Unique continuation in *CR* manifolds and in hypo-analytic structures

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Introduction

There are two basic "unique continuation" properties for a holomorphic function h in a connected complex manifold $\mathfrak{M}$: h will vanish identically in $\mathfrak{M}$ in either one of the following two cases: I) when $h \equiv 0$ on a totally real submanifold X of $\mathfrak{M}$ such that $\dim_{\mathbf{R}} X = \dim_{\mathbf{C}} \mathfrak{M}$; II) when h vanishes to infinite order at a single point.

It is natural to ask whether such properties can be generalized to an arbitrary *locally integrable structure* on a manifold $\mathfrak{M}$. By this we mean the datum of a complex vector subbundle T' of the complexified cotangent bundle $\mathbf{C}T^*\mathfrak{M}$ such that locally, T' is generated by exact differentials. On this subject and on the concept of *hypo-analytic structures* which is used below, we refer the reader to Sect. 4 of the present article, also to [BCT] and [T]. In a locally integrable structure the role of

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holomorphic functions is played by what we call the *solutions,* for want of a better term: these are the functions whose differentials are sections of the bundle T'.

Unique continuation in Case I generalizes to all locally integrable structures as an immediate consequence of the Approximation Formula in [BT] (see also [T], p. 29). In trying to extend Case II the first problem is to identify the class of submanifolds that could play the role played by points in a complex structure. Natural candidates are the *noncharacteristic submanifolds:* these are the submanifolds Σ whose conormal bundle $N^*\Sigma$ does not intersect, off the zero section, the *characteristic set* T^0 of the locally integrable structure (T^0 is equal to the the intersection of T' with the real cotangent bundle $T^*\mathfrak{M}$).

There is no indication, so far, of the validity of unique continuation for submanifolds that are merely noncharacteristic. Recently (see [R]) J.-P. Rosay has shown that the property is valid when the base manifold $\mathfrak{M}$ is an embedded real hypersurface in complex space $\mathbf{C}^{n+1}$ and Σ is equal to the transverse intersection of $\mathfrak{M}$ with a holomorphic curve in $\mathbf{C}^{n+1}$ (*i.e.,* a complex submanifold of complex dimension one). Heie, of course, $\mathfrak{M}$ inherits its structure, which is a Cauchy—Riemann (abbreviated henceforth to CR) structure, from the ambient complex space.

Actually, the structure that the hypersurface $\mathfrak{M}$ inherits from $\mathbf{C}^{n+1}$ is more than locally integrable; it is a particular case of a hypo-analytic structure. And the particular kind of submanifolds Σ to which Rosay's result applies are the *hypo-analytic noncharacteristic submanifolds* of $\mathfrak{M}$ (see Sect. 4). All this leads naturally to the

Conjucture. In any hypo-analytic manifold $\mathfrak{M}$, if a solution h in $\mathfrak{M}$ (endowed with a modicum of regularity — in the present article it will be Lipschitz continuity) vanishes to infinite order on a hypo-analytic noncharacteristic submanifold Σ then it vanishes identically in an open neighborhood of Σ.

The present work falls short of proving the conjecture (about whose validity the authors have doubts). What this article does is to present a generalization of Rosay's result to a large class of hypo-analytic manifolds (in particular, of generic submanifolds of $\mathbf{C}^{n+d}$ whose codimension is equal to $d\geqq 1$). Our methods are quite different from Rosay's, which are based on holomorphic extension and use of the Bochner—Martinelli formula to insure the "right" kind of polynomial approximation. Our proof combines a "miniversion" of the Approximation Formula of [BT] with a hypo-analytic change of variable that transforms the given solution into one with compact support. The latter ingredient is an adaptation of ideas in the work [BZ]; and as a matter of fact, in the case where $\mathfrak{M}$, its hypo-analytic structure and the submanifold Σ are all real-analytic, unique continuation is a direct consequence of the main theorem in [BZ].

In the present article we reason under the hypothesis that $\mathfrak{M}$ and its structure are of class $\mathscr{C}^1$. One noteworthy case where unique continuation holds is when the submanifold Σ is analytic in a suitable sense. When $\mathfrak{M}$ is a generic submanifold, of class $\mathscr{C}^1$, of codimension d, in $\mathbf{C}^{n+d}$, this means that Σ is a d-dimensional real-analytic submanifold of $\mathbf{C}^{n+d}$ (and is the holomorphic-transverse intersection of $\mathfrak{M}$ with a d-dimensional holomorphic submanifold $\mathscr{H}$ of $\mathbf{C}^{n+d}$; see Sect. 1).

The main result of the paper, Th. 3.1, concerns generic submanifolds of complex space. The article is self-contained, except for the use, in the proof of Th. 4.1 (which generalizes Th. 3.1 to hypo-analytic manifolds), of the uniqueness in the Cauchy problem, which, as we have said at the beginning, is a consequence of the Approximation Formula.

1. Holomorphic-transversal intersection of a generic submanifold with a holomorphic submanifold

Throughout the present section $\mathfrak{M}$ will denote a *generic* submanifold of $\mathbf{C}^{n+d}$ of class $\mathscr{C}^1$ and codimension d.* It means that, locally, $\mathfrak{M}$ is defined by a set of equations

$$\varrho_j(z,\bar{z}) = 0, \quad j = 1, \ldots, d, \tag{1.1}$$

where the ϱ_j are real-valued functions of class $\mathscr{C}^1$ such that

$$\partial\varrho_1 \wedge \ldots \wedge \partial\varrho_d \neq 0. \tag{1.2}$$

We have used the customary notation $\partial f = \sum_{j=1}^{n+d} (\partial f/\partial z_j)\, dz_j$. Needless to say, Conditions (1.1) and (1.2) do not depend on the choice of the defining functions ϱ_j.

Let J denote the complex structure on the *real* tangent spaces to $\mathbf{C}^{n+d}$; if p is an arbitrary point of $\mathfrak{M}$, $T_p\mathfrak{M} \cap JT_p\mathfrak{M}$ is a complex subspace of $T_p\mathbf{C}^{n+d}$ of complex dimension n, which we denote by $\mathscr{T}_p\mathfrak{M}$.

Let $\mathscr{H}$ be a holomorphic submanifold of $\mathbf{C}^{n+d}$ with $\dim_{\mathbf{C}} \mathscr{H} = d$, which intersects $\mathfrak{M}$ at a point p_0. We shall say that the intersection of $\mathfrak{M}$ and $\mathscr{H}$ is *holomorphic-transversal* (or that $\mathfrak{M}$ and $\mathscr{H}$ are holomorphic-transversal to one another) at the point p_0 if

$$T_p\mathbf{C}^{n+d} = \mathscr{T}_p\mathfrak{M} \oplus T_p\mathscr{H} \quad (\oplus\text{: direct sum}) \tag{1.3}$$

when $p = p_0$. Then property (1.3) holds at all points p in a full neighborhood of p_0 in $\mathfrak{M} \cap \mathscr{H}$.

Suppose that $\mathscr{H}$ is defined, in some open neighborhood of p_0, by n (independent)

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* If properly interpreted all the statements in this article remain valid when either n or d are equal to zero. In these cases, however, they become uninteresting. The reader should therefore assume that both n and d are $\geqq 1$.

holomorphic equations

$$h_j(z) = 0, \quad j = 1, \ldots, n. \tag{1.4}$$

Then condition (1.3) is equivalent to the property that

$$\partial h_1 \wedge \ldots \wedge \partial h_n \wedge \partial \varrho_1 \wedge \ldots \wedge \partial \varrho_d \neq 0 \tag{1.5}$$

in a subneighborhood of p_0. If this is so then, near p_0, $\Sigma = \mathfrak{M} \cap \mathscr{H}$ is a $\mathscr{C}^1$ submanifold and $\dim_{\mathbf{R}} \Sigma = d$. We say that $\mathfrak{M}$ and $\mathscr{H}$ have a holomorphic-transversal intersection if they do at every point of Σ.

