Eigenfunction expansions for the Schrödinger operator

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Abstract

We obtain an eigenfunction expansion for the operator $-\Delta+V$ under assumptions (1.2)—(1.5) given below.

1. Introduction

The purpose of this note is to obtain an eigenfunction expansion for the operator

$$H' = -\Delta + V(x) \tag{1.1}$$

under conditions on V which are weaker than previously allowed (Δ is the Laplacian in E^n and $V(x)$ is a real valued function). Before stating our assumptions we introduce some notation. Put

$$\begin{aligned} \omega_\beta(x) &= |x|^{\beta-n}, & 0<\beta<n \\ &= -\log|x|, & \beta = n \\ &= 1, & \beta > n, \end{aligned}$$

$$M_{\beta,p,\delta,x}(V) = \int_{|x-y|<\delta} |V(y)|^p \omega_\beta(x-y)\,dy$$

$$M_{\beta,p,\delta}(V) = \sup_x M_{\beta,p,\delta,x}(V),\ M_{\beta,p}(V) = M_{\beta,p,1}(V),\ \hat{V}(x) = M_{n+1,1,1,x}(V).$$

Let $M_{\beta,p}$ be the set of those g such that $M_{\beta,p}(g)<\infty$, and let $M^{\Omega}_{\beta,p}$ be the set of those g such that $\chi_\Omega g \in M_{\beta,p}$, where χ_Ω is the characteristic function of the set $\Omega \subset E^n$. Our assumptions are

$$M_{2,1,\delta}(V) \to 0 \quad \text{as} \quad \delta \to 0. \tag{1.2}$$

$$\hat{V}(x) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{1.3}$$

(1.4) For each bounded Ω there is a $\beta<4$ such that $V\in M^{\Omega}_{\beta,2}$.

(1.5) $$\varrho^{\alpha}\hat{V}\in L^p = L^p(E^n), \quad \varrho(x) = 1+|x|,$$

for some $\alpha\geqq 0$, $1\leqq p\leqq\infty$ such that

(1.6) $$\alpha > 1-(n/2p).$$

If $2n/(n+1)\leqq p\leqq\frac{1}{2}n$, we require $\alpha>0$.

Theorem 1.1. *Assume that $V(x)$ satisfies* (1.2)—(1.5). *Then there are a self adjoint realization H_1 of H', a closed set $e\subset E^1$ of measure* 0 *and two families $\Phi_{\pm}(x, s, w)$ defined on $E^n\times(E^1\setminus e)\times\Sigma$, where Σ is the unit sphere $|\xi|=1$ in E^n, with the following properties:*

(a) $\Phi_{\pm}\in L^2_{\text{loc}}(E^n\times(E^1\setminus e)\times\Sigma)$.

(b) *For fixed x, s, $\Phi_{\pm}\in\mathscr{C}=L^2(\Sigma)$.*

(c) *For fixed s, ω, $\Phi_{\pm}$ is locally Hölder continuous in x and has derivatives up to order* 2 *in L^2_{loc}. It is a solution of*

(1.7) $$(s-H')u(x) = 0.$$

(d) *If $V\in L^1$, then $\Phi_{\pm}$ is locally Hölder continuous in x, s, ω together.*

(e) *If $h\in\mathscr{C}$ and*

(1.8) $$\Phi_{\pm}h = \int h(\omega)\Phi_{\pm}(x, s, \omega)\,d\omega,$$

then $\Phi_{\pm}$ is locally Hölder continuous in x, s and is a solution of (1.7).

(f) *If $\{E_1(\lambda)\}$ is the spectral family of H_1, then the limits*

(1.9) $$F_{\pm}u(s, \omega) = \lim_{R\to\infty}\int_{|x|<R} u(x)\Phi_{\pm}(x, s, \omega)\,dx,$$

exist for each $u\in L^2$ and

(1.10) $$\|F_{\pm}u\|_{L^2(I,\mathscr{C})} = \|E_1(I)u\|$$

for each closed Borel set I which does not intersect e. Moreover, for each such I, $F_{\pm}$ maps $E_1(I)L^2$ onto $L^2(I, \mathscr{C})$.

