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Radial rearrangement, harmonic measures and extensions of Beurling's shove theorem

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Abstract. Let I be a union of finitely many closed intervals in $[-1,0)$. Let $I^{\leftarrow}$ be a single interval of the form $[-1,-a]$ chosen to have the same logarithmic length as I. Let $\mathbf{D}$ be the unit disc. Then, Beurling [8] has shown that the harmonic measure of the circle $\partial\mathbf{D}$ at the origin in the slit disc $\mathbf{D}\backslash I$ is increased if I is replaced by $I^{\leftarrow}$. We prove a number of cognate results and extensions. For instance, we show that Beurling's result remains true if the intervals in I are not just one-dimensional, but if they in fact constitute polar rectangles centred on the negative real axis and having some fixed constant angular width. In doing this, we obtain a new proof of Beurling's result. We also discuss a conjecture of Matheson and Pruss [25] and some other open problems.

1. Beurling's shove theorem and introduction

Let I be a union of finitely many closed intervals in $[-1,0)$. Write

$$\varrho(I)=\int_{-1}^{0}\frac{1_I(x)}{|x|}\,dx,$$

for the *logarithmic length* of I, where 1_I is the indicator function of I. Let $I^{\leftarrow}$ be the interval $[-1,-e^{-\varrho(I)}]$. Note that $\varrho(I^{\leftarrow})=\varrho(I)$. The effect of transforming I into $I^{\leftarrow}$ is to dilate all the intervals of I together until they touch the point -1 and then to individually dilate each interval until it touches the interval to the left of it, proceeding until a single interval is formed.

Given a domain D and a Borel set $A\subseteq\partial D$, write $\omega(z,A;D)$ for the harmonic measure of A at z in D. This is the value at z of the solution to the Dirichlet problem on D with boundary value 1 on A and 0 outside A. Equivalently, $\omega(z,A;D)$ is the

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probability that a Brownian motion, started at z and stopped as soon as it leaves D, first hits the set A before hitting any other part of the boundary of D.

Beurling in his thesis [8] has shown that if $\mathbf{D}$ is the unit disc, while $\mathbf{T}=\partial\mathbf{D}$ and I is a union of finitely many closed intervals in $[-1,0)$, then

$$\omega(0,\mathbf{D}\backslash I;\mathbf{T})\leq\omega(0,\mathbf{D}\backslash I^{\leftarrow};\mathbf{T}). \tag{1.1}$$

This result can be called Beurling's shove theorem. The name "shove theorem" was suggested by Albert Baernstein II because of the way that the intervals in I are "shoved" to the left to form $I^{\leftarrow}$. An account of the theorem may be found in Beurling's original thesis [8, pp. 58–62] or in Nevanlinna's book [26, §IV.5.4]. Essén and Haliste [15] have generalized Beurling's theorem to higher dimensions and have also managed to replace the unit disc $\mathbf{D}$ in (1.1) by some more general domains. Pruss [31] has extended Beurling's theorem to some even more general domains, but only in two dimensions.

The main purpose of the present paper is to discuss extensions of Beurling's shove theorem, all the time working in two dimensions. For instance, we will show that Beurling's theorem extends in a natural way to the case where I consists of a finite union of polar rectangles centred on the negative real axis and having some fixed constant angular width (Theorem 2.1, below). We shall formulate a number of conjectures, discuss their relation to questions of Sakai [32] and Matheson and Pruss [25], and present some evidence for the conjectures.

The reader who would like to know more about symmetrization methods may wish to examine the work of Baernstein [3], [5] as well as [7], [12], [17], [19], [27], [28]. Further results on some of the topics in the present paper can also be found in [31].

We shall use the terms "increasing", "decreasing", "positive", etc., in their non-strict senses, i.e., "non-decreasing", "non-increasing", "non-negative", etc., respectively. We write $\mathbf{D}(r)=\{z\in\mathbf{C}:|z|<r\}$ and put $\mathbf{T}(r)=\{z\in\mathbf{C}:|z|=r\}$. Note that if $r>0$ then $\mathbf{T}(r)=\partial\mathbf{D}(r)$, while $\mathbf{T}(0)=\{0\}$. We then define $\mathbf{D}=\mathbf{D}(1)$ and $\mathbf{T}=\mathbf{T}(1)$.

Henceforth all domains are assumed to be connected open Greenian subsets of $\mathbf{C}$, i.e., connected open sets on which there exist nonconstant positive superharmonic functions. In fact, in the cases which interest us most, the domains will have finite area, and such domains are well known to be automatically Greenian. Likewise, domains which are simply connected proper subsets of $\mathbf{C}$ are automatically Greenian.

2. Marcus' radial rearrangement and some results

Let D be a domain in the plane containing the origin and choose $\varepsilon>0$ such that $\mathbf{D}(\varepsilon)\subseteq D$. Define

$$R_\varepsilon(\theta; D)=\int_\varepsilon^\infty 1_{\{\varrho e^{i\theta}\in D\}}\varrho^{-1}\,d\varrho,$$

where $1_{\{P\}}$ is 1 when P is true and is 0 when P is false. The quantity $R_\varepsilon(\theta; D)$ is then the *logarithmic length* of $\{\varrho\in[\varepsilon,\infty):\varrho e^{i\theta}\in D\}$. Note that logarithmic length is dilation invariant in the sense that $R_{\lambda\varepsilon}(\theta;\lambda D)=R_\varepsilon(\theta; D)$ for all $\lambda>0$.

The set

$$D^\star=\left\{re^{i\theta}:\int_\varepsilon^r \varrho^{-1}\,d\varrho<R_\varepsilon(\theta; D), r\geq 0\right\}$$

is then called the (Marcus) *radial rearrangement* of D [22]. Note that $\mathbf{D}(\varepsilon)\subseteq D^\star$ and that if D is open, then $\theta\mapsto R_\varepsilon(\theta; D)$ is lower semicontinuous and hence $D^\star$ is open. It is easy to verify that $D^\star$ is independent of the choice of ε (subject to the constraint $\mathbf{D}(\varepsilon)\subseteq D$) and that $\mathrm{Area}(D^\star)\leq\mathrm{Area}(D)$ with equality if and only if $D^\star$ and D coincide almost everywhere with respect to Lebesgue area measure.

A domain D containing the origin is said to be *star-shaped* if for every $z\in D$, the line segment joining z with 0 lies inside D. Note that $D^\star$ is always star-shaped and that $D^\star=D$ if and only if D is star-shaped. A star-shaped domain is automatically simply connected and it is the simple connectivity of $D^\star$ which makes for much of its interest to us, since simply connected domains are often much easier to work with than multiply connected ones because of the Riemann mapping theorem.

Note that if I is a finite union of closed intervals as in Beurling's shove theorem, then

$$(\mathbf{D}\backslash I)^\star=\mathbf{D}\backslash I^{\leftarrow}.$$

It is this connection that lets us say that the transformation $D\mapsto D^\star$ is a generalization of Beurling's shoving operation $I\mapsto I^{\leftarrow}$.

Example 2.1. At first one might confidently conjecture that inequality (1.1) in Beurling's theorem generalizes to the inequality

$$\omega(0,\mathbf{T}\cap\overline{D}; D)\leq\omega(0,\mathbf{T}\cap\overline{D^\star}; D^\star), \tag{2.1}$$

whenever D is a domain containing the origin and contained in $\mathbf{D}$. However, this conjecture would be false. Let $D\subseteq\mathbf{D}$ be any domain with the following properties for some positive δ, where ε is such that $\mathbf{D}(\varepsilon)\subseteq D$:

(i) the harmonic measure of $\mathbf{T}\cap\overline{D}$ at zero in D does not vanish;

(ii) $R_\varepsilon(\theta; D)\leq\int_\varepsilon^{1-\delta}\varrho^{-1}\,d\varrho$ for every θ.

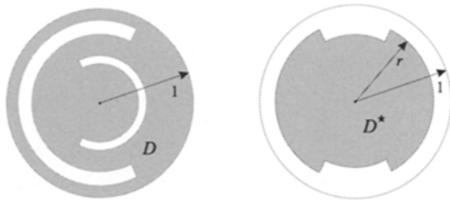

Figure 2.1. A multiply connected domain D for Example 2.1, together with its radial rearrangement $D^\star$.

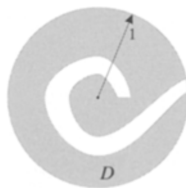

Figure 2.2. A simply connected domain for Example 2.1. The radial rearrangement of this domain will be contained in the disc $\mathbf{D}(r)$ for some $r \in (0,1)$.

Such a domain can easily be exhibited; see Figure 2.1 (left) for a multiply-connected example, and Figure 2.2 for a simply-connected example.

Given such a domain, (ii) implies that we will have $D^\star \subseteq \mathbf{D}(r)$, where $r = 1-\delta$ (see Figure 2.1, right), and hence

$$\omega(0, \mathbf{T} \cap \overline{D^\star}; D^\star) = 0.$$

But (i) guarantees that $\omega(0, \mathbf{T} \cap \overline{D}; D) > 0$, and so (2.1) cannot be true.