Remark 1.1. In general, for $\mathfrak{M}$ and $\mathscr{H}$ to be holomorphic-transversal at p_0 is *not* the same as to be transversal at p_0, as shown by the following example: take $\mathfrak{M} = \mathbf{C}^2 \times \mathbf{R}^2$ defined in $\mathbf{C}^4$ by $\operatorname{Im} z_j = 0, j = 3, 4$, and let $\mathscr{H}$ be defined by the equations $z_3 = z_1$, $z_4 = \sqrt{-1}\, z_1$. Indeed, $\mathfrak{M}$ and $\mathscr{H}$ intersect transversally, and their intersection is the z_2-plane $z_1 = z_3 = z_4 = 0$; but they are not holomorphic-transversal.

However, as the reader will easily ascertain, the two notions coincide when the codimension of $\mathfrak{M}$ is equal to one, *i.e.*, when $\mathfrak{M}$ is a real hypersurface in $\mathbf{C}^{n+1}$. □

In the sequel we shall always reason about a central point of $\mathfrak{M}$ which we take as the origin in $\mathbf{C}^{n+d}$. The choice of coordinates will presently be modified, and the last d coordinates will not any more be called $z_{n+1}, \ldots, z_{n+d}$; instead we shall denote them by $w_1, \ldots, w_d$.

Proposition 1.1. *Let $\mathfrak{M}$ be a generic submanifold and $\mathscr{H}$ be a holomorphic submanifold of $\mathbf{C}^{n+d}$, with $\dim_{\mathbf{R}} \mathfrak{M} = 2n + d$, $\dim_{\mathbf{C}} \mathscr{H} = d$.*

If $\mathfrak{M}$ and $\mathscr{H}$ have a holomorphic transversal intersection at the origin, then the coordinates $z_1, \ldots, z_n, w_1, \ldots, w_d$ in $\mathbf{C}^{n+d}$ can be chosen in such a way that, in an open neighborhood $\mathscr{O}$ of the origin in $\mathbf{C}^{n+d}$, $\mathscr{H}$ will be defined by the equation $z = 0$, whereas $\mathfrak{M}$ will be defined by the equation

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w), \tag{1.6}$$

with φ a $\mathbf{C}^1$ map of an open neighborhood of the origin in $\mathbf{R}^{2n+d}$ into $\mathbf{R}^d$, satisfying:

$$\varphi(0) = 0, \quad d\varphi(0) = 0. \tag{1.7}$$

Conversely, if $\mathfrak{M}$ is defined near the origin by the equations (1.6) *with φ as described, then $\mathfrak{M}$ and the w-subspace (defined by $z = 0$) have a holomorphic-transversal intersection.*

Proof. It is well known, and elementary to check, that $\mathfrak{M}$ can be defined by the equations (1.6) with the map φ fulfilling the requirements of the statement, in particular (1.7). From (1.5) and (1.6) we derive that, if a holomorphic submanifold $\mathscr{H}$ of $\mathbf{C}^{n+d}$ such that $\dim_{\mathbf{C}} \mathscr{H}=d$, is holomorphic-transversal to $\mathfrak{M}$ in a neighborhood of 0 it must be defined, there, by equations of the kind $z=g(w)$, with g a holomorphic map of an open neighbourhood of the origin in $\mathbf{C}^d$ into $\mathbf{C}^n$. Let us then perform the biholomorphic change of coordinates $(z, w)\to(z-g(w), w)$. In the new coordinates the equations (1.6) read

$$\text{(1.8)} \qquad \operatorname{Im} w-\varphi\big(z+g(w), \bar{z}+\bar{g}(\bar{w}), \operatorname{Re} w\big) = 0.$$

But by virtue of (1.7) the Jacobian matrix with respect to $\operatorname{Im} w$, of the left-hand side in (1.8), is equal, at the origin, to the $d\times d$ identity matrix. We can therefore solve (1.8) with respect to $\operatorname{Im} w$, which yields an equation of the same kind as (1.6), with a new map φ that has the same properties, in particular (1.7), as the old one.

The proof of the converse is immediate, and is left to the reader. □

Proposition 1.2. *Let $\mathfrak{M}$ be a generic $\mathscr{C}^1$ submanifold of $\mathbf{C}^{n+d}$ with $\operatorname{codim}_{\mathbf{R}} \mathfrak{M}=d$. Let Σ be a C^1 submanifold of $\mathfrak{M}$, with $\dim_{\mathbf{R}} \Sigma=d$, having the following property:*

Every point p of Σ has an open neighborhood $\mathcal{O}_p$ in $\mathbf{C}^{n+d}$ which contains a holomorphic submanifold $\mathscr{H}_p$ whose complex dimension is equal to d, and whose intersection with $\mathfrak{M}$ is holomorphic-transversal and is equal to $\Sigma\cap\mathcal{O}_p$.

Then there is a holomorphic submanifold $\mathscr{H}$ of $\mathbf{C}^{n+d}$, with $\dim_{\mathbf{C}} \mathscr{H}=d$, whose intersection with $\mathfrak{M}$ is holomorphic transversal and is equal to Σ.

Proof. We select a set S of points p in Σ such that $\{\mathcal{O}_p\}_{p\in S}$ forms a locally finite open covering of Σ. Possibly after replacing each $\mathcal{O}_p$ by a smaller open set we may reason under the following two hypotheses: i) for each $p\in S$ the submanifold $\mathscr{H}_p$ is the zero-set of an ideal $\mathscr{I}_p$ of holomorphic functions in $\mathcal{O}_p$; ii) for every pair of points $p, q\in S$ either the holomorphic submanifold $\mathscr{H}_p\cap\mathcal{O}_q$ is empty, or else $\Sigma\cap\mathcal{O}_p\cap\mathcal{O}_q\neq\emptyset$ and in this case, $\mathscr{H}_p\cap\mathcal{O}_q$ is connected. Assume $\Sigma\cap\mathcal{O}_p\cap\mathcal{O}_q\neq\emptyset$. It is readily checked that $\Sigma\cap\mathcal{O}_p\cap\mathcal{O}_q$ is a totally real submanifold of $\mathscr{H}_p\cap\mathcal{O}_q$. It has real dimension d and, as a consequence, any holomorphic function in $\mathcal{O}_q$ which vanishes on $\Sigma\cap\mathcal{O}_p\cap\mathcal{O}_q$ must vanish identically on $\mathscr{H}_p\cap\mathcal{O}_q$. We see thus that the zero set in $\mathcal{O}_p\cap\mathcal{O}_q$ of the elements of $\mathscr{I}_p$ is contained in the zero set of the elements of $\mathscr{I}_q$. By symmetry this shows that $\mathscr{H}_p\cap\mathcal{O}_p\cap\mathcal{O}_q=\mathscr{H}_q\cap\mathcal{O}_p\cap\mathcal{O}_q$. The union $\mathscr{H}=\bigcup_{p\in S} \mathscr{H}_p$, has the properties required of it in the statement. □

Let us introduce the *characteristic set* T^0 of the CR manifold $\mathfrak{M}$; a point (p, θ) of the (real) cotangent bundle $T^*\mathfrak{M}$ belongs to T^0 if the covector θ is orthogonal to $\mathscr{T}_p\mathfrak{M}$. Since $\dim_{\mathbf{R}} \mathscr{T}_p\mathfrak{M}=2n$ we see that T^0 is a vector subbundle of $T^*\mathfrak{M}$ whose fibre dimension is equal to d. A submanifold Σ of $\mathfrak{M}$ is called *noncharacteristic if*

$$\text{(1.9)} \qquad N^*\Sigma\cap(T^0|_\Sigma) = 0$$

($N^*\Sigma$: conormal bundle of Σ). Notice that (1.9) sets a lower bound on $\dim \Sigma$; it must be $\geqq d$. By duality (1.9) is equivalent to

$$T\mathfrak{M}|_\Sigma = T\Sigma + \mathscr{T}\mathfrak{M}|_\Sigma \tag{1.10}$$

where $+$ stands for the fibrewise vector sum, not necessarily direct.

