Connections between two set functions

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1. Introduction

The two set functions are the Hausdorff measure and a set function denoted by μ. Both set functions are defined for all point sets in the plane (or in Euclidean n-space). The definitions are here given in sections 2 and 3. The set function μ first appeared in connection with some problems for exceptional sets for entire functions and meromorphic functions, [3, 4, 5].

The results in this paper (Theorem 1 and Theorem 2, section 4) are given in a unified (simplified) form. The results in S. J. Taylor's [6] are here presented in the same (simplified) form, in Theorem A and Theorem B below. The similarities are obvious, and the conclusion must be that the set functions μ and Hausdorff measure are (essentially) different in the same way as the set functions generalized capacity and Hausdorff measure are essentially different.

For the three set functions considered here, there is a real function associated to each one of them. This real function is like a parameter, and what we have called a set function is actually a whole family of set functions.

The Hausdorff measure, here denoted by h^*, is defined with respect to a real measure function h, so that for $h(t)=t^\alpha$, the corresponding Hausdorff measure is an α-dimensional outer measure.

The generalized capacity, here denoted by C, is defined with respect to a kernel function φ, which for non-generalized capacity is $\varphi(t)=-\ln t$.

The set function μ is defined by means of a real function g. In this paper, as well as in existing applications [3, 4, 5] this real function is $g(x)=\exp(-\exp(x))$.

The connections between the set functions Hausdorff measure and generalized capacity, as given in the four theorems in S. J. Taylor's paper [6] imply the following Theorem A and Theorem B. The functions h and φ must satisfy reasonable regularity assumptions, see [6].

Theorem A. *Let the functions h and φ be given. Then*

$$\liminf_{t\to 0} h(t)\varphi(t) > 0$$

if and only if for all bounded sets A

$$h^*(A) = 0 \Rightarrow C(A) = 0.$$

Theorem B. *Let the functions h and φ be given. Then*

$$-\int_0 h(t)\, d\varphi(t) < \infty$$

if and only if for all bounded sets B

$$C(B) = 0 \Rightarrow h^*(B) = 0.$$

The function theoretic application of the set function μ can be illustrated in this way: Let U be any set of complex numbers. For all entire functions f, and for all $a \in U$, consider the lower order of the entire function $f(z)\exp(az)$, take the supremum over a and the infimum over f. This gives a real number, t, $0 \leqq t \leqq 1$. The main result in [4] is that $\mu(U) = -1/\ln t$. (With $\mu(U) = +\infty$ if $t=1$ and $\mu(U)=0$ if $t=0$.)

An application to value distribution theory for meromorphic functions is the following: For a meromorphic function of finite order in the plane, the set U of those complex numbers a which are deficient in the sense of Valiron, i.e. $\limsup\,(T(r,f)-N(r,a))/T(r,f) > t$, $0<t<1$, is a set with $\mu(U)<\infty$, and each set U_1 with $\mu(U_1)<\infty$ is contained in such a set of deficient values. This is a simplified form of the result in [5]. As a consequence of the results in this paper, the above set of Valiron deficient values can not be equally well described in terms of the set function Hausdoff measure. The results in [3] on properties of the set function μ show that the set functions μ and generalized capacity are essentially different. Therefore the best description of the above set of Valiron deficient values for a function of finite order, can not be given in terms of some generalized capacity.

For meromorphic functions of unrestricted growth (infinite order), the situation is different. The set of Valiron deficient values can be much larger than what was possible with finite order. The appropriate set function is the logarithmic capacity. This was proved by W. K. Hayman, [2].

The material in this paper is organized in 10 sections. Definitions are given in sections 2 and 3. The results are given in section 4 and proved in sections 6, 7, 8 and 9. Some comments and discussion of related topics are found in sections 1, 5 and 10. The concept of an approximating sequence in metric space, studied by R. J. Gardner in [1], is here discussed in section 10.

