On star polynomials of complements of graphs

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1. Introduction

The graphs considered here will be finite, undirected, and will contain no loops nor multiple edges. Let G be such a graph. An *m-star* in G is a subgraph of G which is a tree with $m+1$ nodes and containing a node of valency m, called the *centre* of the m-star. A *1-star* is an edge and a *0-star*, a node. A *star cover* of G is a spanning subgraph of G, in which every component is a star.

Let us associate with every m-star S_m in G, an indeterminate or *weight* w_{m+1}, and with each star cover C in G with r components; $S_{m_1}, S_{m_2}, \ldots, S_{m_r}$ — the weight

$$w(C) = \prod_{i=1}^{r} w_{m_i}.$$

Then, the *star polynomial* of G is

$$E(G; \underline{w}) = \sum w(C),$$

where the summation is taken over all the star covers C of G, and $\underline{w}$ is a vector of indeterminates w_1, w_2, etc.

The star polynomial of a graph was introduced in Farrell [1]. The basic properties of $E(G; \underline{w})$ are given in [1]. In this paper, we will obtain a formula for the star polynomial of the complement $\bar{G}$ of a graph G, in terms of the star polynomial of G. This will yield a useful result on *costar graphs* (graphs with the same star polynomial). We will then derive various formulae for certain coefficients of $E(\bar{G}; w)$. A formula will also be deduced for the number of spanning stars in $\bar{G}$. Finally, we will use our results to deduce analogous results for the matching polynomial of the complement of a graph.

Throughout this paper, we will assume that a graph has p nodes and q edges, unless otherwise specified. We will denote the complete graph with p nodes by K_p. The number of spanning stars in G will be denoted by $\Gamma(G)$. Since the same weight vector $\underline{w}=(w_1, w_2, \ldots)$ will be used throughout the paper, we will abbreviate $E(G; \underline{w})$ to $E(G)$.

2. The main theorem

Theorem 1. *Let*

$$E(G) = \sum_k A_k w_1^{n_{1,k}} w_2^{n_{2,k}} \dots w_r^{n_{r,k}},$$

where $\sum_{i=1}^r i n_{i,k} = p$ *— the number of nodes in* G. *Let* $N_k = \sum_{i=1}^r (i-1) n_{i,k}$. *Then*

$$E(\bar{G}) = \sum_k A_k (-1)^{N_k} \sum \binom{n_{1,k}}{s} E(K_s) \prod_{i=2}^r \left(\prod_{j=1}^{n_{1,k}} \varepsilon_{i,j} w_{i+\delta_{i,j}}\right),$$

where the second summation is taken over all non-negative integral solutions of $s + \sum_{u=2}^r \sum_{j=1}^{n_{u,k}} \delta_{u,j} = n_{1,k}$ *and*

$$\varepsilon_{i,j} = \begin{cases} 1 & \text{if } i \neq 2 \text{ or if } i = 2 \text{ and } \delta_{ij} = 0. \\ 2 & \text{otherwise} \end{cases}$$

Proof. The result can be established, by using the Principle of Inclusion and Exclusion. Let the edge set of G be $\{e_1, e_2, \dots, e_q\}$. We will consider G to be a subgraph of K_p. A cover of K_p will have property i if it contains the edge e_i. The covers of $\bar{G}$ will then be those covers with none of the q properties.

Consider a cover C of G defined by the monomial $w_1^{n_{1,k}} w_2^{n_{2,k}} \dots w_r^{n_{r,k}}$. A cover C^* of K_p having C as a subgraph will contain $N_k = \sum_{i=1}^r (i-1) n_{i,k}$ edges of G, and therefore N_k properties. We can construct all such possible covers C^* of K_p as follows. Take a subset of s of the $n_{1,k}$ isolated nodes and form all possible combinations of stars. The remaining $n_{1,k} - s$ nodes can then be used to form (possibly) bigger stars from the existing stars in C.

The weight of all the possible combinations of stars formed with the s isolated nodes is $E(K_s)$, and the s nodes can be chosen in $\binom{n_{1,k}}{s}$ ways. For any 1-star in C, we may either (i) leave it unchanged or (ii) choose one of the two nodes as a centre, then join it to $\delta_{2,j}$ of the remaining nodes to form the star with weight $w_{2+\delta_{2,j}}$. The contribution of these stars to weight of C^* will be

$$\prod_{j=1}^{n_{2,k}} \varepsilon_{i,j} w_{2+\delta_{2,j}},$$

where $\varepsilon_{i,j} = 1$, if the 1-star is unchanged and

$\varepsilon_{i,j} = 2$, if nodes are added to it, to create larger star. For the proper star (i.e. an i-star when $i > 1$) with i nodes, we can join its centre to any number $\delta_{i,j}$ of isolated nodes, to form a larger star with weight $w_{i+\delta_{i,j}}$. The contribution of these new stars to the weight of C^* will be

$$\prod_{j=1}^{n_{i,j}} \varepsilon_{i,j} w_{i+\delta_{i,j}},$$

where

$$\varepsilon_{i,j} = 1.$$

The cover C^* of K_p will have N_k properties. Therefore we multiply the weight $W(C^*)$ by $(-1)^N k$ in accordance with the Principle. It is clear that the result follows from the Principle of Induction. □