Proposition 1.3. *Let $\mathfrak{M}$ and $\mathscr{H}$ be as in Prop.* 1.1. *If $\mathfrak{M}$ and $\mathscr{H}$ are holomorphic-transversal then $\Sigma = \mathfrak{M} \cap \mathscr{H}$ is noncharacteristic (and does not contain any non-characteristic submanifold of strictly lower dimension, since* $\dim \Sigma = d$*).*

Proof left to the reader.

Proposition 1.4. *Let $\mathfrak{M}$ be a generic submanifold of $\mathbf{C}^{n+d}$, of class $\mathscr{C}^1$, of codimension d, and let $\Sigma \subset \mathfrak{M}$ be a real-analytic submanifold of $\mathbf{C}^{n+d}$ which is noncharacteristic in $\mathfrak{M}$ and whose real dimension is equal to d. Then there is a holomorphic submanifold $\mathscr{H}$ of $\mathbf{C}^{n+d}$ with $\dim_{\mathbf{C}} \mathscr{H} = d$, whose intersection with $\mathfrak{M}$ is holomorphic-transversal and equal to Σ.*

Proof. By Prop. 1.2 it suffices to show that the assertion is valid locally. We reason near the origin and suppose that $\mathfrak{M}$ is defined by (1.6), and that (1.7) is true. By hypothesis Σ is equal to the image of a real-analytic map (λ, μ) from an open neighborhood of 0 in $\mathbf{R}^d$ into $\mathbf{R}^{2n} \times \mathbf{R}^{2d}$. Thus, near the origin, Σ is defined by the parametric equations

$$z = \lambda(s), \quad w = \mu(s). \tag{1.11}$$

Due to (1.7) the fibre of $\mathscr{T}\mathfrak{M}$ at 0, $\mathscr{T}_0\mathfrak{M}$, is spanned by the vectors $\partial/\partial x_i$, $\partial/\partial y_j$ $(1 \leqq i, j \leqq n)$. Then (1.10) demands that the Jacobian determinant of μ does not vanish at 0. But then extending μ holomorphically to the complex values of s enables us to solve with respect to s the second set of equations (1.11), getting thus $s = \chi(w)$. Now extending also λ holomorphically and setting $g = \lambda \circ \chi$ shows that, near 0, Σ lies on the holomorphic submanifold $\mathscr{H}_0$ of $\mathbf{C}^{n+d}$ defined by the equation $z = g(w)$. We may as well make the change of variables $(z, w) \to (z - g(w), w)$, which does not change the equation for $\mathfrak{M}$ (near 0) in any essential manner, as already indicated in the proof of Prop. 1.1. In the new coordinates $\mathscr{H}$ is defined by the equation $z = 0$ and is obviously holomorphic-transversal to $\mathfrak{M}$ in some neighborhood of the origin. $\square$

2. Condition ($\mathscr{A}$)

In the present section $\mathfrak{M}$ shall denote a generic $\mathscr{C}^1$ submanifold of $\mathbf{C}^{n+d}$ and Σ a $\mathscr{C}^1$ submanifold of $\mathfrak{M}$, such that $\operatorname{codim}_{\mathbf{R}} \mathfrak{M} = \dim_{\mathbf{R}} \Sigma = d$. We shall always reason under the hypothesis that

(2.1) there is a holomorphic submanifold $\mathcal{H}$ of $\mathbf{C}^{n+d}$, with $\dim_{\mathbf{C}} \mathcal{H} = d$, whose intersection with $\mathfrak{M}$ is holomorphic transversal and equal to Σ.

We shall say that the submanifold Σ satisfies *Condition* $(\mathcal{A})$ at the point $p_0 \in \Sigma$ if the following is true:

$(\mathcal{A})$ Given any open neighborhood $\mathcal{V}$ of p_0 in Σ there is a holomorphic function F in an open subset of $\mathcal{H}$ containing $\mathcal{V}$, such that $F(p_0) \neq 0$ and that the connected component of p_0 in the set $\{p \in \mathcal{V};\ F(p) \neq 0\}$ has compact closure contained in $\mathcal{V}$.

It is clear that Condition $(\mathcal{A})$ is invariant under local biholomorphic transformations.

Let us take p_0 to be the origin and use local representations of $\mathfrak{M}$ and of $\mathcal{H}$ as in Prop. 1.1. Thus we identify the submanifold $\mathcal{H}$ to the w-subspace $z=0$, in which Σ is defined by the equation

$$\operatorname{Im} w = \varphi(0, 0, \operatorname{Re} w). \tag{2.2}$$

As $0 < \varepsilon \to 0$ the sets

$$\mathcal{V}_\varepsilon = \{w \in \mathbf{C}^d;\ \exists s \in \mathbf{R}^d,\ |s| < 2\varepsilon, \quad \text{such that} \quad w = s + \sqrt{-1}\,\varphi(0, 0, s)\}$$

form a basis of neighborhoods of 0 in Σ. Let $F(w)$ be the holomorphic function in Condition $(\mathcal{A})$. We must have $F(0) \neq 0$ and the closure of the connected component of the origin in the set

$$\{s \in \mathbf{R}^d;\ |s| < 2\varepsilon,\ F(s + \sqrt{-1}\,\varphi(0, 0, s)) \neq 0\},$$

must be a compact set contained in the open ball $\{s \in \mathbf{R}^d;\ |s| < 2\varepsilon\}$.

Let us show right-away that Condition $(\mathcal{A})$ is no restriction at all when $\mathfrak{M}$ is a real hypersurface.

Proposition 2.1. *If $d=1$ the submanifold Σ of $\mathfrak{M}$ satisfies Condition $(\mathcal{A})$ at every point.*

Proof. Take $F(w) = [w - \varepsilon - \sqrt{-1}\,\varphi(0, 0, \varepsilon)][w + \varepsilon - \sqrt{-1}\,\varphi(0, 0, -\varepsilon)]$ and observe that $F(s + \sqrt{-1}\,\varphi(0, 0, s)) = 0$ if and only if $s = \pm\varepsilon$. □

The next statement is of interest when $d > 1$:

Proposition 2.2. *Suppose that $\varphi(0, 0, s)$ is a real-analytic function of s in some ball $\{s \in \mathbf{R}^d;\ |s| < \varepsilon_0\}$. Then Σ has Property $(\mathcal{A})$ at the origin.*

Proof. First suppose that $\varphi(0, 0, s) = 0$ if $|s| < \varepsilon_0$. Then, for any $\varepsilon < \varepsilon_0/2$, $F(w) = \varepsilon^2 - (w_1^2 + \ldots + w_d^2)$ satisfies the requirements above, in the neighborhood of 0 in Σ, $\mathcal{V}_\varepsilon$.

Consider now the general case. Let $G(w)$ denote the solution, in a suitably small neighborhood of the origin in $\mathbf{C}^d$, of the equation

$$w = G+\sqrt{-1}\,\varphi(0, 0, G),$$

such that $G(0)=0$. At the origin the Jacobian matrix of $G(w)$ is equal to the $d\times d$ identity matrix. Furthermore we have $G(s+\sqrt{-1}\,\varphi(0, 0, s))\equiv s$, which shows that the holomorphic change of variables $(z, w)\to(z, G(w))$ preserves $\mathscr{H}$ and transforms Σ into real space $\mathbf{R}^d$. Near the origin we may take $\tilde{s}^k=\operatorname{Re} G^k(s+\sqrt{-1}\,\varphi(z, \bar{z}, s))$ $(k=1, \dots, d)$ as the coordinates in $\mathbf{R}^d$, and check that $G(s+\sqrt{-1}\,\varphi(z, \bar{z}, s))=\tilde{s}+\sqrt{-1}\,\tilde{\varphi}(z, \bar{z}, \tilde{s})$ has the same properties as $s+\sqrt{-1}\,\varphi(z, \bar{z}, s)$ (in particular, $\tilde{\varphi}$ satisfies (1.7)). □

Corollary 2.1. *Let $\Sigma\subset\mathfrak{M}$ be a real-analytic submanifold of $\mathbf{C}^{n+d}$, with $\dim_{\mathbf{R}} \Sigma=d$. If Σ is noncharacteristic in $\mathfrak{M}$ then Σ has Property ($\mathscr{A}$) at every one of its points.*

According to Prop. 1.4 the hypotheses in Cor. 2.1 imply that Condition (2.1) is satisfied.