On the other hand, we can obtain a few positive results. Recall that a polar rectangle of angular width θ_0 centred on the negative real axis is a set of the form

$$\{re^{i\theta} : r_1 \le r \le r_2,\ |\theta - \pi| \le \tfrac{1}{2}\theta_0\},$$

where $0 \le r_1 \le r_2 < \infty$. Given any domain D, we write

$$w_r(D) = \omega(0, \mathbf{T}(r) \cap \overline{D}; \mathbf{D}(r) \cap D).$$

Note that if $D \subseteq \mathbf{D}$, then $w_1(D)$ coincides with $\omega(0, \mathbf{T} \cap \overline{D}; D)$, a quantity that we have already studied.

The following result then generalizes Beurling's shove theorem. In fact, the shove theorem is the special case $\theta_0 = 0$ of this result.

Theorem 2.1. *Fix $\theta_0 \in [0, 2\pi)$. Let H be a finite union of polar rectangles of angular width θ_0 centred on the negative real axis. Put $D = \mathbf{D} \setminus H$. Then,*

$$w_1(D) \le w_1(D^\star).$$

A more general result will be given as Corollary 7.1 in Section 7.2, below, and will provide a new proof of Beurling's shove theorem. Note that according to the referee, the weaker inequality $w_1(D) \le (\text{const.}) w_1(D^\star)$ under the conditions of Theorem 2.1 follows from [13].

The following result is not a generalization of Beurling's shove theorem because Beurling's theorem does not appear to follow from it, but it is very much in the same spirit.

Theorem 2.2. *Let D be a simply connected domain which is reflection symmetric about the real axis and which contains the interval $[0, R)$. Then,*

$$w_r(D) \le w_r(D^\star),$$

whenever $0 < r \le R$.

The proof will be given in Section 8.1 as a consequence of a more general result, and will proceed by the method of Haliste [17, Proof of Theorem 4.1].

3. Least harmonic majorants

Let $\mathcal{F}$ be the collection of all continuous functions $\Phi: [0, \infty) \to [-\infty, \infty)$ for which $t \mapsto \Phi(e^t)$ is convex and where $\Phi(x) > -\infty$ for all $x > 0$. (Note that the convexity and continuity conditions together with the requirement that $\Phi(0) < \infty$ imply that Φ is increasing.) Then, $\mathcal{F}$ coincides with the collection of all functions Φ on $[0, \infty)$ for which $z \mapsto \Phi(|z|)$ is subharmonic on $\mathbf{C}$.

Given $\Phi \in \mathcal{F}$, and given a domain D, let h be the least harmonic majorant of $z \mapsto \Phi(|z|)$ on D. Put

$$\Gamma_\Phi(D) = h(0).$$

An alternate description of Γ_Φ is that it is the value at 0 of the solution to the Dirichlet problem in D with boundary values $\Phi(|z|)$. These kinds of functionals were studied by, e.g., Burkholder [9], Essén, Haliste, Lewis and Shea [16], Sakai [32], and Essén [14].

We then have the following result of the same type as the shove theorem.

Theorem 3.1. *Fix $\theta_0 \in [0, 2\pi)$. Let H be a finite union of polar rectangles of angular width θ_0 centred on the negative real axis. Put $D = \mathbf{D} \setminus H$. Then,*

$$\Gamma_\Phi(D) \leq \Gamma_\Phi(D^\star).$$

While formally this is apparently new even if $\theta_0 = 0$, in that case it can be proved by the methods used in Beurling's original proof [8, pp. 58–62] (see also [26, §IV.5.4]) of his shove theorem. See also [21] for a closely related result. Theorem 3.1 follows from the more general Corollary 7.1 in Section 7.2, below.

Example 2.1 can also be used to show that $\Gamma_\Phi(D)$ can sometimes be strictly larger than $\Gamma_\Phi(D^\star)$. (In the setting of that example, just let $\Phi(t) = \max(0, t - r)$ and note that $\Phi(r) = 0$ so that $\Gamma_\Phi(D^\star) = 0$ while $\Gamma_\Phi(D) > 0$ since $\Phi(1) > 0$.)

4. Circular symmetry and some conjectures

Let D be a domain in the plane. Fix $r \in [0, \infty)$ and if $\mathbf{T}(r) \subset D$, then let $\theta(r; D) = \infty$, while otherwise let $\theta(r; D) = |\{\theta \in [0, 2\pi) : re^{i\theta} \in D\}|$, where $|\cdot|$ indicates Lebesgue measure on $\mathbf{R}$. Let

$$D^\circledcirc = \{re^{i\theta} : |\theta| < \tfrac{1}{2}\theta(r; D)\}$$

be the *circular symmetrization* of D. Note that $\mathrm{Area}(D^\circledcirc) = \mathrm{Area}(D)$. Furthermore,

$$\Gamma_\Phi(D) \leq \Gamma_\Phi(D^\circledcirc) \tag{4.1}$$

for every $\Phi \in \mathcal{F}$; this can be proved directly by mirroring the proof of Baernstein [3, Theorem 7], though it is also a consequence of [3, Theorem 5] together with the well-known Theorem 6.1, below. Baernstein has also proved [3, Theorem 7] that

$$w_r(D) \leq w_r(D^\circledcirc) \tag{4.2}$$

for all $r > 0$.

We say that a domain D is *circularly symmetric* if $D^\circledcirc = D$. Note that if D is circularly symmetric then so is $D^\star$ though the converse does not in general hold. It is easy to see that no domain $D \subseteq \mathbf{D}$ with $\mathbf{D}(\varepsilon) \subseteq D$ and satisfying (i) and (ii) of Example 2.1 can be circularly symmetric. The domains $\mathbf{D} \setminus H$ in Theorems 2.1 and 3.1 are circularly symmetric. Any bounded circularly symmetric simply connected domain containing the origin satisfies the hypotheses of Theorem 2.2 with $R = \sup\{x > 0 : x \in D\}$. In light of these observations, we make the following two conjectures.

Conjecture 4.1. *Confined to the class of circularly symmetric domains, radial rearrangement increases the functionals* Γ_Φ *for* $\Phi\in\mathcal{F}$*, i.e.,* $\Gamma_\Phi(D)\le\Gamma_\Phi(D^\star)$ *whenever* D *is a circularly symmetric domain containing the origin and* $\Phi\in\mathcal{F}$*.*

Conjecture 4.2. *Let* D *be a circularly symmetric domain with* $0\in D$*. Then* $w_r(D)\le w_r(D^\star)$ *for every* $r>0$*.*

Remark 4.1. In fact, to prove Conjecture 4.2 in general, it suffices to prove that $w_r(D)\le w_r(D^\star)$ whenever D is a circularly symmetric domain with $0\in D$ and $D\subseteq\mathbf{D}(r)$. To see this, assume that we do have $w_r(D)\le w_r(D^\star)$ under these circumstances, and suppose that D is now a general circularly symmetric domain containing the origin. If we define $U_r=\mathbf{D}(r)\cap U$ for any domain U, then $(D_r)^\star\subseteq D^\star\cap\mathbf{D}(r)=(D^\star)_r$ and so

$$w_r(D)=w_r(D_r)\le w_r((D_r)^\star)\le w_r(D^\star),$$

since $D_r\subseteq\mathbf{D}(r)$, and where we have used the definition of w_r and the maximum principle.

In fact, as will be seen in Section 6, Conjecture 4.2 is a special case of Conjecture 4.1. Unfortunately, both conjectures remain open, but they do provide prototypes for the kinds of results that we are looking for. In light of (4.1) and (4.2), Conjectures 4.1 and 4.2 would also imply the inequalities

$$\Gamma_\Phi(D)\le\Gamma_\Phi((D^{\circledcirc})^\star)\quad\text{for all }\Phi\in\mathcal{F},\tag{4.3}$$

and

$$w_r(D)\le w_r((D^{\circledcirc})^\star)\quad\text{for all }r>0,\tag{4.4}$$

respectively, for any domain D containing the origin, whether circularly symmetric or not.

