{R}}$ of conformal maps of the open unit disc $\varDelta$ onto itself such that for ϕ in $H^\infty(\mathbf{T})$ and t in $\mathbf{R}$

$$(\tilde{T}_t\phi)(z) = \phi(\alpha_t(z)) \quad \text{a.e. } \mu \tag{4.1}$$

where μ is a normalized Lebesgue measure on $\mathbf{T}$. Since $\overline{H^\infty(\mathbf{T})} + H^\infty(\mathbf{T})$ is weak-$*$ dense in $L^\infty(\mathbf{T})$ equation (4.1) holds for all ϕ in $L^\infty(\mathbf{T})$ as well. That is, $\{\tilde{T}_t\}_{t\in\mathbf{R}}$ is implemented by the transformation group $\{\alpha_t\}_{t\in\mathbf{R}}$ restricted to $\mathbf{T}$.

The hypothesis that X is separable now allows us to apply a theorem of Mackey [9, Theorem 2] (see [12] also) to conclude that there are invariant null sets N_1 and N_2 in $\mathbf{T}$ and X respectively and a Borel isomorphism Φ from $\mathbf{T}\setminus N_1$ onto $X\setminus N_2$ which carries μ to a measure equivalent to m such that for all z in $\mathbf{T}\setminus N_1$ and for all t in $\mathbf{R}$, $\Phi(\alpha_t(z)) = \Phi(z) - t$. Thus to show that m is carried on an orbit (which must be $X\setminus N_2$), it suffices to show that $\{\alpha_t\}_{t\in\mathbf{R}}$ restricted to $\mathbf{T}$ acts transitively on $\mathbf{T}\setminus N_1$. There are two cases to consider.

Case I. Some point in $\varDelta$ is fixed by $\{\alpha_t\}_{t\in\mathbf{R}}$.

After a conformal change of variables, if necessary, we need only consider the case when the fixed point is the origin. But, then, it is readily verified that there is a real θ such that $\alpha_t(z) = e^{it\theta}z$ for all t in $\mathbf{R}$ and all z. Whence, in this case, N_1 is actually the empty set, $\{\alpha_t\}_{t\in\mathbf{R}}$ acts transitively on $\mathbf{T}$, and $X\setminus N_2$ is a periodic orbit.

Case II. No point in $\varDelta$ is fixed by $\{\alpha_t\}_{t\in\mathbf{R}}$.

First note that since $\{\alpha_t\}_{t\in\mathbf{R}}$ is a commutative group of fractional linear transformations, the set $\mathfrak{F}$ of common fixed points for $\{\alpha_t\}_{t\in\mathbf{R}}$ coincides with the set of fixed points for any particular α_t which is not the identity transformation. It follows that $\mathfrak{F}$ is nonempty and consists of at most two points; moreover, the hypothesis implies that $\mathfrak{F}$ is a subset of $\mathbf{T}$. Select a point from $\mathfrak{F}$ and let τ be the fractional linear transformation which maps $\varDelta$ to the upper half plane and carries the selected point to ∞. Then if $\tilde{\alpha}_t = \tau\alpha_t\tau^{-1}$, $\{\tilde{\alpha}_t\}_{t\in\mathbf{R}}$ is a continuous one-parameter group of fractional linear transformations of the upper half plane onto itself which fixes ∞. Therefore, for each t in $\mathbf{R}$, there is an $a_t > 0$ and a b_t in $\mathbf{R}$ such that $\tilde{\alpha}_t(z) = a_tz + b_t$ for all z; i.e., $\{\tilde{\alpha}_t\}_{t\in\mathbf{R}}$ is a one-parameter subgroup of the well known "$ax + b$ group". A moment's reflection directed toward the exponential map on the Lie algebra of the "$ax+b$ group" reveals that γ and β exist in $\mathbf{R}$ such that $\tilde{\alpha}_t(z)=e^{t\gamma}z+\beta((e^{t\gamma}-1)/\gamma)$ for all t in $\mathbf{R}$ and all z, where the expression in parentheses is to be interpreted as t if $\gamma=0$. We assert that $\gamma=0$. For if not, then $\{\tilde{\alpha}_t\}_{t\in\mathbf{R}}$ fixes the point $-\beta/\gamma$ on the real axis and so leaves invariant each of the infinite rays on either side of $-\beta/\gamma$. Back on the disc this implies that μ is not ergodic under $\{\alpha_t\}_{t\in\mathbf{R}}$ which in turn implies that m is not ergodic and this is contrary to Theorem VI of [11]. Hence $\gamma=0$, and so $\beta\neq 0$ for otherwise each α_t would be the identity. Thus we find in this case, that $N_1=\mathfrak{P}=\{\tau^{-1}(\infty)\}$, and that $\{\alpha_t\}_{t\in\mathbf{R}}$ acts transitively and freely on $\mathbf{T}\setminus N_1$; i.e., no α_t, $t\neq 0$, fixes any point in $\mathbf{T}\setminus N_1$.