Proof. We apply Prop. 1.1 and use the local equations $z=0$ for $\mathscr{H}$ and (1.6) for $\mathfrak{M}$; we assume that (1.7) holds. The hypothesis that Σ is real-analytic implies that this is true of the function $\varphi(0, 0, s)$ in some open neighborhood of the origin in $\mathbf{R}^d$, whence the result by Prop. 2.2. □

Remark 2.1. One can easily produce examples of d-dimensional totally real submanifolds Σ of $\mathscr{H}=\mathbf{C}^d$ (with $d>1$) that have Property ($\mathscr{A}$) without being real-analytic. Suppose, for instance, that Σ is defined by the equations (2.2), and that

$$\varphi(0, 0, s) = (\psi_1(s_1), \dots, \psi_d(s_d)).$$

Then, as suggested by the proof of Prop. 2.1, we can take

$$F(w) = \prod_{k=1}^{d} [w_k-\varepsilon-\sqrt{-1}\,\psi_k(\varepsilon)][w_k+\varepsilon-\sqrt{-1}\,\psi_k(-\varepsilon)],$$

for $\varepsilon>0$ suitably small. Subtler examples can also be produced. □

We do not know of an example of a d-dimensional totally real submanifold Σ of $\mathbf{C}^d$ that does not have Property ($\mathscr{A}$), but we believe that such examples exist.

3. Unique continuation in a generic submanifold of $\mathbf{C}^{n+d}$

As before, $\mathfrak{M}$ will be a generic submanifold of $\mathbf{C}^{n+d}$, of class $\mathscr{C}^1$, of codimension $d\geqq 1$; Σ is a d-dimensional submanifold of $\mathfrak{M}$, also of class $\mathscr{C}^1$, equal to the holomorphic-transversal intersection (see Sect. 1) of $\mathfrak{M}$ with a holomorphic submanifold $\mathscr{H}$ of $\mathbf{C}^{n+d}$, with $\dim_{\mathbf{C}} \mathscr{H}=d$.

Let p_0 be a point of Σ. We shall say that there is *unique continuation on Σ at p_0* when the following is true:

($\mathscr{U}\mathscr{C}$) Give any open neighborhood U of p_0 in $\mathfrak{M}$ there is an open neighborhood $V \subset U$ of p_0 in $\mathfrak{M}$ such that the folloving is true:

If aLipschitz-continuous CR function h in U vanishes to infinite order on $\Sigma \cap U$, then $h \equiv 0$ in V.

What we mean by saying that h vanishes to infinite order on $\Sigma \cap U$ is that, given any compact subset K of U and any integer $N \geqq 0$, there is a constant $C_{K,N} > 0$ such that

$$|h(p)| \leqq C_{K,N}[\text{dist}(p, \Sigma)]^N, \quad \forall p \in K.$$

We may now state and prove our main result:

Theorem 3.1. *If Σ satisfies Condition ($\mathscr{A}$) at one of its points, p_0, then there is unique continuation on Σ at p_0.*

Proof. We use the local representations of $\mathfrak{M}$, $\mathscr{H}$ and Σ of Prop. 1.1; p_0 will be the origin of $\mathbf{C}^{n+d}$ and $z = x + \sqrt{-1}y$, $w = s + \sqrt{-1}t$, with $x, y \in \mathbf{R}^n$, $s, t \in \mathbf{R}^d$. Our hypothesis is that Condition ($\mathscr{A}$) holds at the origin. Take the neighborhood U of 0 in the product form: $U = U_1 \times U_2$, with U_1 (resp., U_2) an open neighborhood of 0 in z-space (resp., in s-space). Let then $F(w)$ be a holomorphic function in an open neighborhood in $\mathbf{C}^d$ of the image of $\mathscr{C}\ell U_2$ (the closure of U_2) under the map $s \to s + \sqrt{-1}\varphi(0, 0, s)$, having the following properties: $F(0) \neq 0$; there is a compact neighborhood of 0 in $\mathbf{R}^d$, $K \subset U_2$, such that

$$F(s + \sqrt{-1}\varphi(0, 0, s)) = 0, \quad \forall s \in \partial K. \tag{3.1}$$

Consider now the following equation in the unknown z:

$$z = F(s + \sqrt{-1}\varphi(z, \bar{z}, s))\zeta. \tag{3.2}$$

Whatever $s \in \mathscr{C}\ell U_2$, the Jacobian of the right-hand side with respect to (x, y) vanishes when $\zeta = 0$. Recalling that the function φ is of class $\mathscr{C}^1$ we can apply the implicit function theorem and solve (3.2) by

$$z = G(\zeta, s) \tag{3.3}$$

for all $(\zeta, s) \in \Delta \times U_2$, with Δ a sufficiently small open polydisk centered at the origin in $\mathbf{C}^n$. We have

$$G(0, s) \equiv 0, \quad \forall s \in U_2; \quad G(\zeta, s) \equiv 0, \quad \forall \zeta \in \Delta, \quad s \in \partial K. \tag{3.4}$$

Indeed, either when $\zeta = 0$ or, by virtue of (3.1), when $s \in \partial K$, Equation (3.2) has the obvious solution $z = 0$, which is perforce unique.

The neighborhood $\Delta\times U_2$ (regarded as an open subset of (ζ, s)-space $\mathbf{C}^n\times\mathbf{R}^d$) is diffeomorphic to a generic submanifold $\mathfrak{M}_0$ of $\mathbf{C}^{n+d}$ via the map $(\xi, \eta, s)\to(\zeta, \tau)$, where

$$\zeta = \xi+\sqrt{-1}\,\eta, \quad \tau = s+\sqrt{-1}\,\psi(\zeta, \bar\zeta, s), \tag{3.5}$$

$$\psi(\zeta, \bar\zeta, s) = \varphi\big(G(\zeta, s), \overline{G(\zeta, s)}, s\big). \tag{3.6}$$

Clearly, the submanifold $\mathfrak{M}_0$ is of class $\mathscr{C}^1$ and $\mathrm{codim}_{\mathbf{R}}\,\mathfrak{M}_0=d$.

Let now h be a Lipschitz-continuous CR function in $U_1\times U_2$ which vanishes to infinite order at $z=0$. We regard the pull-back to $\Delta\times U_2$,

$$H(\zeta, s) = h\big(G(\zeta, s), s\big)$$

as a function on $\mathfrak{M}_0$. We contend that H is a CR function, naturally Lipschitz continuous. This is based on the following observation: Regard the map $(\zeta, s)\to(z, s)$, with z given by (3.3), as a map from $\mathfrak{M}_0$ into $\mathfrak{M}$. Then it extends as a *holomorphic* map of an open neighborhood of the origin in $\mathbf{C}^{n+d}$ into another such neighborhood, namely as the map $(\zeta, \tau)\to(z, w)$ with $z=F(\tau)\zeta$ and $w=\tau$. Because of this, if the differential dh belongs to the span of the dz_j, dw_k, then dH must belong to the span of the $d\zeta_j$, $d\tau_k$ $(1\leqq j\leqq n,\ 1\leqq k\leqq d)$. Actually, we shall regard H as a CR function in $\Delta\times U_2$ equipped with the CR structure pulled back from $\mathfrak{M}_0$.

The new feature, here, is that the CR function $H(\zeta, s)$ vanishes to infinite order not only at $\zeta=0$ whatever $s\in U_2$, but also when $s\in\partial K$ whatever $\zeta\in\Delta$. Define $\tilde H(\zeta, s)$ as being equal to $H(\zeta, s)$ when $s\in K$, and to zero when $s\in U_2\setminus K$. If we show that $\tilde H\equiv 0$ in $\Delta\times U_2$, it will entail that $H(\zeta, s)$, and therefore also $h(z, s)$, vanish identically in a full neighborhood of the origin (in $\Delta\times U_2$ and in $U_1\times U_2$ respectively).