2. The Hausdorff measure

The words covering and majorizing are here used in a rather strange way. A given point set B in the complex plane is said to be *covered (majorized)* by a given sequence $D=\{d_n\}_{n=1}^{\infty}$ of positive numbers if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers, so that each $a\in B$ satisfies $|a-a_n|<d_n$ for some (infinitely many) n. The sequence D is called a covering (majorizing) sequence for the point set B. A majorizing sequence is also a covering sequence for the same set. An infinite disjoint union of covering sequences is a majorizing sequence. Throughout this paper, all covering or majorizing sequences are assumed to be decreasing (non-increasing). (This assumption is technically convenient, but otherwise irrelevant.) All non-trivial covering or majorizing sequences have limit zero.

The Hausdorff measure of a point set is defined by means of covering sequences for that set. In the definition of the set function μ, majorizing sequences are used. This indicates that the two set functions are different.

The Hausdorff measure $h^*(B)$ of a point set B is defined as the limit when $\max_n d_n (=d_1)$ tends to zero of the infimum of $\sum_{n=1}^{\infty} h(d_n)$ over all covering sequences with given d_1, for the point set B.

The assumptions on the measure function $h\colon R^+\to R^+$ are:

(1) h is continuous and increasing,

(2) the inverse function h^{-1} exists,

(3) the function $t\mapsto t^{-1}h(t)$ is decreasing,

(4) $h(t)\to 0$ when $t\to 0$,

(5) $t^{-1}h(t)\to +\infty$ when $t\to 0$.

A function denoted by h or h_1 is called admissible if it satisfies the above five assumptions.

3. The set function μ

The word majorizing is defined in section 2 of this paper. For the definition of the set function μ, a real function g is needed. We here simply put

$$g(x) = \exp(-\exp(x)), \quad x > 0. \tag{6}$$

(With minor changes the proofs in this paper are valid for more general g. The function g just needs to be smooth and rapidly decreasing.) This function g is here called admissible.

For a given point set B in the complex plane, the value $\mu(B)$ of the set function is defined as the lower bound of positive numbers $1/k$ for which $\{g(nk)\}_{n=1}^{\infty}$ majorizes the set B. If no such majorizing sequence exists, (the set B is too large) then $\mu(B)=+\infty$.

The set function μ is defined for all point sets in the plane (and similarly in R^n). It is countably subadditive, but not strongly subadditive. It is translation invariant in a strong sense. It is finite only for very small sets, but of course a set B with $\mu(B)=0$ need not be finite or countable.

If we prefer not to use the word majorize, the definition of the set function μ can be written: For a given point set U in the complex plane, the value $\mu(U)$ of the set function μ is defined as the lower bound of $1/k>0$ for which there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers so that $|a-a_n|<g(nk)$ for infinitely many n whenever $a\in U$.

4. Results

In sections 6 and 7 we prove:

Theorem 1. *For given admissible real functions h and g and corresponding set functions h^* and μ, the following three statements are equivalent:*

(d) *For arbitrary point sets B in the plane, $h^*(B)=0 \Rightarrow \mu(B)<\infty$,*

(e) $\liminf_{n\to\infty} nh(g(n))>0$,

(f) *For arbitrary point sets B in the plane, $h^*(B)<\infty \Rightarrow \mu(B)=0$.*

The corresponding results in the other direction are proved in sections 8 and 9. They are:

Theorem 2. *For given admissible real functions h and g and corresponding set functions h^* and μ, the following three statements are equivalent:*

(i) *For arbitrary point sets B in the plane, $\mu(B)=0 \Rightarrow h^*(B)<\infty$,*

(j) $\sum_{n=1}^{\infty} h(g(n))<\infty$,

(k) *For arbitrary point sets B in the plane, $\mu(B)<\infty \Rightarrow h^*(B)=0$.*

5. Comments

One consequence of the results in Theorems 1 and 2 is that

$$(h^* = 0 \Rightarrow \mu < \infty) \Rightarrow (h^* < \infty \Rightarrow \mu = 0),$$

and

$$(\mu = 0 \Rightarrow h^* < \infty) \Rightarrow (\mu < \infty \Rightarrow h^* = 0).$$

This means that the set functions h^* and μ are fundamentally different. Precise information about one set function implies only vague information about the other set function, and from the theorems in this paper we know exactly how vague. One other reason (which seems to be sufficient but not necessary) for the set functions to be fundamentally different, is that they have entirely different (additivity) properties.

