Analytic approximability of solutions of partial differential equations

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1. Introduction

Let $P(x, D_x)$ be a linear partial differential operator with analytic coefficients defined in a neighborhood of a point $x_0 \in \mathbf{R}^n$. We shall call P *locally approximable* at x_0 if for any distribution u for which $Pu \equiv 0$ in a neighborhood of x_0, there is a neighborhood $\mathcal{U}$ of x_0 and a sequence of distributions u_j real analytic in $\mathcal{U}$ such that

$$u_j \to u \quad \text{in} \quad \mathcal{U},$$
$$Pu_j \equiv 0 \quad \text{in} \quad \mathcal{U}.$$

The property of local approximability was studied by Baouendi and Treves [2], who showed that P is locally approximable at x_0 if its complex characteristics at x_0 are simple. Métivier [7] has proved approximability for a class of first order nonlinear equations. Baouendi and the second author [1] showed that any left invariant differential operator on a Lie group is locally approximable.

The class of locally approximable differential operators contains that of analytic hypoelliptic differential operators. (Recall that P is analytic hypoelliptic at x_0 if Pu real analytic in a neighborhood of x_0 implies that u is real analytic near x_0.) The notion of analytic hypoellipticity has been microlocalized in an obvious way, but the notion of microlocal approximability is less clear. In § 2 we give a definition of microlocal approximability and also extend the definition of local approximability to pseudodifferential operators. These definitions are based on the constants for the Fourier—Bros—Iagolnitzer transform of a distribution (see e.g. [11]). We show that when $\operatorname{char}_{x_0} P$ is contained in a line then local approximability is equivalent to microlocal approximability in all directions.

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In § 3 we follow the method of Sjöstrand [10] and Helffer [4] (see also [3] as well as the references to Grušin's work given in [10]) to study a class of differential operators with symplectic characteristic variety. For these we show that the question of microlocal approximability for P is equivalent to that of a system of analytic pseudodifferential operators in one variable. In § 4 we use the machinery developed by Métivier [6] to prove the analyticity of the operators defined in § 3. We refer the reader to [12] and [6], respectively for the definitions of classical analytic pseudodifferential operators of type (1/2, 1/2).

In § 5 we give the first example of a differential operator, not totally characteristic at x_0, which is not locally approximable at x_0. This operator is

$$P=\frac{\partial^2}{\partial t^2}+t^2\frac{\partial^2}{\partial y^2}+i\frac{\partial}{\partial y}+y$$

on $\mathbf{R}^2$ with $x_0=(0,0)$. The proof of non approximability uses the reduction to a pseudodifferential operator in one variable given in § 3 and § 4, and the connection between microlocal and local approximability proved in § 2.

2. Microlocal analytic approximability

We shall microlocalize the definition of analytic approximability. Recall that for a distribution u defined near $x_0\in\mathbf{R}^n$, an *FBI (Fourier—Bros—Iagolnitzer) transform* of u is an integral of the form

$$\mathscr{F}u(x,\xi)=\int e^{i(x-y)\cdot\xi-(x-y)^2|\xi|}\chi(y)u(y)\,dy, \tag{2.1}$$

where $\chi\in C_0^\infty(\mathbf{R}^n)$, $\chi\equiv 1$ near x_0, Then u is *analytic* at (x_0,ξ_0), $\xi_0\neq 0$, or $(x_0,\xi_0)\notin WF_a u$, if there is a conic neighborhood Γ of ξ_0, a neighborhood $\mathscr{U}$ of x_0 and a constant C such that

$$|\mathscr{F}u(x,\xi)|\leqq Ce^{-|\xi|/C}\quad\text{for all}\quad (x,\xi)\in\mathscr{U}\times\Gamma. \tag{2.2}$$

We shall write $u\sim 0$ at (x_0,ξ_0) if $(x_0,\xi_0)\notin WF_a u$. If $\{u_j\}$ is a sequence of distributions, we write $\{u_j\}\sim 0$ at (x_0,ξ_0) if (2.2) holds for all u_j with the same C, $\mathscr{U}$ and Γ.

(2.3) *Definition.* A classical analytic pseudodifferential operator Q defined in a conic neighborhood of (x_0,ξ_0) is *microlocally approximable* at (x_0,ξ_0) if for any distribution v for which $Qv\sim 0$ at (x_0,ξ_0) there is a sequence of distributions v_j such that

i) $v_j\to v$
ii) $\{Qv_j\}\sim 0$ at (x_0,ξ_0)
iii) $v_j\sim 0$ at $(x_0,\xi_{0},)$ with the conic neighborhood independent of j.

In order to show that these definitions make sense we need the following lemma. For an analytic pseudodifferential operator (either classical or of type $(\frac{1}{2}, \frac{1}{2})$) we write $R \sim 0$ at (x_0, ξ_0) if R is of order $-\infty$ in a conic neighborhood of (x_0, ξ_0).

(2.4) Lemma. *If R is an analytic pseudodifferential operator and $\{w_j\}$ is a sequence of distributions with $\{w_j\} \sim 0$ at (x_0, ξ_0) then $\{Rw_j\} \sim 0$ at (x_0, ξ_0).* If $R \sim 0$ *and $\{w_j\}$ is any bounded sequence of distributions, then $\{Rw_j\} \sim 0$ at (x_0, ξ_0).*

Proof. The second statement may be proved by following the constants in the FBI transform. The first is also refinement of the statement that an analytic pseudodifferential operator does not extend the wave front set.

By abuse of notation we shall also write $Q_1 \sim Q_2$ at (x_0, ξ_0) if $Q_1 - F_1 Q_2 F_2$ is of order $-\infty$ at (x_0, ξ_0), where F_1 and F_2 are elliptic pseudodifferential operators. By the above it follows that Q_1 is microlocally approximable at (x_0, ξ_0) if and only if Q_2 is.

We may also generalize local approximability to locally defined analytic pseudodifferential operators as follows.

(2.5) *Definition.* If Q is an analytic pseudodifferential operator defined in a neighborhood of x_0 then Q is *locally approximable* at x_0 if for every distribution v for which Qv is real analytic in a neighborhood $\mathscr{U}$ of x_0 there is a sequence v_j of functions, real analytic in a neighborhood $\mathscr{U}'$ of x_0, such that

i) $v_j \to v$

ii) $\{Qv_j\}$ extends to a convergent sequence of holomorphic functions in a neighborhood of x_0 in $\mathbf{C}^n$.

It is easy to check, using the Cauchy—Kovalevsky Theorem, that if Q is an analytic differential operator not totally characteristic at x_0, then this definition agrees with that of Baouendi and Treves [2].

We write $w \sim 0$ or $\{w_j\} \sim 0$ or $Q \sim 0$ at x_0 if the equivalence defined above holds for every $\lambda \in \mathbf{R}^n \setminus 0$. We note that if $\{w_j\} \sim 0$ at x_0, then (see, e.g., [11]) there is a neighborhood of x_0 in $\mathbf{C}^n$ to which all the w_j extend as uniformly bounded holomorphic functions W_j. Hence, by the well known theorem for holomorphic functions, the W_j have a convergent subsequence. This is the connection between the two conditions (ii) in the local and microlocal definitions of analytic approximability.

In a special case microlocal approximability in all directions is equivalent to local approximability.