Finally, we wish to indicate a modified version of Conjecture 4.2 which is false. Given a domain D and $r>0$, let

$$v_r(D)=\omega(0,\partial D\backslash\mathbf{D}(r);D).\tag{4.5}$$

In general $v_r(D)\le w_r(D)$ by the maximum principle. If $D\subseteq\mathbf{D}(r)$ then $v_r(D)=w_r(D)$. If $D\not\subseteq\mathbf{D}(r)$ and $w_r(D)>0$, then in fact $v_r(D)<w_r(D)$. One might conjecture that

$$v_r(D)\le v_r(D^\star)\tag{4.6}$$

for all circularly symmetric domains D and all $r>0$. As a piece of positive evidence one might cite the fact that this inequality is indeed true for all $r>0$ if $D=\mathbf{D}\setminus H$ where H is a polar rectangle (this follows from Lemma 8.1 in Section 8.2, below). To see that (4.6) *cannot* be true for all circularly symmetric domains D, let

$$D=\mathbf{D}\setminus\big[\big(-1,-\tfrac{1}{3}\big]\cup\mathbf{T}\big(\tfrac{1}{2}\big)\big].$$

Evidently D is circularly symmetric and $D^\star=\mathbf{D}\setminus\big(-1,-\frac{1}{3}\big]$. Set $r=\frac{1}{2}$. Clearly,

$$v_{1/2}(D)=\omega\big(0,\mathbf{T}\big(\tfrac{1}{2}\big);\mathbf{D}\big(\tfrac{1}{2}\big)\setminus\big(-\tfrac{1}{2},-\tfrac{1}{3}\big]\big).$$

Define

$$\alpha(z)=\omega\big(z,\mathbf{T}\big(\tfrac{1}{2}\big);\mathbf{D}\big(\tfrac{1}{2}\big)\setminus\big(-\tfrac{1}{2},-\tfrac{1}{3}\big]\big),$$

and

$$\beta(z)=\omega\big(z,\partial D^\star\setminus\mathbf{D}\big(\tfrac{1}{2}\big);\mathbf{D}\setminus\big(-1,-\tfrac{1}{3}\big]\big).$$

Then, it is easy to see that $\beta(z)<1$ for every $z\in\mathbf{T}\big(\frac{1}{2}\big)\setminus\big\{-\frac{1}{2}\big\}$ and that $\beta(z)=0$ for every $z\in\big(-\frac{1}{2},-\frac{1}{3}\big]$. On the other hand $\alpha(z)=1$ for each $z\in\mathbf{T}\big(\frac{1}{2}\big)$ and again $\alpha(z)=0$ for $z\in\big(-\frac{1}{2},-\frac{1}{3}\big]$. Hence the maximum principle applied in the domain $\mathbf{D}\big(\frac{1}{2}\big)\setminus\big(-\frac{1}{2},-\frac{1}{3}\big]$ implies that $\alpha(z)>\beta(z)$ for every z in this domain. In particular $v_{1/2}(D)=\alpha(0)>\beta(0)=v_{1/2}(D^\star)$. Of course, one might say that this is not really a counterexample to (4.6) since D is not connected and domains are taken to be connected in this paper. But we can make D connected! Fix a small $\varepsilon>0$. Let

$$D_\varepsilon=\mathbf{D}\setminus\big[\big(-1,-\tfrac{1}{3}\big]\cup\big\{\tfrac{1}{2}e^{i\theta}:\varepsilon\le\theta\le 2\pi-\varepsilon\big\}\big].$$

The D_ε are connected (in fact, also simply connected) and circularly symmetric for every positive ε. But as $\varepsilon\to 0$, we have $v_{1/2}(D_\varepsilon)\to v_{1/2}(D)$, and certainly $D_\varepsilon^\star=\mathbf{D}\setminus\big(-1,-\frac{1}{3}\big]=D^\star$, so that for sufficiently small ε we have $v_{1/2}(D_\varepsilon)>v_{1/2}(D^\star)=v_{1/2}(D_\varepsilon^\star)$ and we truly have a counterexample to (4.6).

5. Consequences of conjectures

Conjectures 4.1 and 4.2 would have a number of interesting consequences because they would allow one to reduce some questions about arbitrary domains to questions about simply connected domains.

Let $\mathfrak{B}$ be the unit ball of the Dirichlet space, i.e., let $\mathfrak{B}$ be the set of all holomorphic functions f on $\mathbf{D}$ with $f(0)=0$ for which

$$\frac{1}{\pi}\iint_{\mathbf{D}}|f'(x+iy)|^2\,dx\,dy\le 1.$$

Note that this inequality is equivalent to asserting that the area of the image of f, *counting multiplicities*, does not exceed π. If f is univalent and $f(0)=0$, then it follows that $f\in\mathfrak{B}$ if and only if $\operatorname{Area} f[\mathbf{D}]\leq\pi$. Given a measurable function Φ on $[0,\infty)$, let

$$\Lambda_\Phi(f)=\frac{1}{2\pi}\int_0^{2\pi}\Phi(|f(e^{i\theta})|)\,d\theta$$

for $f\in\mathfrak{B}$. For some concrete functions Φ, these Λ_Φ functionals on $\mathfrak{B}$ have been studied to various degrees by Beurling [8], Chang and Marshall [10] (see also [23]), Andreev and Matheson [2], and Cima and Matheson [11]. For more general classes of functions Φ, the functionals have been studied by Matheson and Pruss [25].

Remark 5.1. A crucial and well-known result (Essén [14] gives it in a concrete case) connecting the Γ_Φ and Λ_Φ functionals is that if $\Phi\in\mathcal{F}$, then

$$\Gamma_\Phi(f[\mathbf{D}])\geq\Lambda_\Phi(f), \tag{5.1}$$

and that equality holds if f is univalent (see the discussion following Open Problem 1 in [25]).

Now recall the following conjecture.

Conjecture 5.1. (Matheson and Pruss [25]) *For any domain D containing the origin and having finite area there exists a star-shaped domain $\widetilde{D}$ with* $\operatorname{Area}\widetilde{D}\leq\operatorname{Area}D$ *such that $\Gamma_\Phi(\widetilde{D})\geq\Gamma_\Phi(D)$ for every $\Phi\in\mathcal{F}$.*

Letting $\widetilde{D}=(D^{\circledcirc})^\star$ (which is always simply connected and satisfies the inequality $\operatorname{Area}[(D^{\circledcirc})^\star]\leq\operatorname{Area}D$), based on the discussion around the conjectural inequality (4.3) we see that Conjecture 5.1 is a special case of Conjecture 4.1. A weaker conjecture but still very much of interest would be as follows.

Conjecture 5.2. *Given a domain D containing the origin and having finite area and given a function $\Phi\in\mathcal{F}$, there exists a simply connected $\widetilde{D}$ with* $\operatorname{Area}\widetilde{D}\leq\operatorname{Area}D$ *and $\Gamma_\Phi(\widetilde{D})\geq\Gamma_\Phi(D)$.*

This conjecture is still open, too. It is weaker first of all because any star-shaped domain is necessarily simply connected (but the converse is obviously false), and secondly because the domain $\widetilde{D}$ is allowed to depend on Φ.

A slightly stronger version of Conjecture 5.2 would be to require this $\widetilde{D}$ to also be star-shaped (but continuing to allow it to depend on Φ). There is actually some indirect evidence for this slightly stronger version. Star-shapedness is the same thing as radial convexity for a domain containing the origin, where we say that a domain D is *radially convex* if the line segment joining z with w lies inside D whenever z and w are in D and lie on the same ray from the origin. Note that Γ_Φ was defined as the value at 0 of the least harmonic majorant of a subharmonic function $\Phi(|z|)$ which changes only radially. Suppose we wanted to look at least harmonic majorants of subharmonic functions which change only horizontally. These last would be functions of the form $\phi(\operatorname{Re} z)$ for a convex ϕ on $\mathbf{R}$. We could then define $\Gamma_\phi^{\mathrm{horiz}}(D)$ to be the value at 0 of the least harmonic majorant of $\phi(\operatorname{Re} z)$ on D. The analogue of our slightly stronger version of Conjecture 5.2 would be that for a domain D of finite area and any given convex ϕ on $\mathbf{R}$, there exists a horizontally convex domain $\widetilde{D}$ of no bigger area than D and such that $\Gamma_\phi^{\mathrm{horiz}}(\widetilde{D})\geq\Gamma_\phi^{\mathrm{horiz}}(D)$. In fact this analogue is true [30, Theorem 3.1]. (Here, a domain D is said to be *horizontally convex* if given any two points z and w in D which lie on the same horizontal line, the line segment joining z and w is contained in D.) The correspondence between Γ_Φ and radial convexity on the one hand and $\Gamma_\phi^{\mathrm{horiz}}$ and horizontal convexity on the other hand then provides some evidence for our slightly stronger version of Conjecture 5.2, and in a very indirect way for the much stronger Conjecture 4.1.

Let $\mathcal{B}$ be the set of all domains of area at most π which contain 0.

Proposition 5.1. *Suppose that Conjecture* 5.2 *holds for some* $\Phi\in\mathcal{F}$ *such that* $\Phi(t)=o(e^{t^2})$, *as* $t\to\infty$. *Then* Γ_Φ *attains its supremum over* $\mathcal{B}$. *Moreover, there exists a simply connected extremal domain in* $\mathcal{B}$ *at which* Γ_Φ *is maximized.*

Proof. Applying the Chang–Marshall inequality [10] as well as [25, Corollary 2] with $\Psi(t)=e^{t^2}$ and the methods of [25, Section 1], it follows that there exists an $f\in\mathfrak{B}$ such that $\Lambda_\Phi(f)\geq\Lambda_\Phi(g)$ for all $g\in\mathfrak{B}$. Let $U=f[\mathbf{D}]$. Let D be an arbitrary domain in $\mathcal{B}$. Then by Remark 5.1 and Conjecture 5.2 we have

$$\Gamma_\Phi(D)\leq\Gamma_\Phi(\widetilde{D})=\Lambda_\Phi(g),$$

where g is a Riemann map from $\mathbf{D}$ onto $\widetilde{D}$ with $g(0)=0$. But Area $\widetilde{D}\leq\pi$ so that $g\in\mathfrak{B}$ (using the univalence of g) and hence $\Lambda_\Phi(g)\leq\Lambda_\Phi(f)$. By (5.1) we then have $\Lambda_\Phi(f)\leq\Gamma_\Phi(U)$, and so

$$\Gamma_\Phi(D)\leq\Lambda_\Phi(g)\leq\Lambda_\Phi(f)\leq\Gamma_\Phi(U)$$

for all $D\in\mathcal{B}$. Hence Γ_Φ attains its maximum over $\mathcal{B}$ at U. Moreover, if it attains this maximum at U, it likewise attains it at $\widetilde{U}$ and hence there exists an extremal simply connected domain. $\square$

Sakai [32] had conjectured that Γ_{Φ_p} attains a maximum over $\mathcal{B}$ where $\Phi_p(t)=t^p$ for $0<p<\infty$. Hence, an affirmative answer to Conjecture 5.2 implies an answer to Sakai's conjecture.