(g) *Let P_{ac} be the projection onto L^2_{ac}, the subspace of absolute continuity of H_1* (cf. [17] p. 516). *Then*

(1.11) $$P_{ac}u(x) = \lim_{R\to\infty}\int_{|s|<R} F_{\pm}u(s, \omega)\overline{\Phi_{\pm}(x, s, \omega)}\,ds\,d\omega$$

and

(1.12) $$\|P_{ac}u\| = \|F_{\pm}u\|_{L^2(e', \mathscr{C})},$$

where $e'=E^1\setminus e$. The $F_{\pm}$ are surjective.

(h) *If $W_{\pm}$ are the wave operators given by*

$$W_{\pm}u = \lim_{t\to\pm\infty} e^{itH_1}e^{-itH}u,$$

where H is the self adjoint realization of $-\Delta$ (the wave operators are known to exist [18]*), then*

(1.13) $$F_\pm W_\pm = F,$$

where F is the Fourier transform.

(i) *If $f(s)$ is any Borel function on E^1, then*

(1.14) $$f(s)F_\pm = F_\pm f(H_1).$$

(j) *If α, p satisfy*

(1.15) $$\alpha > 1-[2n/(n+1)p],$$

then e is a discrete set having no accumulation point except possibly 0 *or* ∞.

The proof of Theorem 1.1 will be given in Section 3 using several lemmas proved in the next section. The methods of this paper can be used to obtain the same results for operators of the form

(1.16) $$H'' = P(D)+V(x),$$

where $P(D)$ is a constant coefficient elliptic operator of order m. In this case (1.2) can be weakened to

(1.17) $$M_{m,1,\delta}(V) \to 0 \quad \text{as} \quad \delta \to 0,$$

and (1.4) need only be assumed for $\beta < 2m$. In the case when (1.15) holds there may be a finite number of points which are accumulation points of eigenvalues of H'. These are the points λ for which there exist $\xi \in E^n$ such that grad $P(\xi)=0$ and $P(\xi)=\lambda$. The proof of these results follow mutatis mutandis the proof of Theorem 1.1. Similar results can be obtained for operators of the form

(1.18) $$H''' = P(D)+Q(x, D),$$

where $Q(x, D)$ is an operator of order $\leqq m$ with variable coefficients.

Eigenfunction expansions applied to Schrödinger operators seem to have originated with the work of Povzner [2] and Ikebe [1] and have been applied to various aspects of scattering theory by many authors (cf. [1—16, 22, 23] and the references quoted in them). The approach we have taken is that of Agmon [10] with several changes. Rather than working with the $L^{2,s}$ spaces as he does, we prefer to factor the perturbation (in fact, Agmon's use of these spaces constitutes a particular type of factorization which we show is not optimal). Our assumptions are too weak for essential self adjointness, and we are forced to use a forms extension ([21, 22]).

Another feature of our approach is that the existence of the eigenfunctions is established via the scattering theory techniques of [19] together with straightforward local estimates of elliptic partial differential operators. To obtain the refined smoothness of the eigenfunctions we use Agmon's method.

Agmon's results [10] were obtained under the assumption

$$\varrho(x)^{\alpha} M_{\beta,2,1,x}(V) \in L^{\infty} \tag{1.19}$$

for some $\alpha > 2$ and $\beta < 4$. Although (1.19) is significantly stronger than (1.2)—(1.5), it is much weaker than the assumptions of previous authors. This indicates the strength of his approach. We would like to thank S. Agmon for making his results available to us before publication and for some very useful conversations.

In Theorem 1.1, assumption (1.2) was used to guarantee the existence of a forms extension. It can be weakened as was shown in [21]. Assumptions (1.3) and (1.5) can be replaced by any one of five other sets of conditions used in [18]. March 10, 1975.