In order to compare Theorems A and B with Theorems 1 and 2 we introduce the set function C_1, the capacity with respect to the kernel function φ_1,

$$\varphi_1(t) = \ln \ln 1/t = g^{-1}(t).$$

For the kernel function φ_1 we obtain equivalence between the conditions in Theorems A and 1 as well as in Theorems B and 2.

$$\liminf_{t\to 0} h(t)\varphi_1(t) > 0 \Leftrightarrow \liminf_{n\to\infty} nh(g(n)) > 0,$$

$$-\int_0 h(t)\, d\varphi_1(t) < \infty \Leftrightarrow \sum_{n=1}^{\infty} h(g(n)) < \infty.$$

The following equivalent conditions can be added in the theorems:

In Theorem 1; for arbitrary bounded sets B,

$$h^*(B) = 0 \Rightarrow C_1(B) = 0.$$

In Theorem 2; for arbitrary bounded sets B,

$$C_1(B) = 0 \Rightarrow h^*(B) = 0.$$

The remaining part of this paper consists of the proofs of Theorems 1 and 2.

The proofs of (d)$\Rightarrow$(e) and (i)$\Rightarrow$(j) are constructions, similar to that of the Cantor set. A set is defined so that one set function (h^* or μ) is large and the other set function is small. The details of these proofs seem to have negligible independent interest.

In the proofs of (e)$\Rightarrow$(f) and (j)$\Rightarrow$(k) we use properties of majorizing and covering sequences.

The remaining steps, (f)$\Rightarrow$(d) and (k)$\Rightarrow$(i) are trivial.

6. Proof of (d)$\Rightarrow$(e) in Theorem 1

It is here proved that if admissible functions h and g are given so that

$$\liminf_{n\to\infty} nh(g(n)) = 0, \tag{7}$$

then there exists a point set A, so that for the corresponding set functions h^* and μ

$$h^*(A) = 0, \quad \mu(A) = \infty. \tag{8}$$

This means not (e) implies not (d), which is the same as (d)$\Rightarrow$(e).

For admissible real functions h and g satisfying (7), there will be defined a sequence $\{N_p\}_{p=1}^{\infty}$ and a sequence $\{O_p\}_{p=1}^{\infty}$ of open sets, so that (8) can be proved for the set $A = \lim O_p$.

The assumption (5) is used only in this proof of (d)$\Rightarrow$(e), i.e. to obtain (9) and (12). Similarly, the assumption (3) on the function h is used only in the proof of (i)$\Rightarrow$(j), to obtain (36).

The assumption (5) can be written:

$$\lim_{T\to+\infty} Th^{-1}(1/T) = 0. \tag{9}$$

Let $N_1=1$. For $p>1$, let $n_p>1$ be an integer so that the inequalities (10) and (12) are satisfied for $N_p=n_pN_{p-1}$. (Thus $N_p \geqq 2^{p-1}$.)

$$p^{-2}N_p h\big(g(p^{-2}N_p)\big) < p^{-3} \qquad \text{(cf. (7),)} \tag{10}$$

i.e.

$$g^{-1}\left(h^{-1}\left(\frac{1}{pN_p}\right)\right) < \frac{N_p}{p^2}, \tag{11}$$

$$pN_p h^{-1}\left(\frac{1}{pN_p}\right) < N_{p-1}h^{-1}\left(\frac{1}{(p-1)N_{p-1}}\right). \qquad \text{(cf. (9).)} \tag{12}$$

These n_p and N_p exist as desired, since (12) holds for all sufficiently large $pN_p(=T)$ because of (9), and (10) holds for an unbounded set of admissible $p^{-2}N_p(=n)$, as a consequence of (7).