Proposition 5.2. *Let $\Phi_p(t)=t^p$. Then*

$$\Gamma_{\Phi_p}(D) \le \Gamma_{\Phi_p}(\mathbf{D})$$

for every $D\in\mathcal{B}$ providing $p\in[0,2]$. If Conjecture 5.2 *holds for Φ_p then this inequality is also true for $p\in(2,4]$.*

Proof. The case of $p\in[0,2]$ is the well-known Alexander–Taylor–Ullman inequality [1] (see Kobayashi [20] for another proof). (More precisely, the Alexander–Taylor–Ullman inequality is the case $p=2$, and, as Sakai [32] notes, the case $p<2$ follows from Hölder's inequality.)

The case $p\in(2,4]$ follows from Conjecture 5.2 and the fact that the inequality is valid for simply connected domains D. To see the validity for simply connected domains, it suffices to use Remark 5.1 and the fact that $\Lambda_{\Phi_p}(f)\le\Lambda_{\Phi_p}(\mathrm{Id})$ for $p\in[0,4]$, where Id is the identity function on $\mathbf{D}$ and f is any function in $\mathfrak{B}$. This latter inequality has been proved by Matheson [24]. Professor Makoto Sakai has kindly informed the author that the desired inequality in the simply connected case was also obtained by Professors N. Suita and S. Kobayashi. $\square$

The inequality in the above proposition was conjectured for $p\in(2,4]$ by Sakai [32]. Hence, an affirmative answer to Conjecture 5.2 would imply an affirmative answer to yet another conjecture of Sakai. It would also simplify the proof of the Alexander–Taylor–Ullman inequality, since the inequality $\Lambda_{\Phi_p}(f)\le\Lambda_{\Phi_p}(\mathrm{Id})$ for $f\in\mathfrak{B}$ is quite easy to prove for $p\in[0,2]$ using the fact that $\mathfrak{B}$ consists precisely of the functions of the form $f(z)=\sum_{n=1}^{\infty} a_n z^n$ where $\sum_{n=1}^{\infty} n|a_n|^2\le 1$.

Remark 5.2. If $p>4$ then there *do* exist simply connected domains $D\in\mathcal{B}$ with $\Gamma_{\Phi_p}(D)>\Gamma_{\Phi_p}(\mathbf{D})$ [32, Proposition 7.4].

Finally, we note that if Conjecture 5.2 were to hold, then the Essén inequality [14]

$$\sup_{D\in\mathcal{B}} \Gamma_\Phi(D) < \infty,$$

where $\Phi(t)=e^{t^2}$, would follow from the Chang–Marshall inequality [10] (see also Marshall [23]) which says that

$$\sup_{f\in\mathfrak{B}} \Lambda_\Phi(f) < \infty,$$

also with $\Phi(t)=e^{t^2}$. Indeed, it would follow from the Chang–Marshall inequality for univalent functions f. Since the proof of the Chang–Marshall inequality given by Marshall [23] is simpler than the proof of Essén's inequality [14] and since the proof in [23] itself simplifies in the univalent case, we see that an affirmative answer to Conjecture 5.2 would imply simpler proofs of the Essén and Chang–Marshall inequalities.

Actually, Conjecture 4.2, despite being weaker than Conjecture 4.1, is also interesting in connection with the Essén inequality, because Essén's proof [14] would also be greatly simplified if one could prove that for any domain D there is a simply connected domain $\widetilde{D}$ such that $w_r(\widetilde{D})\geq w_r(D)$ for all $r>0$ and $\operatorname{Area}\widetilde{D}\leq\operatorname{Area}D$.

6. Green's functions and least harmonic majorants

Given a (Greenian) domain D, let $g(z,w;D)$ be Green's function for D. Then, for fixed $w\in D$, the function $z\mapsto g(z,w;D)$ is harmonic on $D\setminus\{w\}$ and the function $z\mapsto g(z,w;D)-\log(1/|z-w|)$ is harmonic in a neighbourhood of w. Moreover, $g(z,w;D)$ vanishes unless both z and w are in D. Now we would like to note a well-known result which was alluded to before.

Theorem 6.1. *For any pair of domains D and D', the following are equivalent:*

(a) *For every $\Phi\in\mathcal{F}$ we have $\Gamma_\Phi(D')\geq\Gamma_\Phi(D)$.*

(b) *For every $r>0$ we have*

$$\int_0^{2\pi} g(re^{i\theta},0;D')\,d\theta\geq\int_0^{2\pi} g(re^{i\theta},0;D)\,d\theta.$$

Proof. The Riesz decomposition theorem for subharmonic functions (see, e.g., [18, Theorem 5.25]) implies that if h_D is the least harmonic majorant of $\Phi(|z|)$ on D where $\Phi\in\mathcal{F}$, then

$$h_D(w)=\Phi(|w|)+\iint_{\mathbf{C}} g(z,w;D)\,d\mu_\Phi(z),$$

where μ_Φ is the Riesz measure corresponding to the subharmonic function $z\mapsto\Phi(|z|)$. But μ_Φ is rotation invariant since $\Phi(|z|)$ is rotation invariant, and hence we may write $d\mu_\Phi(re^{i\theta})=d\theta\,d\nu_\Phi(r)$ for some measure ν_Φ on $[0,\infty)$. Then,

$$h_D(w)=\Phi(|w|)+\int_0^\infty\int_0^{2\pi} g(re^{i\theta},w;D)\,d\theta\,d\nu_\Phi(r).$$

Thus,

$$\Gamma_\Phi(D)=\Phi(0)+\int_0^\infty\int_0^{2\pi} g(re^{i\theta},0;D)\,d\theta\,d\nu_\Phi(r). \tag{6.1}$$

In exactly the same way we see that

$$\Gamma_\Phi(D')=\Phi(0)+\int_0^\infty\int_0^{2\pi} g(re^{i\theta},0;D')\,d\theta\,d\nu_\Phi(r). \tag{6.2}$$

Since ν_Φ is a positive measure, it follows from (6.1) and (6.2) that if (b) holds then (a) must likewise hold.

Conversely, if (a) holds then we may fix r and let $\Phi(t)=\max(0,\log t-\log r)$. Then, the support of μ_Φ lies on the circle $\mathbf{T}(r)$ because $\mu_\Phi(z)=(2\pi)^{-1}\Delta\Phi(|z|)$ on any open set on which $z\mapsto\Phi(|z|)$ lies in C^2 while our choice of Φ satisfies $\Delta\Phi(|z|)=0$ for $z\in\mathbf{C}\setminus\mathbf{T}(r)$. Therefore ν_Φ is a measure concentrated at the one point r so that

$$\Gamma_\Phi(D)=c\int_0^{2\pi} g(re^{i\theta},0;D)\,d\theta,$$

where $c=\nu_\Phi(\{r\})>0$. The same expression holds with D' in place of D and so (a) implies (b) as desired. $\square$

It might be the case that in the setting of Conjecture 4.1 it would be easier to prove condition (b) of Theorem 6.1 with $D'=D^\star$ than to prove condition (a). Nonetheless, Conjecture 4.1 remains open, even given this reformulation.

Remark 6.1. To see that Conjecture 4.1 implies Conjecture 4.2, note first that by Remark 4.1 we may assume that $D\subseteq\mathbf{D}(r)$. Then, harmonic measures on the boundary of a domain correspond to normal derivatives of Green's functions, so that (at least if $D\subseteq\mathbf{D}(r)$ is sufficiently nice)

$$\frac{2\pi}{r}w_r(D)=-\int_0^{2\pi}\frac{\partial g(re^{i\theta},0;D)}{\partial r}\,d\theta=\lim_{r'\to r-}(r-r')^{-1}\int_0^{2\pi} g(r'e^{i\theta},0;D)\,d\theta, \tag{6.3}$$

as noted in [4, p. 146]. Now if Conjecture 4.2 holds, then by Theorem 6.1 we have an inequality between the right-hand side of (6.3) and the same right-hand side but with D replaced by $D^\star$, so that the desired inequality $w_r(D)\le w_r(D^\star)$ follows.

