A shortcut to weighted representation formulas for holomorphic functions

Mats Andersson and Mikael Passare

0. Introduction

In [4] a method was given for generating weighted solution kernels to the $\bar\partial$-equation, i.e. kernels K such that

$$\bar\partial \int K \wedge w = w, \quad \text{if} \quad \bar\partial w = 0.$$

As a by-product a variety of projection kernels (P such that $f = \int Pf$, for f holomorphic) were obtained. These kernels give representation formulas for holomorphic functions which in general consist of an integral over the whole domain and a boundary integral. The projection part and the corresponding representation formulas have proved to be quite fruitful. They have been used by several authors (see e.g. [2], [3], [5] and [10]) to obtain explicit solutions to division and interpolation problems.

The purpose of this paper is to give a short proof of a generalization of the representation formulas in [3] and [4] without making the détour to the $\bar\partial$-problem and the kernels K.

We derive in § 1 a quite general formula (Theorem 1) which is then turned into a more tangible one for bounded domains (Theorem 2). Using logarithmic residues we also obtain weighted versions of certain formulas in [13] and [15]. In § 2 we give a few examples and comments.

To motivate what follows, let us take a brief look at the case $n=1$. Let f be holomorphic in a domain $\Omega \subset \mathbf{C}$ and suppose that $Q \in C^1(\bar\Omega \times \bar\Omega)$. We then have

$$\int_\Omega f \frac{\partial Q}{\partial \bar\zeta} d\bar\zeta \wedge d\zeta = \int_{\partial\Omega} fQ\, d\zeta$$

and by the Cauchy formula it follows that

$$f(z) = \frac{1}{2\pi i} \int_\Omega f(\zeta) \frac{\partial Q}{\partial \bar\zeta}(\zeta, z)\, d\bar\zeta \wedge d\zeta + \frac{1}{2\pi i} \int_{\partial\Omega} f(\zeta) \frac{1+(z-\zeta)Q(\zeta, z)}{\zeta - z}\, d\zeta.$$

In certain cases $1+(z-\zeta)Q(\zeta, z)=0$ for $\zeta\in\partial\Omega$ so that the boundary integral disappears. If Q is holomorphic in z then so is the kernel. We have now essentially proved our theorems in case $n=1$. The proof for general n is similar except that it requires a certain amount of algebraic organization.

1. The general formulas

Let us start out by formulating our basic result, a rather general representation formula for holomorphic functions.

Theorem 1. *Suppose that the function f is holomorphic in some domain $\Omega\subset\mathbf{C}^n$ and continuous up to the boundary. Let there be given*
i) *continuously differentiable functions*

$$Q^k\colon \bar{\Omega}\times\bar{\Omega}\to\mathbf{C}^n, \quad k=1, \dots, p,$$

ii) *a function G of p complex variables, holomorphic in a neighborhood of the image of $\bar{\Omega}\times\bar{\Omega}$ by the mapping $(z-\zeta)Q(\zeta, z)$ defined by*

$$(\zeta, z)\mapsto\left(\sum(z_j-\zeta_j)Q_j^1(\zeta, z), \dots, \sum(z_j-\zeta_j)Q_j^p(\zeta, z)\right),$$

and satisfying $G(0)=1$.
Finally, assume that for $|\alpha|\leqq n$ the functions $D^\alpha G=D^\alpha G((z-\zeta)Q(\zeta, z))$, obtained by composing $(z-\zeta)Q(\zeta, z)$ with derivatives of G, for each fixed $z\in\Omega$ have compact support contained in Ω when considered as functions of ζ. In that case the following formula holds for all $z\in\Omega$:

(1) $$f(z)=(2\pi i)^{-n}\int_\Omega f(\zeta)\sum_{|\alpha|=n}\frac{D^\alpha G}{\alpha!}(\bar{\partial}q)^\alpha,$$

where $q^k(\zeta, z)=\sum Q_j^k(\zeta, z)\,d\zeta_j$ and we have used the shorthand notation $\alpha!=\alpha_1!\dots\alpha_p!$ and $(\bar{\partial}q)^\alpha=(\bar{\partial}q^1)^{\alpha_1}\wedge\dots\wedge(\bar{\partial}q^p)^{\alpha_p}$.

Specific choices of $Q=(Q^1, \dots, Q^p)$ and G will be given below.