Let $l_p=h^{-1}(1/pN_p)$. This definition of l_p gives

$$N_p h(l_p) = \frac{1}{p} \to 0 \quad \text{as} \quad p\to\infty. \tag{13}$$

The inequality (12) can be written

$$pn_pl_p < l_{p-1}. \tag{14}$$

The set O_p will be defined as the union of N_p disjoint open intervals of length l_p on the real axis. If this construction is performed in a natural way, then we obtain (8). The details of the construction are:

Let O_1 be the interval (O, l_1). For $p>1$ the set $O_p \subset O_{p-1}$ is defined so that each component of O_{p-1} contains n_p components of O_p. Let the distance between any two components of O_p be at least

$$\Delta_p = (l_{p-1}-n_pl_p)/(n_p-1). \tag{15}$$

This means that the gaps in each O_p are equal $(=\Delta_p)$. It follows from (14) that l_p is small enough, so that $\Delta_p>0$. Let $A=\lim_{p\to\infty} O_p$. This defines the set A and (8) is now to be proved. From (13) we obtain $h^*(A)=0$, and it remains to prove that $\mu(A)=\infty$. The definition in section 3 of the set function μ says that we shall prove the following: For given $k>0$ is $\{g(nk)\}_{n=1}^{\infty}$ not a majorizing sequence for the set A.

To see why or how this majorization fails, we study for given p the set of those n for which

$$l_{p-1} \geqq g(nk) > l_p.$$

Let s_p be the number of those n.

$$s_p \leqq \frac{1}{k} g^{-1}(l_p) = \frac{1}{k} g^{-1}\left(h^{-1}\left(\frac{1}{pN_p}\right)\right) < \frac{N_p}{kp^2}, \tag{16}$$

where the last part of the estimate is (11). We also need an estimate for the sum of $2g(nk)$ over those n. This sum gives an upper bound for the total length of that part of the real axis that can be contained in the corresponding discs. This sum is no greater than $4l_{p-1}$ if p is sufficiently large, since g is rapidly decreasing. (In the case of a more general g, this is one place in the proof where assumptions on g are needed.)

The total number m_p of components of O_p intersected by the union of these discs can be estimated. The last term in (17) corresponds to the number of covered gaps in O_p, each gap of length Δ_p.

$$m_p \leqq s_p + \frac{4l_{p-1}}{\Delta_p}. \tag{17}$$

To obtain an estimate for m_p/N_p we use (17), (16), (15)+(14), $p \geqq 2$ and $N_p \geqq 2^{p-1}$.

$$\frac{m_p}{N_p} \leqq \frac{s_p}{N_p} + \frac{4l_{p-1}}{\Delta_p N_p} < \frac{1}{kp^2} + \frac{4l_{p-1}(n_p-1)}{\left(l_{p-1} - \frac{1}{p} l_{p-1}\right) N_p} \leqq \frac{1}{kp^2} + \frac{8}{N_{p-1}} \leqq \frac{1}{kp^2} + 2^{5-p}.$$

The sum $\sum_{p=1}^{\infty} m_p/N_p$ is convergent. For some p_0 is $\sum_{p=p_0}^{\infty} m_p/N_p < 1$.

To see if majorization is possible, we need only consider what happens for $p > p_0$. (Any finite number of discs may be disregarded in this context.) Let a positive unit mass be uniformly distributed over A so that each component of O_p has the mass $1/N_p$. Then the mass in the union of discs of radius $g(nk) < l_{p_0}$ is no greater than the corresponding sum of m_p/N_p which is less than the total mass of A. There is always some point in A that is not contained in sufficiently (infinitely) many discs. Therefore the majorization fails and the desired result (d)$\Rightarrow$(e) follows.

