A remark on embedding theorems for Banach spaces of distributions

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1. Introduction and results

This note is more or less an appendix to the paper [9]. We use the notions of [9] and recall some of them. R_n is the n-dimensional Euclidean space: $x = (x_1, \ldots, x_n) \in R_n$. $S(R_n)$ is the Schwartz space of rapidly decreasing (complex) infinitely differentiable functions, $S'(R_n)$ is the dual space of tempered distributions (with the strong topology). F is the Fouriertransformation in $S'(R_n)$, F^{-1} the inverse Fouriertransformation. We use special systems of functions $\{\varphi_k\}_{k=0}^{\infty}$ (see [9], 4.2.1) with

1. $\varphi_k(x) \in S(R_n)$, $F\varphi_k(x) \geqq 0$; $k = 0, 1, 2, \ldots$
2. $\exists N$; $N = 1, 2, \ldots$; with $\operatorname{supp} F\varphi_k \subset \{\xi | 2^{k-N} \leqq |\xi| \leqq 2^{k+N}\}$ for $k = 1, 2, \ldots$; $\operatorname{supp} F\varphi_0 \subset \{\xi | |\xi| \leqq 2^N\}$; (supp denotes the support of a function).
3. $\exists\, c_1 > 0$ with $c_1 \leqq (\sum_{j=0}^{\infty} F\varphi_j)(\xi)$;
4. $\exists\, c_2 > 0$ with $|(D^\alpha F\varphi_k)(\xi)| \leqq c_2 |\xi|^{-|\alpha|}$ for $0 \leqq |\alpha| \leqq [n/2] + 1$; $k = 1, 2, \ldots$

The most important system of functions of this type is the following. We consider a function $\varphi(x) \in S(R_n)$; $(F\varphi)(x) \geqq 0$;

$$\operatorname{supp} F\varphi \subset \{\xi | 2^{-N} \leqq |\xi| \leqq 2^N\}; \quad (F\varphi)(\xi) > 0 \ \text{ for } \ 1/\sqrt{2} \leqq |\xi| \leqq \sqrt{2}\,. \tag{1}$$

It is not difficult to see that the functions $\varphi_k(x)$ with

$$(F\varphi_k)(\xi) = (F\varphi)(2^{-k}\xi); \quad k = 1, 2, \ldots; \tag{2}$$

by suitable choice of $\varphi_0(x)$ are a system of the described type.

Now we define the spaces $F^s_{p,q} = F^s_{p,q}(R_n)$ and $B^s_{p,q} = B^s_{p,q}(R_n)$. Let $-\infty < s < \infty$; $1 < p < \infty$; $1 < q < \infty$; $\{\varphi_k\}_{k=0}^{\infty}$ is a system of the described type. We set

$$F^s_{p,q} = \left\{ f | f \in S'(R_n),\ \|\{f * \varphi_k\}\|_{L_p(l^s_q)} = \left[\int_{R_n} \left(\sum_{k=0}^{\infty} 2^{sqk} |(f * \varphi_k)(x)|^q \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} < \infty \right\}. \tag{3}$$

$f * \varphi_k = (2\pi)^{n/2} F^{-1}(F\varphi_k Ff)$ is the convolution of f and φ_k. In the same way we define for $-\infty < s < \infty$; $1 < p < \infty$; $1 \leqq q \leqq \infty$;

$$B^s_{p,q} = \left\{ f | f \in S'(R_n),\ \|\{f * \varphi_k\}\|_{l^s_q(L_p)} = \left(\sum_{k=0}^{\infty} 2^{sqk} \|f * \varphi_k\|^q_{L_p}\right)^{\frac{1}{q}} < \infty \right\}, \tag{4}$$