Before we embark on the proof proper we digress a little to give some background to our method. In one complex variable the simple formula

(2) $$\varphi(z)=\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{|\zeta-z|=\varepsilon}\frac{\varphi(\zeta)\,d\zeta}{\zeta-z}$$

holds for any continuous function φ. This merely expresses the fact that $1/\pi\zeta$ is a fundamental solution to the $\bar{\partial}$-equation. Assuming φ to have compact support we may introduce the following notation:

$$\left[\frac{1}{\zeta-z}\right](\varphi(\zeta)\,d\bar{\zeta}\wedge d\zeta) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0}\int_{|\zeta-z|>\varepsilon} \frac{\varphi(\zeta)\,d\bar{\zeta}\wedge d\zeta}{\zeta-z}$$

and, consequently:

$$\bar{\partial}\left[\frac{1}{\zeta-z}\right](\varphi(\zeta)\,d\zeta) = -\lim_{\varepsilon\to 0}\int_{|\zeta-z|>\varepsilon} \frac{\bar{\partial}\varphi(\zeta)\wedge d\zeta}{\zeta-z} = \lim_{\varepsilon\to 0}\int_{|\zeta-z|=\varepsilon} \frac{\varphi(\zeta)\,d\zeta}{\zeta-z}.$$

Equation (2) thus assumes the form

$$\varphi(z) = \frac{1}{2\pi i}\bar{\partial}\left[\frac{1}{\zeta-z}\right](\varphi(\zeta)\,d\zeta).$$

In the more general case of several variables we consider tensor products such as

$$\left[\frac{1}{\zeta_1-z_1}\right]\bar{\partial}\left[\frac{1}{\zeta_2-z_2}\right]\wedge\bar{\partial}\left[\frac{1}{\zeta_3-z_3}\right].$$

This convenient formalism has its origin in the theory of meromorphic currents (cf. [11], [12]), which of course in general is far more delicate than are the simple cases we are dealing with here. What matters in this context is that computations like

$$\bar{\partial}\left(\left[\frac{1}{\zeta_1-z_1}\right]\bar{\partial}\left[\frac{1}{\zeta_2-z_2}\right]\right) = \bar{\partial}\left[\frac{1}{\zeta_1-z_1}\right]\wedge\bar{\partial}\left[\frac{1}{\zeta_2-z_2}\right]$$

and

$$(\zeta_1-z_1)\left[\frac{1}{\zeta_1-z_1}\right] = 1$$

actually hold.

In analogy to the iterated Cauchy formula we have

(3) $$\varphi(z) = c_n\bar{\partial}\left[\frac{1}{\zeta_1-z_1}\right]\wedge\ldots\wedge\bar{\partial}\left[\frac{1}{\zeta_n-z_n}\right](\varphi(\zeta)\omega(\zeta)),$$

with $c_n=(-1)^{(1/2)n(n-1)}(2\pi i)^{-n}$ and $\omega(\zeta)=d\zeta_1\wedge\ldots\wedge d\zeta_n$. Now we are all set to give a quick proof of our theorem.

Proof of Theorem 1. We start by taking $\varphi=fG((z-\zeta)Q)$ in formula (3). Since $\varphi(z)=f(z)G(0)=f(z)$ we get

$$f(z) = c_n\bar{\partial}\left[\frac{1}{\zeta_1-z_1}\right]\wedge\ldots\wedge\bar{\partial}\left[\frac{1}{\zeta_n-z_n}\right](f(\zeta)G((z-\zeta)Q)\omega(\zeta))$$

$$= (-1)^n c_n\left[\frac{1}{\zeta_1-z_1}\right]\bar{\partial}\left[\frac{1}{\zeta_2-z_2}\right]\wedge\ldots\wedge\bar{\partial}\left[\frac{1}{\zeta_n-z_n}\right](f(\zeta)\bar{\partial}G((z-\zeta)Q)\wedge\omega(\zeta)).$$

Now $\bar\partial G((z-\zeta)Q)=\sum_{k=1}^{p}\sum_{j=1}^{n} D_k G((z-\zeta)Q)(z_j-\zeta_j)\bar\partial Q_j^k$, and since

$$(z_j-\zeta_j)\bar\partial\left[\frac{1}{\zeta_j-z_j}\right]=\bar\partial(-1)=0$$

only terms containing $z_1-\zeta_1$ give any contribution. We are thus left with

$$f(z)=(-1)^{n-1}c_n\bar\partial\left[\frac{1}{\zeta_2-z_2}\right]\wedge\ldots\wedge\bar\partial\left[\frac{1}{\zeta_n-z_n}\right]\left(f(\zeta)\sum_{k=1}^{p} D_k G\bar\partial Q_1^k\wedge\omega(\zeta)\right)$$

$$=(-1)^{2(n-1)}c_n\left[\frac{1}{\zeta_2-z_2}\right]\bar\partial\left[\frac{1}{\zeta_3-z_3}\right]\wedge\ldots$$

$$\ldots\wedge\bar\partial\left[\frac{1}{\zeta_n-z_n}\right]\left(f(\zeta)\sum_{k=1}^{p}\bar\partial D_k G\wedge\bar\partial Q_1^k\wedge\omega(\zeta)\right).$$