Notation. $[A]$, A^*, $D(A)$, $R(A)$ will denote the closure, adjoint, domain and range of an operator A. $\Omega \subset\subset \Omega_1$ means that the closure $\bar{\Omega}$ of Ω is compact and contained in Ω_1. C_0^{∞} is the set of test functions (infinitely differentiable with compact supports). $\|f\|_p^{\Omega}$ is the $L^p(\Omega)$ norm of f and $\|f\|$ is its norm in $L^2 = L^2(E^n)$. $L^2(I, \mathscr{C})$ is the set of $\mathscr{C}$ valued functions which are square integrable over I. $B(\mathscr{C})$ denotes the set of bounded operators on $\mathscr{C}$. $H^k(\Omega)$ denotes the set of functions having all derivatives up to order k in $L^2(\Omega)$.

2. Some lemmas

We let H be the closure in L^2 of $-\Delta$ on C_0^{∞}, and we put $R(z) = (z-H)^{-1}$. We let $\{E(s)\}$ be the spectral family of H.

Lemma 2.1. *Suppose the functions $A(x)$, $B(x)$ satisfy*

$$M_{2,2,\delta}(|A|+|B|) \to 0 \quad \text{as} \quad \delta \to 0 \tag{2.1}$$

and there are constants α, p satisfying (1.6) *such that*

$$\varrho(x)^{\alpha} \int_{|x-y|<1} (|A(y)|^2 + |B(y)|^2)\, dy \in L^p, \tag{2.2}$$

then the operators $Q(z) = [AR(z)B]$ and $G(z) = I - Q(z)$ are in $B(L^2)$ for nonreal z and $G(s \pm ia) \to G_{\pm}(s)$ in norm as $0 < a \to 0$. The limits are Hölder continuous functions of s. If in addition

$$\int_{|x-y|<1} (|A(y)|^2 + |B(y)|^2)\, dy \to 0 \quad \text{as} \quad |x| \to \infty, \tag{2.3}$$

then there is a closed set $e \subset E^1$ of measure 0 such that $G_{\pm}(s)$ has a bounded inverse $G_{1\pm}(s)$ for $s \notin e$. If α, p satisfy (1.15) *then the set e is discrete having only 0 or ∞ as possible accumulation points.*

Lemma 2.2. *If A, B satisfy* (2.1), (2.2), *there is a Hölder continuous map $N(s)$ from the positive reals to $B(L^2)$ such that*

(2.4) $$d(E(s)Au, Bv)/ds = (N(s)u, v) \quad \text{a.e.,} \quad u, v \in C_0^\infty.$$

In particular, there is such a map $M(s)$ satisfying

(2.5) $$d(E(s)Au, Av)/ds = (M(s)u, v), \quad \text{a.e.,} \quad u, v \in C_0^\infty.$$

Let Σ denote the sphere $|\xi|=1$ in E^n, put $\mathscr{C}=L^2(\Sigma)$ and

(2.6) $$T_s(u)\omega = 2^{-1/2}s^{(n-2)/4}F(Au)(s^{1/2}\omega), \quad \omega \in \Sigma,$$

where F is the Fourier transform. Then

(2.7) $$(M(s)u, v) = (T_s u, T_s v)_{\mathscr{C}} \quad \text{a.e.,} \quad u, v \in C_0^\infty.$$

Lemma 2.3. *If $V(x)$ satisfies* (1.2), *then there is a self adjoint operator H_1 on L^2 such that $D(|H_1|^{1/2})=D(|H|^{1/2})=D$ and*

(2.8) $$(H_1 u, v) = (Hu, v) - (Vu, v), u, v \in D.$$

If $V=AB$, where A, B satisfy (2.1)—(2.3), *then*

(2.9) $$d(E_1(s)Au, Av)/ds = (T_{\pm s}u, T_{\pm s}v)_{\mathscr{C}},$$

where $T_{\pm s}=T_s G_{1\pm}(s)^*$ and $\{E_1(s)\}$ denotes the spectral family of H_1.

The proofs of Lemmas 2.1—2.3 can be found in [18—20].