7. Proof of (e)$\Rightarrow$(f) in Theorem 1

Let admissible real functions h and g be given so that

$$\liminf_{n \to \infty} nh(g(n)) > 0. \tag{18}$$

Let there be given a point set A so that

$$h^*(A) < \infty. \tag{19}$$

(In the proof of (e)$\Rightarrow$(f), the set denoted by A is just an arbitrary point set in the plane with the property (19). It must not be confused with sets A in other proofs in this paper.)

We here prove that (19) and (18) imply that $\mu(A)=0$, which is equivalent to proving (e)$\Rightarrow$(f).

An auxiliary set function h_* (which will turn out to be trivial) is introduced.

Definition. Let B be a point set with $h^*(B)<\infty$. Then $h_*(B)$ is defined as the infimum of $\sup_n nh(d_n)$ over all sequences $\{d_n\}_{n=1}^{\infty}$ that cover the set B and have $\sum_{n=1}^{\infty} h(d_n)<\infty$.

If $h^*(B)=\infty$, then $h_*(B)$ is not defined. Monotonicity gives $0<\sup_n nh(d_n)\leqq \leqq\sum h(d_n)$ and $0\leqq h_*(B)\leqq h^*(B)$.

We first prove that h_* is identically zero. Then $\mu(A)=0$ is proved, i.e. (e)$\Rightarrow$(f).

(Proof of $h_*=0$.)

Let B and C be given point sets with

$$h^*(B)<\infty, \quad h^*(C)<\infty.$$

Let $\varepsilon>0$. The definition of the set function h_* implies that the set B has a covering sequence $\{d_n'\}_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} h(d_n')<\infty$ (which gives (21) because of monotonicity) and

$$nh(d_n') < h_*(B)+\varepsilon \quad \text{for all} \quad n. \tag{20}$$

There exists $d>0$ so that

$$nh(d_n') < \varepsilon \quad \text{for} \quad d_n' < d. \tag{21}$$

For the point set C we use the definition of the set function h^*, and this gives a covering sequence $\{d_n''\}_{n=1}^{\infty}$ for the set C so that $d_n''<d$ for all n, and

$$\sum_{n=1}^{\infty} h(d_n'') < h^*(C)+\varepsilon.$$

Monotonicity gives

$$nh(d_n'') < h^*(C)+\varepsilon \quad \text{for all} \quad n. \tag{22}$$

These two covering sequences together make one new monotone sequence $\{d_n'''\}_{n=1}^{\infty}$ which becomes a covering sequence for the union $B\cup C$. We can estimate $nh(d_n''')$ for small n, (23) and for large n, (24).

For $d_n'''\geqq d$ is $d_n'''=d_n'$ (since $d_n''<d$) and the inequality (20) gives

$$nh(d_n''') < h_*(B)+\varepsilon. \tag{23}$$

For $d_n'''<d$ it works like adding the two inequalities (21) and (22), since it is the number of elements of any given magnitude that counts, and those numbers are added when sequences are put together. The resulting inequality is

$$nh(d_n''') < h^*(C)+2\varepsilon. \tag{24}$$

For arbitrary n is (because of (23) and (24),)

$$nh(d_n''') < \max\big(h_*(B)+\varepsilon,\ h^*(C)+2\varepsilon\big).$$

Therefore, for arbitrary sets B and C of finite Hausdorff measure

$$h_*(B\cup C) < \max\big(h_*(B),\ h^*(C)\big).$$

Repeated use of this inequality gives the formula

$$h_*(A_1\cup A_2\cup\ldots\cup A_n) < \max\big(h_*(A_1),\ h^*(A_2),\ \ldots,\ h^*(A_n)\big).$$

Since a set of finite Hausdorff measure can be written as a finite union of sets of arbitrarily small Hausdorff measure, it follows that the set function h_* is identically zero,

$$h^*(A) < \infty \Rightarrow h_*(A) = 0.$$

The set A is given so that $h^*(A)<\infty$. For $m=1, 2, 3, \ldots$ we use the fact that $h_*(A)<2^{-m}$ to obtain a covering sequence $D_m=\{d_{m,n}\}_{n=1}^{\infty}$ for the set A so that $\sum_{n=1}^{\infty} h(d_{m,n})<\infty$ and $nh(d_{m,n})<2^{-m}$ for all n and m.