As before we have

$$\bar\partial D_k G((z-\zeta)Q)=\sum_{l=1}^{p}\sum_{j=1}^{n} D_l D_k G((z-\zeta)Q)(z_j-\zeta_j)\bar\partial Q_j^l,$$

and we find again that terms involving $z_j-\zeta_j$ for $j=3,\ldots,n$ do not contribute. Moreover,

$$\sum_{k=1}^{p}\sum_{l=1}^{p}\bar\partial Q_1^l\wedge\bar\partial Q_1^k=0$$

by anticommutativity, so terms corresponding to $j=1$ may also be neglected. What remains is

$$f(z)=(-1)^{(n-1)+(n-2)}c_n\bar\partial\left[\frac{1}{\zeta_3-z_3}\right]\wedge\ldots$$

$$\ldots\wedge\bar\partial\left[\frac{1}{\zeta_n-z_n}\right]\left(f(\zeta)\sum_{k,l=1}^{p} D_l D_k G\bar\partial Q_2^l\wedge\bar\partial Q_1^k\wedge\omega(\zeta)\right).$$

It is clear that this procedure may be repeated n times to yield

$$(4)\qquad f(z)=(-1)^{(n-1)+\cdots+1}c_n\left(f(\zeta)\sum_{k_r=1}^{p} D_{k_n}\ldots D_{k_1}G\bar\partial Q_n^{k_n}\wedge\ldots\wedge\bar\partial Q_1^{k_1}\wedge\omega(\zeta)\right)$$

$$=(2\pi i)^{-n}\int_\Omega f(\zeta)\sum_{k_r=1}^{p} D_{k_n}\ldots D_{k_1}G\bar\partial Q_n^{k_n}\wedge\ldots\wedge\bar\partial Q_1^{k_1}\wedge\omega(\zeta).$$

In order to simplify this formula, notice that

$$\bar\partial Q_n^{k_n}\wedge\ldots\wedge\bar\partial Q_1^{k_1}\wedge d\zeta_1\wedge\ldots\wedge d\zeta_n=\bar\partial Q_1^{k_1}\wedge d\zeta_1\wedge\bar\partial Q_n^{k_n}\wedge\ldots\wedge\bar\partial Q_2^{k_2}\wedge d\zeta_2\wedge\ldots\wedge d\zeta_n=\ldots$$

$$\ldots=\bar\partial Q_1^{k_1}\wedge d\zeta_1\wedge\ldots\wedge\bar\partial Q_n^{k_n}\wedge d\zeta_n,$$

so that on writing

$$q^k(\zeta,z)=\sum_{j=1}^{n} Q_j^k(\zeta,z)d\zeta_j$$

and keeping in mind that 2-forms do commute we obtain

$$\sum_{k_r=1}^{p} \bar\partial Q_n^{k_n} \wedge \ldots \wedge \bar\partial Q_1^{k_1} \wedge \omega(\zeta) = \sum_{k_r=1}^{p} \bar\partial Q_1^{k_1} \wedge d\zeta_1 \wedge \ldots \wedge \bar\partial Q_n^{k_n} \wedge d\zeta_n$$

$$= \sum_{|\alpha|=n} \frac{1}{\alpha_1!\ldots\alpha_p!} (\bar\partial q^1)^{\alpha_1} \wedge \ldots \wedge (\bar\partial q^p)^{\alpha_p}, \quad \text{with} \quad |\alpha| = \alpha_1 + \ldots + \alpha_p.$$

It follows that the last member in (4) is actually equal to the right hand side in (1) and the proof is complete.

Remark. A special choice of G is

$$G = G_0 \cdot G_1$$

with G_0 being a function of one single variable. Notice that in this case it suffices to assume the functions $D^l G_0$ to be of compact support. It is this situation that we will exploit in proving our next theorem, a representation formula with boundary terms.

Theorem 2. *Suppose that the function f is holomorphic in some bounded domain $\Omega \subset \mathbf{C}^n$ and continuous up to the boundary. Let there be given*

i) *continuously differentiable functions*

$$Q^k\colon\ \bar\Omega \times \bar\Omega \to \mathbf{C}^n, \quad k = 1, \ldots, p,$$

ii) *a function G of p complex variables, holomorphic in a neighborhood of the image of $\bar\Omega \times \bar\Omega$ by the mapping $(z-\zeta)Q(\zeta, z)$ and satisfying $G(0)=1$,*

iii) *a smooth map $S\colon \partial\Omega \times \bar\Omega \to \mathbf{C}^n$ such that*

$$\sum (z_j - \zeta_j) S_j(\zeta, z) \neq 0,$$

unless $z=\zeta$.