(with the usual modification for $q = \infty$). $L_p = L_p(R_n)$ is the usual space of Lebesgue-measurable complex functions with $|f|^p$ integrable. In [9], theorem 4.2.2, it is shown that the spaces $F^s_{p,q}$ (and $B^s_{p,q}$) with the norms $\|\{f * \varphi_k\}\|_{L_p(l^s_q)}$ (and $\|\{f * \varphi_k\}\|_{l^s_q(L_p)}$) are Banach spaces, and independent of the choice of the system $\{\varphi_k\}$. At least for $s > 0$ the spaces $B^s_{p,q}$ are the usual Besov spaces introduced by Besov [1], see also Nikol'skij [5] and Taibleson [8]. The equivalence of the usual definitions and the definition (4) is proved in [9], see also [10]. The idea of using definitions of type (4) is due to Nikol'skij [5] and Peetre [6, 7]. The spaces $F^s_{p,q}$ are introduced by the author in [9]. Special cases are the well-known Lebesgue spaces

$$F^s_{p,2} = H^s_p = \{f | f \in S'(R_n),\ \ F^{-1}(1 + |\xi|^2)^{\frac{s}{2}} Ff \in L_p(R_n)\}. \tag{5}$$

(See [9], theorem 4.2.6). Further we define the spaces $C^t = C^t(R_n)$; $t \geqq 0$. $C = C^0$ is the set of all complex continuous functions $f(x)$ in R_n with $f(x) \to 0$ for $|x| \to \infty$. Let t be an integer. Then is

$$C^t = \{f | D^\alpha f \in C \ \text{ for } \ |\alpha| \leqq t\}.$$

(We use the usual notation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}; \ \ \alpha = (\alpha_1, \ldots, \alpha_n); \ \ |\alpha| = \sum_{j=1}^{n} \alpha_j; \ \ \alpha_j \ \text{ integers} \geqq 0\Big).$$

C^t with the norm

$$\|f\|_{C^t} = \sum_{|\alpha| \leqq t} \max_{x \in R_n} |D^\alpha f(x)|$$

becomes a Banach space. Let be $t \neq$ integer. We set

$$t = [t] + \{t\}; \ \ [t] \ \text{ integer}; \ \ 0 < \{t\} < 1;$$

and define

$$C^t = \left\{ f | f \in C^{[t]},\ \sup_{\substack{x \neq y \\ x, y \in R_n}} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|^{\{t\}}} < \infty \ \text{ for all } \ \alpha \ \text{ with } \ |\alpha| = [t] \right\}.$$

C^t with the norm

$$\|f\|_{C^t} = \|f\|_{C^{[t]}} + \sum_{|\alpha|=[t]} \sup_{\substack{x \neq y \\ x, y \in R_n}} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{\{t\}}}$$

becomes a Banach space.

The aim of this paper is the proof of the following theorem.

THEOREM. (a) *Let* $\infty > q \geqq p > 1;\ 1 \leqq r \leqq \infty;\ -\infty < t \leqq s < \infty;$ *and*

$$s - n/p = t - n/q\,. \tag{6}$$

Then holds

$$B^s_{p,r} \subset B^t_{q,r}\,. \tag{7}$$

(b) *Let* $\infty > q \geqq p > 1;\ 1 < r < \infty;\ -\infty < t \leqq s < \infty;$ *and*

$$s - n/p = t - n/q\,.$$

Then holds

$$F^s_{p,r} \subset F^t_{q,r}\,. \tag{8}$$

(c) *Let* $1 < p < \infty;\ t \geqq 0;\ 1 \leqq r \leqq \infty$. *Then holds*

$$B^{\frac{n}{p}+t}_{p,1} \subset C^t \tag{9}$$

and

$$B^{\frac{n}{p}+t}_{p,r} \subset C^t \quad for \quad t \neq integer\,. \tag{10}$$

(d) *Let* $1 < p < \infty;\ 1 < r < \infty;\ 0 < t \neq$ *integer. Then holds*

$$F^{\frac{n}{p}+t}_{p,r} \subset C^t\,. \tag{11}$$

The first part is well known, see for instance [5]. We give two independent proofs of (7). The first proof is very short and uses the definition (4). A similar proof is given by Peetre [6]. The second proof is inspired by a paper of Yoshikawa [11]. Perhaps it will be interesting from the methodical point of view. A special case of theorem (b) is (see (5))

$$H^s_p \subset H^t_q;\ 1 < p \leqq q < \infty;\ s - n/p = t - n/q\,.$$

This relation is also known [5]. The embedding theorems (9) and (10) are also known. A special case of (11) is

$$H^{\frac{n}{p}+t}_p \subset C^t;\ 1 < p < \infty;\ 0 < t \neq \text{integer.}$$