Step II. Let $\mathfrak{O}$ be the orbit upon which m is concentrated and assume first that $\mathfrak{O}$ is not periodic. Then there is a one to one continuous function η from $\mathbf{R}$ onto $\mathfrak{O}$ such that $\eta(s+t)=\eta(s)+t$ for all s and t in $\mathbf{R}$. Since m is quasi-invariant [11, Theorem III], m is equivalent to Lebesgue measure on $\mathbf{R}$ transplanted to $\mathfrak{O}$ via η because, as is well known, every non-zero quasi-invariant measure on $\mathbf{R}$ is equivalent to Lebesgue measure. From this it follows that there is a ξ in $\mathfrak{O}$ and a positive nonvanishing function h in $L^1(\mathbf{R})$ such that $m=\delta_\xi * h$. When $H^\infty(\mathbf{R})$ is transplanted to $\mathfrak{O}$ via η, we may regard it as a proper weak-$*$ closed subalgebra of $L^\infty(m)$ which contains $\mathfrak{A}$. On the other hand, as we pointed out earlier, $H^\infty(m)$ is a maximal weak-$*$ closed subalgebra of $L^\infty(m)$ and so it must coincide with the transplant of $H^\infty(\mathbf{R})$ to $\mathfrak{O}$. Hence it follows that the measure hdt, where dt is Lebesgue measure on $\mathbf{R}$, is multiplicative on $H^\infty(\mathbf{R})$. But it is well known that this implies that there is a z in the upper half plane such that $h(t)=P_z(t)$. Therefore, if $\tau=\mathrm{Re}\,(z)$, $y=\mathrm{Im}\,(z)$, and if $x=\xi+\tau$, then $\delta_\xi * P_z=(T^*_{-\tau}\delta_\xi)*P_{iy}=\delta_x * P_{iy}$ as was to be shown.

If, on the other hand, $\mathfrak{O}$ is a periodic orbit, then since the hypothesis of the theorem excludes the possibility that $\mathfrak{O}$ reduces to a point, we may find a homeomorphism η' from $\mathbf{T}$ onto $\mathfrak{O}$ such that for t in $\mathbf{R}$ and z in $\mathbf{T}$, $\eta'(e^{it}z)=\eta'(z)+t$. It follows that η' implements an isomorphism between the algebra $\mathfrak{A}|_{\mathfrak{O}}$ obtained by restricting the functions in $\mathfrak{A}$ to $\mathfrak{O}$ and the disc algebra on $\mathbf{T}$. Because of this, the well known expression for the representing measures for the points in the maximal ideal space of the disc algebra, and the hypothesis that m is *not* invariant, we may assert that there is an ξ in $\mathfrak{O}$ and a z in the upper half plane such that $m=\delta_\xi * P_z$. Translating ξ by the real part of z if necessary, we arrive again at the desired conclusion.

As for the uniqueness of the representation of m as $\delta_x * P_{iy}$, suppose $\delta_x * P_{iy}=\delta_{x_1} * P_{iy_1}$. Then these two measures must be concentrated on the same orbit. Consequently, there exists a t in $\mathbf{R}$ such that $x_1=x+t$. But then it follows easily that the two measures $P_{iy}dt$ and $P_{t+iy_1}dt$ on $\mathbf{R}$ are equal and so $t=0$ and $y=y_1$. This shows that $x=x_1$ as well and we may conclude that the proof of Theorem I is complete.