(2.6) Theorem. *Let Q be a classical analytic pseudodifferential operator for which* $\text{char}_{x_0} Q$ *is contained in a line. Then Q is locally approximable at x_0 if and only if Q is microlocally approximable at (x_0, ξ) for all $\xi \in \mathbf{R}^n \setminus \{0\}$.*

Proof. We may assume that at x_0, Q is elliptic away from $\{(0, \ldots, 0, \xi_n), \xi_n \neq 0\}$. Suppose first that Q is microlocally approximable at (x_0, ξ) for all ξ and that $Qv \sim 0$ at x_0. Then v is analytic at (x_0, ξ), $\xi \neq (0, \ldots, 0, \pm 1)$ and by using appropriate cut-off functions we may write $v = v_1 + v_2$, $v_1 \sim 0$ except at $(x_0, 0, \ldots, 0, \xi_n)$, $\xi_n > 0$ and $v_2 \sim 0$ except at $(x_0; 0, \ldots, 0, \xi_n)$, $\xi_n < 0$. Since $Qv_1 + Qv_2 \sim 0$ at x_0 and $Qv_2 \sim 0$ at $(x_0; 0, \ldots, 0, 1)$ we have $Qv_1 \sim 0$. Hence we may assume $v \sim 0$ at (x_0, ξ), $\xi \neq (0, \ldots, 0, 1)$. By assumption of microlocal approximability there is a sequence $\{v'_j\}$ such that $v'_j \to v$, $\{Qv'_j\} \sim 0$ and $v'_j \sim 0$ in a conic neighborhood $\mathcal{U} \times \Gamma$ of $(x_0; 0, \ldots, 0, 1)$. We claim first that $\{v'_j\} \sim 0$ at (x_0, ξ), $\xi \neq (0, \ldots, 0, \pm 1)$. Indeed, since Q is elliptic at such points, there is an analytic pseudodifferential operator P such that $PQ \sim I$ at (x_0, ξ). The claim then follows by Lemma (2.4). Now we may find an analytic pseudodifferential operator $\Psi(D)$ such that $\Psi(D) \sim 0$ at (x_0, ξ) if $\xi_n < 0$ and $\Psi(D) \sim I$ at (x_0, ξ), ξ near $(0, \ldots, 0, 1)$. Let $v_j = \Psi(D)v'_j + (I - \Psi(D))v$. Then $v_j \to v$, since $\Psi(D)v'_j - \Psi(D)v \to 0$. Also, $\Psi(D)v'_j \sim 0$ for all j and $(I - \Psi(D))v \sim 0$, since $\Psi(D)v \sim v$ near $(x_0; 0, \ldots, 0, 1)$ and $v \sim 0$ at (x_0, ξ), $\xi \neq (0, \ldots, 1)$. Finally,

$$\{Qv_j\} = \{Q\Psi(D)v'_j\} + \{Q(I - \Psi(D))v\},$$

so it suffices to show $\{Q\Psi(D)v'_j\} \sim 0$. We have

$$Q\Psi v'_j = \Psi Q v'_j + [Q, \Psi]v'_j$$

and since $\{Qv'_j\} \sim 0$, $\{\Psi Qv'_j\} \sim 0$ by Lemma (2.4), while $\{[Q, \Psi]v'_j\} \sim 0$ also by the lemma, because $[Q, \Psi]$ is of order $-\infty$ near $\mathrm{char}_{x_0}(Q)$ and $\{v_j\} \sim 0$ away from that set. Hence, Q is locally approximable at x_0.

For the converse, assume Q is locally approximable at x_0 and let v be such that $Qv \sim 0$ at $(x_0; 0, \ldots, 0, 1)$. We write $v = v_1 + v_2$ as above. Then $Qv_1 \sim 0$ at x_0, so there exists a sequence $v'_j \to v_1$ so that $v'_j \sim 0$ at x_0 and $\{Qv'_j\} \sim 0$ at x_0. Since $v_2 \sim 0$ at $(x_0; 0, \ldots, 0, 1)$ we may take $v_j = v'_j + v_2$ which proves Q is microlocally approximable at $(x_0; 0, \ldots, 0, 1)$. The proof of microlocal approximability at $(x_0; 0, \ldots, 0, -1)$ is the same, and it is clear that Q is microlocally approximable at any noncharacteristic point.

3. A criterion of microlocal approximability for some differential operators

We consider here a differential operator of degree $m > n$ in the variables $(t, y) \in \mathbf{R}^n \times \mathbf{R}$ of the form

(3.1) $$\sum_{|\alpha|+|\beta| \leq m} a_{\alpha\beta}(t, y, D_t, D_y) t^\alpha D_y^{|\alpha|} D_t^\beta$$

where

$$a_{\alpha\beta}(t, y, D_t, D_y) = \sum_{|\gamma|+|\delta| \leq (m-|\beta|-|\beta|)/2} a_{\alpha\beta\gamma\delta}(t, y) D_y^\gamma D_t^\delta$$

is an analytic differential operator of degree $\leqq (m-|\alpha|-|\beta|)/2$ with $a_{\alpha\beta\gamma 0}$ constant for $|\alpha|+|\beta|\leqq m$ and $|\gamma|=(m-|\alpha|-|\beta|)/2$. We let

$$P_m = \sum_{\substack{|\alpha|+|\beta|\leqq m \\ |\gamma|=(m-|\alpha|-|\beta|)/2}} a_{\alpha\beta\gamma 0} D_y^\gamma t^\alpha D_y^{|\alpha|} D_t^\beta .$$

We shall assume the condition of transversal ellipticity, i.e., that

$$\sum_{|\alpha|+|\beta|=m} a_{\alpha\beta 00} t'^\alpha \tau'^\beta \neq 0 \quad \text{for all} \quad (t', \tau')\in \mathbf{R}^{2n}\setminus\{0\}. \tag{3.2}$$

The operators considered here are in a more restricted class than those studied in [4], [6] and [10]. Following the approach of Grušin and that of [4] and [10] (for the C^∞ case) we shall reduce the question of microlocal analytic hypoellipticity and microlocal approximability to that of a system of pseudodifferential operators.

We first use a result from [3] applied to P_m. We fix a point $(0; 0, ..., \eta_0)$ in char P_m (determined by $\eta_0>0$ or $\eta_0<0$) and let

$$P_m(\eta) = \sum_{\substack{|\alpha|+|\beta|\leqq m \\ |\gamma|=(m-|\alpha|-|\beta|)/2}} a_{\alpha\beta\gamma 0} \eta^{|\alpha|+|\gamma|} t^\alpha D_t^\beta .$$