Thus the proof of Th. 3.1 will be complete if we prove the following

Lemma 3.1. *If Δ and U_2 are sufficiently small, then any Lipschitz-continuous CR function $\tilde H$ in $\Delta\times U_2$ which vanishes to infinite order at $\zeta=0$ and is such that $\tilde H(\zeta, s)=0$ for all $(\zeta, s)\in\Delta\times U_2$, $s\notin K$, for some compact subset K of U_2, must vanish identically in $\Delta\times U_2$.*

Lemma 3.1 will be a consequence of the following

Lemma 3.2. *Let $f(\zeta, s)$ be a Lipschitz-continuous CR function in $\Delta\times U_2$ which vanishes for all $(\zeta, s)\in\Delta\times(U_2\setminus K)$. Then the integral*

$$\int_{U_2} f\,d\tau = \int_{U_2} f(\zeta, s)\det[I+\sqrt{-1}\,\psi_s(\zeta, \bar\zeta, s)]\,ds \tag{3.7}$$

is a holomorphic function of ζ in Δ.

Proof that Lemma 3.2 implies Lemma 3.1. First we choose U_2 small enough that $|\nabla_s\varphi(0, 0, s)|<1/2$ for all $s\in U_2$. Thanks to (3.4) and (3.6) this enables us to choose Δ small enough that $|\nabla_s\psi(\zeta,\bar\zeta,s)|<3/4$ for all $(\zeta,s)\in\Delta\times U_2$. Next, we apply Lemma 3.2 to the function

$$f(\zeta,s)=\tilde{H}(\zeta,s)E_\nu[\tau_0-s-\sqrt{-1}\psi(\zeta,\bar\zeta,s)],$$

where τ_0 is an arbitrary (but fixed) point in $\mathbf{C}^d$ and

$$E_\nu(\tau)=(\nu/\pi)^{d/2}\exp\left(-\nu\sum_{k=1}^d\tau_k^2\right).$$

Call $I_\nu(\tau_0,\zeta)$ the corresponding integral (3.7); $I_\nu(\tau_0,\zeta)$ vanishes to infinite order at $\zeta=0$; therefore it vanishes identically in Δ. But it is well-known, and readily checked (by making the change of variables $s\to s_0-s/\sqrt{\nu}$), that, when $\nu\to+\infty$, $I_\nu(s_0+\sqrt{-1}\,\psi(\zeta,\bar\zeta,s_0),\zeta)$, converges uniformly to $\tilde{H}(\zeta,s_0)$ on any compact subset of $\Delta\times U_2$. □

Proof of Lemma 3.2. By Hartog's theorem it suffices to show that the integral (3.7) is separately holomorphic in each variables ζ_j $(j=1,\dots,n)$. Write $\Delta=\Delta_1\times\dots\times\Delta_n$ and fix arbitrarily $\zeta_j\in\Delta_j$ for all $j\neq i$. Then $(\zeta_i,s)\to\tilde{H}(\zeta,s)$ is a CR function in $\Delta_i\times U_2$ for the CR structure defined by the functions ζ_i and $\tau_k=s_k+\sqrt{-1}\,\psi_k(\zeta,\bar\zeta,s)$ $(k=1,\dots,d)$, in which ζ_j has been fixed for all $j\neq i$. This CR function vanishes identically when $s\notin K$ (the special properties of ψ will not be needed here).

Let therefore $n=1$; Δ is a disk centered at 0 in the ζ-plane. We have:

$$d[f\,d\zeta\wedge d\tau]=df\wedge d\zeta\wedge d\tau, \tag{3.8}$$

where $d\tau=d\tau_1\wedge\dots\wedge d\tau_d$. The left-hand side exterior derivative must be understood in the distribution sense, since the coefficients of the d-form $d\tau$ are of class $\mathscr{C}^0$. Those of the 1-form df belong to $L^\infty(\Delta\times U_2)$. Now, formula (3.8) is valid when τ is a $\mathscr{C}^\infty$ function of (ξ,η,s) in $\Delta\times U_2$. It remains true when τ is a $\mathscr{C}^1$ function, as one sees by taking the limit along a sequence of regularizations of τ.

Suppose now that $f\equiv0$ when $s\notin K$. Let γ be a simple closed, smooth curve in Δ, and call Ω its interior. If we integrate the right-hand side in (3.8) over $\Omega\times U_2$ we get, by Stokes' theorem,

$$\int_\gamma\int_{U_2}f\,d\zeta\wedge d\tau=\int_\Omega\int_{U_2}df\wedge d\zeta\wedge d\tau. \tag{3.9}$$

But the fact that f is a CR function entails that the coefficients of the $(d+2)$-form $df\wedge d\zeta\wedge d\tau$ vanish almost everywhere, since df is a linear combination of $d\zeta$, $d\tau_1,\dots,d\tau_d$ with coefficients in $L^\infty(\Delta\times U_2)$. We reach the conclusion that

$$\int_\gamma\left\{\int_{U_2}f\,d\tau\right\}d\zeta=0.$$

The simple closed curve γ is arbitrary and $\int_{U_2}f\,d\tau$ is a continuous function of ζ in Δ. Thus the assertion in Lemma 3.2 is a consequence of Morera's theorem. □

Corollary 3.1. *Let $\mathfrak{M}$ be a hypersurface in $\mathbf{C}^{n+1}$ and Σ be the transversal intersection of $\mathfrak{M}$ with a holomorphic curve $\mathscr{H}$. Then there is unique continuation on Σ at every one of its points.*

Proof. Combine Th. 3.1 with Remark 1.1 and Prop. 2.1. □

Corollary 3.2. *Let $\mathfrak{M}$ be a generic submanifold of $\mathbf{C}^{n+d}$ of codimension d. Let Σ be a noncharacteristic submanifold of $\mathfrak{M}$ of dimension d which is a real-analytic submanifold of $\mathbf{C}^{n+d}$. Then there is unique continuation on Σ at every one of its points.*

Proof. Combine Th. 3.1 with Prop. 1.4 and Cor. 2.1. □

4. Unique continuation in hypo-analytic structures

Let now $\mathfrak{M}$ denote an "abstract" $\mathscr{C}^1$ manifold. We write $\dim \mathfrak{M}=m+n$ (with $m\geqq 1$, $n\geqq 0$). *A hypo-analytic* structure on $\mathfrak{M}$ is a collection $\mathscr{F}$ of pairs (U, Z) consisting of an open subset U of $\mathfrak{M}$ and of a $\mathscr{C}^1$ map $Z=(Z_1, \ldots, Z_m)\colon U\to\mathbf{C}^m$, submitted to the following conditions:

(4.1) As (U, Z) ranges over $\mathscr{F}$ the open sets U from a covering of $\mathfrak{M}$.

(4.2) Whatever $(U, Z)\in\mathscr{F}$, the differentials $dZ_1, \ldots, dZ_m$ are $\mathbf{C}$-linearly independent at every point of U.

(4.3) Whatever the pair of elements (U, Z) and (U', Z') of $\mathscr{F}$ such that $U\cap U'\neq\emptyset$ there is a biholomorphic map H of an open neighborhood of $Z(U\cap U')$ *in* $\mathbf{C}^m$ onto one of $Z'(U\cap U')$ such that $Z'=H\circ Z$ in $U\cap U'$.

This concept generalizes that of an analytic manifold, as well as the concept of the structure of an embedded generic submanifold of $\mathbf{C}^{n+d}$. In the latter case $m=n+d$ and the family $\mathscr{F}$ consists of a single element, $(\mathfrak{M}, Z)$, where Z is the natural injection of $\mathfrak{M}$ into $\mathbf{C}^{n+d}$.