2. First proof of theorem (a)

We choose two systems of functions $\{\varphi_k(x)\}_{k=0}^{\infty}$ and $\{\varrho_k(x)\}_{k=0}^{\infty}$ of type (1), (2) with

$$(F\varrho)(\xi) = 1 \quad \text{for} \quad \xi \in \operatorname{supp} F\varphi .$$

Let be $f \in B^s_{p,r}$. With

$$1/\sigma = 1 - 1/p + 1/q \tag{12}$$

follows from Young's inequality for convolutions that

$$\|f * \varphi_k\|_{L_q} = \|f_k * \varphi * \varrho_k\|_{L_q} \leqq \|\varrho_k\|_{L_\sigma}\|f * \varphi_k\|_{L_p} . \tag{13}$$

We have

$$\varrho_k(x) = 2^{kn}\varrho(2^k x); \quad k = 1, 2, \ldots;$$

and

$$\|\varrho_k\|_{L_\sigma} = 2^{kn\left(1-\frac{1}{\sigma}\right)}\|\varrho\|_{L_\sigma}; \quad k = 1, 2, \ldots .$$

We obtain from (12), (13) and the definition (4)

$$\|f\|_{B^t_{q,r}} \leqq c_1\|\{f * \varphi_k\}\|_{l^t_r(L_q)} \leqq c_2\|\{f * \varphi_k\}\|_{l_r^{t+n\left(\frac{1}{p}-\frac{1}{q}\right)}(L_p)} \leqq c_3\|f\|_{B^s_{p,r}} .$$

This proves theorem (a).

3. Proof of theorem (b)

Let $\{\varphi_k(x)\}_{k=0}^{\infty}$ and $\{\varrho_k(x)\}_{k=0}^{\infty}$ be the same systems of functions as in Section 2. We consider the matrix $\{K_{k,j}(x)\}_{-\infty<k,j<\infty}$ with

$$(FK_{k,k})(x) = |x|^t 2^{-kt}(F\varrho_k)(x); \quad k = 1, 2, \ldots; \quad K_{k,j}(x) = 0 \quad \text{otherwise.}$$

It is not difficult to see that the assumptions of the multiplier theorem 3.5 (b) of [9] hold. This shows

$$\|[\sum_{k=1}^{\infty} |F^{-1}[|x|^t F(f * \varphi_k)]|^r]^{\frac{1}{r}}\|_{L_q} = \|[\sum_{k=1}^{\infty} |F^{-1}[|x|^t 2^{-kt} F\varrho_k 2^{kt} F(f * \varphi_k)]|^r]^{\frac{1}{r}}\|_{L_q}$$

$$\leqq c\|[\sum_{k=1}^{\infty} (2^{kt}|f * \varphi_k|)^r]^{\frac{1}{r}}\|_{L_q} .$$

For the »inverse» multiplier $\{\tilde{K}_{k,j}(x)\}_{-\infty<k,j<\infty}$

$$F\tilde{K}_{k,k}(x) = |x|^{-t}2^{kt}(F\varrho_k)(x); \quad k = 1, 2, \ldots; \quad \tilde{K}_{k,j}(x) = 0 \quad \text{otherwise;}$$

the assumptions of theorem 3.5 (b) of [9] are also true. So we can prove the opposite direction of the last inequality. It follows

$$\|f\|_{F^t_{q,r}} \sim \|f * \varphi_0\|_{L_q} + \|[\sum_{k=1}^{\infty} |F^{-1}[|x|^t F(f * \varphi_k)]|^r]^{\frac{1}{r}}\|_{L_q} . \tag{14}$$