Assuming $D\subseteq\mathbf{D}(r)$, in light of (6.3), Conjecture 4.2 is in effect an inequality between $g(\,\cdot\,,0;D)$ and $g(\,\cdot\,,0;D^\star)$ near $\mathbf{T}(r)$. Let us consider such inequalities near 0. Given a domain D containing the origin, its Green's function with pole at 0 can be written in the form

$$g(z,0;D)=\log\frac{1}{|z|}+\log\varrho+o(1), \quad \text{as } z\to 0,$$

where $\varrho=\varrho(D)$ is a constant known as the *inner radius* of D (about 0). Then, asymptotic inequalities between $g(\,\cdot\,,0;D)$ and $g(\,\cdot\,,0;D^\star)$ at the origin correspond to inequalities between $\varrho(D)$ and $\varrho(D^\star)$. Marcus [22, Theorem 3] had shown that for any domain D (not necessarily simply connected or circularly symmetric) we have

$$\varrho(D) \le \varrho(D^\star). \tag{6.4}$$

(See [31] for some generalizations of this fact.)

By Theorem 2.2, then, in the simply connected circularly symmetric case (assuming $D \subseteq \mathbf{D}(r)$) we know that we do have the correct inequality between $g(\,\cdot\,,0;D)$ and $g(\,\cdot\,,0;D^\star)$ near $\mathbf{T}(r)$ and near 0; for general domains we only know that we have it near 0.

7. The cutting procedure

7.1. Procedure and conjectures

Given a domain D and given numbers $0<r_1<r_2<\infty$, we now define a new domain $\operatorname{Cut}(D;r_1,r_2)$ which is obtained by cutting out the ring $\mathbf{D}(r_2)\setminus\overline{\mathbf{D}}(r_1)$ and then dilating $D\cap\mathbf{D}(r_1)$ by a factor of r_2/r_1 in order to fill up the gap. More precisely,

$$\operatorname{Cut}(D;r_1,r_2) = \operatorname{Int}[(D\setminus\mathbf{D}(r_2)) \cup (r_2/r_1)(D\cap\mathbf{D}(r_1))],$$

where $\operatorname{Int} S$ indicates the topological interior of a set $S\subseteq\mathbf{C}$ and where $\lambda S = \{\lambda z : z\in S\}$.

Remark 7.1. Using the dilation invariance of logarithmic length, it is not difficult to verify that in general

$$\operatorname{Cut}(D;r_1,r_2)^\star \supseteq D^\star.$$

Moreover, if the annulus $\mathbf{D}(r_2)\setminus\overline{\mathbf{D}}(r_1)$ is contained in D, then

$$\operatorname{Area}\operatorname{Cut}(D;r_1,r_2) \le \operatorname{Area} D$$

and

$$\operatorname{Cut}(D;r_1,r_2)^\star = D^\star.$$

Finally, it is easy to see that $\operatorname{Cut}(D;r_1,r_2)$ will be circularly symmetric whenever D is circularly symmetric.

In general the set $\operatorname{Cut}(D;r_1,r_2)$ may be disconnected even if D is connected. This will be the case, for instance, if D is as on the left-hand side of Figure 2.1, while r_1 is the outermost radius of the inner hole and r_2 is the innermost radius of the outer hole. In the same example we would have $w_r(\operatorname{Cut}(D;r_1,r_2))=0<w_r(D)$. However, if D is circularly symmetric and connected, then it easily follows that $\operatorname{Cut}(D;r_1,r_2)$ must be connected.

Conjecture 7.1. *For any $\Phi \in \mathcal{F}$ and any $0<r_1<r_2<\infty$ we have*

$$\Gamma_\Phi(\mathrm{Cut}(D;r_1,r_2)) \geq \Gamma_\Phi(D),$$

if D is circularly symmetric.

Conjecture 7.2. *For any $0<r_1<r_2<\infty$ and any $r>0$ we have*

$$w_r(\mathrm{Cut}(D;r_1,r_2)) \geq w_r(D),$$

if D is circularly symmetric.

Remark 7.2. Conjecture 7.2 implies Conjecture 4.2. To see this, note that by approximation it suffices to prove Conjecture 4.2 for a bounded finitely connected and circularly symmetric domain D. Assume Conjecture 7.2 and proceed by induction on the number n of components in $\mathbf{C}\setminus D$. If $n=1$ then D is simply connected and we are done by Theorem 2.2. Suppose $n>1$ and that the result has been proved for $n-1$. Let

$$r_2 = |\inf\{x : x \in (-\infty, 0] \cap D\}|.$$

The infimum is finite since D is bounded. Because of circular symmetry, it is easy to see that $n-1$ is equal to the number of components of $(-r_2, 0]\setminus D$. Since $n>1$ it follows that $(-r_2, 0]$ cannot be contained inside D. Let

$$r_1 = |\inf\{x : x \notin D,\ -r_2 < x < 0\}|.$$

We then have $0<r_1<r_2<\infty$ and the interval $(-r_2,-r_1)$ is contained in D. Circular symmetry implies that the whole annulus $\mathbf{D}(r_2)\setminus\overline{\mathbf{D}}(r_1)$ must be contained in D so that

$$\mathrm{Cut}(D;r_1,r_2)^\star = D^\star, \tag{7.1}$$

by Remark 7.1. Moreover, $-r_1$ and $-r_2$ lie in the boundary of D. Let $D'=\mathrm{Cut}(D;r_1,r_2)$. The construction of D' then implies that the number of components of $\mathbf{C}\setminus D'$ is one less than the number of components of $\mathbf{C}\setminus D$. This is because the component containing $-r_1$ has been dilated so as to meet and touch the component containing $-r_2$.

Thus $\mathbf{C}\setminus D'$ has $n-1$ components, and so

$$w_r((D')^\star) \geq w_r(D') \tag{7.2}$$

by the induction hypothesis. If Conjecture 7.2 is true, then

$$w_r(D') \geq w_r(D). \tag{7.3}$$

Combining (7.1), (7.2) and (7.3) we see that $w_r(D^\star)\geq w_r(D)$ as desired.

Remark 7.3. Conjecture 7.1 implies Conjecture 5.2. The argument here is slightly involved because of a difficulty with handling domains D for which $D\cap(-\infty,0)$ is of fractal-type. However, if $\mathbf{C}\backslash D$ has finitely many components and D is connected, then the argument in Remark 7.2 shows that by a finite number of cutting operations we may reduce D to a simply connected domain D' with $(D')^\star=D$ and, as can be easily verified (cf. Remark 7.1), also with $\operatorname{Area} D'\leq\operatorname{Area} D$. We now prove our remark. Fix a domain D containing the origin, as in Conjecture 5.2. Without loss of generality, $\operatorname{Area} D=\pi$.

Let $\mathcal{B}$ as before denote all domains of area at most π which contain the origin. Let $\mathcal{S}$ be the subclass of all domains from $\mathcal{B}$ which are circularly symmetric and simply connected. Finally, let $\mathcal{B}_1$ be the collection of all domains U in $\mathcal{B}$ which are circularly symmetric and for which $\mathbf{C}\backslash U$ has finitely many components. Because circular symmetrization increases the Γ_Φ and because one may approximate an arbitrary circularly symmetric domain in $\mathcal{B}$ by ones in $\mathcal{B}_1$, we have

$$\sup_{U\in\mathcal{B}}\Gamma_\Phi(U)=\sup_{U\in\mathcal{B}_1}\Gamma_\Phi(U), \tag{7.4}$$

whenever $\Phi\in\mathcal{F}$. The above-mentioned construction of $D'\in\mathcal{S}$ given $D\in\mathcal{B}_1$ shows that

$$\sup_{U\in\mathcal{B}_1}\Gamma_\Phi(U)=\sup_{U\in\mathcal{S}}\Gamma_\Phi(U). \tag{7.5}$$

Moreover,

$$\sup_{U\in\mathcal{S}}\Gamma_\Phi(U)=\sup_{f\in\mathfrak{U}}\Lambda_\Phi(f), \tag{7.6}$$

by Remark 5.1, where $\mathfrak{U}$ consists of all univalent functions in $\mathfrak{B}$. If Φ is such that Λ_Φ attains its maximum over $\mathfrak{U}$, then we may suppose it attains this maximum at f and let $U=(f[\mathbf{D}])^{\circledcirc}$. Then, Γ_Φ attains its supremum over $\mathfrak{B}$ at U by (7.4)–(7.6) together with Remark 5.1 and (4.1), while U is simply connected since $f[\mathbf{D}]$ is. This is the desired conclusion. Note that Λ_Φ does attain its maximum over $\mathfrak{U}$ if $\Phi(t)=o(e^{t^2})$, as $t\to\infty$. This is so because in that case Λ_Φ is weak upper semicontinuous on $\mathfrak{B}$ by [25, Corollary 2] together with the Chang–Marshall inequality [10], while $\mathfrak{B}$ is weakly compact and $\mathfrak{U}\cup\{0\}$ is weakly closed by a normal families argument since weak convergence in $\mathfrak{B}$ implies uniform convergence on compact subsets of $\mathbf{D}$ (this last fact follows from a much more general result of Cima and Matheson [11]).

If $\Phi(t)$ is not of the form $o(e^{t^2})$ then a careful and somewhat involved approximation argument is needed. Such an argument is given in a rather similar setting in [30].

7.2. A partial result on the cutting procedure

We are able to prove the following partial result.