An illustration

Let G be the cycle with 5 nodes. Then it can be easily verified that

$$E(G) = w_1^5 + 5w_1^3 w_2 + 5w_1^2 w_3 + 5w_1 w_2^2 + 5w_2 w_3.$$

We will tabulate the contributions of the various covers of G.

Term in $E(G)$	Contribution to $E(\bar{G})$
w_1^5	$E(K_5) = w_1^5 + 10w_1^3 w_2 + 30w_1^2 w_3 + 15w_1 w_2^2 + 20w_1 w_4 + 30w_2 w_3 + 5w_5$
$5w_1^3 w_2$	$-5\left\{(w_1^3 + 3w_1 w_2 + 3w_3) w_2 + \binom{3}{2} 2(w_1^2 + w_2) w_3 + \binom{3}{3} 2 \cdot w_1 w_4 + 2w_5\right\}$
$5w_1^2 w_3$	$5\left\{(w_1^2 + w_2) w_3 + \binom{2}{1} w_1 w_4 + w_5\right\}$
$5w_1 w_2^2$	$5(w_1 w_2^2 + 2 \cdot 2w_2 w_3)$
$5w_2 w_3$	$-5w_2 w_3$

Hence $E(\bar{G}) = w_1^5 + 5w_1^3 w_2 + 5w_1^2 w_3 + 5w_1 w_2^2 + 5w_2 w_3$.

The following corollary gives a useful result for costar graphs. It confirms an observation made during a computer generation of catalogues of star polynomials of graphs with up to 7 nodes [3].

Corollary 1.1. *If two graphs are costar, then so also are their complements.*

Proof. This is immediate from the theorem. □

3. Some deductions for the coefficients of $E(\bar{G})$

The following definitions will be relevant to this section.

Definitions. Let G be a graph with p nodes. A *simple m-cover* of G is a star cover consisting of an m-star and $p-m-1$ isolated nodes. It is clear that the cover consisting of p isolated nodes (i.e. a 0-star and $p-1$ isolated nodes) and the cover consisting of a spanning star i.e. a $(p-1)$-star together with 0 isolated nodes) are simple covers. The weight of a simple m-cover i.e. $w_1^{p-m-1}w_{m+1}$ will be called a *simple term* of $E(G)$. The number of simple m-covers, or the coefficient of $w_1^{p-m-1}w_{m+1}$, will be called a *simple coefficient* and will be denoted by $c_m(G)$ (or simply by c_m, when G is understood).

It is clear that for any graph G, $c_0(G)=1$ and $c_1(G)=q$, the number of edges in G. The following lemmas give some properties of c_m. They can be easily proved.

Lemma 1. *Let G be a graph with p nodes. Let the partition of G be*

$$(n^{b_n}, \dots, 2^{b_2}, 1^{b_1}, 0^{b_0}),$$

where k^{b_k} denotes $b_k, b_k, \dots, b_k$ (k times) $(0 \leqq n \leqq p-1)$. *Then for* $m>1$,

$$c_m = \sum_{r=m}^{n} \binom{r}{m} b_r.$$

Lemma 2. *Let n be the highest valency of a node in G. Then $E(G)$ contains all the simple terms $w_1^{p-r-1}w_{r+1}$ $(0 \leqq r \leqq n)$, with non-zero coefficients* i.e. $c_r \neq 0$, *for* $0 \leqq r \leqq n$.

Lemma 3.

$$c_0(K_p)=1,\ c_1(K_p)=\binom{p}{2} \quad \textit{and} \quad c_m(K_p)=p\binom{p-1}{m}, \quad \textit{for} \quad m>1.$$

It is clear from Theorem 1, that a simple term in $E(\bar{G})$ can only result from a simple term in $E(G)$. Let $c_k(G)w_1^{p-k-1}w_{k+1}$ be a simple term in $E(G)$. The associated terms in $E(\bar{G})$ will be

$$c_k(G)(-1)^k \sum_{r=0}^{p-k-1} \binom{p-k-1}{r} E(K_{p-k-r-1})\, \varepsilon_{k,r} w_{k+r+1} \quad (k>0).$$

The resulting contribution to the simple terms in $E(\bar{G})$ will therefore be

$$(1) \qquad \gamma_k = c_k(G)(-1)^k \sum_{r=0}^{p-k-1} \binom{p-k-1}{r} \varepsilon_{k,r} w_1^{p-k-r-1} w_{k+r+1} \quad (k>0)$$