With the aid of this equivalent norm in $F^t_{q,r}$ it is not difficult to prove theorem (b). It is known that

$$F^{-1}|x|^{-\frac{n}{\varkappa}} = c_\varkappa |x|^{-\frac{n}{\varkappa'}};\ 1 < \varkappa < \infty;\ \frac{1}{\varkappa} + \frac{1}{\varkappa'} = 1 ;$$

see [3]. Let be $f \in S(R_n)$ and $q > p$. The last relation and (6) show

$$F^{-1}[|x|^t F(f * \varphi_k)](x) = c \int_{R_n} (F^{-1}|\xi|^{-(s-t)})(x - y) \cdot F^{-1}[|\xi|^s F(f * \varphi_k)](y) dy$$

$$= c' \int_{R_n} |x - y|^{-n\left(1 - \frac{1}{p} + \frac{1}{q}\right)} F^{-1}[|\xi|^s F(f * \varphi_k)](y) dy .$$

With the aid of the generalized triangle inequality we find

$$(\sum_{k=1}^{\infty} |F^{-1}[|x|^t F(f * \varphi_k)](x)|^r)^{\frac{1}{r}}$$

$$\leqq c \int_{Rn} |x - y|^{-n\left(1 - \frac{1}{p} + \frac{1}{q}\right)} (\sum_{k=1}^{\infty} |F^{-1}[|\xi|^s F(f * \varphi_k)](y)|^r)^{\frac{1}{r}} dy .$$

With the aid of the Hardy-Littlewood-Sobolev inequality, see [3], follows

$$\|(\sum_{k=1}^{\infty} |F^{-1}[|x|^t F(f * \varphi_k)]|^r)^{\frac{1}{r}}\|_{L_q} \leqq c \|(\sum_{k=1}^{\infty} |F^{-1}[|x|^s F(f * \varphi_k)]|^r)^{\frac{1}{r}}\|_{L_p} .$$

Together with (14) this shows

$$\|f\|_{F^t_{q,r}} \leqq c \|f\|_{F^s_{p,r}} ,\ f \in S(R_n) .$$

$S(R_n)$ is dense in $F^s_{p,r}$, [9]. This proves theorem (b).

4. Proof of theorem (c), (d)

First we prove (9). Let m be an integer; $m \geqq 0$. Let $f \in B^{\frac{n}{p}+m}_{p,1}$. We choose a system $\{\varphi_k\}_{k=0}^{\infty}$ of type (1), (2) with

$$\sum_{k=0}^{\infty} (F\varphi_k)(\xi) = (2\pi)^{-n/2};\ \xi \in R_n .$$

$\varrho_k(x)$ has the same meaning as in Section 2. Then holds for $|\alpha| \leqq m$

$$D^\alpha f \underset{S'}{=} \sum_{k=0}^{\infty} D^\alpha f * \varphi_k \underset{S'}{=} \sum_{k=0}^{\infty} f * \varphi_k * D^\alpha \varrho_k .$$

($\underset{S'}{=}$: convergence in S', see [9]). We have

$$(D^\alpha \varrho_k)(x) = 2^{kn+|\alpha|k} (D^\alpha \varrho)(2^k x) .$$

With the aid of Young's inequality follows in the same way as in the second section, $(1/p + 1/p' = 1)$,

$$\sum_{k=0}^{\infty} \|D^\alpha f * \varphi_k\|_{L_\infty} \leqq \sum_{k=0}^{\infty} \|D^\alpha \varrho_k\|_{L_{p'}} \|f * \varphi_k\|_{L_p} \leqq c \sum_{k=0}^{\infty} 2^{kn - \frac{kn}{p'} + |\alpha|k} \|f * \varphi_k\|_{L_p} \leqq c' \|f\|_{B_{p,1}^{\frac{n}{p}+m}} .$$

The last estimate shows the convergence of $\sum_{k=0}^{\infty} D^\alpha f * \varphi_k$ in $L_\infty(R_n)$. On the other hand the sum converges in $S'(R_n)$ to $D^\alpha f$. So we obtain

$$\sum_{|\alpha| \leqq m} \sup_{x \in R_n} |D^\alpha f(x)| \leqq c \|f\|_{B_{p,1}^{\frac{n}{p}+m}} .$$

(9) with $t = m$ follows now from the fact that $C_0^\infty(R_n)$ (the set of all complex infinitely differentiable functions with compact support in R_n) is dense in $B_{p,1}^{\frac{n}{p}+m}$, [9].