Put $Q_\pm(s)=\lim_{0<a\to 0} Q(s\pm ia)$. We have

Lemma 2.4. *If A^{-1} is locally bounded and $f=A^{-1}Q_\pm(\lambda)g$, then f is a weak solution of*

(2.10) $$(\lambda - H)f = Bg.$$

Proof. *If $v\in C_0^\infty$* and $z = \lambda \pm ia$,

$$(f, (\lambda-H)v) = \lim([R(z)B]g, (z-H)v) = (g, Bv). \qquad \blacksquare$$

Lemma 2.5. *Let Ω be a bounded domain, and suppose $g\in L^p(\Omega)$ for some $p\geqq 2$ and $B\in M^\Omega_{\beta,2}$ for some $\beta<4$. If f is a weak solution of* (2.10), *then $f\in L^q(\Omega_1)$ for each $\Omega_1 \subset\subset \Omega$ and each q such that $1/p-1/q<(4-\beta)/2n$. Moreover*

(2.11) $$\|f\|_q^{\Omega_1} \leqq C(\|f\|_2^\Omega + M^\Omega_{\beta,2}(B)^{1/2}\|g\|_p^\Omega),$$

where the constant depends only on p, q, β, Ω, Ω_1.

The proof of Lemma 2.5 can be found in [10, Appendix C]. If a function u is Hölder continuous in Ω with exponent θ, put

$$\|u\|_{C^\theta(\Omega)} = \|u\|_\infty^\Omega + \sup_{x,y\in\Omega} |x-y|^{-\theta}|u(x)-u(y)|.$$

The following lemma is also given in [10].

Lemma 2.6. *If $B\in M^{\Omega}_{\beta,2}$ for some $\beta<4$ and f is a weak solution of $(\lambda-H)f=B$, then f is locally Hölder continuous in Ω with exponent $\theta>0$, and for $\Omega_1\subset\subset\Omega$*

(2.12) $$\|f\|_{C^{\theta}(\Omega_1)}\leqq C(\|f\|^{\Omega}+M^{\Omega}_{\beta,2}(B)^{1/2}),$$

where the constant depends only on β, Ω, Ω_1.

Lemma 2.7. *Assume A, A^{-1}, v are bounded in Ω, $B\in M^{\Omega}_{\beta,2}$ for some $\beta<4$, $u\in L^2$ and that*

(2.13) $$G_{\pm}(\lambda)u=v$$

in Ω. Then u is locally bounded in Ω and for $\Omega_1\subset\subset\Omega$

(2.14) $$\|u\|^{\Omega_1}_{\infty}\leqq C(\|u\|+\|v\|^{\Omega}_{\infty})$$

where the constant depends only on A, B, β, Ω, Ω_1.

Proof. Let Ω_2 be such that $\Omega_1\subset\subset\Omega_2\subset\subset\Omega$, and put $f=A^{-1}(v-u)$. By Lemma 2.4, f is a weak solution of (2.10) with $g=u$. By Lemma 2.5, $f\in L^q(\Omega_2)$ for any q such that $1/2-1/q<(4-\beta)/2n$. Thus

$$\|u\|^{\Omega_2}_{q}\leqq C(\|u\|+\|v\|^{\Omega}_{\infty}).$$

Insert a domain Ω_3 between Ω_1 and Ω_2 and reapply Lemma 2.5. This gives

$$\|u\|^{\Omega_3}_{r}\leqq C(\|u\|+\|v\|^{\Omega}_{\infty}),$$

where $1/q-1/r<(4-\beta)/2n$. In a finite number of steps we obtain (2.14). ■

Lemma 2.8. *Under the same hypotheses, if $A^{-1}v$ is Hölder continuous in Ω, then the same is true of $A^{-1}u$.*

Proof. By Lemma 2.7, u is locally bounded in Ω. In view of Lemmas 2.4 and 2.6, $f=A^{-1}(v-u)$ is Hölder continuous in Ω. ■

Lemma 2.9. *Assume A, A^{-1}, $v\in L^{\infty}(\Omega)$ and that $f\in L^2(\Omega)$ is a weak solution of*

(2.15) $$(\lambda-H')f=v$$

in Ω. If $B\in M^{\Omega}_{\beta,2}$ for some $\beta<4$, then for each $\Omega_1\subset\subset\Omega$