In that case the following formula holds for all $z \in \Omega$:

$$f(z) = (2\pi i)^{-n} \int_\Omega f(\zeta) \sum_{|\alpha|=n} \frac{D^\alpha G}{\alpha!} (\bar\partial q)^\alpha \tag{5}$$

$$+ (2\pi i)^{-n} \int_{\partial\Omega} f(\zeta) \sum_{\alpha_0+|\alpha|=n-1} \frac{s \wedge (\bar\partial s)^{\alpha_0}}{\langle \zeta - z, S\rangle^{\alpha_0+1}} \wedge \frac{D^\alpha G}{\alpha!} (\bar\partial q)^\alpha,$$

where $s(\zeta, z) = \sum S_j(\zeta, z)\, d\zeta_j$ and $\langle \zeta - z, S\rangle = \sum (\zeta_j - z_j) S_j(\zeta, z)$.

Proof. The idea is to choose the weights so that G_0 becomes (almost) equal to χ_Ω, the characteristic function of Ω. To do this we first fix $z \in \Omega$ and extend the mapping S smoothly to all of Ω. Then we pick a smooth function $\chi\colon \bar\Omega \to [0, 1]$, vanishing near $\partial\Omega$ and such that $\chi(\zeta)=1$ whenever $\langle \zeta - z, S\rangle = 0$.

Next, define a smooth function $Q^0\colon \bar\Omega \times \bar\Omega \to \mathbf{C}^n$ by

$$Q^0(\zeta, z) = (1 - \chi(\zeta)) \frac{S(\zeta, z)}{\langle \zeta - z, S\rangle}.$$

Since z is fixed we interpret Q^0 as being constant with respect to the second set of variables.

Finally we put $G_0(t_0)=(t_0+1)^N$, with $N>n$. It follows that

$$D^l G_0((z-\zeta)Q^0) = \frac{N!}{(N-l)!}\chi^{N-l}(\zeta)$$

so that on extending Q to (Q^0, Q) and replacing $G(t_1, \ldots, t_p)$ by $G_0(t_0)G(t_1, \ldots, t_p)$ we find that the conditions in Theorem 1 are fulfilled.

Notice that the map S simply serves to allow us to divide by $\zeta-z$. We get

$$f(z) = (2\pi i)^{-n}\int_\Omega f(\zeta)\sum_{\alpha_0+|\alpha|=n}\frac{D^{\alpha_0}G_0}{\alpha_0!}(\bar\partial q^0)^{\alpha_0}\wedge\frac{D^\alpha G}{\alpha!}(\bar\partial q)^\alpha. \tag{6}$$

Now

$$(\bar\partial q^0)^{\alpha_0} = \left[(1-\chi)\bar\partial\left(\frac{s}{\langle\zeta-z, S\rangle}\right)-\frac{\bar\partial\chi\wedge s}{\langle\zeta-z, S\rangle}\right]^{\alpha_0}$$

$$= (1-\chi)^{\alpha_0}\left[\bar\partial\left(\frac{s}{\langle\zeta-z, S\rangle}\right)\right]^{\alpha_0}-\alpha_0(1-\chi)^{\alpha_0-1}\frac{\bar\partial\chi\wedge s\wedge(\bar\partial s)^{\alpha_0-1}}{\langle\zeta-z, S\rangle^{\alpha_0}},$$

the latter equality stemming from the fact that $s\wedge s=0$. We are going to let χ approach the characteristic function of Ω (in any suitable way). Clearly then

$$\chi^{N-\alpha_0}(1-\chi)^{\alpha_0}\to 0, \quad \text{for} \quad \alpha_0>0,$$

so on recalling that

$$D^{\alpha_0}G_0 = \frac{N!}{(N-\alpha_0)!}\chi^{N-\alpha_0},$$

we see that (6) reduces to

$$f(z) = (2\pi i)^{-n}\int_\Omega f(\zeta)\sum_{|\alpha|=n}\frac{D^\alpha G}{\alpha!}(\bar\partial q)^\alpha \tag{7}$$

$$-(2\pi i)^{-n}\int_\Omega f(\zeta)\sum_{\substack{\alpha_0+|\alpha|=n\\ \alpha_0>0}}\binom{N}{\alpha_0}\alpha_0\chi^{N-\alpha_0}(1-\chi)^{\alpha_0-1}\bar\partial\chi\wedge\frac{s\wedge(\bar\partial s)^{\alpha_0-1}}{\langle\zeta-z, S\rangle^{\alpha_0}}\wedge\frac{D^\alpha G}{\alpha!}(\bar\partial q)^\alpha.$$

All that remains is to see what becomes of (7) as we let χ tend to χ_Ω. First notice that for any continuously differentiable form ψ of bidegree $(n, n-1)$ we have

$$\int_\Omega\bar\partial\chi\wedge\psi = \int_{\partial\Omega}\chi\psi-\int_\Omega\chi\bar\partial\psi.$$

Since $\chi=0$ on $\partial\Omega$ it follows that

$$\int_\Omega\bar\partial\chi\wedge\psi\to-\int_\Omega\chi_\Omega\bar\partial\psi = -\int_{\partial\Omega}\psi.$$

Observe also that we could have used any positive power of χ in this argument.

