Some lacunary conditions for Fourier—Stieltjes transforms

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Let $\mathbf{T}$ denote the circle group, $\mathbf{Z}$ the ring of integers and $M(\mathbf{T})$ the usual convolution algebra of measures on $\mathbf{T}$. The Fourier—Stieltjes coefficients $\hat{\mu}(n)$ of the measure $\mu \in M(\mathbf{T})$ are defined by

$$\hat{\mu}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\, d\mu(\theta) \quad (n \in \mathbf{Z}).$$

Denote by $M_a(\mathbf{T})$ those $\mu \in M(\mathbf{T})$ which are absolutely continuous with respect to Lebesgue measure on $\mathbf{T}$, $M_0(\mathbf{T})$ those $\mu \in M(\mathbf{T})$ such that $\hat{\mu}$ vanishes at infinity, and $M_s(\mathbf{T})$ the set of $\mu \in M(\mathbf{T})$ which are singular, i.e., concentrated on sets of Lebesgue measure zero.

An increasing sequence $\langle n_k \rangle_1^\infty$ with $n_k \in \mathbf{Z}^+$ (the positive integers) is said to satisfy the gap condition $(\mathbf{F}_p)$, if there is a $p \in \mathbf{Z}^+$ such that

$$\lim_{k \to \infty} (n_{k+p} - n_k) = \infty.$$

Using a theorem of Mahler which is based on a p-adic version of the Thue—Siegel theorem, we prove in Section 1 that finite unions of sets of the form

$$\mathbf{S}_j = \{r^j : r \in \mathbf{Z}^+\} \quad (j = 2, 3, \ldots)$$

satisfy $(\mathbf{F}_1)$. It then follows from an extension of a theorem of Wallen that if $\mu_i \in M(\mathbf{T})$ $(i=1, 2)$ and supp $\hat{\mu}_i \subset \mathbf{Z}^- \cup E$ where E is any finite union of sets $\mathbf{S}_j$ then $|\mu_1| * |\mu_2| \in \in M_a(\mathbf{T})$. Here $|\mu_i|$ is (of course) the usual total variation measure.

In section 2 we investigate a weaker lacunary condition than $(\mathbf{F}_p)$ which we now define:

A subset $E \subset \mathbf{Z}^+$ is said to satisfy the condition $(\mathscr{P})$ if for every increasing sequence $n_1, n_2 \ldots \in E$

$$\mathbf{Z}^+ \cap \underline{\lim}\, (E - n_j) \text{ is finite.}$$

Our main result is that if E satisfyes $(\mathscr{P})$ and if supp $\hat{\mu} \subset \mathbf{Z}^- \cup E$ then $\mu \in M_0(\mathbf{T})$.

§ 1. Convolution products and gap conditions

A subset $\mathscr{R}\subset\mathbf{Z}$ is called a Riesz set if $\mu\in M(\mathbf{T})$ and supp $\hat{\mu}\subset\mathscr{R}\Rightarrow\mu\in M_a(\mathbf{T})$. The F. and M. Riesz theorem states that both $\mathbf{Z}^+$ and $\mathbf{Z}^-$ are Riesz sets.

A subset $S\subset\mathbf{Z}$ is said to have property (**M**), if for any Riesz set $\mathscr{R}$ the union of $\mathscr{R}$ with S is again a Riesz set. Any strong Riesz set S has property (**M**); see [1] and [4] for examples. Furthermore, it is known that any Sidon set has property (**M**); see [7] and [8]. Whether or not sets $E\subset\mathbf{Z}^+$ which satisfy the gap condition ($\mathbf{F}_p$) for some p possess property (**M**) is an open question. What we do know is the following extension of a theorem of Wallen [10]:

Theorem 1. *Let $\mu_i\in M(\mathbf{T})$ $(i=1, 2, \ldots, p+1)$ with $p\in\mathbf{Z}^+$. Let E satisfy* ($\mathbf{F}_p$) *and suppose supp $\hat{\mu}_i\subset\mathbf{Z}^-\cup E$ for all i. Then*

$$|\mu_1|*|\mu_2|*\ldots*|\mu_p|*|\mu_{p+1}|\in M_a(\mathbf{T}).$$

Proof. The Theorem is a simple variant on that of Theorem 2 of [6]. The proof is obtained by iterating the method of proof of Theorem 2 of [6]. We leave the details to the reader.