Let $\{d_n\}_{n=1}^{\infty}$ be the union of this countable collection of covering sequences, thus a majorizing sequence for the set A. We apply obvious additivity rules for sup and lim sup to the following estimates:

$$\limsup_{n\to\infty} nh(d_{m,n}) = 0 \qquad m = 1, 2, \ldots, M$$

(which holds for the same reason as (21),) and

$$\sup_n nh(d_{m,n}) < 2^{-m} \qquad m = M+1,\ M+2, \ldots.$$

The sum over $m>0$ gives

$$\limsup_{n\to\infty} nh(d_n) < 2^{-M}$$

where M is any positive integer.

The assumption (18) implies that for arbitrary $k>0$

$$\liminf_{n\to\infty} nh\big(g(nk)\big) > 0.$$

As a result of these two inequalities, for large enough n

$$d_n < g(nk).$$

The conclusion is that $\{g(nk)\}_{n=1}^{\infty}$ is a majorizing sequence for the point set A, since $\{d_n\}_{n=1}^{\infty}$ is majorizing. Therefore $\mu(A)\leqq 1/k$. Since $k>0$ is arbitrary, $\mu(A)=0$ and we have established (e)$\Rightarrow$(f).

8. Proof of (i)⇒(j) in Theorem 2

It is here proved that if admissible functions h and g are given so that

$$\sum_{n=1}^{\infty} h(g(n)) = \infty, \tag{25}$$

then there exists a point set A with

$$\mu(A) = 0, \quad h^*(A) = \infty. \tag{26}$$

This is equivalent to proving (i)⇒(j).

The assumption (25) can be improved in two ways, via (27) to (29). (These two steps together are roughly equivalent to (i)⇒(k).) There exists an unbounded increasing sequence $\{k_n\}_{n=1}^{\infty}$ of positive numbers (e.g. $k_{m+1}=\sum_{n=1}^{m} h(g(nk_n))$,) so that

$$\sum_{n=1}^{\infty} h(g(nk_n)) = \infty. \tag{27}$$

There also exists a function h_1, admissible in the same sense as the function h and such that

$$\lim_{t\to 0} \frac{h(t)}{h_1(t)} = \infty, \tag{28}$$

and

$$\sum_{n=1}^{\infty} h_1(g(nk_n)) = \infty. \tag{29}$$

The main partial result in this proof is:

Lemma. *Let h_1 be a given admissible real function. Let $\{l_n\}_{n=1}^{\infty}$ be a sequence of positive numbers so that $\sum_{n=1}^{\infty} h_1(l_n)=\infty$. Then the sequence $\{l_n\}_{n=1}^{\infty}$ majorizes some point set A with $h_1^*(A)>0$.*

The set function h_1^* is the Hausdorff measure with respect to the measure function h_1.

We first show that this Lemma implies (i)⇒(j). For k_n as above, apply the Lemma to $l_n=g(nk_n)$. The point set A, which exists according to the Lemma, is majorized by $\{l_n\}_{n=1}^{\infty}=\{g(nk_n)\}_{n=1}^{\infty}$ and therefore, for arbitrary N, it is also majorized by $\{g(nk_N)\}_{n=1}^{\infty}$. This gives $\mu(A)\leqq 1/k_N$ and $\mu(A)=0$. The remaining part of (26) follows since (28) implies $h_1^*(A)>0 \Rightarrow h^*(A)=\infty$.