Now, in (7) we are faced with integrals of the form

(8) $$\int_\Omega \chi^{N-\alpha_0}(1-\chi)^{\alpha_0-1}\bar\partial\chi\wedge\psi.$$

Using the fact that

$$\chi^M\bar\partial\chi = \frac{1}{M+1}\bar\partial\chi^{M+1}$$

and the reasoning above, we find that (8) tends to

$$-C_{N,\alpha_0}\int_{\partial\Omega}\psi,$$

with the constant given by

$$C_{N,\alpha_0} = \int_0^1 t^{N-\alpha_0}(1-t)^{\alpha_0-1}\,dt$$

$$= \frac{\alpha_0-1}{N-\alpha_0+1}\int_0^1 t^{N-\alpha_0+1}(1-t)^{\alpha_0-2}\,dt = \ldots = \left[\binom{N}{\alpha_0}\alpha_0\right]^{-1}.$$

Consequently, (7) becomes

$$f(z) = (2\pi i)^{-n}\int_\Omega f(\zeta)\sum_{|\alpha|=n}\frac{D^\alpha G}{\alpha!}(\bar\partial q)^\alpha$$

$$+(2\pi i)^{-n}\int_{\partial\Omega} f(\zeta)\sum_{\substack{\alpha_0+|\alpha|=n\\ \alpha_0>0}}\frac{s\wedge(\bar\partial s)^{\alpha_0-1}}{\langle\zeta-z, S\rangle^{\alpha_0}}\wedge\frac{D^\alpha G}{\alpha!}(\bar\partial q)^\alpha$$

and the theorem follows.

Remarks. If $Q\equiv 0$, (5) reduces to the classical Cauchy—Fantappiè formula. From our proof it is immediate that the formula is not affected if S is multiplied by a scalar function, one just looks at the expression for Q^0.

It is possible to improve on the above theorems by letting more general holomorphic mappings play the rôle of the coordinate functions $\zeta_j - z_j$. This leads to the following result.

Theorem 3. *Suppose that the function f is holomorphic in some bounded domain $\Omega\subset\mathbf{C}^n$ and continuous up to the boundary. Let there be given*

i) *a holomorphic map $g\colon \bar\Omega\to\mathbf{C}^n$ such that $g^{-1}(0)$ is a finite subset of Ω,*

ii) *continuously differentiable functions*

$$Q^k\colon \bar\Omega\times\bar\Omega\to\mathbf{C}^n,\quad k=1,\ldots,p,$$

iii) *a function G of p complex variables, holomorphic in a neighborhood of the image of $\bar\Omega\times\bar\Omega$ by the mapping $-g(\zeta)Q(\zeta,z)$ and satisfying $G(0)=1$,*

iv) *a smooth map $S\colon \partial\Omega\times\bar\Omega\to\mathbf{C}^n$ such that $\langle g(\zeta), S(\zeta,z)\rangle\neq 0$.*

In that case the following formula holds:

(9) $$\sum_{z\in g^{-1}(0)} m_z(g)f(z) = (2\pi i)^{-n}\int_\Omega f(\zeta)\sum_{|\alpha|=n}\frac{D^\alpha G}{\alpha!}(\bar\partial q(g))^\alpha$$

$$+(2\pi i)^{-n}\int_{\partial\Omega} f(\zeta)\sum_{\alpha_0+|\alpha|=n-1}\frac{s(g)\wedge(\bar\partial s(g))^{\alpha_0}}{\langle g, S\rangle^{\alpha_0+1}}\wedge\frac{D^\alpha G}{\alpha!}(\bar\partial q(g))^\alpha,$$

where $m_z(g)$ denotes the multiplicity of the zero z, $D^\alpha G = D^\alpha G(-g(\zeta)Q(\zeta, z))$, $q^k(g) = \sum Q_j^k\, dg_j$ and $s(g) = \sum S_j\, dg_j$.

Remark. The multiplicity $m_z(g)$ may be defined in several equivalent ways, see e.g. [1, § 2], [8, p. 663].

Proof of Theorem 3. The logarithmic residue current

$$\bar\partial\left[\frac{dg_1}{g_1}\right]\wedge\ldots\wedge\bar\partial\left[\frac{dg_n}{g_n}\right] = (-1)^{(1/2)n(n-1)}\bar\partial\left[\frac{1}{g_1}\right]\wedge\ldots\wedge\bar\partial\left[\frac{1}{g_n}\right]\wedge\omega(g)$$

has the following structure (cf. [6, p. 52]):

(10) $$\frac{(-1)^{(1/2)n(n-1)}}{(2\pi i)^n}\bar\partial\left[\frac{1}{g_1}\right]\wedge\ldots\wedge\bar\partial\left[\frac{1}{g_n}\right](\varphi\omega(g)) = \sum_{z\in g^{-1}(0)} m_z(g)\varphi(z),$$

for any continuous function φ.