Then $P_m^*(\eta)P_m(\eta)$ has a kernel in $L^2(\mathbf{R}^n)$ of finite dimension q_1 and $P_m(\eta)P_m^*(\eta)$ a kernel of dimension q_2. In [3] it is shown that there exist microlocal systems of analytic pseudodifferential operators J_1, J_2, Q, L defined

$$J_i\colon \mathscr{E}'(\mathbf{R}_y)^{q_i} \to \mathscr{D}'(\mathbf{R}^{n+1}_{t,y}), \quad i = 1, 2$$

$$Q\colon \mathscr{E}'(\mathbf{R}^{n+1}_{t,y}) \to \mathscr{D}'(\mathbf{R}^{n+1}_{t,y})$$

$$L\colon \mathscr{E}'(\mathbf{R}_y)^{q_1} \to \mathscr{D}'(\mathbf{R}_y)^{q_2}$$

such that

$$\begin{pmatrix} P_m & J_2 \\ J_1^* & 0 \end{pmatrix}\begin{pmatrix} Q & J_1 \\ J_2^* & -L \end{pmatrix} \sim \begin{pmatrix} I & 0 \\ 0 & I_{q_1} \end{pmatrix} \tag{3.3}$$

and

$$\begin{pmatrix} Q & J_1 \\ J_2^* & -L \end{pmatrix}\begin{pmatrix} P_m & J_2 \\ J_1^* & 0 \end{pmatrix} \sim \begin{pmatrix} I & 0 \\ 0 & I_{q_2} \end{pmatrix} \tag{3.4}$$

near $(0; 0, ..., \eta)$. Here the analyticity of the above systems means that if $u\in\mathscr{E}'(\mathbf{R}^{n+1}_{t,y})$ and $v_i\in\mathscr{E}'(\mathbf{R}_y)^{q_i}$ then

$$WF_a(Qu) \subset WF_a(u),$$

$$WF_a(Lv_1) \subset WF_a(v_1),$$

$$WF_a(J_i v_i) \subset \{(t, y;\, \tau, \eta) | t = \tau = 0,\ (y, \eta)\in WF_a(v)\},$$

$$WF_a(J_i^* u) \subset \{(y, \eta) | (0, y;\, 0, \eta)\in WF_a(u)\}.$$

By the construction of Helffer [4], following Sjöstrand [10], there exist C^∞-microlocal operators E, E^+, E^- and $E^\pm$ so that if

$$\mathscr{E} = \begin{pmatrix} E & E^+ \\ E^- & E^\pm \end{pmatrix}, \quad \mathscr{P} = \begin{pmatrix} P & J_2 \\ J_1^* & 0 \end{pmatrix}$$

then $\mathscr{E}\mathscr{P}\sim I$, $\mathscr{P}\mathscr{E}\sim I$ in the C^∞ sense. In fact, these operators may be obtained from

$$\mathscr{E}_0=\begin{pmatrix} Q & J_1 \\ J_2^* & -L \end{pmatrix}$$

by a Neumann series: by (3.3) $\mathscr{P}\mathscr{E}_0=\mathscr{I}+\mathscr{A}$, where

$$\mathscr{A}=\begin{pmatrix} (P-P_m)Q & (P-P_m)J_1 \\ 0 & 0 \end{pmatrix}$$

and formally

(3.5)
$$(\mathscr{I}-\mathscr{A})^{-1}=\begin{pmatrix} \sum_{j=0}^{\infty}(-1)^j((P-P_m)Q)^j & -\sum_{j=0}^{\infty}(-1)^j((P-P_m)Q)^j(P-P_m)J_1 \\ 0 & I_{q_1} \end{pmatrix}$$

so, $\mathscr{E}=\mathscr{E}_0(\mathscr{I}-\mathscr{A})^{-1}$. It is shown in [4] that the operators E, E^+, E^-, $E^\pm$ can be obtained from (3.5) in this fashion and that they have the appropriate C^∞ behavior. In particular, $E^\pm$ is semiclassical and E is of type $(\frac{1}{2}, \frac{1}{2})$. We shall show, using the machinery of Métivier [6] that they are also analytic.

(3.6) Theorem. *Let P be of the form* (3.1), *transversally elliptic and such that $a_{\alpha\beta\gamma 0}$ is constant for $|\alpha|+|\beta|\leqq m$, $|\gamma|=(m-|\alpha|-|\beta|)/2$. Let $\omega=(0; 0, \dots, 0, \eta)$. Then the operators E, E^+, E^-, $E^\pm$ are all microlocally analytic at ω. P is microlocally analytic hypoelliptic at ω if and only if $E^\pm$ is at $\omega'=(0;\eta)\in\mathbf{R}\times(\mathbf{R}\diagdown 0)$. Furthermore, P is microlocally approximable at ω if and only if $E^\pm$ is microlocally approximable at ω'.*

Proof. The first statement will be proved in § 4. We shall assume it here. The statement about microlocal analytic hypoellipticity is proved by standard arguments (see [1], [4] and [10]). For the statement about microlocal approximability, suppose first that P is microlocally approximable at ω and that $E^\pm v\sim 0$ at ω'. If $w=E^+v$ then $Pw\sim 0$ at ω since $PE^++J_2E^\pm\sim 0$. Let $\{w_j\}$ be such that $w_j\to w$ and $\{Pw\}_j\sim 0$ at ω, and set $v_j=J_1^*w_j$. Since $E^-P+E^\pm J_1^*\sim 0$ and $\{Pw_j\}\sim 0$, $\{E^\pm J_j^*w_j\}\sim 0$, that is, $\{E^\pm v_j)\sim 0$ at ω'. Finally, if $v_0=v-J_1^*E^+v$ then $v_0\sim 0$ at ω', $v_j+v_0\sim 0$ at ω' and $v_j+v_0\to v$.

Conversely, if $E^\pm$ is microlocally approximable at ω' and $Pu\sim 0$ at ω, let $v=J_1^*u$. Then $Pu\sim 0$ at ω and $E^-P+E^\pm J_1^*\sim 0$ imply $E^\pm v\sim 0$ at ω'. Let $v_j\to v$, $\{E^\pm v_j\}\sim 0$ at ω', $u_j=E^+v_j$. Then $\{Pu_j\}\sim 0$ at ω since $PE^++J_2E^\pm\sim 0$. Since $EP+E^+J_1^*\sim I$, $E^+J_1^*u=u-u_0$ with $u_0\sim 0$ at ω so $u_j+u_0=E^+v_j+u_0\to u$. This completes the proof of theorem.

4. Proof of the analyticity of E, E^+, E^-, $E^\pm$

We shall base the proof of the analyticity of the operator $\mathscr{E}$ on Métivier [6], where it is proved an operator of the form (3.1) (in fact, those in a more general class) has a left analytic parametrix if $\ker P_m(\eta)$ is trivial. In what follows we use the material contained in Chapter II and III of [6] and will recall only essential definitions and results as needed to make this section readable.

After some analysis of the Neumann series expansion of $(\mathscr{P}\mathscr{E}_0)^{-1}$ we see that the only thing to do is find the sequence of operators Q_j, $j=0, 1, \dots$ defined by taking $Q_0=Q$, $Q_j=-Q\tilde{P}Q_{j-1}$, $j\geqq 1$, where $\tilde{P}=P(t, y, D_t, D_y)-P_m(D_y)(t, D_t)$. If this is done and we let $E=\Sigma Q_j$, $E^+=-EPJ_1+J_1$, $E^-=J_2^*-J_2^*\tilde{P}E$ and $E^\pm=E^-\tilde{P}J_1$ then using (3.3) and the definition of E one sees that at least formally the matrix $\mathscr{E}$ thus obtained is a right inverse for $\mathscr{P}$. We shall show that E can be found in this way and that it is an analytic operator.