§ 5. In this section we present our characterization of the maximal ideal space of $\mathfrak{A}$. As we noted in the introduction we must consider two cases. The first is when the unique invariant measure μ on X is not a point mass on X. In this case we let $\mathbf{D}$ denote the quotient space obtained from $X \times [0, 1]$ by identifying the slice $X \times \{0\}$ to a point. The point in $\mathbf{D}$ which is the image of $X \times \{0\}$ under the quotient map will be denoted by $\hat{0}$ and the points in $X \times (0, 1]$ will be identified with their images under the quotient map. The space $\mathbf{D}$ may be regarded as a big disc with boundary X and origin $\hat{0}$ and it is possible to think of a pair (x, r) in $X \times (0, 1]$ as the polar coordinates for the point in $\mathbf{D} \setminus \{0\}$ to which it corresponds under the quotient map. In the second case, which is when μ is a point mass δ_{x_0} on X (so that x_0 is fixed under the action of $\mathbf{R}$ on X), we let $\mathbf{H}$ denote the quotient space obtained from $X \times [0, 1]$ by identifying the closed set $(X \times \{0\}) \cup (\{x_0\} \times [0, 1])$ to a point. We shall identify the points (x, r) in $(X \setminus \{x_0\}) \times (0, 1]$ with their images in $\mathbf{H}$ under the quotient map and we shall denote the image of $(X \times \{0\}) \cup (\{x_0\} \times [0, 1])$ in $\mathbf{H}$ by $\widehat{\infty}$. The choice of the symbols $\mathbf{H}$ and $\widehat{\infty}$ is motivated by the observation that if X is the one point compactification of the real line then $\mathbf{H}$ is homeomorphic to the (closed) upper half plane with $\widehat{\infty}$ corresponding to the point at infinity. Of course in this case $\mathbf{H}$ is conformally equivalent to the closed unit disc but in general it is not possible to make such an identification between $\mathbf{H}$ and $\mathbf{D}$.

Recall that if C_0 denotes the closure in $C(X)$ of the space of functions with positive spectra, then because the flow $(X, \mathbf{R})$ is strictly ergodic, C_0 is a maximal ideal in $\mathfrak{A}$ and μ is its representing measure (see [11, Theorem V]). If μ is not a point mass on X, then we define a map Γ from $\mathbf{D}$ into $\mathcal{M}_{\mathfrak{A}}$ as follows:

i) $\Gamma(\hat{0}) = C_0$;

ii) $\Gamma(x, 1) = x$ for all x in X; and

iii) if $0 < r < 1$, then $\Gamma(x, r)$ is the point represented by the measure $\delta_x * P_{iy}$ where $y = -\log r$.

On the other hand, if μ is a point mass δ_{x_0} on X, so that $C_0 = x_0$, then we define a map Γ' from $\mathbf{H}$ to $\mathcal{M}_{\mathfrak{A}}$ as follows:

i) $\Gamma'(\widehat{\infty}) = x_0$;

ii) $\Gamma'(x, 1) = x$, if $x \neq x_0$;

iii) if $0 < r < 1$, and if $x \neq x_0$, then $\Gamma'(x, r)$ is the point represented by the measure $\delta_x * P_{iy}$ where $y = -\log r$.

Observe that the maps Γ and Γ' are well defined on all of $\mathbf{D}$ and $\mathbf{H}$, respectively.

THEOREM II. A. *If μ is not a point mass on X, then the map Γ defined above is a homeomorphism from $\mathbf{D}$ onto $\mathcal{M}_{\mathfrak{A}}$. Moreover, for each r, $0 < r < 1$, the point*

$\Gamma(x, r)$ lies in a nontrivial Gleason part and the Gleason part containing C_0 is nontrivial if and only if μ is concentrated on an orbit.

B. *If μ is a point mass δ_{x_0} on X, then the map Γ' defined above is a homeomorphism from $\mathbf{H}$ onto $\mathcal{M}_{\mathfrak{A}}$ and for each point (x, r) in $\mathbf{H}$, $x \neq x_0$, $0 < r < 1$, $\Gamma'(x, r)$ lies in a nontrivial Gleason part.*

Proof. We prove A; the proof of B is similar and so will be omitted. Proposition 3.1 and Theorem I imply that Γ is bijective and continuous at each point in $\mathbf{D}\diagdown\{0\}$. Since $\mathbf{D}$ is compact and $\mathcal{M}_{\mathfrak{A}}$ is Hausdorff it suffices to check the continuity of Γ at 0 in order to show that Γ is a homeomorphism. To this end let $\{(x_n, r_n)\}_{n=1}^{\infty}$ be a sequence converging to $\hat{0}$ in $\mathbf{D}$ and let $y_n = -\log r_n$ so that $\lim_{n\to\infty} y_n = \infty$. (Note that it suffices to consider sequences since X is separable.) Then since any weak-$*$ cluster point of the sequence $\{\delta_{x_n} * P_{iy_n}\}_{n=1}^{\infty}$ is μ by Proposition 3.1 (ii) and the strict ergodicity of the flow, it follows that $\lim_{n\to\infty} \Gamma(x_n, r_n) = C_0 = \Gamma(\hat{0})$. Whence Γ is continuous at $\hat{0}$ and the first part of the proof is complete.