Henceforth we shall refer to ($\mathbf{F}_1$) as the Faber-gap condition. We make the following observation:

If $E=E_1\cup\ldots\cup E_p=\{n_1<n_2<\ldots\}$ where each E_i satisfies the Faber-gap condition, then we have $n_{k+p}-n_k\to\infty$. Suppose not and say $n_{k_i+p}-n_{k_i}<C$ for some constant C and some infinite subsequence of E. Then there is an E_{i_0} such that for infinitely many n_{k_i} we have that at least two members of E_{i_0} are in the set $\{n_{k_i}, n_{k_i+1}, \ldots, n_{k_i+p}\}$. If n'_{k_i} is the first member of E_{i_0} in this set and m'_{k_i} is the second, then $m'_{k_i}=n'_{k_i}<C$, which contradicts E_{i_0} satisfying the Faber-gap condition.

On the other hand, if $E=\{n_1<n_2<\ldots\}$ satisfies $n_{k+p}-n_k\to\infty$ for some p then $E=E_1\cup\ldots\cup E_p$, where $E_i=\{n_{k_i}:k_i\equiv i \pmod p\}$ and each E_i clearly satisfies the Faber-gap condition.

In view of the preceding our next result is therefore somewhat surprising.

Theorem 2. *Let E by any finite union of the sets $\mathbf{S}_j$, then E satisfies the Faber-gap condition.*

Proof. Let $k_1, k_2, \ldots, k_n$ be distinct integers greater than 1. Let $\mathbf{K}_i=\mathbf{S}_{k_i}=\{r^{k_i}:r\in\mathbf{Z}^+\}$ for $i=1, 2, \ldots, n$. Put

$$E=\bigcup_{i=1}^{n}\mathbf{K}_i=\{s_1<s_2<\ldots\}.$$

We prove that E satisfies the Faber-gap condition.

Given $x \in \mathbf{Z}^+$ we claim there is an a_0 such that if $s_p \in E$ and $s_p \geqq a_0$ then $s_{p+1} - s_p > x$. By a theorem of Mahler [3] it follows that if $(z, w) \leqq x$, $ab \neq 0$, $g \geqq 2$, and $h \geqq 3$ (or $h \geqq 2$ and $g \geqq 3$) then there is an integer $N^{g,h}$ such that the largest prime divisor of $az^g - bw^h$ is greater than x if $\max\{|z|, |w|\} > N^{g,h}$. Also there is an integer c such that if $q^2 \geqq c$ then $q^2 + x < (q+1)^2$.

Let (k_i, k_j) run through the collection of all ordered pairs where $k_i > 2$ or $k_j > 2$. We thus generate a collection of forms

$$z^{k_i} - w^{k_j}$$

and corresponding to these forms we obtain the numbers N^{k_i, k_j}.

Let $k_0 = \max\{k_1, \ldots, k_n\}$ and let

$$a_1 = \max\{(N^{k_i, k_j})^{k_0}\}.$$

Finally let $a_0 = \max\{c, a_1\}$.

Now, if $s_p \geqq a_0$, say $s_{p+1} = l^{k_i}$ and $s_p = f^{k_j}$. Suppose $k_i > 2$ or $k_j > 2$. If $s_{p+1} - s_p = x' \leqq x$, we will derive a contradiction. Set $d = (l, f)$. Since $l^{k_i} - f^{k_j} = x'$ we have $d | x'$ and so $d \leqq x$. Then

$$f = s_p^{1/k_j} \geqq a_0^{1/k_j} \geqq \left((N^{k_i, k_j})^{k_0}\right)^{1/k_j} \geqq N^{k_i, k_j}.$$

It then follows that the greatest prime divisor of x' is greater than x, which is a contradiction.

Finally, if $k_i = 2 = k_j$, then since $s_p \geqq c$ we have $s_p + x < s_{p+1}$ which completes the proof.

Remark. A refinement of the proof of Theorem 2 allows us to replace the set $\mathbf{S}_{k_i}$ $(k_i > 2)$ by any set $\bigcup_{a=1}^{c} \{ar^{k_i} : r \in \mathbf{Z}^+\}$ $(c \in \mathbf{Z}^+)$. Also, we may replace the set $\mathbf{S}_2$ by any of the sets $\{ar^2 : r \in \mathbf{Z}^+\}$ $(a \in \mathbf{Z}^+)$. However, the theory of the Pell equation forbids us from replacing $\mathbf{S}_2$ by any of the sets $\bigcup_{a=1}^{c} \{ar^2 : r \in \mathbf{Z}^+\}$ $(c \geqq 2)$.