We begin by recalling from [6] that for a nonnegative integer k, the space $\mathscr{H}^k(\mathbf{R}^n)$ consists of those $u\in H^k(\mathbf{R}^n)$ such that $\hat{u}\in H^k(\mathbf{R}^n)$, where H^k is the usual Sobolev space based on $L^2(\mathbf{R}^n)$. In $\mathscr{H}^k(\mathbf{R}^n)$ we have the norm $\| \ \|_k$ given by $\|u\|_k=\sup|T_I u|$, where the sup is taken over all $T_I=T_1\dots T_k$, each T_i being either a multiplication by a t_j or a partial derivative ∂_{t_j}. The dual of $\mathscr{H}^k$ is $\mathscr{H}^{-k}$. For a positive real number R and a nonnegative integer m $\mathscr{L}_R^m(\mathbf{R}^n)$ is the space of operators $K\colon \mathscr{S}'\to\mathscr{S}'$ such that for $p=0, \dots, m+|\gamma|$ $(\operatorname{ad} T)^\gamma K\colon \mathscr{H}^{-p}\to\mathscr{H}^{-p+m+|\gamma|}$ with norm $\leqq CR^{|\gamma|}|\gamma|!$ for some C independent of γ and p. The best of such constants C is denoted by $\|K\|_{\mathscr{L}_R^m}$. In this definition, γ is a multiindex $(\gamma_1, \dots, \gamma_{2n})$, $\operatorname{ad} T^\gamma K=\operatorname{ad} t_1^{\gamma_1}\dots \operatorname{ad} t_n^{\gamma_n} \operatorname{ad}\partial_{t_1}^{\gamma_{n+1}}\dots$ $\dots \operatorname{ad}\partial_{t_n}^{\gamma_{2n}} K$ and as usual $(\operatorname{ad} T)(K)=TK-KT$.

One has (see [6])

$$\|\operatorname{ad} T^\gamma K\|_{\mathscr{L}_{2R}^m} \leqq \sqrt{|\gamma|!}\,(2R)^{|\gamma|}\|K\|_{\mathscr{L}_R^m}.$$

With this one can show: If $K\in\mathscr{L}_R^m$ then its left symbol $\sigma(K)(r, \varrho)=e^{-ir\varrho}K(e^{ir\varrho})$ satisfies

$$(4.1)\qquad |r^\alpha\varrho^\beta\partial_r^\gamma\partial_\varrho^\delta\sigma(K)(r, \varrho)| \leqq C_0(CR)^{|\alpha|+|\beta|+|\gamma|+|\delta|}\sqrt{\alpha!\,\beta!\,\gamma!\,\delta!}\,\|K\|_{\mathscr{L}_R^m}$$

if $|\alpha|+|\beta|\leqq m+|\gamma|+|\delta|$, for some C_0 and C.

Now let $\varphi_i\colon \mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ be given by $\varphi_i(t, s)=t_i^2+s_i^2$ if $t_is_i\leqq 0$, $\varphi_i(t, s)=|t_i^2-s_i^2|$ if $t_is_i\geqq 0$. For $\varepsilon>0$ B_ε is the space of operators $K\colon L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)$ whose Schwartz kernel, denoted as $K(t, s)$, satisfies

$$(4.2)\qquad \|e^{\varepsilon\varphi_j(t,s)}K(t, s)\|_{L^2}<\infty,\quad \|e^{\varepsilon\varphi_j(\tau,\sigma)}\tilde{K}(\tau, -\sigma)\|_{L^2}<\infty,\quad j=1, \dots, n.$$

$\tilde{K}(\tau, -\sigma)$ is essentially the Schwartz kernel of $\tilde{K}$, the operator given by $\tilde{K}\hat{u}=(Ku)\hat{}$. The norm $\|K\|_\varepsilon$ of K is the maximum of the numbers in (4.2).

Proposition 2.10 of [6] states that if $0<\varepsilon'<\varepsilon\leqq 1$ then $K\in B_\varepsilon$ implies $(\operatorname{ad} T)K\in B_{\varepsilon'}$, where T is either t_j or ∂_{t_j}, and

$$\|(\operatorname{ad} T)K\|_{\varepsilon'}\leqq\left(\frac{M_0}{\varepsilon-\varepsilon'}\right)^{1/2}\|K\|_{B_\varepsilon} \tag{4.3}$$

with $M_0\geqq 1$ independent of ε, ε' and K. It follows from this, Plancherel's theorem and the fact that $B_\varepsilon\hookrightarrow L^2(\mathbf{R}^{2n})$ with norm independent of ε that if $K\in B_\varepsilon$, $\varepsilon\leqq 1$ and $\varepsilon'<\varepsilon$, the left symbol

$$\sigma(K)(r,\varrho)=e^{-ir\varrho}\int e^{is\varrho}K(r,s)\,ds$$

of K satisfies

$$\|\partial_r^\alpha\partial_\varrho^\beta\sigma(K)\|_{L^2}\leqq C\left(\frac{M_0}{\varepsilon}\right)^{(|\alpha|+|\beta|)/2}(|\alpha|+|\beta|)^{(|\alpha|+|\beta|)/2}\|K\|_\varepsilon$$

with C independent of α, β, ε, ε' and K. Therefore

$$|\partial_r^\alpha\partial_\varrho^\beta\sigma(K)(r,\varrho)|\leqq C\left(\frac{M_0}{\varepsilon}\right)^{n+1+(|\alpha|+|\beta|)/2}(|\alpha|+|\beta|)^{(|\alpha|+|\beta|)/2}\|K\|_\varepsilon \tag{4.4}$$

with another constant C.

Let Ω be a complex neighborhood of $(0,y_0)\in\mathbf{R}^n\times\mathbf{R}$ and $\Gamma_0\subset\mathbf{R}^n\times\mathbf{R}\setminus 0$ a conic neighborhood of $(0,\eta_0)$ such that $(\tau,\eta)\in \mathrm{h}_0\Rightarrow|(\tau,\eta)|\sim|\eta|$. We shall assume $\eta_0>0$. For $c>0$ small let $\Gamma=\{(\tau,\eta)\in\mathbf{C}^n\times\mathbf{C}\colon \operatorname{Re}(\tau,\eta)\in\Gamma_0,\ |\operatorname{Im}(\tau,\eta)|<c|\operatorname{Re}(\tau,\eta)|\}$. Modifying slightly the definition given in [6] we let $G_\varepsilon^\mu(\Omega\times\Gamma)$ be the space of holomorphic functions $a\colon\Omega\times\Gamma\to B_\varepsilon$ such that

$$\|a(t,y,\tau,\eta)\|_\varepsilon\leqq C|\operatorname{Re}(\tau,\eta)|^\mu\quad\text{if}\quad(t,y,\tau,\eta)\in\Omega\times\Gamma. \tag{4.5}$$

Similarly $F_R^\mu(\Omega\times\Gamma)$ consists of the holomorphic functions $a\colon\Omega\times\Gamma\to\mathscr{L}_R^0$ such that (4.5) holds with the $\mathscr{L}_R^0$-norm.