The fact that each $\Gamma(x, r)$, $0 < r < 1$, lies in a nontrivial Gleason part follows from Corollary 3.3. Suppose that C_0 also lies in a nontrivial Gleason part and suppose m represents some other point in the part containing C_0. Then since m is not invariant, because $m \neq \mu$, and since m is not a point mass, m is concentrated on an orbit by Theorem I. Since m and μ are mutually absolutely continuous [7, p. 143] we may conclude that μ is also concentrated on an orbit. Conversely, if μ is concentrated on an orbit $\mathfrak{O}$, then it is easy to see that for each x in $\mathfrak{O}$ and each $y > 0$, the measures $\delta_x * P_{iy}$ and μ are mutually absolutely continuous. Consequently the Gleason part containing C_0 is nontrivial [7, p. 144] and we may conclude the proof is complete.

One noteworthy corollary to Theorem II is the fact that $\mathcal{M}_{\mathfrak{A}}$ *is contractible* and consequently the invertible elements in $\mathfrak{A}$ have logarithms [7, p. 91]. This certainly is not obvious a priori.

§ 6. In this section we discuss possibilities of extending the analysis presented above to more general situations. In one rather special case it is possible to apply the above arguments to characterize the maximal ideal space of $\mathfrak{A}$ when the flow $(X, \mathbf{R})$ is not strictly ergodic. This is the case when it is possible to fiber X smoothly over a space Y whose points are closed invariant subsets X_y of X, $y \in Y$, such that on each X_y the flow is strictly ergodic. It develops that $\mathcal{M}_{\mathfrak{A}}$ may be fibered similarly where the fibers are the maximal ideal spaces for the algebras $\mathfrak{A}_y$ associated with the flows $(X_y, \mathbf{R})$. In general, of course, no such fibering of X exists.

One of the principal obstacles which we have encountered in trying to extend our results is the problem of deciding when $\mathfrak{A}$ belongs to any of the well known classes of abstract function algebras. In particular we would like to know conditions other than the strict ergodicity of the flow under which every point in $\mathcal{M}_{\mathfrak{A}}$ has a unique representing measure. In this direction we are able to prove the following curious fact: Suppose $(X, \mathbf{R})$ is arbitrary and that σ is a point in $\mathcal{M}_{\mathfrak{A}} \setminus X$. If each representing measure for σ is singular with respect to *every* finite invariant measure on X, then, in fact, σ has a unique representing measure and, assuming that X is separable, it may be written as $\delta_x * P_{iy}$ for some x in X and $y > 0$. What the situation is regarding the other points in $\mathcal{M}_{\mathfrak{A}}$ remains a mystery to us. Of course the nature of the flow has some bearing on this problem and one class of flows for which this problem seems tractible is the class of distal flows. As a test question we ask: If the flow $(X, \mathbf{R})$ is distal, is $\mathfrak{A}$ a Dirichlet algebra on X? One reason for suspecting that the answer is yes is that all distal flows can be built up in a very explicit manner from almost periodic flows (see [4]) and the algebra of analytic functions associated with an almost periodic flow is, after all, a Dirichlet algebra. On the other hand, distal flows, even minimal ones, need not be strictly ergodic (see [3]) and consequently there is reason to suspect that the answer to the question is no. Nonetheless, because of the way distal flows are constructed, it should be possible, if the answer is no, to decide precisely why $\mathfrak{A}$ fails to be a Dirichlet algebra.

We conclude by indicating another proof of part of Theorem II which is valid even when the space X is not separable and which may indicate a way of removing all considerations of separability from our arguments. We assert that if the flow $(X, \mathbf{R})$ is strictly ergodic where X need not be separable, and if the Gleason part containing C_0 is nontrivial, then the unique invariant probability measure μ is concentrated on an orbit. For if Z is the Wermer imbedding function (see [7, p. 133]), then it is possible to show that the sequence $\{Z^n\}_{n=-\infty}^{\infty}$ constitutes an orthonormal basis for $L^2(\mu)$ such that for some nonzero real λ,

$$Z^n(x+t) = e^{in\lambda t} Z^n(x) \quad \text{a.e.} \quad \mu$$

for each integer n and each t in $\mathbf{R}$. This means that the unitary representation of $\mathbf{R}$ on $L^2(\mu)$ induced by the action of $\mathbf{R}$ on X has pure point spectrum consisting of integral multiples of λ. The proof is completed by appealing to an argument of Rohlin [13, p. 227] which he used to obtain a sharpened form of a well known theorem of von Neumann concerning ergodic $\mathbf{R}$-actions with pure point spectrum.

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