With (10) as the starting point instead of (3) the theorem is proved precisely as our previous ones except that $\zeta_j - z_j$ has to be replaced by $g_j(\zeta)$ and $d\zeta_j$ by $dg_j(\zeta)$. The fact that the necessary computational rules still are true is a consequence of g being a complete intersection (see [12]).

Remarks. With the weight factors removed, i.e. $Q=0$, (9) becomes a formula obtained by Roos in [13]. If we also set $S=\bar g$ we arrive at Yuzhakov's generalization of the Bochner—Martinelli formula [15].

2. Some applications

We give here a few concrete examples to show how the above formulas can be used. For further applications see e.g. the references mentioned in the introduction.

Example 1. Let Ω be a strictly pseudoconvex domain with C^{k+2}-boundary and let ϱ be a defining function for Ω. There exist functions $H_1, \ldots, H_n$ in $C^{k+1}(\bar\Omega\times\bar\Omega)$, holomorphic in z and such that

(11) $$2\,\mathrm{Re}\langle H, \zeta - z\rangle \geqq \varrho(\zeta) - \varrho(z) + \delta|\zeta - z|^2,$$

for some $\delta > 0$, see [7].

If Ω is strictly convex we take

$$H_j(\zeta, z) = (\partial\varrho/\partial\zeta_j)(\zeta).$$

Choosing $S=H$ in (5) and Q holomorphic in z, we obtain a representation formula with holomorphic kernel.

For instance, if Ω is the unit ball $B=\{|\zeta|^2-1<0\}$ and $Q=0$ one gets the familiar Szegő representation

$$f(z) = (2\pi i)^{-n}\int_{\partial B}\frac{f(\zeta)\partial|\zeta|^2\wedge(\bar\partial\partial|\zeta|^2)^{n-1}}{(1-\bar\zeta\cdot z)^n} = \frac{(n-1)!}{2\pi^n}\int_{\partial B}\frac{f(\zeta)\,d\sigma(\zeta)}{(1-\bar\zeta\cdot z)^n}.$$

In general, if $Q=0$, Theorem 2 gives integrals over $\partial\Omega$. If one instead applies Theorem 1 as in the proof of Theorem 2, without taking limits, the representation occurs as an integral over a neighborhood of $\partial\Omega$, i.e. a kind of thickened boundary integral but still with holomorphic kernel.

Now we put $G_0(t_0)=(t_0+1)^{-r}$, $r>0$, $Q_j^0=\dfrac{H_j(\zeta, z)}{\varrho(\zeta)-\varepsilon}$ and G_0G instead of G in (5). When $\varepsilon\to 0$ the boundary integral vanishes and if we set $h=\sum H_j\,d\zeta_j$ we obtain the weighted formula

$$f(z) = (2\pi i)^{-n}\int_\Omega f(\zeta)\sum\nolimits_{\alpha_0+|\alpha|=n} C_{\alpha_0,r}\frac{(-\varrho)^{r-1}}{(\langle H, \zeta-z\rangle-\varrho)^{r+\alpha_0}} \tag{12}$$

$$\wedge(\varrho\bar\partial h-\alpha_0\bar\partial\varrho\wedge h)\wedge(\bar\partial h)^{\alpha_0-1}\wedge\frac{D^\alpha G}{\alpha!}(\bar\partial q)^\alpha,$$

where

$$C_{\alpha_0,r} = \frac{-r(r+1)\dots(r+\alpha_0-1)}{\alpha_0!}.$$

Notice that, by virtue of (11), f may be allowed to grow somewhat near $\partial\Omega$.

If $\Omega=B$ and $Q=0$, (12) is the Bergman integral representation with respect to the weight $(1-|\zeta|^2)^{r-1}$.

As r approaches zero (12) tends to (5) (with $S=H$) and in Example 3 we shall see what happens when $r\to\infty$.

Particular choices of G and Q may be used to obtain weighted solution formulas for certain division problems, cf. [3], [10].

Example 2. Although Theorem 1 (and 2) does not give solution formulas for the $\bar\partial$-equation in general we can easily obtain a formula for the boundary values of a solution to $\bar\partial u=w$, w being a $\bar\partial$-closed (0, 1)-form (or current) in a strictly pseudoconvex domain Ω.