Returning to the general hypo-analytic structure $\mathscr{F}$ on the manifold $\mathfrak{M}$ we define a *hypo-analytic function* in an open subset Ω of $\mathfrak{M}$ as a function $f\colon \Omega\to\mathbf{C}$ having the following property: *Given any point $p_0\in\Omega$ and any pair $(U, Z)\in\mathscr{F}$ such that $p_0\in U$ there is a holomorphic function $\tilde{f}$ in an open neighborhood of $Z(p_0)$ in $\mathbf{C}^m$ such that $f=\tilde{f}\circ Z$ in a neighborhood of p_0 in Ω.*

We can now define a *hypo-analytic chart* in $\mathfrak{M}$: it is any pair (U, Z) consisting of an open subset U of $\mathfrak{M}$ and of a map $Z\colon U\to\mathbf{C}^m$ which satisfies (4.2) and whose components Z_i $(i=1, \ldots, m)$ are hypo-analytic functions in U. All elements of $\mathscr{F}$ are hypo-analytic charts, but a hypo-analytic chart need not belong to $\mathscr{F}$.

By the *structure bundle* of the hypo-analytic structure we mean the vector subbundle T' of the complexified cotangent bundle $\mathbf{C}T^*\mathfrak{M}$ whose local sections are the differentials of the hypo-analytic functions: if (U, Z) is a hypo-analytic chart, $T'|_U$ is spanned by $dZ^1, \ldots, dZ^m$. Thus the fibre dimension of T' is equal to m. Its orthogonal for the duality between tangent and cotangent vectors is the vector subbundle $\mathscr{V}$ of the complexified tangent bundle $\mathbf{C}T\mathfrak{M}$ whose local sections are the complex vector fields L such that $Lh=0$ whatever the hypo-analytic function h.

Two different hypo-analytic structures can have the same structure bundle T'. It is more precise to say that T' defines the *locally integrable structure* underlying the hypo-analytic structure under consideration.

When $\mathbf{C}T^*\mathfrak{M}=T'\oplus\bar{T}'$ ($\bar{T}'$: complex conjugate of T') the locally integrable structure is a *complex* structure, in the customary sense of the word. When $\mathbf{C}T^*\mathfrak{M}=T'+\bar{T}'$ it is a CR structure. When $T'\cap\bar{T}'=0$ it is an *elliptic* structure; when $T'=\bar{T}'$ it is *essentially real* (see [T]).

The *characteristic set* of the hypo-analytic structure of $\mathfrak{M}$ is the subset T^0 of the real cotangent bundle $T^*\mathfrak{M}$ equal to the intersection $T'\cap T^*\mathfrak{M}$. In general it is not a vector bundle, *i.e.*, the dimension of its fibres may vary. However it is a vector bundle when the structure is either CR or essentially real. When it is elliptic (and *a fortiori* when it is a complex structure) we have $T^0=0$. A point $(p, \theta)\in T^*\mathfrak{M}$ belongs to T^0 if $\langle\theta, \mathrm{Re}\, v\rangle=0$ whenever $(p, v)\in\mathscr{V}$. Thus its orthogonal is the subset $\mathrm{Re}\,\mathscr{V}$ of $T\mathfrak{M}$, the image of $\mathscr{V}$ under the map $(p, v)\to(p, \mathrm{Re}\, v)$.

A $\mathscr{C}^1$ submanifold Σ of $\mathfrak{M}$ is said to be *noncharacteristic* if (1.9) holds or, equivalently, if

$$T\mathfrak{M}|_\Sigma = T\Sigma+\mathrm{Re}\,\mathscr{V}|_\Sigma. \tag{4.4}$$

(cf. (1.10)).

The proof of the following statement is immediate:

Proposition 4.1. *Let Σ be a $\mathscr{C}^1$ submanifold of $\mathfrak{M}$, whose codimension is even and equal to $2\varkappa$, and which has the following property:*

(4.5) *Each point $p\in\Sigma$ has an open neighborhood U_p in $\mathfrak{M}$ in which there are $\varkappa$ hypo-analytic functions h_j $(j=1, \ldots, \varkappa)$ such that, in the set U_p,*

(4.6) *$\Sigma\cap U_p$ is defined by the equations $h_1=\ldots=h_\varkappa=0$;*

(4.7) *$dh_1, \ldots, dh_\varkappa, d\bar{h}_1, \ldots, d\bar{h}_\varkappa$ are $\mathbf{C}$-linearly independent* $\mathrm{mod}\, T'\cap\bar{T}'$.
Then Σ is noncharacteristic.

We shall say that Σ is a *hypo-analytic noncharacteristic submanifold* of $\mathfrak{M}$ if it has Property (4.5).

When $\mathfrak{M}$ is an embedded generic submanifold of $\mathbf{C}^{n+d}$ and $\mathrm{codim}\,\mathfrak{M}=d$, a d-dimensional submanifold Σ of $\mathfrak{M}$ is hypo-analytic noncharacteristic if locally, Σ

is equal to the holomorphic-transverse intersection of $\mathfrak{M}$ with a holomorphic submanifold $\mathscr{H}$ of $\mathbf{C}^{n+d}$ such that $\dim_{\mathbf{C}} \mathscr{H}=d$ (see Sect. 1).

Let (U, Z) be a hypo-analytic chart. After a $\mathbf{C}$-linear substitution we may assume that the following property, stronger than (4.2), holds:

(4.8) the differentials $d(\operatorname{Re} Z_1), \ldots, d(\operatorname{Re} Z_m)$ are linearly independent at every point of U.

After contracting U about one of its points, p_0, we may take the functions $x_i=\operatorname{Re} Z_i$ as part of a coordinate system in U whose remaining coordinates we shall provisionally denote by $y_1, \ldots, y_n$. We may and shall assume that these coordinates, as well as the functions Z_i themselves, all vanish at p_0. For this reason we shall often refer to p_0 as *the origin* and denote it by 0. We now have

$$Z_i = x_i+\sqrt{-1}\,\varphi_i(x, y), \quad i = 1, \ldots, m, \tag{4.9}$$

with the φ_i real-valued. Further contractions of U and $\mathbf{C}$-linear substitutions of the Z_i allows us to assume (cf. (1.6), (1.7)) that

$$\nabla_x \varphi_i(0, 0) = 0, \quad i = 1, \ldots, m. \tag{4.10}$$

By (4.2) we know that the rank at the origin of the map Z is $\geqq m$; denote it by $m+\nu$. By (4.9) and (4.10) ν must be equal to the rank at 0 of $d_y\varphi_1, \ldots, d_y\varphi_m$. We can carry out an $\mathbf{R}$-linear substitution of the Z_i so as to achieve that

(4.11) $d_y\varphi_1, \ldots, d_y\varphi_\nu$ are linearly independent at 0; $d\varphi_{\nu+1}, \ldots, d\varphi_n$ vanish at 0.

Further contracting of U about 0 allows us to take $\varphi_1, \ldots, \varphi_\nu$ as the first ν coordinates y_j. In order to bring our notation closer to the one used in the CR case we shall make the following changes:

We write W_k instead of $Z_{\nu+k}$, s_k instead of $x_{\nu+k}$ and $\varphi_k(x, y, s, t)$ instead of $\varphi_{\nu+k}(x, y)$ for $k=1, \ldots, d=m-\nu$. We write t_ℓ instead of $y_{\nu+\ell}$ for $\ell=1, \ldots, d'=n-\nu$.

We end up with the following "representation"

$$\begin{aligned} Z_j &= x_j+\sqrt{-1}\,y_j (= z_j), \quad j = 1, \ldots, \nu; \\ W_k &= s_k+\sqrt{-1}\,\varphi_k(x, y, s, t), \quad k = 1, \ldots, d. \end{aligned} \tag{4.12}$$

Moreover, the functions φ_k are real-valued and

$$\varphi_k|_0 = 0, \quad d\varphi_k|_0 = 0, \quad k = 1, \ldots, d. \tag{4.13}$$

Below we write $\varphi=(\varphi_1, \ldots, \varphi_d)\colon U\to\mathbf{R}^d$.

When $d=d'=0$ (*i.e.*, $\nu=m=n$ and $\dim \mathfrak{M}=2n$) the underlying locally integrable structure is a complex structure over the set U. When $d'=0$ it is a *CR* structure. When $d=0$ it is elliptic. When $\nu=0$ (in which case $d=m$) and $\varphi_k\equiv 0$ for all $k=1, \ldots, d$, the structure is essentially real.