Theorem 7.1. *Fix* $0<r_1<r_2<\infty$. *Let* D *be a circularly symmetric domain such that* $\mathbf{D}(r_2)\cup D$ *is star-shaped. Then for any* $\Phi\in\mathcal{F}$ *we have*

$$\Gamma_\Phi(\mathrm{Cut}(D;r_1,r_2))\geq\Gamma_\Phi(D), \tag{7.7}$$

and for any $r>0$ *we have*

$$w_r(\mathrm{Cut}(D;r_1,r_2))\geq w_r(D). \tag{7.8}$$

The proof will be given in Section 8.2. We now note that Theorems 2.1 and 3.1 follow from Theorem 7.1. In fact, we shall inductively prove a somewhat more general result which is a corollary of Theorem 7.1.

Corollary 7.1. *Let* $H\subseteq\mathbf{D}$ *be a finite union of polar rectangles of fixed angular width* θ_0 *centred on the negative real axis. Let* $U\subseteq\mathbf{D}$ *be a circularly symmetric star-shaped domain with the property that there exists an* $r_0\in(0,1]$ *such that*

$$\mathbf{D}(r_0)\subseteq U \tag{7.9}$$

and

$$\{re^{i\theta}:r_0\leq r<1,\ |\theta-\pi|\leq\tfrac{1}{2}\theta_0\}\subseteq\mathbf{D}\setminus U. \tag{7.10}$$

Put $D=U\setminus H$. *Then for any* $\Phi\in\mathcal{F}$ *we have*

$$\Gamma_\Phi(D^\star)\geq\Gamma_\Phi(D), \tag{7.11}$$

and for any $r>0$ *we have*

$$w_r(D^\star)\geq w_r(D). \tag{7.12}$$

Letting $U=\mathbf{D}$ and $r_0=1$, Theorems 2.1 and 3.1 follow from this result. The proof of the corollary uses the methods of Remark 7.2.

Proof of Corollary 7.1. Without loss of generality we can assume $H\subseteq\overline{\mathbf{D}}(r_0)$, since otherwise we may replace H by $H\cap\overline{\mathbf{D}}(r_0)$ and this will not change $U\setminus H$ in light of (7.10).

Let n be the number of components in H. If $n=0$ then $D=U$ is star-shaped so that $D^\star=D$ and the result is trivial. Proceed by induction, assuming that $n>0$ and that the result has been proved for $n-1$. Let $r_2=r_0$. Let

$$r_1=|\inf\{x:x\in(-\infty,0]\cap H\}|.$$

This is finite as $n>0$. Then the annulus $\mathbf{D}(r_2)\setminus\overline{\mathbf{D}}(r_1)$ is contained in D so that

$$\mathrm{Cut}(D;r_1,r_2)^\star = D^\star \tag{7.13}$$

by Remark 7.1. Moreover, (7.7) and (7.8) hold by Theorem 7.1. Let H_1 be the component of H containing $-r_1$ and let $H_2=H\setminus H_1$. We then have

$$\mathrm{Cut}(D;r_1,r_2) = U\setminus\lambda H = U'\setminus\lambda H_2, \tag{7.14}$$

where $U'=U\setminus\lambda H_1$ and $\lambda=r_2/r_1$. Note that by (7.10), since H_1 is a polar rectangle of angular width θ_0 centred on the negative real axis and containing the point $-r_1$, it follows that U' is star-shaped. Moreover, if U satisfies (7.9) and (7.10), then U' will likewise also satisfy them with

$$r_0' \stackrel{\text{def}}{=} |\inf\{x : x\in(-\infty,0]\cap\lambda H_2\}|$$

in place of r_0. Moreover, λH_2 has $n-1$ components so that

$$\Gamma_\Phi(U'\setminus\lambda H_2) \le \Gamma_\Phi((U'\setminus\lambda H_2)^\star) = \Gamma_\Phi(\mathrm{Cut}(D;r_1,r_2)^\star) = \Gamma_\Phi(D^\star), \tag{7.15}$$

by the induction hypothesis and by (7.13) and (7.14), where $\Phi\in\mathcal{F}$. But by (7.7) and (7.14) we have

$$\Gamma_\Phi(D) \le \Gamma_\Phi(\mathrm{Cut}(D;r_1,r_2)) = \Gamma_\Phi(U'\setminus\lambda H_2).$$

By (7.15) we then obtain (7.11). The proof of (7.12) is analogous. □

8. Proofs and generalizations

8.1. The simply connected cases

Theorem 2.2 follows from the following more general result, the proof of which takes only a little more effort. Given $z\in\mathbf{C}$, write $[0,z)=\{tz:t\in[0,1)\}$.

Theorem 8.1. *Let D be a simply connected domain which is symmetric with respect to reflection about the real axis and satisfies $[\alpha,r)\subset D\subseteq\mathbf{D}(r)$ for some $\alpha\le 0$. Let C be a symmetric arc in $\mathbf{T}(r)\cap\partial D$, centred on the point r. Let*

$$C' = \{z\in C : [0,z)\subset D\}.$$

Assume C' is connected. Then

$$\omega(\alpha,C;D) \le \omega(\alpha,C';D^\star). \tag{8.1}$$

It is easy to see that $C' \subseteq \bar{D} \cap \partial D^\star$. Theorem 2.2 then follows by applying Theorem 8.1 to the domain $\mathbf{D}(r) \cap D$ with $C = \mathbf{T}(r) \cap \partial D$ and $\alpha = 0$ and by the reasoning of Remark 4.1.

It is easy to see that (8.1) need no longer hold if $\alpha > 0$. To see this, consider the circularly symmetric and simply connected domains

$$D_\varepsilon = \mathbf{D} \setminus \big[(-1, \tfrac{1}{2}] \cup \{re^{i\theta} : \tfrac{1}{4} \le r \le \tfrac{1}{2},\ \varepsilon \le \theta \le 2\pi - \varepsilon\}\big]$$

for $\varepsilon \in (0, \pi)$ and note that

$$D_\varepsilon^\star = \mathbf{D} \setminus \big[(-\tfrac{1}{2}, -\tfrac{1}{4}] \cup \{re^{i\theta} : \tfrac{1}{2} \le r < 1,\ \varepsilon \le \theta \le 2\pi - \varepsilon\}\big].$$

Fix $\alpha \in (\frac{1}{2}, 1)$. Then, it is easy to see that $\omega(\alpha, \mathbf{T} \cap \partial D_\varepsilon; D_\varepsilon)$ is bounded away from zero, as $\varepsilon \to 0$, while $\omega(\alpha, \mathbf{T} \cap \partial D_\varepsilon^\star; D_\varepsilon^\star) \to 0$, as $\varepsilon \to 0$, so that (8.1) will not hold if ε is sufficiently small.

Finally, note that C' will automatically be connected in Theorem 8.1 if D is circularly symmetric and simply connected.

Now we proceed to the proof of Theorem 8.1. First however we need some background results so that we can use a special case of a quite general result of Baernstein [5]. The reader interested in the many different kinds of rearrangements which all yield analogous results is referred to [5]. Let I be the interval $(-\pi, \pi]$, and let F be a positive Lipschitz function on $I \times \mathbf{R}$. Recall that the *Steiner symmetrization about the real axis* $S^\boxminus$ of a set $S \subseteq I \times \mathbf{R}$ was defined by

$$S^\boxminus = \{(x, y) : x \in I,\ |y| < \tfrac{1}{2} |\{t : (x, t) \in S\}|\},$$

where $|\{t : (x, t) \in S\}|$ indicates the Lebesgue measure of $\{t : (x, t) \in S\}$. Given a function F on $I \times \mathbf{R}$ we may define the *Steiner symmetrization* $F^\boxminus$ of F by

$$F^\boxminus(x, y) = \sup\{\lambda : (x, y) \in (F_\lambda)^\boxminus\},$$

where

$$F_\lambda = \{(x, y) : F(x, y) > \lambda\}.$$

Note that $F(x, \cdot)$ and $F^\boxminus(x, \cdot)$ are equimeasurable whenever $x \in I$ is fixed.

Then we have the following result.

Theorem 8.2. (Baernstein [5, Corollary 3]) *Let F be Lipschitz and positive on $I \times \mathbf{R}$ and assume that for every fixed $x \in I$ we have $F(x, y) \to 0$, as $y \to \pm\infty$. Then $F^\boxminus$ is also Lipschitz and for any convex increasing function Φ we have*

$$\iint_{I \times \mathbf{R}} \Phi(|\nabla F^\boxminus|) \le \iint_{I \times \mathbf{R}} \Phi(|\nabla F|).$$

The integrals in the theorem are taken with respect to Lebesgue area measure on $\mathbf{R}^2$. For $\Phi(t)=t^2$, they are known as *Dirichlet integrals.* These kinds of isoperimetric inequalities go back to Pólya and Szegő [27].

As a corollary, we obtain the following modified version of a result of Marcus [22, Theorem 1]. For a positive function f which is Lipschitz on $\bar{\mathbf{D}}$ and which satisfies $f(0)=0$, and for any $(x,y)\in I\times\mathbf{R}$, we define $F(x,y)=f(e^{-|y|+ix})$. It is easy to verify that this is Lipschitz on $I\times\mathbf{R}$. Then, identifying $\mathbf{R}^2$ with $\mathbf{C}$, let $\log z$ be the branch of the logarithm with $\operatorname{Im}\log z\in I$, and set $f_\star(z)=F^{\ominus}(-i\log z)$ for $z\in\mathbf{D}$. Then, $f_\star$ will be radially increasing on $\mathbf{D}$.