(4.6) Lemma. *For $j=0,1,\dots$ let $A_j\in F_R^{m-j/2}(\Omega\times\Gamma)$ (respectively $G_\varepsilon^{m-j/2}(\Omega\times\Gamma)$) be such that*

$$\|A_j(t,y,\tau,\eta)\|\leqq C_0R_0^j j^{j/2}|\operatorname{Re}(\tau,\eta)|^{m-j/2} \tag{4.7}$$

where the norm is that of $\mathscr{L}_R^0$ (respectively B_ε) and let $a_j(t,y,\tau,\eta,r,\varrho)=\sigma(A_j(t,y,\tau,\eta))(r,\varrho)$. Then there exist C_1, R_1 such that for $(t,y)\in\Omega$, $(\tau,\eta)\in\Gamma_1\subset\subset\Gamma_0$, $(r,\varrho)\in\mathbf{R}^{2n}$

$$|r^\alpha\varrho^\beta\partial_r^\gamma\partial_\varrho^\delta\partial_\tau^\theta\partial_\eta^\mu a_j(t,y,\tau,\eta,r,\varrho)|\leqq C_1\sqrt{\alpha!\beta!\gamma!\delta!}\,\theta!\,\mu!\times R_1^{|\alpha+\beta+\gamma+\delta+\theta|+\mu}|\eta|^{m-j/2-|\theta|-\mu} \tag{4.8}$$

for any $\alpha,\beta,\dots,\mu$ with $|\alpha+\beta|\leqq|\gamma+\delta|$ (respectively $\alpha=\beta=0$). If $\chi_j\in C^\infty(\mathbf{R}^{n+1}_{\tau,\eta})$ is a sequence such that $0\leqq\chi_j\leqq 1$, $\chi_j(\tau,\eta)=0$ if $|(\tau,\eta)|<j$, $\chi_j(\tau,\eta)=1$ if $|(\tau,\eta)|>2j$

and $|\partial_\tau^\theta \partial_\eta^\mu \chi_j(\tau, \eta)| \leqq C^{|\theta|+|\mu|}$ *when* $|\theta|+|\mu| \leqq j$ *for some* C *independent of* j *and if* $a = \sum \chi_j((\tau, \eta)/\lambda) a_j$ *then for some* C_2, λ, R_2 *and indices* $\alpha, \ldots, \mu$ *as above (respectively* $\alpha=\beta=0$*)*

$$(4.9) \qquad |r^\alpha \varrho^\beta \partial_r^\gamma \partial_\varrho^\delta \partial_\tau^\theta \partial_\eta^\mu a(t, y, \tau, \eta, r, \varrho)| \leqq C_2 \sqrt{\alpha!\beta!\gamma!\delta!}\,\theta!\mu! \times R_2^{|\alpha+\beta+\gamma+\delta+\theta|\mu} |\eta|^{m-|\theta|-\mu}$$

if $(t, y) \in \Omega$, $(\tau, \eta) \in \Gamma_1$ *and* $|(\tau, \eta)| > R_2(|\theta|+\mu)$.

For the proof of the lemma we only point out that (4.8) is a consequence of (4.7), (4.1) (respectively (4.4)) and Cauchy's integral formula, and (4.9) follows from (4.8) by the argument given in the proof of Lemma 3.2 of [6].

(4.10) Lemma. *If* $a: \Omega \times \Gamma_1 \to \mathbf{C}$ *satifies* (4.9) *for some* C_2, R_2 *and all* $\alpha, \ldots, \mu$ *with* $|\alpha|+|\beta| \leqq |\gamma|+|\delta|$ *(respectively* $\alpha=\beta=0$*) then there are constants* C_3, R_3 *such that for* $|\tau|+|\eta| > R_3(|\theta|+\mu)$

$$(4.11) \qquad |\partial_\varrho^\alpha \partial_\tau^\theta \partial_\eta^\mu \partial_\lambda^\nu a(t, y, \tau, \eta, \lambda^{1/2} r, \lambda^{-1/2}\varrho)| \leqq C_3 \sqrt{\alpha!}\,\theta!\mu!\nu! \times R_3^{|\theta|+\mu+\nu} e^{|\mathrm{Im}\, r|^2 \lambda/R_3} |\eta|^{m-\mu-\theta_\lambda-\nu-|\alpha|/2}$$

if $(t, y) \in \Omega$, $(\tau, \eta) \in \Gamma_1$, $\lambda > 0$, $r \in \mathbf{C}^n$ *and* $|\varrho| < c\lambda$ *for some small* c. *(Respectively*

$$(4.12) \qquad |\partial_\tau^\theta \partial_\eta^\mu a(t, y, \tau, \eta, \eta^{1/2} r, \eta^{-1/2}\tau)| \leqq C_3 \sqrt{\theta!\mu!} \times R_3^{|\theta|+\mu} e^{|\mathrm{Im}\, r|^2 |\eta|/R_3} |\eta|^{m-\mu/2-|\theta|/2}$$

if $(t, y) \in \Omega$, $(\tau, \eta) \in \Gamma$, $r \in \mathbf{C}^n$ *and* $|\mathrm{Re}\, r| < 1$*). In particular, for* $(t, y, \tau, \eta) \in \mathrm{Re}\, \Omega \times \Gamma_1$ $a(t, y, \tau, \eta, \eta^{1/2} t, \eta^{-1/2}\tau)$ *is an analytic symbol of type* $(1/2, 1/2)$.

This Lemma is a direct consequence of the estimates (4.9).

Let us now consider the operator

$$(4.13) \qquad P_m(\eta)(r, D_r) = \sum_{\substack{|\alpha|+|\beta| \leqq m \\ |\gamma|=(m-|\alpha|-|\beta|)/2}} a_{\alpha\beta\gamma 0} \eta^{|\alpha|+|\gamma|} r^\alpha D_r^\beta$$

of the previous section with the ellipticity condition (3.2). It satisfies $P_m(\eta)(r, D_r) = \eta^{m/2} P_m(1)(\eta^{1/2} r, \eta^{-1/2} D_r)$. Let $\Pi_1: L^2 \to L^2$ be the orthogonal projection onto the kernel of $P_m(1)$ as an operator on L^2 and let $I - \Pi_2: L^2 \to L^2$ be the orthogonal projection onto the range, which is known to be closed; Π_1 and Π_2 have finite range (see [3] and references given there) because of the ellipticity of $P_m(1)$, which implies also the existence of an operator $K: L^2 \to \mathscr{H}^m$ such that $P_m K = I - \Pi_2$, $K P_m = I - \Pi_1$. $K \in \mathscr{L}_R^m$ for some $R > 0$ so $K \in B_{\varepsilon_0}$ for some $\varepsilon_0 > 0$ (see [6]) since $m > n$. Let $k(r, \varrho)$ be the symbol of K. Then $\eta^{-m/2} k(\eta^{1/2} t, \eta^{-1/2}\tau)$ is an analytic symbol of type $(1/2, 1/2)$ and order $-m/2$.

We will now begin the construction of the operators Q_j mentioned at the beginning of this section. Let $g(\tau, \eta) \in C^\infty(\mathbf{R}^{n+1}_{\tau,\eta})$ be supported in a conic neighborhood of $(0, \eta_0)$ contained in Γ_0 such that $g(\tau, \eta) = 1$ if $|(\tau, \eta)| \geqq 1$ and $(\tau, \eta) \in \Gamma_1 \subset\subset \Gamma_0$, another conic neighborhood of $(0, \eta_0)$, vanishing near 0 and such that

$$|\partial_\tau^\theta \partial_\eta^\mu g(\tau, \eta)| \leqq C^{|\theta|+\mu}(|\alpha|/|(\tau, \eta)|)^{\varrho(|\theta|+\mu)} \tag{4.14}$$

as in Lemma 3.1 of [6].