The formulas (4.12) make clear that, in general, the "hypo-analytic" map $(Z, W)\colon U\to \mathbf{C}^{\nu+d}$ is *not* an embedding. It is one when there are no variables t, *i.e.*, when the structure induced on U is a *CR* structure. Otherwise the pre-image of a point (z, w) under that map is the set $\{(x, y, s, t);\ x+\sqrt{-1}\,y=z,\ s=\operatorname{Re} w,\ \varphi(x, y, s, t)=\operatorname{Im} w\}$, which, in general, will consist of more than one point.

Over U the vector bundle $\mathscr{V}$ (see above) is spanned by $n=\nu+d'$ vector fields

$$
(4.14)\qquad
\begin{aligned}
L_j &= \partial/\partial \bar z_j+\textstyle\sum_{k=1}^{d}\lambda_{j,k}\partial/\partial s_k, \quad j = 1, \ldots, \nu,\\
L_{\nu+\ell} &= \partial/\partial t_\ell+\textstyle\sum_{k=1}^{d}\lambda_{\nu+\ell,k}\partial/\partial s_k, \quad \ell = 1, \ldots, d',
\end{aligned}
$$

in which the coefficients $\lambda_{j,k}$ are determined by the requirement that $L_j W_k=0$ for all j, k, $1\leqq j\leqq n$, $1\leqq k\leqq d$. It follows from (4.13) that all these coefficients vanish at the origin. It follows that the fibre of $\mathscr{V}$ at 0 is spanned (over $\mathbf{C}$) by the tangent vectors $\partial/\partial \bar z_j$, $\partial/\partial t_\ell$ ($j=1, \ldots, \nu$, $\ell=1, \ldots, d'$). From this, or directly from (4.12)—(4.13), it follows that the fibre at the origin of the characteristic set T^0 is spanned (over $\mathbf{R}$) by the ds_k, $k=1, \ldots, d$. Thus the dimension of the fibre of T^0 at 0 is equal to d. This, combined with (1.9), demands that the codimension of any noncharacteristic submanifold of $\mathfrak{M}$, Σ, passing through the origin, be $\leqq m+n-d=n+\nu$, *i.e.*, $\dim \Sigma \geqq d$.

We shall denote by $(U, (Z, W))$ any hypo-analytic chart in which the "basic" hypo-analytic functions are given by (4.12). If moreover (4.13) hold true we shall say that $(U, (Z, W))$ is a *distinguished* hypo-analytic chart. It ought to be kept in mind, however, that the integers ν, d, d' may vary from one distinguished hypo-analytic chart to another.

The proof of the next statement is left as an exercise to the reader:

Proposition 4.2. *Let Σ be a hypo-analytic noncharacteristic submanifold of $\mathfrak{M}$ such that* $\operatorname{codim} \Sigma=2\varkappa$. *Then every point of Σ lies in the domain U of a distinguished hypo-analytic chart $(U, (Z, W))$ such that $\Sigma\cap U$ is defined in U by the equations*

$$
(4.15)\qquad Z_j = 0, \quad j = 1, \ldots, \varkappa.
$$

(Thus we must have $\varkappa\leqq \nu$.)

Conversely, given any distinguished hypo-analytic chart $(U, (Z, W))$, the equations (4.15) *define a hypo-analytic noncharacteristic submanifold of U.*

We shall say that the distinguished hypo-analytic chart $(U, (Z, W))$ is *adapted* to the manifold Σ if $\Sigma \cap U$ is defined in U by the equations (4.15).

In the hypo-analytic structure on the manifold $\mathfrak{M}$ the *solutions* play the role that CR functions play on a generic submanifold of $\mathbf{C}^{n+d}$. Here we shall be interested in Lipschitz-continuous solutions h in some open subset Ω of $\mathfrak{M}$. This means that h is a Lipschitz-continuous function in Ω whose differential is an L^∞ section of the vector bundle T'. In any hypo-analytic local chart (U, Z) with $U \subset \Omega$, dh is a linear combination of $dZ_1, \ldots, dZ_m$ with coefficients in $L^\infty(U)$. This is equivalent to saying that, given any continuous section L of $\mathscr{V}$ over Ω, we have $Lh=0$. In this last characterization lies the motivation for the name "solution".

Any hypo-analytic function in Ω is a solution in Ω but, in general, there are solutions which are not hypo-analytic.

When $\mathfrak{M}$ is a generic submanifold of $\mathbf{C}^{n+d}$ inheriting its CR structure from the ambient complex space, the sections of $\mathscr{V}$ are the tangential Cauchy—Riemann vector fields; and the solutions are the CR functions. The hypo-analytic functions are those functions which can be extended holomorphically to an open neighborhood in $\mathbf{C}^{n+d}$ of every point of their domain of definition.

Let Σ be a hypo-analytic noncharacteristic submanifold of U and p_0 a point of Σ. We shall say that Σ satisfies *Condition* $(\mathscr{B})$ at p_0 if there is a distinguished hypo-analytic chart $(U, (Z, W))$ centered at p_0 (*i.e.*, in which p_0 becomes the origin), with Z and W given by (4.12), adapted to the manifold Σ and such, furthermore, that the following is true:

(4.16) The submanifold Σ_0 *of* $\mathbf{C}^d$ defined by the equations

$$\operatorname{Im} w = \varphi(0, 0, \operatorname{Re} w, 0)$$

satisfies Condition $(\mathscr{A})$ at the origin (see Sect. 2).

Theorem 4.1. *Let Σ be a hypo-analytic noncharacteristic submanifold of $\mathfrak{M}$. Suppose that Σ satisfies Condition $(\mathscr{B})$ at one of its points, p_0.*

Then, to each open neighborhood U of p_0 in $\mathfrak{M}$ there is another one, $V \subset U$, such that every Lipschitz-continuous solution h in U which vanishes to infinite order on $\Sigma \cap U$ also vanishes identically in V.

Proof. After contracting U about p_0 we may assume that U is the domain of a distinguished hypo-analytic chart $(U, (Z, W))$ centered at p_0 and adapted to Σ. Suppose that Z and W are given by (4.12) and that (4.13) holds. Let U_0 denote the subset of U defined by $t=0$. The map

(4.17) $\qquad (x, y, s) \to (z, w), \quad z = x + \sqrt{-1}\, y, \quad w = s + \sqrt{-1}\, \varphi(x, y, s, 0),$

is a $\mathscr{C}^1$ diffeomorphism of U_0 onto a generic submanifold $\mathfrak{M}_0$ of $\mathbf{C}^{\nu+d}$ whose codimension is equal to d. Let Σ_0 denote the submanifold of $\mathfrak{M}_0$ defined by $z=0$. It is equal to the holomorphic-transversal intersection of $\mathfrak{M}_0$ with a holomorphic submanifold $\mathscr{H}_0$ of $\mathbf{C}^{\nu+d}$, with $\dim_{\mathbf{C}} \mathscr{H}_0=d$. By hypothesis it has Property $(\mathscr{A})$. Therefore, by Th. 3.1, there is unique continuation in $\mathfrak{M}_0$, on Σ_0, at the origin.