The Lemma remains to be proved. It can be assumed that $\sum_{n=1}^{\infty} l_n<\infty$, since the Lemma is trivial otherwise (let A be an interval on the real axis). The sequence $\{l_n\}_{n=1}^{\infty}$ can be assumed non-increasing. The point set A is to be defined. An auxiliary sequence $\{f_n\}_{n=1}^{\infty}$ of continuous piecewise linear real functions is introduced. For some sequence $\{a_n\}_{n=1}^{\infty}$ of positive numbers, not yet defined, let

$$f_n(x) = \begin{cases} 0 & \text{if} \quad x \leqq a_n \\ \text{linear} & \text{if} \quad a_n \leqq x \leqq a_n+l_n \\ h(l_n) & \text{if} \quad a_n+l_n \leqq x. \end{cases} \tag{30}$$

For some positive integers $K_m < L_m < K_{m+1} < L_{m+1} < \dots$ we denote

$$F_m(x) = \sum_{n=K_m}^{n=L_m} f_n(x) \qquad m = 1, 2, \dots. \tag{31}$$

The sequence $\{a_n\}_{n=1}^{\infty}$ will be given indirectly, by means of conditions on the pointwise convergence of F_m.

The intervals (a_n, a_n+l_n) on which F_m is non-constant are assumed to be disjoint, $a_n+l_n<a_{n+1}$. These intervals are also assumed to be contained in the (union of the) corresponding intervals for F_{m-1}, $\operatorname{supp} dF_m \subset \operatorname{supp} dF_{m-1}$. Let $F_0(x)=x$ for $0 \leqq x \leqq 50$. Let $0<\varepsilon_m<\frac{1}{2}\varepsilon_{m-1}<2^{-m}$, $m=2, 3, \dots$, and let the ε_m be as small as will be required in the sequel. For $m=1, 2, \dots$ it is assumed that

$$|F_m(x)-F_{m-1}(x)| < \varepsilon_m, \quad 0 \leqq x \leqq 50. \tag{32}$$

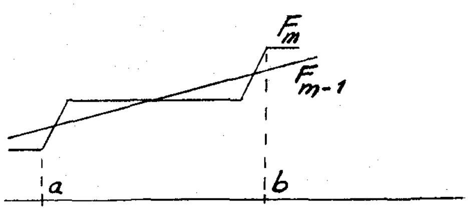
The next assumption on the F_m is stronger and more complicated. For each interval (a, b) which intersects only one component of $\operatorname{supp} dF_{m-1}$ but intersects at least two components of $\operatorname{supp} dF_m$, it is required that

$$F_m(b)-F_m(a) < 10\big(F_{m-1}(b)-F_{m-1}(a)\big), \tag{33}$$

and it is assumed that $\varepsilon_{m+1}+\varepsilon_{m+2}+\dots$ is so small that (33) implies

$$F(b)-F(a) < 20\big(F_{m-1}(b)-F_{m-1}(a)\big). \tag{34}$$

The inequality (33) will follow if the a_n are chosen in an adequate way, e.g. $F_m(a_n+\frac{1}{2}l_n)=F_{m-1}(a_n+\frac{1}{2}l_n)$. (It is sufficient to have (33) at the critical points $a=a_n$, $b=a_{n+1}+l_{n+1}$ for all admissible n.)



Let $A=\lim_{m\to\infty} \operatorname{supp} dF_m = \limsup_{n\to\infty} [a_n, a_n+l_n]$. (It is convenient to assume that all n are not used, i.e. $L_m \ll K_{m+1}$. The sum of $h_1(l_n)$ is divergent like $O(m)$.) The sequence $\{l_n\}_{n=1}^{\infty}$ is a majorizing sequence for the point set A, as was required in the Lemma. What remains to prove in the Lemma is $h_1^*(A)>0$. For this purpose we estimate what part of the set A can be contained in a circle C_i of radius r_i. The intersection of the circle C_i with the real axis (which contains A) is an interval of length at most $2r_i$. The end points of this interval are called a and b, and can be identified with a and b in (33) and (34). This defines m as above. (If C_i intersects A, which may be assumed, then the above conditions on the number of intersected components of $\operatorname{supp} dF_m$ and $\operatorname{supp} dF_{m-1}$ determine m.)