To this end (using the same notation as in Example 1) we first take $g\in C^1(\bar\Omega)$

and define

$$Pg(z) = \frac{C_{n,r}}{(2\pi i)^n}\int_\Omega g(\zeta)\frac{(-\varrho)^{r-1}}{(\langle H, \zeta-z\rangle-\varrho)^{r+n}}(\varrho\bar\partial h - n\bar\partial\varrho\wedge h)\wedge(\bar\partial h)^n.$$

According to (12), $Pg=g$ if g is holomorphic.

Now put $M_j(\zeta, z)=-H_j(z,\zeta)$ and $Q_j=(g(z)-g(\zeta))\dfrac{M_j}{\langle M,\zeta-z\rangle-\varrho(z)}$. Then Q_j is smooth for $z\in\Omega$ and, since M_j is holomorphic in ζ, we have

$$\bar\partial q = \frac{m}{\langle M, \zeta-z\rangle-\varrho(z)}\wedge\bar\partial g, \quad \text{where} \quad m = \sum M_j d\zeta_j.$$

We next observe that if $G(0)=c$ then all our formulas get multiplied by c. In particular, if $f=1$ and $G(t)=t+g(z)$ in (12), then the resulting integral equals $g(z)$. Letting $z\to\partial\Omega$ we get in fact that (12) becomes

$$g(z) = Pg(z)+K\bar\partial g(z),$$

where

(13)
$$K\bar\partial g(z) =$$

$$= \frac{C_{n-1,r}}{(2\pi i)^n}\int_\Omega\frac{(-\varrho)^{r-1}}{(\langle H,\zeta-z\rangle-\varrho)^{r+n-1}}\frac{m}{\langle M,\zeta-z\rangle}(\varrho\bar\partial h-(n-1)\bar\partial\varrho\wedge h)\wedge(\bar\partial h)^{n-2}\wedge\bar\partial g.$$

If w is any $\bar\partial$-closed smooth (0, 1)-form it follows that $\bar\partial Kw=w$. Using the explicit expression (13) and an appropriate choice of r it is now easy to obtain the L^1-estimates on $\partial\Omega$, originally given by Henkin [9] and Skoda [14].

Example 3. Here we consider the case $\Omega=\mathbf{C}^n$. Let us assume that the function ϱ is strictly convex in $\mathbf{C}^n$ and that $D^\alpha G((z-\zeta)Q)$ is defined on $\mathbf{C}^n\times\mathbf{C}^n$. Applying formula (12) with $H_j=\dfrac{\partial\varrho}{\partial\zeta_j}$ in $\Omega=\{\varrho-r<0\}$ we get after an easy rewriting

(14)
$$f(z) = (2\pi i)^{-n}\int_{\varrho<r}\sum\nolimits_{\alpha_0+|\alpha|=n}\frac{-C_{\alpha_0,r}}{(r-\varrho)^{\alpha_0}}\left(1-\frac{\left\langle\frac{\partial\varrho}{\partial\zeta}, z-\zeta\right\rangle}{r-\varrho}\right)^{-r-\alpha_0}$$
$$\left[(\bar\partial\partial\varrho)^{\alpha_0}+\alpha_0\frac{\bar\partial\varrho\wedge\partial\varrho\wedge(\bar\partial\partial\varrho)^{\alpha_0-1}}{r-\varrho}\right]\wedge\frac{D^\alpha G}{\alpha!}(\bar\partial q)^\alpha.$$

Recalling that $-C_{\alpha_0,r}=\dfrac{r^{\alpha_0}}{\alpha_0!}+0(r^{\alpha_0-1})$ and letting $r\to\infty$ one obtains

(15)
$$f(z) = (2\pi i)^{-n}\int_{\mathbf{C}^n}f(\zeta)\exp\left(\left\langle\frac{\partial\varrho}{\partial\zeta}, z-\zeta\right\rangle\right)\sum\nolimits_{\alpha_0+|\alpha|=n}\frac{(\bar\partial\partial\varrho)^{\alpha_0}}{\alpha_0!}\wedge\frac{D^\alpha G}{\alpha!}(\bar\partial q)^\alpha.$$

Evidently we have to restrict the rate of growth of the entire function f. In view of the strict convexity of ϱ it is enough to have $|f(\zeta)|\leqq\text{const}\exp(\varrho(\zeta)/2)$. Of course formula (15) can be derived directly from Theorem 1 by choosing $G_0(t_0)=\exp t_0$, but we wanted to emphasize the connection between the representations (12) and (15).

Example 4. We conclude by presenting formulas for vector-valued functions. They come out easily by the technique of this paper, whereas it is not clear how to obtain them by the methods of [4].

Let r be an integer, f an r-column of holomorphic functions and $Q_1^0, \dots, Q_n^0$, $r\times r$-matrices of functions in $C^1(\bar\Omega\times\bar\Omega)$.