Now, given any Lipschitz-continuous solution h in U, the transfer of $h|_{t=0}$ to $\mathfrak{M}_0$ via the map (4.17) is a Lipschitz-continuous CR function $\tilde{h}_0$ on $\mathfrak{M}_0$. It follows from Th. 3.1 that $\tilde{h}_0 \equiv 0$ in a neighborhood $\tilde{V}_0$ of the origin in $\mathbf{C}^{2\nu+d}$ ($\tilde{V}_0$ can be taken independently of h). By pull-back under (4.17) we obtain that $h\equiv 0$ in a full neighborhood V_0 of 0 in the subspace $t=0$ of U. Th. 4.1 follows then from the uniqueness in the Cauchy problem which is one of the consequences of the Approximation Formula in locally integrable structures (see [T], p. 29*). □

The analogues of Corollaries 3.1, 3.2 are valid here:

Corollary 4.1. *Let Σ be a hypo-analytic noncharacteristic submanifold of $\mathfrak{M}$ passing through a point p_0 at which the fibre of the characteristic set T^0 of $\mathfrak{M}$ has dimension $\leqq 1$. Then the conclusion of Th.* 4.1 *is valid.*

Proof. In the distinguished hypo-analytic chart $(U, (Z, W))$ given by (4.12) and adapted to Σ, the hypothesis means that $d\leqq 1$. When $d=0$ (*i.e.*, there are no variables s) every solution is a holomorphic function of z (independent of t) and the assertion is immediate. When $d=1$, Prop. 2.1 entails that Σ satisfies Condition $(\mathscr{B})$ at p_0. □

We leave to the reader the statement and the proof of the analogue of Cor. 3.2.

Remark 4.1. The proof of Th. 4.1 has made use solely of the fact that the solution under consideration vanishes to infinite order on the submanifold of U defined by $Z=0$, $t=0$ (which we may identify to the submanifold Σ_0 of $\mathbf{C}^d$ in Condition (4.16)). But in fact this does not imply any loss of generality. Indeed, observe first that the restriction of any Lipschitz-continuous solution h to the subspace $Z=0$ defines a Lipschitz-continuous solution h_0 in an open neighborhood of the

* In the proof of the Approximation Formula that have been published so far the regularity assumptions on the basic "hypo-analytic" functions Z_j are fairly strong (at least $\mathscr{C}^2$). Actually by the same argument used in the proof of Lemma 3.2 the formula can be proved under the hypothesis that the Z_j are of class $\mathscr{C}^1$ and the solution is Lipschitz-continuous.

origin, in (s, t)-space $\mathbf{R}^{d+d'}$, for the hypo-analytic structure defined by the functions

$$w_k = s_k + \sqrt{-1}\,\varphi_k(0, 0, s, t), \quad k = 1, \ldots, d. \tag{4.18}$$

The uniqueness in the Cauchy problem, already used at the end of the proof of Th. 4.1, implies that if h_0 vanishes (to first order) on the subspace $t=0$, then it vanishes in a full neighborhood (independent of h_0) of that subspace.

Of course there is no greater generality to be gained by looking at the traces of solutions on submanifolds of the kind $t=f(s)$ since a change of variables $t \to t-f(s)$ can always bring us back to the case $t=0$. □

5. Two extraneous examples of unique continuation

Let us return to the embedded CR case: $\mathfrak{M}$ is a generic $\mathscr{C}^1$ submanifold and $\mathscr{H}$ a holomorphic submanifold of $\mathbf{C}^{n+d}$ such that $\operatorname{codim}_{\mathbf{R}} \mathfrak{M} = \dim_{\mathbf{C}} \mathscr{H} = d$; the intersection $\mathfrak{M} \cap \mathscr{H}$ is holomorphic-transversal and equal to Σ.

It is sometimes possible to prove unique continuation on Σ, at a point p_0, even if we cannot prove that Σ satisfies Condition ($\mathscr{A}$) at p_0 (Sect. 2). Let us give two examples, one quite trivial, the other one less so:

Example 5.1. Assume that the following property holds:

(5.1) Given any open neighborhood U of p_0 in $\mathfrak{M}$ there is an open neigborhood $\mathcal{O}$ of p_0 in $\mathbf{C}^{n+d}$ such that $\mathcal{O} \cap \mathfrak{M} \subset U$ and such that the following is true:
To each Lipschitz-continuous CR function h in U there is a holomorphic function $\tilde{h}$ in $\mathcal{O}$ such that $h = \tilde{h}$ in $\mathcal{O} \cap \mathfrak{M}$.

In other words, *every germ of CR function at p_0 is hypo-analytic at p_0* (*cf.* Sect. 4). Then obviously there is unique continuation on Σ at p_0 (*i.e.*, Property ($\mathscr{UC}$) holds).

Condition (5.1) is satisfied, in particular, when the Levi form of $\mathfrak{M}$ has at least one eigenvalue <0 at every characteristic cotangent vector to $\mathfrak{M}$ at the point p_0 (see [BCT], Cor. 6.1).

Example 5.2. Suppose $n=1$ and $0 \in \mathfrak{M}$. Suppose moreover that $\mathfrak{M}$ is defined, in some open neighborhood of 0 in $\mathbf{C}^{1+d}$, by the equations

$$\operatorname{Im} w_k = \varphi_k(|z|, \operatorname{Re} w), \quad k = 1, \ldots, d. \tag{5.2}$$

Let Σ be defined by the equation $z=0$. Then ($\mathscr{UC}$) holds at the origin.

Proof. Because of the special form of the defining equations it is convenient to use polar coordinates r, θ in the complex z-plane, in particular in representing the tangential Cauchy—Riemann vector field on $\mathfrak{M}$ away from Σ. Thus we may take it to have the form

$$L = \partial/\partial r + \sqrt{-1}\, r^{-1}\partial/\partial\theta + \sum_{k=1}^{d} \lambda_k(r, s)\partial/\partial s_k.$$

Let now $U = \Delta\times\mathscr{B}$ be a neighborhood of 0 in (z, s)-space $\mathbf{C}\times\mathbf{R}^d$; Δ is an open disk in the z-plane, $\mathscr{B}$ an open ball in $\mathbf{R}^d$, both centered at the origin. Let $\tilde{U}$ be the image of U under the map $(z, s)\to(z, w)$ with $w=(w_1, \dots, w_d)$ given by (5.2), and let $\tilde{h}$ be any Lipschitz-continuous CR function in $\tilde{U}$ which vanishes on Σ. Denote by $h(r, \theta, s)$ its pull-back to U and set

$$h_0(r, s) = (2\pi)^{-1}\int_0^{2\pi} h(r, \theta, s)\, d\theta.$$

By integrating with respect to θ over $(0, 2\pi)$ the equation $Lh=0$, we obtain

$$\partial h_0/\partial r + \sum_{k=1}^{d} \lambda_k(r, s)\,\partial h_0/\partial s_k = 0. \tag{5.3}$$

Since $h_0(0, s)\equiv 0$ for all $s\in\mathscr{B}$ we may extend h_0 to $r<0$ by setting it equal to zero there, thus getting a Lipschitz-continuous function in $U_0=(-r_0, r_0)\times\mathscr{B}$ (r_0: radius of Δ). We see that, in U_0, $h_0(r, s)$ is a solution for the hypo-analytic structure defined by the functions $w_k=s_k+\sqrt{-1}\,\varphi_k(r, s)$, $k=1, \dots, d$. Uniqueness in the Cauchy problem, with Cauchy data on the hypersurface $r=0$, holds for the vector field L (by the Approximation Formula, see [T], p. 29), and therefore $h_0\equiv 0$ in an open neighborhood V_0 of $\{0\}\times\mathscr{B}$ in U_0. Furthermore V_0 can be chosen independently of h.

Suppose now that $\tilde{h}$ vanishes *to infinite order* on Σ. The preceding reasoning may now be applied not merely to $\tilde{h}$ but to $\tilde{h}/z^\mu$ whatever the integer $\mu\geqq 0$ or <0. We conclude that, whatever $\mu\in\mathbf{Z}$,

$$h_\mu(r, s) = (2\pi)^{-1}\int_0^{2\pi} h(r, \theta, s)e^{-\sqrt{-1}\mu\theta}\, d\theta$$

vanishes for all $(r, s)\in V_0$. But of course this means that $h(r, \theta, s)\equiv 0$ for all $(r, s)\in V_0$, whereby our contention is proved. □

Added in the proofs. After completion of this paper Howard Jacobowitz gave an example of a two-dimensional totally real submanifold of $\mathbf{C}^2$ that does not have property ($\mathscr{A}$) (cf. end of Sect. 2). See H. Jacobowitz, "On the intersection of varieties with a totally real submanifold", *Proc. Am. Math. Soc.* **101** (1987), 127—130.

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