Next we prove (10). Let be $0 < t \neq$ integer. We choose an integer m with $t < m$. Then holds

$$C^t = (C, C^m)_{\frac{t}{m}, \infty} . \tag{15}$$

$(\cdot, \cdot)_{\Theta, r}$ denotes the interpolation spaces in the sense of Lions-Peetre [4], see also [2]. We sketch a short proof of the last relation. The operator

$$A_j f = \frac{\partial f}{\partial x_j}$$

with the domain of definition

$$D(A_j) = \left\{ f | f \in C, \frac{\partial f}{\partial x_j} \in C \right\}$$

is the infinitesimal generator of the semigroup in C

$$G_j(\tau) f = f(x_1, \ldots, x_{j-1}, x_j + \tau, x_{j+1}, \ldots, x_n); \quad j = 1, \ldots, n .$$

(15) follows now from the interpolation theory for commutative semigroups [2, 4] and the theory of equivalent norms in these spaces, [10]. (10) follows now from (9), (15), and the general interpolation theory [4, 9],

$$B_{p,r}^{\frac{n}{p}+t} = (B_{p,1}^{\frac{n}{p}}, B_{p,1}^{\frac{n}{p}+m})_{\frac{t}{m},r} \subset (C, C^m)_{\frac{t}{m},r} \subset (C, C^m)_{\frac{t}{m},\infty} = C^t .$$

We obtain (11) from (10) and the inclusion property, [9], theorem 5.2.3,

$$F_{p,r}^{\frac{n}{p}+t} \subset B_{p,\max(p,r)}^{\frac{n}{p}+t} .$$

5. Second proof of theorem (a)

5.1. *A special semigroup of operators*

We consider the homogeneous polynomial of degree $2m$ with real coefficients

$$a(x) = \sum_{|\alpha|=2m} a_\alpha x^\alpha; \quad x = (x_1, \ldots, x_n) \in R_n ;$$

$\alpha = (\alpha_1, \ldots, \alpha_n)$ multiindex; $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.
Let for a suitable number $c > 0$

$$(-1)^m \sum_{|\alpha|=2m} a_\alpha x^\alpha \geqq c|x|^{2m}, \quad x \in R_n . \tag{16}$$

It is easy to see that $G(\tau)$; $\tau \geqq 0$;

$$(G(\tau)f)(x) = e^{(-1)^{(m+1)}\tau a(x)} f(x); \quad f \in L_p(R_n) ;$$

is a strong continuous semigroup of operators in $L_p(R_n)$; $1 < p < \infty$; with the infinitesimal generator A,

$$(Af)(x) = (-1)^{m+1} a(x) f; \quad D(A) = \{f | (1 + |a(x)|) f \in L_p\} .$$

($D(A)$ denotes the domain of definition of the operator A). We define $\hat{G}(\tau)$ by

$$\hat{G}(\tau) f = F^{-1} G(\tau) F f; \quad 0 \leqq \tau < \infty; \quad f \in S(R_n) . \tag{17}$$

We set

$$h(\xi) = (F^{-1} e^{(-1)^{m+1} a(x)})(\xi) \in S(R_n) . \tag{18}$$

Then holds

$$h_\tau(\xi) = F^{-1}(e^{(-1)^{m+1}\tau a(x)})(\xi) = \tau^{-\frac{n}{2m}} h(\tau^{-\frac{1}{2m}} \xi) \in S(R_n) . \tag{19}$$