Theorem 8.3. *Let f be Lipschitz and positive on $\bar{\mathbf{D}}$ and assume that $f(0)=0$. Then $f_\star$ is Lipschitz on compact subsets of $\bar{\mathbf{D}}\setminus\{0\}$, satisfies $f_\star(0)=O(z)$, as $z\to 0$, and has*

$$\iint_{\mathbf{D}} |\nabla f_\star|^2 \le \iint_{\mathbf{D}} |\nabla f|^2.$$

This follows immediately from Theorem 8.2 with $\Phi(t)=t^2$ and from the well-known conformal invariance of Dirichlet integrals, where we use the conformal map $-i\log z$ from $\mathbf{D}\setminus(-1,0]$ onto the upper half of $I\times\mathbf{R}$, and then note that the Dirichlet integral for $F^{\ominus}$ over the lower half of $I\times\mathbf{R}$ is the same as that over the upper half of it. The only subtlety is with proving the Lipschitz character of $f_\star$. From Theorem 8.2 we find that $f_\star$ is Lipschitz on compact subsets of $\bar{\mathbf{D}}\setminus[-1,0]$. Rotational symmetry in the definition of $f_\star$ (i.e., applying the above to $f^\varphi(z)\stackrel{\text{def}}{=}f(e^{i\varphi}z)$ and noting that $(f^\varphi)_\star(z)=f_\star(e^{i\varphi}z)$) shows that in fact it must be Lipschitz on compact subsets of all of $\bar{\mathbf{D}}\setminus\{0\}$. Now $f(0)=0$ and so I claim that for every $r>0$ we have

$$\sup_{z\in\mathbf{D}(r)} f(z) \ge \sup_{z\in\mathbf{D}(r)} f_\star(z).$$

Thus $f_\star(z)=O(z)$, as $z\to 0$, since $f(z)=O(z)$, as $z\to 0$. The claim follows immediately from the fact that

$$\{z\in\mathbf{D}: f(z)<\lambda\}^\star = \{z\in\mathbf{D}: f_\star(z)<\lambda\} \tag{8.2}$$

for any $\lambda>0$. To prove (8.2) it suffices to show that equality holds when we intersect both sides with a ray starting from the origin, and to do this one needs to note that the logarithmic metric defining radial rearrangement precisely corresponds to the composition with $-i\log z$ in the definition of $f_\star$.

We now proceed to the proof of our Theorem 8.1 by the method of Haliste [17, Proof of Theorem 4.1].

Proof of Theorem 8.1. Without loss of generality set $r=1$. As usual, by an internal exhaustion like the one in [3, Proof of Theorem 7] we may assume that all our domains have nice boundaries. We shall assume for now that $\alpha=0$ and at the end of the proof we discuss the minor modifications necessary to take care of the case $\alpha<0$. We now use the method of Haliste [17]. Let U be any subdomain of $\mathbf{D}$ such that $[0,1)\subset U$ while U is symmetric under reflection in the real axis. Let E be a symmetric arc about 1 in $\mathbf{T}\cap\partial U$. Let f be the (unique) holomorphic map of U onto the disc $\mathbf{D}$ with $f(0)=0$ and $f'(0)>0$. Then $f(E)$ is a symmetric arc of $\mathbf{T}$ centred about 1, and its harmonic measure at 0 in $\mathbf{D}$ equals its normalized Lebesgue measure. By conformal invariance, this normalized harmonic measure also equals $\omega(0,E;U)$. Now, as in [17] (but for convenience with reversed boundary values so that Theorem 8.3 would work better), let $u=u_{E,U}$ be the solution of the following mixed Dirichlet–Neumann problem on $\mathbf{D}\backslash[0,1)$:

$$u(z)=0,\quad z\in[0,1), \tag{8.3a}$$

$$u(z)=1,\quad z\in\mathbf{T}\backslash f(E), \tag{8.3b}$$

$$\frac{d}{dn}u(z)=0,\quad z\in f(E), \tag{8.3c}$$

$$\Delta u(z)=0,\quad z\in\mathbf{D}\backslash[0,1), \tag{8.3d}$$

where d/dn denotes a normal derivative. Let

$$\phi(E,U)=\iint_{\mathbf{D}\backslash[0,1)}|\nabla u_{E,U}|^2.$$

Then, gluing two copies of $\mathbf{D}$ together along the arc $f(E)$ to form a Riemann surface, and applying the Dirichlet and maximum principles on it, we easily see that $\phi(E,U)$ must be strictly decreasing with respect to the length of the arc $f(E)$. But since the length of this arc is proportional to $\omega(0,E;U)$, there must be a strictly decreasing function $\psi\colon[0,\infty)\to[0,\infty)$ such that for all E and U as described above we have

$$\psi(\omega(0,E;U))=\phi(E,U). \tag{8.4}$$

Haliste [17, equation (3.6)] gives an explicit expression for ψ in terms of elliptic integrals. Now, by conformal invariance and the fact that f sends 0 to 0, E onto $f(E)$ and $[0,1)$ onto $[0,1)$ (the last assertion being due to the reflection symmetry of U), it follows that we may instead consider the function $s_{E,U}\stackrel{\text{def}}{=}u_{E,U}\circ f$ on U and we will have

$$\phi(E,U)=\iint_{U\backslash[0,1)}|\nabla s_{E,U}|^2, \tag{8.5}$$

while moreover the function $s=s_{E,U}$ will be the solution to the following mixed Dirichlet–Neumann problem on $U\backslash[0,1)$:

$$s(z)=0, \quad z\in[0,1), \tag{8.6a}$$

$$s(z)=1, \quad z\in\partial U\backslash E, \tag{8.6b}$$

$$\frac{d}{dn}s(z)=0, \quad z\in E, \tag{8.6c}$$

$$\Delta s(z)=0, \quad z\in U\backslash[0,1). \tag{8.6d}$$

The proof of Theorem 8.1 is now not very difficult. Take $s=s_{C,D}$ with the above definition and set $s(z)=1$ for $z\in\overline{\mathbf{D}}\backslash D$. Since D has a nice boundary, s has no problem with satisfying the conditions of Theorem 8.3. Hence,

$$\iint_{\mathbf{D}} |\nabla s_\star|^2 \leq \iint_{\mathbf{D}} |\nabla s|^2. \tag{8.7}$$

Now, it is easy to verify that s and $s_\star$ are identically 1 in $\overline{\mathbf{D}}\backslash\overline{D}$ and $\overline{\mathbf{D}}\backslash\overline{D^\star}$, respectively; for $s_\star$ this follows from (8.2) with $\lambda=1$, together with the fact that $\{z\in\mathbf{D}:s(z)<1\}=D$. Thus the integrands in (8.7) are supported on $\overline{D}$ and $\overline{D^\star}$, respectively, so that

$$\iint_{D^\star} |\nabla(s_{C,D})_\star|^2 \leq \iint_{D} |\nabla s_{C,D}|^2, \tag{8.8}$$

since we have a nice boundary which thus has Lebesgue measure zero. Furthermore, if D is symmetric about the real axis and simply connected, then so is $D^\star$, while $s_\star$ is identically 1 on $\partial D^\star\backslash C'$ since for $z\in C'$ we have $s_\star(z)=\max_{[0,z)} s=1$. Clearly, too, $s_\star$ is identically 0 on $[0,1)$. Hence $s_\star=(s_{C,D})_\star$ satisfies the two Dirichlet boundary conditions which would be imposed on $s_{C',D^\star}$ by (8.6a) and (8.6b) (with $U=D^\star$ and $E=C'$), though in general it will fail to satisfy the Neumann condition (8.6c) and the harmonicity condition (8.6d). Then, it follows by the Dirichlet principle with free boundary values that

$$\iint_{D^\star} |\nabla s_{C',D^\star}|^2 \leq \iint_{D} |\nabla(s_{C,D})_\star|^2,$$

which combined with (8.5) and (8.8) yields

$$\phi(C',D^\star)\leq\phi(C,D).$$

By (8.4) it follows that

$$\omega(0,C';D^\star)\geq\omega(0,C;D),$$

since ψ is strictly decreasing. This completes the proof in the case of $\alpha=0$.