The operator Q appearing in $\mathscr{E}_0$ in § 3 is given by

$$Qf(t, y) = (2\pi)^{-n-1} \int e^{i(t-s)\tau + i(y-y')\eta} g(\tau, \eta) \eta^{-m/2} \times k(\eta^{1/2} t, \eta^{-1/2}\tau) f(s, y')\, ds\, dy'\, d\tau\, d\eta \tag{4.15}$$

where g is as in (4.14) with $1/2 < \varrho < 1$. If $\tilde{P}(t, y, D_t, D_y) = P(t, y, D_t, D_y) - P_m(D_y)(t, D_t)$ its left symbol can be written as $\tilde{p}(t, y, \tau, \eta, \eta^{1/2}t, \eta^{-1/2}\tau)$ where $\tilde{p}(t, y, \tau, \eta)(r, \varrho)$ is a polynomial in (r, ϱ) of degree $\leqq m$ with coefficients which are holomorphic functions in a complex conic neighborhood $\Omega \times \Gamma$ of $(0, y_0, 0, \eta_0)$ and bounded there by $C|\mathrm{Re}\,(\tau, \eta)|^{m/2-1}$.

Since the symbol of Q is of type $(1/2, 1/2)$ and that of $\tilde{P}$ of type $(1, 0)$ the usual formula for the symbol of the composition holds (see [6]). Using it we obtain, after some rearrangement of terms, that $Q\tilde{P}$ has a symbol of the form $h(t, y, \tau, \eta, \eta^{1/2}t, \eta^{-1/2}\tau)$ and for (t, y, τ, η) close to $(0, y_0, 0, \eta_0)$

$$h(t, y, \tau, \eta, r, \varrho) \sim \sum_{k=0}^\infty \sigma(H_k(t, y, \tau, \eta))(r, \varrho)$$

where

$$H_k(t, y, \tau, \eta)(r, D_r) = \eta^{-m/2-l} \sum_{2\mu+|\theta|=l} \sum_{j \leqq \mu' \leqq \mu} C_{\theta, \mu\mu', j} \times [(\mathrm{ad}\,(-ir))^\theta L^j K](r, D_r) \circ D_t^\theta D_t^\mu \tilde{p}(t, y, \tau, \eta, r, D_r).$$

Here $L = 1/2(\sum r_j\, \mathrm{ad}\, \partial_{r_j} + i\partial_{r_j}\, \mathrm{ad}\, r_j)$ and $|C_{\theta, \mu, \mu', j}| \leqq 2^{m/2+\lambda+\mu'} \times (\mu - j)!/\theta!\mu!$. Since $K \in \mathscr{L}_R^m$ there exists R_1, C_1 such that $(\mathrm{ad}\, r)^\theta L^j K \in \mathscr{L}_{R_1}^m$ with norm bounded by $C_1\sqrt{\theta!}\, j!(C_1 R_1)^{|\theta|+j} \|K\|_{\mathscr{L}_R^m}$ and with this and the fact that $\tilde{p}$ is a polynomial in (r, ϱ) of degree $\leqq m$ whose coefficients satisfy specific bounds one shows $H_l \in F_{R_1}^{-j/2-1}(\Omega \times \Gamma)$ with

$$\|H_j(t, y, \tau, \eta)\|_{\mathscr{L}_R^0} \leqq C_0 R_0^j \sqrt{j!}\, |\mathrm{Re}\,(\tau, \eta)|^{-j/2-1} \tag{4.16}$$

for some C_0, R_0. Let $W_j \in G^{m'-j/2}(\Omega \times \Gamma)$ be such that

$$\|W_j(t, y, \tau, \eta)\|_\varepsilon \leqq C_0 R_0^j \sqrt{j!}\, |\mathrm{Re}\,(\tau, \eta)|^{m'-j/2}. \tag{4.17}$$

Let $h_j = \sigma(H_j)$, $w_j = \sigma(W_j)$ and set

$$h(t, y, \tau, \eta, r, \varrho) = g(\tau, \eta) \sum \chi_{j+1}(\tau, \eta) h_j(t, y, \tau, \eta, r, \varrho)$$

(likewise $w(t, y, \tau, \eta, r, \varrho)$) with χ_j chosen as in lemma (4.6) and g as in the definition of Q. For a symbol such as h or w let

$$\mathrm{op}\,(h) f(t, y) = (2\pi)^{-n-1} \int e^{i(t-s)\tau + i(y-y')\eta} h(t, y, \tau, \eta, \eta^{1/2}t, \eta^{-1/2}\tau) f(s, y')\, ds\, dy'\, d\tau\, d\eta.$$

We also write $\operatorname{op}(\Sigma H_j)$ for this operator. Of course different realizations of the formal symbol ΣH_j yield operators differing only by an analytic-regularizing operator near $(0, y_0, 0, \eta_0)$.

(4.18) Proposition. *Let h and w be as above. Then $H=\operatorname{op}(h)$ and $W=\operatorname{op}(w)$ are analytic pseudodifferential operators in a neighborhood Ω_0 of $(0, y_0)$ in $\mathbf{R}^{n+1}$. If $\varphi\in C_0^\infty(\Omega_0)$ and $\varphi=1$ near $(0, y_0)$ then $H\varphi W=C$ where C is an operator which in a a neighborhood of $(0, y_0, 0, \eta_0)$ is of the form $\operatorname{op}(c)$ with*

$$c(t, y, \tau, \eta, \eta^{1/2}t, \eta^{-1/2}\tau) = \Sigma \frac{(2\pi)^{-n}}{\alpha!\beta!\gamma!\delta!}\int e^{i(t-s)(\tau-\sigma)}\eta^{|\beta|/2} \times\partial_\eta^\gamma[\eta^{-|\alpha|/2}\partial_\sigma^{\beta+\delta}\partial_\varrho^\alpha h_j(t, y, \tau, \eta, \eta^{1/2}t, \eta^{-1/2}\tau)] \times(D_y^\gamma D_t^{\alpha+\gamma}D_r^\beta W_i)(t, y, \tau, \eta, \eta^{1/2}s, \eta^{-1/2}\sigma)\, ds\, d\tau. \tag{4.19}$$