We used $\tau a(x) = a(\tau^{\frac{1}{2m}} x)$ and a well known (and easily proved) transformation formula for the Fourier transformation. (17) and (19) show

$$(\hat{G}(\tau)f)(x) = (2\pi)^{-n/2} \int_{R_n} h_\tau(x-y) f(y) dy = (2\pi)^{-n/2} \int_{R_n} h(y) f(x - y\tau^{\frac{1}{2m}}) dy . \tag{20}$$

The last formula is also meaningful for $f \in L_p(R_n)$. $\hat{G}(\tau)$ is a continuous semigroup of operators in $L_p(R_n)$; $1 < p < \infty$: That the operators $\hat{G}(\tau)$ are linear and bounded follows from (20). The semigroup property follows from (17). The continuity of the semigroup follows from (20),

$$(2\pi)^{-n/2} \int_{R_n} h(y)dy = (Fh)(0) = e^0 = 1\,,$$

and the usual estimation technique. We want to show that $\hat{A}$,

$$(\hat{A}f)(x) = -\sum_{|\alpha|=2m} a_\alpha D^\alpha f, \quad D(\hat{A}) = W_p^{2m}(R_n)\,, \tag{21}$$

is the infinitesimal generator of $\hat{G}(\tau)$. (W_p^{2m} is the usual Sobolev space). Let $f \in S(R_n)$ and $b(x) = 1 + |x|^{2j}$, where j is a sufficiently large positive integer. Δ denotes the Laplacian, E is the identity. Then we have

$$\begin{aligned}
&\left\| \frac{\hat{G}(\tau) - E}{\tau} f - \hat{A}f \right\|_{L_p} = \left\| F^{-1}\left[\left(\frac{G(\tau) - E}{\tau} - A\right) Ff\right]\right\|_{L_p} \\
&= (2\pi)^{-n/2} \left\| F^{-1}\left[\frac{1}{b(x)}\left(\frac{e^{(-1)^{m+1}\tau a(x)} - 1}{\tau} + (-1)^m a(x)\right)\right] * F^{-1}(b(x)Ff) \right\|_{L_p} \\
&\leqq c \left\| F^{-1}\left[\frac{1}{b(x)}\left(\frac{e^{(-1)^{m+1}\tau a(x)} - 1}{\tau} + (-1)^m a(x)\right)\right]\right\|_{L_1} \\
&\leqq c' \left\| F^{-1}\left\{(1 + (-1)^r \Delta^r)\left[\frac{1}{b(x)}\left(\frac{e^{(-1)^{m+1}\tau a(x)} - 1}{\tau} + (-1)^m a(x)\right)\right]\right\}\right\|_{L_\infty} \\
&\leqq c'' \left\| (1 + (-1)^r \Delta^r)\left[\frac{1}{b(x)}\left(\frac{e^{(-1)^{m+1}\tau a(x)} - 1}{\tau} + (-1)^m a(x)\right)\right]\right\|_{L_1} \\
&\to 0 \quad \text{for} \quad \tau \downarrow 0\,.
\end{aligned}$$

We used Young's inequality for convolutions and known estimates for Fourier-transformations. That $\hat{A}$ is the infinitesimal generator of $\hat{G}(\tau)$ follows now from: (a) the last estimate, (b) $S(R_n)$ is dense in $W_p^{2m}(R_n)$, (c) $\hat{A}$ is closed operator with non empty resolvent set.

We notice an interesting special case. Let be $a(x) = -|x|^2/2$. In this case holds

$$h(\xi) = ce^{-\frac{|\xi|^2}{2}}\,.$$

$h_\tau(\xi)$ are the Gauss-Weierstrass kernels.