(2.16) $$\|f\|_{C^{\theta}(\Omega_1)}\leqq C(\|f\|^{\Omega}+\|v\|^{\Omega}_{\infty}).$$

Proof. Since $v\in L^{\infty}(\Omega)$, there is an $f_1\in C^{\theta}_{\text{loc}}(\Omega)$ such that $(\lambda-H)f_1=v$ and for each $\Omega_2\subset\subset\Omega$

(2.17) $$\|f_1\|_{C^{\theta}(\Omega_2)}\leqq C\|v\|^{\Omega}_{\infty}.$$

Put $u=f-f_1$. Then u is a weak solution of (2.10) with $g=Af\in L^2(\Omega)$. Applying Lemma 2.5, we obtain

$$\|u\|_q^{\Omega_2} \leqq C(\|f\|^{\Omega} + \|v\|_\infty^{\Omega}) \tag{2.18}$$

for q as given there. Thus a similar estimate holds for $\|f\|_q^{\Omega_2}$. Reapplying Lemma 2.5 as many times as necessary, we finally get such an estimate for $\|f\|_\infty^{\Omega_1}$. Now we go back to (2.15) and apply Lemma 2.6. ■

3. The proof

Our proof of Theorem 1.1 is based upon the lemmas of Section 2. Put

$$\tau(x)^2 = e^{2|x|^2}\big(\sup_{|x-y|<1} M_{\beta,2,1,y}(V)+1\big),$$

where β, depending on x, is given in (1.4). We define

$$\begin{aligned} A(x) &= \tau(x), & \tau(x)^2 &< |V(x)| \\ &= |V(x)|^{1/2}, & \tau(x)^{-2} &< |V(x)| \leqq \tau(x)^2 \\ &= 1/\tau(x), & |V(x)| &\leqq \tau(x)^{-2}, \end{aligned}$$

and $B(x)=V(x)/A(x)$. From these definitions it follows that A, A^{-1} are in L^∞_{loc} and that A, B satisfy (2.1)—(2.3) with α, p the same as for V. Let H_1 be the operator described in Lemma 2.3 and let $T_\pm=T_{\pm s}$ be the operators defined there. Let Z be the set of those $\xi\in E^n$ such that $|\xi|^2\in e$, the closed set of measure 0 described in Lemma 2.1, and let e' denote the complement of e in E^1. For $s\in I\subset\subset e'$, Lemmas 2.1 and 2.2 show that there is a constant depending only on I such that

$$\|T_\pm u\|_{\mathscr{C}} \leqq C_I\|u\|. \tag{3.1}$$

For $h\in\mathscr{C}$, put $U_\pm h=T_\pm^* h$. Consequently we have

$$\|U_\pm h\| \leqq C_I\|h\|_{\mathscr{C}}. \tag{3.2}$$

Moreover we have for $\varphi\in C_0^\infty$

$$\big(G_\pm(s)U_\pm h,\ \varphi\big) = (h,\ T_s\varphi)_{\mathscr{C}} = C_s(w^h,\ A\varphi), \tag{3.3}$$

where $C_s=2^{-1/2}s^{(n-2)/4}$ and

$$w^h(x,s) = \int_\Sigma h(\omega)e^{ix\omega s^{1/2}}d\omega. \tag{3.4}$$

Note that w^h is a bounded solution of $(s-H)u=0$. By (3.3) we have

$$G_\pm(s)U_\pm h = C_sAw^h, \tag{3.5}$$

and if we put $\Phi_\pm h = A^{-1}U_\pm h$, we have

(3.6) $$\Phi_\pm h = C_s w^h - A^{-1}Q_\pm(s)U_\pm h.$$

Applying Lemma 2.7 to (3.5), we see that for each bounded Ω

(3.7) $$\|U_\pm h\|_\infty^\Omega \leqq C_{I,\Omega}\|h\|_{\mathscr{C}}$$

Thus for each x, s, $U_\pm h$ is a bounded linear functional on $\mathscr{C}$. Consequently there is a function $U_\pm(x, s, \omega)$ such that