For $k=0$, the contribution of the simple term w_1^p of $E(G)$ will be $E(K_p)$. Therefore the contribution to the simple terms of $E(\bar{G})$ will be (from Lemma 3),

$$(2) \qquad w_1^p + \binom{p}{2} w_1^{p-2}w_2 + \sum_{j=1}^{p-1} p\binom{p-1}{j} w_1^{p-j-1}w_{j+1} = \sum_{s=0}^{p} \varepsilon_{0,s} w_1^{p-s-1} w_{s+1},$$

where $\varepsilon_{0,0}=1$, $\varepsilon_{0,1}=\binom{p}{2}$ and $\varepsilon_{0,s}=p\binom{p-1}{s}$, for $s>1$.

We can combine the contributions given in Equations (1) and (2), to obtain the following lemma.

Lemma 4.

$$\gamma_k = c_k(G)(-1)^k \sum_{r=0}^{p-k-1} \binom{p-k-1}{r} \varepsilon_{k,r} w_1^{p-k-r-1} w_{k+r+1},$$

where

$$\varepsilon_{0,0} = 1;\ \varepsilon_{0,1} = \binom{p}{2};\ \varepsilon_{0,s} = p\binom{p-1}{s},\quad \textit{for all}\quad s > 1;\ \varepsilon_{1,0} = 1,$$

$$\varepsilon_{1,r} = 2,\quad \textit{for}\quad r > 0;\ \varepsilon_{k,r} = 1,\quad \textit{for}\quad k > 1.$$

By considering all the simple terms of $E(G)$, we can obtain the total contribution; which is $\sum_{k=0}^{n} r_k$, where n is the highest valency of a node in G. In order to obtain $c_i(\bar{G})$, we put $k+r+1=i+1. \Rightarrow r=i-k$. We note also that no monomial of a simple cover in G, with an m-star, for $m>i$ can contribute to $c_i(\bar{G})$. Thus we have the following theorem.

Theorem 2.

$$c_i(\bar{G}) = \sum_{k=0}^{i} (-1)^k c_k(G) \binom{p-k-1}{i-k} \varepsilon_{k,i-k},$$

where $\varepsilon_{k,j}$, for all k and j, are as defined in Lemma 4.

The above theorem can be used to obtain a formula for the number of spanning stars in $\bar{G}$. We simply put $i=p-1$ to obtain the following corollary.

Corollary 2.1.

$$\Gamma(\bar{G}) = \sum_{k=0}^{n} (-1)^k C_k(G)\, \varepsilon_k,$$

where $\varepsilon_0=p$; $\varepsilon_1=2$ *and* $\varepsilon_r=1$, *for all* $r>2$.

4. Applications to Matching Polynomials

A matching is a star cover containing nodes and edges only. The matching polynomial of a graph G, written as $M(G;\underline{w})$, was introduced in Farrell [2]. The following lemma gives in a formal manner, the relation between $M(G;\underline{w})$ and $E(G;\underline{w})$.

Lemma 6.

$$M(G;\underline{w}) = E(G;(w_1, w_2, 0, 0, \ldots, 0)).$$

We can use Theorem 1 in order to obtain an analogous result for the matching polynomial of the complement of a graph. In this case, we will assume that

$$M(G;\underline{w}) = \sum_{k=0}^{[p/2]} a_k w_1^{p-2k} w_2^k.$$

Using Theorem 1 with $N_{1,k}=p-2k$, $N_{2,k}=k$ and $r=2$, we get

$$N_k = \sum_{i=1}^{2} (i-1)n_{i,k} = n_{2,k} = k.$$

Also $\delta_{2,j}=0$, for all j, since the largest subscript of w must be 2. $\Rightarrow \varepsilon_{i,j}=1$, for all i. Finally, $s=n_{1,k}=p-2k$. $\Rightarrow \binom{n_{1,k}}{s}=1$. Hence we obtain the following result.

Lemma 7. *Let*

$$M(G) = \sum_{k=0}^{[p/2]} a_k w_1^{p-2k} w_2^k.$$

Then

$$M(\bar{G}) = \sum_{k=0}^{[p/2]} a_k (-1)^k M(K_{p-2k}) w_2^k.$$

By using the explicit formula for $M(K_p)$ given in Theorem 18 of [2] and the above lemma, we obtain the following theorem.

Theorem 3.

$$M(\bar{G}) = \sum_{k=0}^{[p/2]} a_k (-1)^k \sum_{m=k}^{[p/2]} \frac{(p-2k)!\, w_1^{p-2m} w_2^m}{(p-2m)!\,(m-k)!\,2^{m-k}}.$$

References

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