For the further considerations we need an estimate for the operators $\hat{G}(\tau)$. Let

$$\infty \geqq q \geqq p > 1; \quad \frac{1}{\varkappa} = 1 - \frac{1}{p} + \frac{1}{q}\,. \tag{22}$$

Young's inequality for convolutions and (19), (20) show

$$\|\hat{G}(\tau)f\|_{L_q} \leqq \|h_\tau\|_{L_\varkappa}\|f\|_{L_p} = c\tau^{-\frac{n}{2m}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L_p};\ f \in L_p(R_n)\,. \tag{23}$$

c is independent of τ. From the theory of semigroups of operators follows

$$(\hat{A} - \tau E)^{-1}f = \int_0^\infty e^{-\tau\sigma}\hat{G}(\sigma)f d\sigma;\quad \tau \geqq \tau_0\,;$$

see [12]. We obtain with theaid of (23) for sufficiently large m

$$\|(\hat{A} - \tau E)^{-1}f\|_{L_q} \leqq c\int_0^\infty e^{-\tau\sigma}\sigma^{-\frac{n}{2m}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L_p}d\sigma = c'\|f\|_{L_p}\tau^{\frac{n}{2m}\left(\frac{1}{p}-\frac{1}{q}\right)-1}$$

and

$$\|(E - \tau\hat{A})^{-1}f\|_{L_q} \leqq c\|f\|_{L_p}\tau^{-\frac{n}{2m}\left(\frac{1}{p}-\frac{1}{q}\right)};\quad 0 < \tau \leqq \tau_1;\ f \in L_p(R_n)\,. \tag{24}$$

The idea of using inequalities of such type for the proof of embedding theorems is due to Yoshikawa [11].

5.2 *Proof of theorem (a).*

The last estimate gives the possibility of a new proof of theorem (a). s, t, p, q, r have the same meaning as in the theorem. We choose an integer m with

$$2m > s - t\,. \tag{25}$$

Without loss of generality we may assume $t > 2m$. Otherwise we would use the lifting property of the spaces $B^t_{q,r}$ and $B^s_{p,r}$, see [9]. Finally we choose an integer k with

$$2km > s \geqq t > 2m\,. \tag{26}$$

From the interpolation theory of the spaces $B^s_{p,r}$, [9], and the known fact $D(\hat{A}^k) = W^{2km}_p(R_n)$ follows

$$B^s_{p,r} = (L_p, D(\hat{A}^k))_{\frac{s}{2km},r} = (D(\hat{A}), D(\hat{A}^k))_{\frac{s-2m}{2m(k-1)},r} \tag{27}$$

and a similar formula for $B^t_{q,r}$. The interpolation theory for semigroups of operators, [4], shows

$$\|f\|_{B^t_{q,r}} \sim \left(\int_0^\delta \tau^{-\frac{t}{2m}r}\|(\hat{G}(\tau) - E)^k f\|^r_{L_q}\frac{d\tau}{\tau}\right)^{\frac{1}{r}} + \|f\|_{L_q} \tag{28}$$

(with the usual modification for $r = \infty$). $\delta > 0$ is a suitable number. Using (24) we find for $f \in B^s_{p,r} \subset D(\hat{A})$

$$\|(\hat{G}(\tau) - E)^k f\|_{L_q} \leqq \|(E - \tau\hat{A})^{-1}(\hat{G}(\tau) - E)^k f\|_{L_q} + \tau\|(E - \tau\hat{A})^{-1}(\hat{G}(\tau) - E)^k \hat{A}f\|_{L_q}$$
$$\leqq c\tau^{-\frac{n}{2m}\left(\frac{1}{p}-\frac{1}{q}\right)}(\|(\hat{G}(\tau) - E)^k f\|_{L_p} + \tau\|(\hat{G}(\tau) - E)^k \hat{A}f\|_{L_p})$$

and

$$\|f\|_{L_q} \leqq \|(E - \hat{A})^{-1}f\|_{L_q} + \|(E - \hat{A})^{-1}\hat{A}f\|_{L_q} \leqq c(\|f\|_{L_p} + \|\hat{A}f\|_{L_p}) \leqq c'\|f\|_{B^s_{p,r}}.$$

The last two relations together with (27), (28), and (6) show

$$\|f\|_{B^t_{q,r}} \leqq c(\|f\|_{B^s_{p,r}} + \|\hat{A}f\|_{B^{s-2m}_{p,r}}) \leqq c'\|f\|_{B^s_{p,r}}.$$

This proves theorem (a).

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