(3.8) $$U_\pm h(x, s) = \int_\Sigma h(\omega)U_\pm(x, s, \omega)\,d\omega$$

and

(3.9) $$\|U_\pm(x, s, \cdot)\|_{\mathscr{C}} = \sup_h |U_\pm h(x, s)|/\|h\|_{\mathscr{C}}.$$

We shall show that $U_\pm(x, s, \omega)\in L^2_{\text{loc}}(E^n\times\Sigma)$ for each $s>0$. Assuming this for the moment, we note that for $\varphi\in C_0^\infty$

$$(h, T_\pm\varphi)_{\mathscr{C}} = (U_\pm h, \varphi) = \iint_\Sigma h(\omega)\overline{U_\pm(x, s, \omega)\varphi(x)}\,d\omega\,dy.$$

Hence

(3.10) $$T_\pm\varphi = \int\varphi(x)U_\pm(x, s, \omega)\,dx.$$

Put

(3.11) $$F_\pm\varphi = T_\pm A^{-1}\varphi = \int\varphi(x)\Phi_\pm(x, s, \omega)\,dx.$$

By (2.9)

(3.12) $$(F_\pm f, F_\pm g)_{L^2(I,\mathscr{C})} = (E_1(I)f, g).$$

By (3.6) and Lemma 2.4, $\Phi_\pm h$ is a weak solution of (2.10) with $g = U_\pm h$. Consequently it is a weak solution of (1.7).

Now we turn to the regularity properties of the $U_\pm(x, s, \omega)$. We suppress the symbol $\pm$ (in U only). Assume $g\in L^2\cap L^\infty(\Omega)$ satisfies the hypotheses of A. Put

$$X(x, \xi) = C_s\{G_1(s)(g(\cdot)e^{-(i\xi,\cdot)})\}(x), \quad Y(x, \xi) = A^{-1}(x)X(x, \xi)$$

where $s = |\xi|^2\in I\subset\subset e'$. Thus

(3.13) $$\|X(\cdot, \xi)\| \leqq C_I\|g\|.$$

By Lemma 2.7 we have

(3.14) $$\|X(\cdot, \xi)\|_\infty^{\Omega_1} \leqq C(\|g\| + \|g\|_\infty^\Omega).$$

Since $G(s)$ is Hölder continuous in s (Lemma 2.1),

(3.15) $$\|X(\cdot, \xi) - X(\cdot, \eta)\| \leqq C|\xi-\eta|^\theta\|g\|,$$

for some $\theta>0$. Assume that $A^{-1}g$ is constant in Ω. Then by Lemma 2.4, $Y(x, \xi) - Y(x, \eta)$ is a weak solution of (2.15) with $v = (s-|\eta|^2)\Phi(x, \eta)$. By Lemma 2.9, (3.14) and (3.15)

(3.16) $$\|Y(\cdot, \xi) - Y(\cdot, \eta)\|_{C^\theta(\Omega_1)} \leqq C|\xi-\eta|^\theta(\|g\| + \|g\|_\infty^\Omega)$$

for $\Omega_1 \subset\subset \Omega$. This shows that $Y(x, \xi)$ is Hölder continuous in x and ξ in $\Omega \times \Sigma_I$, where $\Sigma_I = \{\xi \mid |\xi|^2 \in I\}$. Put

$$Xh = \int_\Sigma h(\omega) X(x, \xi)\, d\omega, \quad Yh = A^{-1} Xh, \quad \omega = \xi/|\xi|.$$

Then one checks easily that

$$G(s)Xh = c_s g w^h. \tag{3.17}$$

Thus

$$(Xh, \varphi) = (h, F[gG_1(s)\varphi]). \tag{3.18}$$

By Lemma 2.2

$$\|F[gv]\|_{\mathscr{C}}^2 = (M_1(s)v, v),$$

where $M_1(s)$ is a Hölder continuous map from the positive reals to $B(L^2)$. Thus by Lemma 2.2 (cf. [19])

$$\|Xh\| \leqq c_I \|h\|_{\mathscr{C}} \|g\tilde{\|}, \quad \|g\tilde{\|} = \|\varrho^\alpha \widehat{g^2}\|_p^{1/2}. \tag{3.19}$$