If $\alpha<0$ then we proceed just as above, the main difference being that the map f, instead of taking 0 to 0, is now required to take α to 0 (note that the condition $f'(0)>0$ is equivalent to the condition $f'(\alpha)>0$ since f is to be univalent and our domains are symmetric under reflection in the real axis). Then, instead of considering the solution $s=s_{E,U}$ to (8.6a)–(8.6d), we now consider the solution $s=s_{\alpha,E,U}$ to the mixed Dirichlet–Neumann problem (8.6a′)–(8.6d′) obtained from (8.6a)–(8.6d) by replacing $[0,1)$ in (8.6a) and (8.6d) by $[\alpha,1)$. The rest of the proof goes through. For, we still have

$$\psi(\omega(\alpha,E;U))=\iint_{U\setminus[0,1)}|\nabla s_{\alpha,E,U}|^2,$$

with exactly the same function ψ as before. Moreover, if $s=s_{\alpha,E,D}$, then $s_\star$ satisfies the two Dirichlet boundary conditions that would be imposed on $s_{\alpha,E,D^\star}$ by (8.6a′) and (8.6b), where (8.6a′) is (8.6a) with $[0,1)$ replaced by $[\alpha,1)$. The reader waiting to see where the assumption $\alpha\le 0$ is used may be pleased to note that it is used precisely in the assertion that $s_\star$ satisfies (8.6a′) for the rearranged domain $U=D^\star$. $\square$

8.2. The partial result on the cutting procedure

Here we shall supply the proof of Theorem 7.1. Our proof depends on the following elementary lemma together with a result of Baernstein [6]. Recall that v_r was defined by (4.5).

Lemma 8.1. *Let U be a star-shaped domain with $\mathbf{D}(r)\subseteq U$. Assume that H is a closed set in $\overline{U}$ and that $\lambda\ge 1$ is such that $\lambda H\subseteq\overline{U}$. Then,*

$$v_r(U\setminus H)\le v_r(U\setminus\lambda H).$$

Proof. Put $D=U\setminus H$ and $D'=U\setminus\lambda H$. Conformal invariance implies that

$$v_r(D)=v_{\lambda r}(\lambda D). \tag{8.9}$$

It is also evident by the maximum principle that

$$v_{\lambda r}(\lambda D)\le v_r(\lambda D), \tag{8.10}$$

if $\lambda\ge 1$.

Let

$$h(z)=\omega(z,\partial(\lambda D)\backslash\mathbf{D}(r);\lambda D)$$

and put

$$H(z)=\omega(z,\partial D'\backslash\mathbf{D}(r);D').$$

I claim that $h(z)\le H(z)$ for every $z\in D'$. Now $D'\subseteq\lambda D$ as D is star-shaped. Thus, to prove the claim, by the maximum principle it suffices to show that $h(z)\le H(z)$ whenever $z\in\partial D'$. So fix $z\in\partial D'$. If $z\in\partial D'\backslash\mathbf{D}(r)$ then $H(z)=1$ while $h\le 1$ everywhere so we are done. Suppose now that $z\in\partial D'\cap\mathbf{D}(r)$. We have $H(z)=0$ then. Since $\mathbf{D}(r)\subseteq U$ and $z\in\mathbf{D}(r)$, it follows that z does not lie in the boundary of U. But it does lie in the boundary of $D'=U\backslash\lambda H$ and hence it must lie in the boundary of λH. But if $z\in\partial(\lambda H)$ then $z\in\partial(\lambda D)\cap\mathbf{D}(r)$ so that $h(z)=0\le 0=H(z)$ as desired. Thus, indeed $h(z)\le H(z)$ whenever $z\in D'$, and in particular when $z=0$. Hence,

$$v_r(\lambda D)=h(0)\le H(0)=v_r(D').$$

Combining this with (8.9) and (8.10), we see that the proof is complete. □

We now need the following generalization of the Baernstein inequality $\Gamma_\Phi(D)\le\Gamma_\Phi(D^{\circledcirc})$.

Proposition 8.1. (cf. Baernstein [6, Theorem 1]) *Let D and D' be domains in the plane containing the origin. Suppose that D' is circularly symmetric. Assume that whenever $r\in(0,\infty)$, then at least one of the following two conditions holds*:

$$D^{\circledcirc}\cap\mathbf{T}(r)\subseteq D' \tag{8.11}$$

or

$$v_r(D)\le v_r(D'). \tag{8.12}$$

Then

$$\Gamma_\Phi(D)\le\Gamma_\Phi(D')$$

for every $\Phi\in\mathcal{F}$.

If we had made the auxiliary assumption that $A(D)\subseteq A(D')$ where $A(U)=\{r\in(0,\infty):\mathbf{T}(r)\in U\}$ for a domain U, then this would be a direct consequence of Baernstein's [6, Theorem 1] together with our Theorem 6.1. However, while [6, Theorem 1] is stated under the additional assumption that $A(D)\subseteq A(D')$, Baernstein's proof works just as well without the assumption that $A(D)\subseteq A(D')$. Thus Proposition 8.1 is true. (Note, however, that Baernstein phrases the conditions somewhat differently since his version of (8.11) is rephrased in a way that uses the assumption $A(D)\subseteq A(D')$.)

Proof of Theorem 7.1. First note that if (7.7) holds for all $\Phi\in\mathcal{F}$, then (7.8) also holds. To see this, use Theorem 6.1 and the reasoning in Remark 6.1.

We now prove (7.7). Let $D'=\mathrm{Cut}(D;r_1,r_2)$. Assume without loss of generality that $\mathbf{D}(r_2)\setminus\bar{\mathbf{D}}(r_1)\subset D$, since if not then we may replace D by $D\cup(\mathbf{D}(r_2)\setminus\bar{\mathbf{D}}(r_1))$ which will only increase $\Gamma_\Phi(D)$, but which will leave $\mathrm{Cut}(D;r_1,r_2)$ unchanged.

We shall show that the conditions of Proposition 8.1 are satisfied, and (7.7) will immediately follow from the conclusion of that proposition.

Evidently, D' is circularly symmetric. Fix $r\in(0,\infty)$. Let $U=D\cup\mathbf{D}(r_2)$. Let $H=\bar{\mathbf{D}}(r_2)\setminus D$. Then, $D=U\setminus H$. Moreover, our assumptions imply that U is star-shaped. Since $\mathbf{D}(r_2)\setminus\bar{\mathbf{D}}(r_1)\subset D$, we have $D'=U\setminus\lambda H$, where $\lambda=r_2/r_1$. Hence if $r\le r_2$, then (8.12) follows from Lemma 8.1. On the other hand, if $r>r_2$ then $\mathbf{T}(r)\cap D=\mathbf{T}(r)\cap U=\mathbf{T}(r)\cap D'$ and condition (8.11) follows. Hence, the conditions of Proposition 8.1 are indeed satisfied and the proof is complete. $\square$

Appendix: Cylinders, Brownian motion and random walks

Suppose $D\subseteq\mathbf{D}$ is a domain with $0\in D$. Let $\mathbf{V}^-$ be the half-cylinder $(-\infty,0]\times\mathbf{T}$. Let f be the map of $\mathbf{D}\setminus\{0\}$ onto the half-cylinder $\mathbf{V}^-$, with f defined by $f(re^{i\theta})=(\log r,e^{i\theta})$ for $r>0$ and $\theta\in\mathbf{R}$. Then, $f[D\setminus\{0\}]$ is an open subset of $\mathbf{V}^-$, which we will denote by $f[D]$, by a slight abuse of notation. Since f is a conformal map, harmonic measure on subsets of $\mathbf{V}^-$ corresponds to harmonic measure on subsets of $\bar{\mathbf{D}}$. In particular,

$$w_1(D)=\lim_{u\to\infty}\omega((-u,1),\{0\}\times\mathbf{T};f[D]).$$

Thus, $w_1(D)$ is equal to the limit as $u\to\infty$ of the probability that a Brownian motion starting at $(-u,1)\in\mathbf{V}^-$ eventually arrives at some point of the ring $\{0\}\times\mathbf{T}$ before touching any other part of the complement of $f[D]$. We will refer to a Brownian motion starting at $(-u,1)\in\mathbf{V}^-$ with the limit $u\to\infty$ being taken as "a Brownian motion started at $-\infty$".

Observe that

$$f[D^\star]=\{(t,e^{i\theta})\in\mathbf{V}^-:t<-|\{s\in(-\infty,0]:se^{i\theta}\notin f[D]\}|\},$$

where $|\cdot|$ is Lebesgue measure on $\mathbf{R}$. (The naturalness of the right-hand side here provides a justification for the use of the logarithmic metric in the original definition of $D^\star$.)

If D is circularly symmetric, then $f[D^{\star}]$ may be thought of as $f[D]$ with the cross-sections $(\{t\}\times\mathbf{T})\cap f[D]$ (for $-\infty<t<0$) re-sorted left-to-right in decreasing order of one-dimensional measure. Then, Conjecture 4.2 (for $D\subseteq\mathbf{D}$ and $r=1$, which by scaling and Remark 4.1 is the only case that need be considered) is equivalent to the conjecture that if D is circularly symmetric then this sorting of cross-sections $(\{t\}\times\mathbf{T})\cap f[D]$ in decreasing order of one-dimensional measure will increase the probability that a Brownian motion started at $-\infty$ arrives in $\{0\}\times\mathbf{T}$ before touching any other part of the complement of $f[D]$.

The one-dimensional measure of $(\{t\}\times\mathbf{T})\cap f[D]$ can be very roughly intuitively thought of as a measure of the likelihood that a Brownian particle crossing the ring $\{t\}\times\mathbf{T}$ does not touch the complement of $f[D]$. Hence we can take these measures to be analogous to the survival probabilities s_n of the random walk in [29], and [29, Theorem 1] will be analogous to Conjecture 4.2, and will indeed intuitively provide some support for our conjecture.

For further discussion of related matters, see [31].

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