Let $[a_n, a_n+l_n]$ be the component of supp dF_{m-1} that intersects (a, b). In the interval $[a_n, a_n+l_n]$ is the function F_{m-1} linear, $F_{m-1}(x)=f_n(x)+\text{const}$, with derivative $h_1(l_n)/l_n$, because of (30), i.e.

$$F_{m-1}(b)-F_{m-1}(a) \leqq (b-a)\frac{h_1(l_n)}{l_n}. \tag{35}$$

It can be assumed that the circle C_i is not too large, (e.g. $b-a\leqq l_n$) and then (3) and (35) give

$$F_{m-1}(b)-F_{m-1}(a) \leqq h_1(b-a) \leqq h_1(2r_i). \tag{36}$$

(The same conclusion holds if $b-a>l_n$.)

From (34) and (36) and (3) we obtain

$$\frac{1}{20}\big(F(b)-F(a)\big) \leqq h_1(2r_i) \leqq 2h_1(r_i). \tag{37}$$

This estimate shows how much of A can be contained in a circle C_i of radius r_i.

Let $\{r_i\}_{n=1}^{\infty}$ be a covering sequence for the point set A. Then (37) gives

$$\sum_{i=1}^{\infty} h_1(r_i) \geqq \frac{1}{40}\big(F(50)-F(0)\big) > 1.$$

The conclusion is

$$h_1^*(A) \geqq 1.$$

This proves the Lemma, which was sufficient for (i)$\Rightarrow$(j).

9. Proof of (j)$\Rightarrow$(k) in Theorem 2

The set B is given so that $\mu(B)<\infty$. The functions h and g are given so that for $k=1$, $N=1$,

$$\sum_{n=N}^{\infty} h\big(g(nk)\big) < \infty.$$

The functions h and g are monotone, and therefore the above sum is convergent for any $k>0$. Let $1/k>\mu(B)$. For this k and for any $N>0$ the sequence $\{g(nk)\}_{n=N}^{\infty}$ is majorizing and therefore also covering for the point set B. The above sum tends to zero when N increases, i.e. $h^*(B)=0$. This proves (j)$\Rightarrow$(k).

10. Approximating sequences

In [1], R. J. Gardner studies the concept of an approximating sequence for a metric space. This concept is (because of its definition) closely related to the set function μ. Gardner studies the connection with Hausdorff measure. The results in [1] will here be compared with our Theorems 1 and 2.

Let h be a real function, as before in this paper. This is the main definition in [1]: A sequence $\{a_n\}_{n=1}^{\infty}$ of points in a metric space Ω with metric ϱ is called an approximating sequence of order h if each a satisfies $nh(\varrho(a_n, a))<1$ for infinitely many n. With the notation $h^{-1}(1/n)=g(n)$, it follows that the existence of an approximating sequence is roughly equivalent to $\mu(\Omega)\leqq 1$ or $\mu(\Omega)<1$. There is one important distinction, since Gardner requires the points a_n to belong to the set Ω, so that the property of having an approximating sequence is not inherited by subsets of Ω.

Gardner's Theorem 1 corresponds to (e)$\Rightarrow$(f) of this paper, and Gardner's Theorem 2 corresponds to (j)$\Rightarrow$(k). In both cases it is obvious from the proofs in this paper that the assumptions can be reduced to a minimum, the functions h and g need only have $\lim_{t\to\infty} h(t)=0$ and $\lim_{x\to\infty} g(x)=0$ and monotonicity. The space need not be Euclidean or metric, since not all axioms for such a space are used. For the converse results, Examples 1 and 2 in [1], Gardner assumes $\Omega\subset R^1$.

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