Outline of Proof. The analyticity of the operators H and W follows from Lemma (4.10) above and Lemma 3.3 of [6]. The expansion of the symbol of the composition is obtained by taking the Taylor expansion of $w(s, y', \sigma, \eta, r, \varrho)$ in s at $s=t$ and that of $h(t, y, \tau, \xi, \xi^{1/2}t, \xi^{-1/2}\varrho)$ in τ at $\tau=\sigma$ and ξ at $\xi=\eta$. After carrying out the η derivatives in (4.19) we see that $c\sim\sum_0^\infty c_k$ with $c_k(t, y, \tau, \eta, r, \varrho)$ the left symbol of $C_k\in G_{\varepsilon'}^{m'-k/2-1}$ ($\varepsilon'<\varepsilon$, see (4.3)) given by

$$C_k = \Sigma\frac{1}{\alpha!}H_{\alpha\beta\gamma\delta_j}(t, y, \tau, \eta)\circ D_y^\gamma D_t^{\alpha+\beta}(-i\operatorname{ad} r)^\beta W_i(t, y, \tau, \eta) \tag{4.20}$$

where the sum extends over the indices $\alpha, \beta, \gamma, \delta, i, j$ such that $|\alpha|+|\beta|+2(|\gamma|+|\delta|)+i+j=k$ and $H_{\alpha\beta\gamma\delta j}\in F_R^{i/2-k/2-1}(\tilde\Omega\times\tilde\Gamma)$ for some $R>0$ with

$$\|H_{\alpha\beta\gamma\delta_j}(t, y, \tau, \eta)\|_{\mathscr{L}_R^0} \leqq \sqrt{|\alpha|!\,j!}\,C_0^{|\alpha|+|\beta|+|\gamma|+|\delta|+j+1}\times|\operatorname{Re}(\tau, \eta)|^{i/2-k/2-1} \tag{4.21}$$

in $\tilde\Omega\times\tilde\Gamma$, for some C_0, because of (4.16). Here $\tilde\Omega\times\tilde\Gamma$ is any sufficiently small neighborhood of $(0, y_0, 0, \eta_0)$ in $\Omega\times\Gamma$.

Now Proposition 2.9 of [6] states that given R there are ε_0 and $C>0$ such that $H\in\mathscr{L}_R^0$ and $W\in B_\varepsilon$ implies $HW\in B_\varepsilon$ and

$$\|HW\|_\varepsilon\in C\|H\|_{\mathscr{L}_R^0}\|K\|_\varepsilon. \tag{4.22}$$

Using this, the bounds (4.21) for the H_j and (4.17) for the W_j one shows easily that the C_k satisfy the bounds (4.17) also, with $m'-1$ in place of m'.

The expansion (4.19) uses only "$(\varrho, \delta), (\varrho', \delta')$ behavior" with $\varrho'<\delta$, see Lemma (4.10), and because of this, the estimates for the C_k and Lemma (4.10) again, the proofs of Propositions 3.6 and 3.7 of [6] give, with minor modifications, the proof that C has the expansion stated.

We will now find the operator ΣQ_j mentioned at the beginning of this section by finding an operator W such that $(I+Q\tilde{P})W=Q$ in a neighborhood of $(0, y_0, 0, \eta_0)$. Q is given by (4.15). If $Q\tilde{P}=\operatorname{op}(\Sigma H_j)$ where $H_j\in F_R^{-j/2-1}(\Omega\times\Gamma)$ satisfies (4.16) in a neighborhood of $(0, y_0, 0, \eta_0)$ and $W=\operatorname{op}(\Sigma W_i)$ with $W_i\in G_\varepsilon^{-m/2-i/2}$ satisfying (4.17), then $(I+Q\tilde{P})W=Q$ if W_0 is given by

$$W_0(t, y, \tau, \eta) = \eta^{-m/2} K(r, D_r)$$

and W_{k+2} is the operator C_k in (4.21) above, by proposition (4.18). These conditions in turn determine the W_k and we will show using standard techniques and Propositions 2.9 and 2.10 of Métivier [6], that the W_i obtained in this manner satisfy (4.17) in a neighborhood $\Omega\times\Gamma$ of $(0, y_0, 0, \eta_0)$ in which the estimates (4.21) already hold.

Let $M_0\geqq 1$ be the constant in (4.3), $z_0=(0, y_0)$, $0<\varepsilon_0\leqq 1$ such that $K\in B_{\varepsilon_0}$, let $0<r<1/M_0$ be such that the polydisc with radius $\varepsilon_0 r$ and center z_0, $D(z_0, \varepsilon_0 r)$, is contained in Ω and let $\Omega_\varepsilon=D(z_0, \varepsilon r)$. Suppose that

$$\|W_i(t, y, \tau, \eta)\|_\varepsilon \leqq M_1^i \sum_{v+l+i} \frac{(v/2+l)^{v/2+l}}{l^{l/2}} \left(\frac{M_0 e}{\varepsilon_0-\varepsilon}\right)^{v/2+l} |\operatorname{Re}(\tau, \eta)|^{-i/2-m/2} \tag{4.23}$$

holds for some M_1, all $\varepsilon<\varepsilon_0$, all $(t, y, \tau, \eta)\in\Omega_\varepsilon\times\Gamma$ when $i\leqq k$. (4.23) is already true if $i=0$. Using Proposition 2.10 of [6] (outlines in (4.3) above) we get

$$\begin{aligned} &\|\partial_t^{\alpha+\delta}\partial_y^\gamma(\operatorname{ad} r)^\beta W_i\|_\varepsilon \\ &\leqq M_1^i \sum_{v+l\leqq i} \frac{[(v+|\beta|+2(|\gamma|+|\delta|))/2+|\alpha|+l]^{(v+|\beta|+2(|\gamma|+|\delta|)/2+|\alpha|+l}}{l^{l/2}|\beta|^{\beta/2}(|\alpha|+|\gamma|+|\delta|)^{|\alpha|+|\gamma|+|\delta|}} \\ &\quad\times\left(\frac{M_0 e}{\varepsilon_0-\varepsilon}\right)^{(v+|\beta|+2(|\gamma|+|\delta|))/2+|\alpha|+l} |\operatorname{Re}(\tau, \eta)|^{-k/2-1} \end{aligned}$$

in $\Omega_\varepsilon\times\Gamma$ if $\varepsilon<\varepsilon_0$. Using now the definition of W_{k+2}, Proposition 2.10 of [6] (equation (4.22) here) and the estimates (4.21) we obtain

$$\begin{aligned} \|W_{k+2}\|_\varepsilon &\leqq M_1^{k+2} \sum_{\mu+s\leqq k} \left(\frac{M_0 e}{\varepsilon_0-\varepsilon}\right)^{\mu/2+s} \frac{(\mu/2+s)^{\mu/2+s}}{s^{s/2}} \\ &\times\left(C\Sigma \frac{s^{s/2}(2C_0)^{|\alpha|+|\beta|+|\gamma|+|\delta|+j} M_1^{i-k-2}}{|\alpha|^{|\alpha|/2} l^{l/2}|\beta|^{\beta/2}(|\alpha|+|\gamma|+|\delta|)^{|\alpha|+|\delta|+|\gamma|}} \left(\frac{M_0}{\varepsilon_0-\varepsilon}\right)^{-j/2}\right) \end{aligned}$$

where the inner sum extends over the indices α, β, γ, δ, i, j, v, l such that $v+|\beta|+2(|\gamma|+|\delta|)+j=\mu$, $|\alpha|+l=s$, $v+l\leqq i\leqq k-|\alpha|-|\beta|-j-2(|\gamma|+|\delta|)$. The inner sum, divided by

$$\sum_{\theta+p\leqq k+2} \left(\frac{M_0 e}{\varepsilon_0-\varepsilon}\right)^{\theta/2+p} \frac{(\theta/2+p)^{\theta/2+p}}{p^{p/2}}$$

will be less than $1/C$ if M_1 is large compared with C_0, since

$$\frac{s^{s/2}\,p^{p/2}}{|\alpha|^{|\alpha|/2}\,l^{l/2}(\theta/2+p)^{\theta/2+p}} \leqq 1$$

if $s=|\alpha|+l$ ($\leqq k$), $p=s$ and $\theta+p\leqq k+2$, and $M_0 e/(\varepsilon_0-\varepsilon)\geqq e$. This shows that W_k satisfies (4.23) for all k.