In view of Lemma 2.7, (3.17) and (3.19) imply

$$\|Xh\|_\infty^{\Omega_1} \leqq C \|h\|_{\mathscr{C}} (\|g\tilde{\|} + \|g\|_\infty^\Omega). \tag{3.20}$$

This gives

$$\|X(x, \cdot)\|_{\mathscr{C}} \leqq C(\|g\tilde{\|} + \|g\|_\infty^\Omega), \quad x \in \Omega_1. \tag{3.21}$$

For $n>0$ put $g_n = A$ in $|x| \leqq n$, and let it vanish outside. Put $X_n(x, \xi) = c_s\{G_1(s)(g_n(\cdot)e^{i(\xi, \cdot)})\}(x)$ with $X_n h$ defined similarly. For Ω bounded and n sufficiently large, $A^{-1} g_n$ is constant in Ω. Thus by (3.21)

$$\|X_n(x, \cdot) - X_m(x, \cdot)\|_{\mathscr{C}} \leqq \|g_n - g_m\tilde{\|} \to 0 \tag{3.22}$$

uniformly in Ω_1. Thus

$$\int_{\Omega_1} \int_\Sigma |X_n(x, \xi) - X_m(x, \xi)|^2 d\omega\, dx \to 0. \tag{3.23}$$

By (3.5), (3.17), (3.19) and Lemma 2.7

$$\|X_n h - Uh\|_\infty^{\Omega_1} \to 0, \quad h \in \mathscr{C},$$

which implies

$$([X_n(x, \cdot) - U(x, \cdot)], h)_{\mathscr{C}} \to 0, \quad x \in \Omega_1.$$

Thus $X_n(x, \xi) \to U(x, w, \omega)$, $Y_n(x, \xi) \to \Phi(x, s, \omega)$ in $L^2(\Omega_1 \times \Sigma)$. In particular for a.e. ξ these converge in $L^2(\Omega_1)$. Since Y_n is a weak solution of (2.15) with $v=0$, we have by Lemma 2.7

$$\|Y_n(\cdot, \xi)\|_{C^\theta(\Omega_2)} \leqq C \|Y_n(\cdot, \xi)\|^{\Omega_1} \leqq C_\xi$$

for each $\Omega_2 \subset\subset \Omega_1$, with $C_\xi < \infty$ for a.e. ξ. This shows that there is a subsequence converging in a Hölder norm to $\Phi(x, s, \omega)$. Interior L^2 estimates for elliptic equations yield

$$\|Y_n(\cdot, \xi)\|_{H^2(\Omega_2)} \leqq C\|Y_n(\cdot, \xi)\|_\infty^{\Omega_1},$$

again with the right hand side bounded for a.e. ξ. Thus a subsequence converges weakly in $H^2(\Omega_2)$. Thus we have that for a.e. ξ, $\Phi(x, s, \omega)$ is in $C^\theta(\Omega)\cap H^2(\Omega)$ for each bounded Ω. Since each Y_n is a weak solution of (1.7), the same is true of Φ. We have verified (a)—(e) and part of (f). The remainder of (f) follows from Lemma 3.16 of [19]. To prove (1.11) note that $P_{ac}=E_1(e')$. Thus

$$(P_{ac}u, v) = (F_\pm u, F_\pm v)_{L^2(I, \mathscr{C})}.$$

Let v have compact support and substitute for $F_\pm v$ using (1.9). Interchanging the order of integration gives (1.11). (1.12) follows from (1.10). (h) was proved in [19]. To prove (1.14) note that

$$F_\pm f(H_1)W_\pm = F_\pm W_\pm f(H_0) = Ff(H_0) = f(s)F$$

by (1.13) and the intertwining relations (cf. [19]). Thus (1.14) holds on $L^2_{ac}=R(W_\pm)$ by (1.13). Moreover, both sides of (1.14) vanish on (L^2_{ac}) . This proves (i). (j) was proved in [20]. ∎

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Received September 22, 1975

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