On the regularity of the distribution of the Fekete points of a compact surface in R^n

PETER SJÖGREN

University of Göteborg, Sweden

1. Introduction

The question to be considered in this note can be stated in terms of classical physics. Let a finite number of equal point charges find a stable equilibrium distribution within an isolated conductor. Then how well does this distribution approximate the equilibrium distribution of the same total charge?

We let S be a compact $(n-1)$-dimensional surface of class $C^{1,\alpha}$ in $\mathbf{R}^n$, which is supposed to be the common boundary of two domains. Only the case $n \geq 3$ will be considered, although the method of proof works also in the plane for a simple closed curve of the same regularity. For $N > 1$ we let $(z_{N1}, \ldots, z_{NN})$ be a system of Fekete points of S, i.e., an N-tuple of points in S which among all N-tuples of points in S minimizes the energy

$$\frac{1}{N^2} \sum_{i \neq j} \sum \frac{1}{|z_{Ni} - z_{Nj}|^{n-2}}$$

of the mass distribution μ_N consisting of point masses $1/N$ at each z_{Ni}. Here we omit the infinite energy of each point mass. The Fekete points need not be uniquely determined by S and N.

Let λ be the equilibrium distribution of total mass 1 in S, to which μ_N will be compared. In two dimensions several estimates of $\mu_N - \lambda$ have been made. In that case our method yields the result that any subarc B of the curve satisfies

$$|\mu_N(B) - \lambda(B)| \leq \text{const.} \frac{\sqrt{\log N}}{\sqrt{N}} .$$

This should be compared to Kleiner's bound, const. $\log N/\sqrt{N}$ in [1]. For analytic curves Pommerenke has given much sharper estimates in for example [3], where further references can be found.

Kleiner's paper [2] contains as a special case of an auxiliary result an estimate of this kind in more than two dimensions. He assumes that the surface S is of class C^2 and chooses a fixed subdivision $S_1, \ldots, S_m$ of S such that each S_i is the image under a C^2 mapping φ_i of a unit square. As test sets B Kleiner takes the images under every φ_i of all axial subrectangles of the unit square, and he obtains the estimate

$$|\mu_N(B) - \lambda(B)| \leq \text{const.}\, N^{-\frac{1}{3(n-1)}}.$$

We consider a larger class of test sets than Kleiner does, and our estimate is sharper. However, at least for more regular surfaces, it is probably not best possible.

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2. Formulation of the theorem

To define test subsets of S, we need the concept of K-regular boundary, which is defined in Sjögren [4], as follows.

Definition. A set $B \subset S$ is said to have K-regular boundary $(K > 0)$ if for any $d > 0$

$$\int_{B_d^*} dS \leq Kd,$$

where

$$B_d^* = \{x \in S \colon \varrho(x, B^*) \leq d\}.$$

Here B^* is the boundary of B in the relative topology of S, while ϱ means distance in $\mathbf{R}^n$. The measure dS is the area measure of S.

THEOREM. *For any $B \subset S$ having K-regular boundary*

$$|\mu_N(B) - \lambda(B)| \leq CK^{\frac{1}{2}}N^{-\frac{1}{2(n-1)}}, \quad C = C(S).$$

Remark 1. As can be seen geometrically, it is to be expected that the exponent of N depends on the dimension.

Remark 2. If we let B be the intersection of S and suitable balls with centres in S, we can deduce certain estimates of distances from a point in S to the closest Fekete point. For example, the distance d from a Fekete point z_{Ni} to its closest neighbour z_{Nj} satisfies

$$d \leq CN^{-\gamma}, \quad C = C(S),$$

where

$$\gamma = \frac{1}{2(n-1)^2}\,.$$

3. Proof of the theorem

For measures μ and ν with compact support in $\mathbf{R}^n$, we define the potential, mutual energy, and energy norm by, respectively,

$$U^\mu(x) = \int \frac{d\mu(y)}{|x-y|^{n-2}}\,,$$

$$(\mu, \nu) = \int U^\mu d\nu = \int U^\nu d\mu,$$

and

$$\|\mu\| = (\mu, \mu)^{\frac{1}{2}}.$$

The letter C will denote various constants, all of which depend only on S.

We let E_N be the finite part of the energy of μ_N:

$$E_N = \frac{1}{N^2} \sum_{i \neq j}\sum \frac{1}{|z_{Ni} - z_{Nj}|^{n-2}}$$

and start by estimating the potential U^{μ_N} on S. If

$$f_i(z) = \sum_{j \neq i} \frac{1}{|z - z_{Nj}|^{n-2}}\,,$$

the minimum property of $\{z_{Ni}\}$ implies that

$$f_i(z) \geq f_i(z_{Ni})$$

if $z \in S$. Adding these inequalities, we get

$$N(N-1)U^{\mu_N}(z) \geq N^2 E_N. \tag{1}$$

Using some ideas from Kleiner [2], we shall estimate E_N in terms of the energy of λ, which we call E and which is also the value on S of the potential U^λ. Let ν_{Ni} be the uniform distribution of the mass $1/N$ on the sphere $|z - z_{Ni}| = r$, where $r > 0$ will be determined later. We put

$$\nu_N = \sum_i \nu_{Ni}\,.$$

Since

$$U^{\nu_{Ni}}(z) \leq \frac{1}{N} \frac{1}{|z - z_{Ni}|^{n-2}}$$

for all z, we see that for $i \neq j$

$$\frac{1}{N^2} \frac{1}{|z_{Ni} - z_{Nj}|^{n-2}} \geq (\nu_{Ni}, \nu_{Nj}).$$

Adding, we get

$$E_N \geq \|\nu_N\|^2 - \sum_i \|\nu_{Ni}\|^2 = \|\nu_N\|^2 - \frac{1}{Nr^{n-2}} . \tag{2}$$

It follows from Theorem 2.4 in Widman [5] that the gradient of U^λ is bounded near S, so that

$$|(\nu_N, \lambda) - E| = \left| \int (U^\lambda - E) d\nu_N \right| \leq Cr.$$

Therefore

$$\|\nu_N\|^2 = - \|\lambda\|^2 + 2(\nu_N, \lambda) + \|\nu_N - \lambda\|^2 \geq E - Cr. \tag{3}$$

Now (1), (2), and (3) imply

$$U^{\mu_N} \geq E - Cr - \frac{1}{(N-1)r^{n-2}}$$

on S. If we take

$$r = N^{-\frac{1}{n-1}},$$

we see that

$$\sup_S U^{\lambda - \mu_N} \leq CN^{-\frac{1}{n-1}},$$

and the theorem follows from Sjögren [4, Theorem 1].

References

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Peter Sjögren
Department of Mathematics
University of Göteborg
Fack
S-402 20 Göteborg
Sweden