§ 2. A lacunary condition

If (as in [10]) one is mainly intersted in deducing that the transform of a measure vanishes at infinity we can require a weaker lacunary property than $(\mathbf{F}_p)$:

A subset E of $\mathbf{Z}^+$ is said to satisfy the condition $(\mathscr{P})$ if for all sequences $n_1 < n_2 < \ldots \in E$, the set

$$\mathbf{Z}^+ \cap \left(\bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} (E - n_k) \right) \text{ is finite.}$$

Observe that the above definition makes sense in $\mathbf{Z}^2$ where $\mathbf{Z}^2$ is considered as an ordered group dual to $\mathbf{T}^2$. Also, it follows easily from the definition that if a set

E satisfies conditions $(\mathscr{P})$, then E cannot contain any set $A+B$, where A and B are both infinite sets. Before presenting our main Theorem, we give some examples:

(a) Any set satisfying the gap condition $(\mathbf{F}_p)$ satisfies the lacunary condition $(\mathscr{P})$. To see this let E satisfy $(\mathbf{F}_p)$ and let $m_1<m_2<\dots$ be any subsequence of E. Suppose there are infinitely many x's such that $x+m_k\in E$ for all sufficiently large k. Let $x_1, \dots, x_p$ be the first p values of x. Then there is a t_0 such that if $k\geqq t_0$ then $x_i+m_k\in E$, where $i=1, 2, \dots, p$ and $k\in\mathbf{Z}^+$. For $k\geqq t_0$, put $m_k=n_{j_k}$. Then $n_{j_k+p}\leqq \leqq n_{j_k}+x_p$ whence $n_{j_k+p}-n_{j_k}\leqq x_p$. Since there are infinitely many n_{j_k} we see that $n_{j_k+p}-n_{j_k}\not\to\infty$ which is the desired contradiction.

(b) Any Sidon set satisfies $(\mathbf{F}_p)$ for some p([5, p. 194]); hence any Sidon set satisfies condition $(\mathscr{P})$.

(c) In this example we outline a construction of a set of positive integers which has property $(\mathscr{P})$ and yet is not a finite union of sets with the Faber-gap property. Thus property $(\mathscr{P})$ is strictly weaker than the property $n_{k+p}-n_k\to\infty$ for some p.

Choose a suitably "thin" subsequence of the sequence of powers of 3 and call it $A_0=\{a_1<a_2<\dots\}$. Next construct the sequence $A_1=A_0\cup\{a_{2^n}+3: n\in\mathbf{Z}^+\}$. Next construct $A_2=A_1\cup\{a_{3^n}+3^2\}\cup\{a_{3^n}+3^3\}$. In general,

$$A_t = A_{t-1}\cup\{a_{p_t^n}+3^{t'}\}\cup\dots\cup\{a_{p_t^n}+3^{t'+t-1}\}$$

where p_t is the t^{th} prime and $t' = 1+\sum_{i=1}^{t-1} i$.

Let

$$A = \bigcup_{t=0}^{\infty} A_t.$$

Clearly, for any p, there is a constant c such that A contains infinitely many members n_k with $n_{k+p}-n_k<c$. Thus, A is not a finite union of sets with the Faber-gap condition. On the other hand, if A_0 is chosen "thin" enough, the fact that no integer can have two representations of the form $\sum_{i=1}^{m}\pm 3^{j_i}$ leads to the conclusion that for any subsequence $\{n_1<n_2\dots\}$ of A there are only finitely many integers $x\in\mathbf{Z}^+$ with the property that $x+n_k$ is an element of A for all sufficiently large k. In fact, the construction guarantees that the set of such $x's$ is empty unless a tail of the sequence $\{n_k\}$ is chosen from one of the sets $\{a_{p_t^n}: n\in\mathbf{Z}^+\}\cup(A_t\setminus A_{t-1})$, in which case there can be at most t such x's.

Theorem 3. *Let E satisfy $(\mathscr{P})$ and suppose $\mu\in M(\mathbf{T})$ and supp $\hat{\mu}\subset\mathbf{Z}^-\cup E$. Then $\mu\in M_0(\mathbf{T})$.*

Proof. Suppose not. Then there is an increasing sequence $n_1<n_2<\dots\in E$ and an $\varepsilon>0$ such that

$$(1)\qquad |\hat{\mu}(n_j)| \geqq \varepsilon > 0.$$

We shall force a contradiction:

Put $d\nu_j = e^{-in_j\theta}\, d\mu$. Then without loss of generality we may assume

(2) $$\nu_j \to \nu \in M_s(\mathbf{T}) \quad \text{weak-}*$$

The fact that $\nu \in M_s(\mathbf{T})$ is a consequence of the Helson translation lemma [9, p. 66]. From (1) and (2) we conclude that

(3) $$\hat{\nu}(0) \neq 0.$$

On the other hand condition $(\mathscr{P})$ in combination with the F. and M. Riesz theorem implies that $\hat{\nu}(0)=0$. This contradicts (3) and so since $\hat{\mu}$ vanishes at "$+\infty$" it follows from [2] that $\mu \in M_0(\mathbf{T})$.

Finally, we observe that the same proof with simple modifications holds in $\mathbf{T}^2$ regardless of the order chosen for $\mathbf{Z}^2$.

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