Now let $W=\operatorname{op}(\sum_0^\infty W_j)$. Because $(I+Q\tilde{P})W=Q$ we have $W=Q-Q\tilde{P}W$ and $P_m W=(I-\Pi)(I-\tilde{P}W)$ near $(0, y_0, 0, \eta_0)$, where $\Pi=J_2 J_2^*$, because of (3.3). With these properties of the operator W and (3.3) one verifies that if $E=W$, $E^+=J_1-W\tilde{P}J_1$, $E^-=J_2^*-J_2^*\tilde{P}E$ and $E^\pm=-E^-\tilde{P}J_1$ then $\mathscr{E}$ is a right inverse for $\mathscr{P}$ near $(0, y_0, 0, \eta_0)$. These operators are compositions of analytic operators (in the sense of the previous section) so they are analytic. Finally, $\mathscr{P}^*$ also has an analytic right inverse near $(0, y_0, 0, \eta_0)$ which is then given by $\mathscr{E}^*$ near that point. This completes the proof of Theorem (3.6).

5. An example of a locally nonapproximable differential operator

(5.1) Theorem. *The operator*

$$P = \partial^2/\partial t + t^2\partial/\partial y^2 + i\partial/\partial y + y \tag{5.2}$$

is not locally approximable in any neighborhood of 0. *In fact, a solution v of $Pv\sim 0$ is approximable at* 0 *if and only if $v\sim 0$ at* 0.

To prove the theorem we first use Theorem (2.6) to reduce the problem of approximability at 0 to that of microlocal approximability in all directions at 0. We claim that P is not microlocally approximable at $\omega=(0, 0; 0, -1)\in T^*\mathbf{R}^2\setminus 0$. By Theorem (3.6) this will follow if $E^\pm$ is not microlocally approximable at $\omega'=(0; -1)\in T^*\mathbf{R}$. We shall show that $E^\pm$ is equivalent, in a neighborhood of ω', to the operator with symbol y via conjugation by elliptic analytic operators. Assume this has been done. The following lemma will then complete the proof of the first statement of the Theorem.

(5.3) Lemma. *Let Q be an analytic pseudodifferential operator on* $\mathbf{R}$ *with total symbol y on $\eta<0$. If $Qv\sim 0$ at $\omega'=(0; -1)$ and there is a sequence of distributions $\{v_j\}$ such that $v_j\to v$, $v_j\sim 0$, $\{Qv_j\}\sim 0$ at ω' then $v\sim 0$ at ω'. Hence Q is not microlocally approximable at ω.*

The proof is given below. To prove the second statement of the theorem we observe that if $Pv\sim 0$ and v is approximable at 0 then $J_1^* v\sim 0$ at ω' by Lemma (5.3) so $v\sim 0$ at ω. But $v\sim 0$ also at all points $(0, 0; \tau, \eta)$ with $\tau\neq 0$ or $\eta>0$ since P

is analytic hypoelliptic at these points (see Métivier [6] for the points with $\tau=0$). Thus, $v\sim 0$ at 0. This completes the proof of the theorem.

Proof of Lemma (5.3). Let v and v_j be as stated, $w=yv$, $w_j=yv_j$. Then $\{Qv_j\}\sim 0$ at ω' and Lemma (2.4) imply $\{w_j\}\sim 0$ at ω' so there is a neighborhood $\mathcal{U}$ of 0 in $\mathbf{R}$ and $a>0$ such that w and the w_j extend as holomorphic functions W and W_j to $\Omega=\mathcal{U}+i(0,a)\subset\mathbf{C}$, and $|W_j(z)|\leqq C\,|\mathrm{Im}\, z|^{-N}$ for all j and $z\in\Omega$ and some C, N independent of j. Passing to a subsequence we may assume that the W_j converge uniformly locally in Ω, to W since $v_j\to v$. Now each v_j, being analytic at ω', is the boundary value of a holomorphic function defined on a set $\Omega+i(0,a_j)$, $a_j>0$, and $yv_j=w_j$ implies $zV_j=W_j$ on the common domain. Thus the V_j can be taken to be defined on Ω, the sequence $\{V_j\}$ converges uniformly locally there and we have estimates of the form $|V_j(z)|\leqq C|\mathrm{Im}\, z|^{-N-1}$, C, N independent of j. Thus the limit V also satisfies such an estimate, and has as boundary value at $\mathrm{Im}\, z=0$ the distribution v, again because $v_j\to v$. Thus $v\sim 0$ at ω'. This proves the lemma.

(5.4) Lemma. *$E^{\pm}$ is a classical pseudodifferential operator with symbol $y+r(y,\eta)$ on $\eta<0$, where r is of order -1.*

Proof. The analysis of the operator P in the C^∞ category carried out using the techniques of Helffer [4] and Proposition 3.2.2 there give that $E^{\pm}$ is a classical pseudodifferential operator, since $\partial_t^2-t^2\eta^2-\eta$ is a selfadjoint second order operator for η real and the eigenvectors, for $\eta<0$, are even. It only remains to find the principal symbol. For our operator P we have $J_1=J_2=J$, $J_1^*=J_2^*=J^*$, where

$$Jv(t,y)=(2\pi)^{-1}\pi^{-1/4}\int_{\eta<0}e^{i(y-y')\eta}|\eta|^{1/4}e^{-t^2|\eta|/2}v(y')\,dy'\,d\eta$$

$$J^*f(y)=(2\pi)^{-1}\pi^{-1/4}\int_{\eta<0}\int_{\mathbf{R}}e^{i(y-y')\eta}|\eta|^{1/4}e^{-t^2|\eta|/2}f(t,y')\,dt\,dy'\,d\eta$$

if $v\in C_0^\infty(\mathbf{R})$ and $f\in C_0^\infty(\mathbf{R}^2)$. Since $\tilde{P}=y$, $E^{\pm}=J^*yJ-J^*yEyJ$. But J^*yEyJ has order -1 while $J^*yJ=y+[J^*,y]J$ and $[J^*,y]J$ has order -1. Thus $E^{\pm}=y+\mathrm{op}\,(r)$ with r classical of order -1 as stated.

(5.5) Lemma. *Let R be a classical analytic pseudodifferential operator of order -1 in a neighborhood of 0 in $\mathbf{R}$. Then there exist analytic elliptic pseudodifferential operators F_1, F_2 such that*

$$F_1(y+R)F_2\sim y. \tag{5.6}$$

Proof. It follows from a known result [8] in $\mathbf{R}^n$ that one can find F_1', F_2' with $F_1'(y+R)F_2'\sim\mathrm{op}\,(y+c\eta^{-1})$, c constant (see Lebeau [5], Théorème 1.4). Hence it suffices to take $R=\mathrm{op}\,(c\eta^{-1})$. Now take F_1 with symbol η^{-ic} and $F_2=F_1^{-1}$ to get (5.6). This proves the lemma.

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