We also choose $G_0(t_0)=(t_0+1)^m$, $m\in\mathbf{N}$, and to simplify notations we put $A=(z-\zeta)Q^0+1$. In the scalar case Theorem 1 gives

$$f(z) = (2\pi i)^{-n}\int_\Omega f(\zeta)\sum\nolimits_{\alpha_0+|\alpha|=n}\binom{m}{\alpha_0}A^{m-\alpha_0}(\bar\partial q^0)^{\alpha_0}\wedge\frac{D^\alpha G}{\alpha!}(\bar\partial q)^\alpha. \tag{16}$$

Now, if $r>1$ and Q_j^0 are diagonal matrices, the same formula holds and it is obtained just by applying the usual one componentwise. For arbitrary Q_j^0 a similar formula holds but in every term of (16) the factor

$$\binom{m}{\alpha_0}A^{m-\alpha_0}(\bar\partial q^0)^{\alpha_0}$$

must be replaced by

$$\sum\nolimits_{|\beta|=\alpha_0}(\bar\partial q^0)^{\beta_0}A\wedge(\bar\partial q^0)^{\beta_1}A\wedge\dots\wedge(\bar\partial q^0)^{\beta_{m-\alpha_0-1}}A\wedge(\bar\partial q^0)^{\beta_{m-\alpha_0}}.$$

The proof is essentially a repetition of the proof of Theorem 1, but since matrices do not commute in general, each term occurs together with all permutations of its factors.

References

1. AÏZENBERG, L. A. and YUZHAKOV, A. P., *Integral'nye predstavlenija i vyčety v mnogomernom kompleksnom analyse*; "Nauka", Sibirsk. Otdel., Novosibirsk, 1979. English translation: *Integral representations and residues in multidimensional complex variables.* (Translation of Math. monographs 58), Am. Math. Soc., Providence, 1983.
2. BERENSTEIN, C. A. and YGER, A., Le problème de la déconvolution, *J. Funct. Anal.* **54** (1983), 113—160.
3. BERNDTSSON, B., A formula for interpolation and division in $\mathbf{C}^n$, *Math. Ann.* **263** (1983), 399—418.
4. BERNDTSSON, B. and ANDERSSON, M., Henkin—Ramirez formulas with weight factors, *Ann. Inst. Fourier* **32** (1982), 91—110.
5. BONNEAU, P., CUMENGE, A. and ZÉRIAHI, A., Division dans les espaces de Lipschitz de fonctions holomorphes, *C. R. Acad. Sci. Paris* **297** (1983), 517—520.
6. COLEFF, N. R. and HERRERA, M. E., *Les courants résiduels associés à une forme méromorphe,* Lecture Notes in Math. 633, Springer, Berlin, 1978.

7. FORNAESS, J. E., Embedding strictly pseudoconvex domains in convex domains, *Amer. J. Math.* **98** (1976), 529—569.
8. GRIFFITHS, P. A. and HARRIS, J., *Principles of algebraic geometry.* Pure and Applied Math. Wiley-Interscience [John Wiley & Sons], New York. 1978.
9. HENKIN, G. M., Solutions with estimates of the H. Lewy and Poincaré—Lelong equations. The construction of functions of a Nevanlinna class with given zeros in a strictly pseudoconvex domain, *Soviet Math. Dokl.* **16** (1975), 1310—1314.
10. PASSARE (PETTERSSON), M., *Residues, currents, and their relation to ideals of holomorphic functions,* thesis, Uppsala, 1984. (To appear in Math. Scand.)
11. PASSARE M., Produits des courants résiduels et règle de Leibniz, *C. R. Acad. Sci. Paris* **301** (1985), 727—730.
12. PASSARE, M., A calculus for meromorphic currents, to appear.
13. ROOS, G., L'intégrale de Cauchy dans $\mathbf{C}^n$. In: *Fonctions de plusieurs variables complexes* (Sém. François Norguet, 1970—1973; à la mémoire d'André Martineau), pp. 176—195, Lecture Notes in Math. 409, Springer, Berlin, 1974.
14. SKODA, H., Valeurs au bord pour les solutions de l'opérateur d'', et caractérisation des zéros des fonctions de la classe de Nevanlinna, *Bull. Soc. Math. France* **104** (1976), 225—299.
15. YUZHAKOV, A. P. and KUPRIKOV, A. V., O logarifmičeskom vyčete v $\mathbf{C}^n$. In: *Nekotorye svojstva golomorfnych funkcij mnogich kompleksnych peremennych,* pp. 181—191, Inst. Fiz. Sibirsk. Otdel. Akad. Nauk SSSR, Krasnoyarsk, 1973.

Received July 15, 1986

Mats Andersson
Matematiska institutionen
Chalmers tekniska högskola
och Göteborgs universitet
S-412 96 Göteborg
Sweden

Mikael Passare
Matematiska institutionen
Stockholms universitet
Box 6701
S-113 85 Stokholm
Sweden